

# Symmetric Algebraic Circuits and Homomorphism Polynomials

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## Abstract

The central open question of algebraic complexity is whether  $VP \neq VNP$ , which is saying that the permanent cannot be represented by families of polynomial-size algebraic circuits. For symmetric algebraic circuits, this has been confirmed by Dawar and Wilsenach (2020), who showed exponential lower bounds on the size of symmetric circuits for the permanent. In this work, we set out to develop a more general symmetric algebraic complexity theory. Our main result is that a family of symmetric polynomials admits small symmetric circuits if and only if they can be written as a linear combination of homomorphism counting polynomials of graphs of bounded treewidth. We also establish a relationship between the symmetric complexity of subgraph counting polynomials and the vertex cover number of the pattern graph. As a concrete example, we examine the symmetric complexity of immanant families (a generalisation of the determinant and permanent) and show that a known conditional dichotomy due to Curticapean (2021) holds unconditionally in the symmetric setting.

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## 1 Introduction

The study of *algebraic circuit complexity* (also called *arithmetic circuit complexity*) aims to understand the power of circuits to succinctly express (or compute) polynomials. In short, we are interested in establishing how many operations of addition and multiplication are needed in a circuit that computes a polynomial  $p \in \mathbb{F}[\mathcal{X}]$ , for some field  $\mathbb{F}$  and set of variables  $\mathcal{X}$ . We are usually interested in how this complexity grows with  $n$  for a family of polynomials  $(p_n)_{n \in \mathbb{N}}$ . The central conjecture in the field (known as  $VP \neq VNP$ ) is that there are no polynomial-size circuits for the *permanent* in the way that there are for the *determinant*.



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Both the determinant and permanent are examples of polynomials over matrices. That is to say that we can treat the set of variables  $\mathcal{X}$  as the entries  $x_{ij}$  of a square matrix. Moreover, the permanent is *symmetric* in the sense that permuting the rows and columns of the matrix does not change the polynomial. The determinant has a smaller group of symmetries, being invariant under a permutation applied *simultaneously* to the rows and columns. This motivates the study of *symmetric algebraic circuits* introduced in [15], which are circuits where symmetries of the polynomial computed are reflected in the automorphisms of the circuits. In [15], an exponential gap between the symmetric circuit complexity of the determinant and the permanent was established: There are polynomial-size symmetric circuits (in the sense of invariance under simultaneous row and column permutations) that compute the determinant but any family of symmetric circuits computing the permanent is of exponential size. It is the latter, exponential lower bound, that is the main technical achievement of that paper. Notably, this result also opens up a new approach to the question of separating VNP from VP and challenges us to find the minimum group of symmetries for a polynomial for which we can prove unconditional lower bounds. Thus, the study of symmetric circuits is motivated by the possibility of proving strong lower bounds which are presently out of reach without the assumption of symmetry.

Among the restrictions of circuit models that have been extensively studied in the literature, symmetry is arguably one of the newest and most interesting ones: Symmetric circuits not only admit provable lower bounds but are still surprisingly powerful, in that they can efficiently compute the determinant. By contrast, for example the restricted model of *multilinear formulas* requires super-polynomial size for both determinant and permanent [34], and the same is true for *bounded-depth multilinear circuits* [35]. Also for the well-studied class of *monotone* circuits, we have exponential lower bounds for the permanent [25], but the determinant cannot even be expressed in this model because it involves negative coefficients.

In the present paper, we approach the study of symmetric algebraic circuits more systematically than has been done before. We develop a framework that links the expressive power of such circuits to the theory of graph *homomorphism counting*. Using this, we are able to completely characterise – within a large and interesting class of polynomials – which polynomials admit efficient symmetric circuits and which ones do not. This characterisation is the main result of the paper.

To be more precise, the polynomials we consider are over (not necessarily square) matrices of variables, have rational coefficients and are invariant under arbitrary permutations of the rows and columns. These are families  $(p_{n,m})_{n,m \in \mathbb{N}}$  of polynomials where  $p_{n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$ , and  $\mathcal{X}_{n,m}$  is the set of variables  $\{x_{ij} \mid i \in [n], j \in [m]\}$ . The invariance condition we require is that for any pair of permutations  $\pi \in \mathbf{Sym}_n$ ,  $\sigma \in \mathbf{Sym}_m$ , if  $p_{n,m}^{(\pi,\sigma)}$  denotes the polynomial obtained from  $p_{n,m}$  by replacing every occurrence of  $x_{ij}$  with  $x_{\pi(i)\sigma(j)}$ , then  $p_{n,m} = p_{n,m}^{(\pi,\sigma)}$ . Examples of such families are the permanent, for  $m = n$ , or more generally, for  $m \geq n$ , the *rectangular* permanent. The latter has been used recently in a lower bound proof for bounded-depth algebraic circuits [21].

The reason for focussing on this class of *matrix-symmetric* polynomials is that they have a very natural semantic interpretation: Every  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric  $p_{n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$  describes a function from weighted bipartite undirected graphs  $G$  with  $(n, m)$  vertices (i.e.  $G$  has a fixed bipartition  $A \uplus B$  with  $|A| = n$  and  $|B| = m$ ) to  $\mathbb{Q}$ . The resulting value  $p_{n,m}(G) \in \mathbb{Q}$  is the evaluation of  $p_{n,m}$  in the bi-adjacency matrix of  $G$ , whose rows are indexed with the vertices in  $A$ , and columns with the vertices in  $B$ . The symmetry of  $p_{n,m}$  ensures that the value  $p_{n,m}(G)$  does not depend on the row and column ordering of the bi-adjacency matrix, and hence the  $\mathbb{Q}$ -valued function expressed by  $p_{n,m}$  is in fact an isomorphism-invariant graph parameter.

It follows from a classical result of Lovász [30] that the isomorphism-invariant  $\mathbb{Q}$ -valued functions on  $n$ -vertex graphs are precisely the linear combinations of homomorphism counting functions  $\sum \alpha_F \text{hom}(F, -)$  for finitely many graphs  $F$  and rational coefficients  $\alpha_F$ . In other words, the isomorphism-invariant  $\mathbb{Q}$ -valued functions on graphs are precisely the *non-uniform graph motif parameters*, i.e. sequences  $(\sum \alpha_{F,n} \text{hom}(F, -))_{n \in \mathbb{N}}$  of linear combinations of homomorphism counts, one for every input graph size  $n$ . In an influential paper [6], Curticapean, Dell, and Marx studied functions which can be uniformly expressed in this way. Their *uniform graph motif parameters* are the functions of the form  $\sum \alpha_F \text{hom}(F, -)$  whose domain are all graphs regardless of their size. They show that the fixed-parameter counting complexity of such uniform graph motif parameters is governed by the treewidth of the patterns  $F$  with non-zero coefficients  $\alpha_F$ .

**Characterising matrix-symmetric polynomials admitting small symmetric circuits.** We broaden the scope of [6] twofold by precisely characterising the symmetric circuit complexity of all, i.e. not necessarily uniform, graph motif parameters. To that end, we observe that the matrix-symmetric polynomials  $p \in \mathbb{Q}[\mathcal{X}_{n,m}]$  are precisely those that can be written as linear combination of homomorphism polynomials, cf. Lemma 4: For a bipartite multigraph  $F$  with bipartition  $A \uplus B$ , the *homomorphism polynomial*  $\text{hom}_{F,n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$  is defined as

$$\text{hom}_{F,n,m} := \sum_{h: A \uplus B \rightarrow [n] \uplus [m]} \prod_{ab \in E(F)} x_{h(a)h(b)}.$$

The name is justified by the fact that  $\text{hom}_{F,n,m}$  evaluates to the number of homomorphisms from  $F$  to a  $(n, m)$ -vertex bipartite graph  $G$  when substituting the bi-adjacency matrix of  $G$  for the matrix of variables  $\mathcal{X}_{n,m}$ .

Our main result asserts that the symmetric complexity of a family  $(p_{n,m})$  of matrix-symmetric polynomials is completely governed by the treewidth of the graphs whose homomorphism polynomials arise in linear expansions<sup>1</sup> of  $(p_{n,m})$ . By symmetric complexity we mean the smallest possible *orbit size* of a family  $(C_{n,m})_{n,m \in \mathbb{N}}$  of symmetric circuits representing  $(p_{n,m})_{n,m \in \mathbb{N}}$ . This is the number of gates of  $C_{n,m}$  in the largest orbit of the action of  $\mathbf{Sym}_n \times \mathbf{Sym}_m$  on  $C_{n,m}$  (see Section 2). Note that lower bounds on orbit size imply lower bounds on the total number of gates in a circuit but not vice versa. We also show that orbit size, not total size, is the correct measure that allows us to build our theory, cf. Theorem 5.

To state our main result, let  $\mathfrak{T}_{n,m}^k$  denote, for every  $k, n, m \in \mathbb{N}$ , the collection of all polynomials that can be obtained as linear combinations of homomorphism polynomials  $\text{hom}_{F,n,m}$  for graphs  $F$  of treewidth less than  $k$ .

- **Theorem 1.** *For every family of polynomials  $p_{n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$ , the following are equivalent:*
1. *there exists a constant  $k \in \mathbb{N}$  such that  $p_{n,m} \in \mathfrak{T}_{n,m}^k$  for all  $n, m \in \mathbb{N}$ ,*
  2. *the  $p_{n,m}$  admit  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of orbit size polynomial in  $n + m$ .*

It is instructive to consider again the permanent  $\text{perm}_n = \sum_{\pi \in \mathbf{Sym}_n} \prod_{i \in [n]} x_{i\pi(i)}$ . This is not a homomorphism polynomial and not even a uniform graph motif parameter but it can be seen as counting, given a bipartite graph  $G$  instantiated as a bi-adjacency matrix  $\mathcal{X}_{n,n}$ , the number of subgraphs isomorphic to an  $n$ -matching. By Möbius inversion [12, Corollary A.3], subgraph counts can be written as a linear combination of homomorphism

<sup>1</sup> In stark contrast to uniform graph motif parameters studied by [6], non-uniform graph motif parameters generally do not admit a unique expansion as linear combinations of homomorphism counts.

counts. One consequence of Theorem 1 together with the exponential lower bound on symmetric circuits for the permanent [15] is that  $\text{perm}_n$  cannot be expressed as a linear combination of homomorphism polynomials of bounded treewidth. Alternatively, we can understand one direction of Theorem 1 as a vast (in fact, as vast as possible) generalisation of the lower bound on the permanent. Indeed, the theorem captures *all* super-polynomial lower bounds on the orbit size of symmetric circuits for matrix-symmetric polynomials.

**Lower bounds via counting width.** Theorem 1 stipulates that, to derive a super-polynomial lower bound for some family  $p_{n,m}$ , one must show that there is no constant  $k \in \mathbb{N}$  such that  $p_{n,m} \in \mathfrak{T}_{n,m}^k$  for all  $n, m \in \mathbb{N}$ . This, however, seems to be difficult with the presently available techniques. Therefore, our next goal is to give more usable characterisations of  $\mathfrak{T}_{n,m}^k$  via a descriptive complexity measure for graph parameters, namely in terms of counting width. The *counting width* [13, 10] of a graph parameter  $p$  is defined as the smallest  $k \in \mathbb{N}$  such that whenever two graphs  $G, H$  are indistinguishable by the  $k$ -dimensional Weisfeiler-Leman algorithm, then  $p(G) = p(H)$ , see also Definition 6 or [12, Definition 6.1]. The Weisfeiler-Leman indistinguishability on graphs is a standard relaxation of graph isomorphism (see [4, 27]), and coincides with equivalence in the  $(k + 1)$ -variable fragment of first-order logic with counting quantifiers.

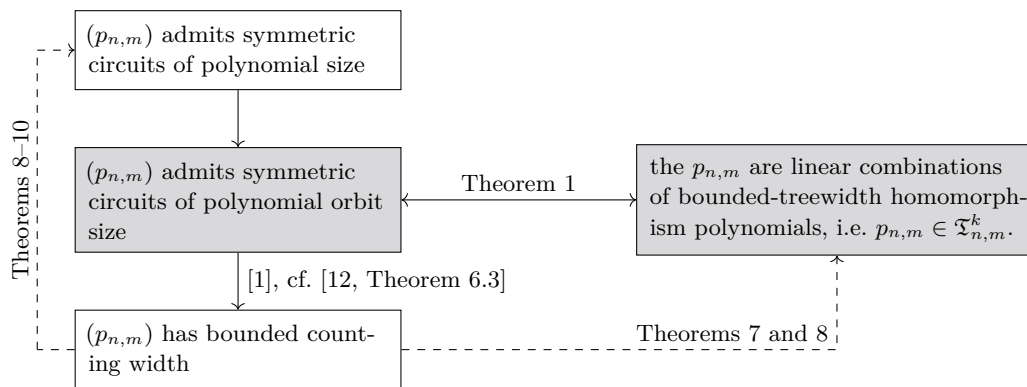
Counting width is well-studied in finite model theory and affords lower bounds, usually via the so-called Cai-Fürer-Immerman construction [4]. For Boolean circuits, a tight relationship between the counting width and orbit size of symmetric circuits is known [1].

The aforementioned exponential lower bound for symmetric circuits for the permanent was in fact established via a counting width lower bound for the number of perfect matchings. In general, unbounded counting width implies super-polynomial symmetric circuit lower bounds (via techniques for example from [1, 15]), so by Theorem 1, polynomials with unbounded counting width cannot be in  $\mathfrak{T}_{n,m}^k$ , for any fixed  $k$ . The question is, does the converse hold: Is bounded counting width a sufficient criterion for the existence of orbit-small symmetric circuits? We answer this question positively, at least in certain restricted, but interesting cases, cf. Figure 1:

1. when the  $p_{n,m}$  are single homomorphism polynomials (rather than linear combinations of such), cf. Theorem 9,
2. when the  $p_{n,m}$  are single subgraph polynomials for graphs of sublinear size, i.e. counting the number of subgraphs isomorphic to the pattern, cf. Theorem 10, and
3. when the  $p_{n,m}$  are linear combinations of homomorphism polynomials of sublinear size, cf. Theorem 8.

Moreover, in the case of Items 1 and 2 and when Item 3 is restricted to linear combinations of polynomially many homomorphism counts, bounded counting width characterises the families of matrix-symmetric polynomials admitting symmetric circuits of polynomial size (rather than orbit size). For Items 1 and 2, we give a combinatorial characterisation of the families of patterns whose homomorphism/subgraph polynomials have bounded counting width in terms of treewidth and vertex cover number, respectively.

Finally, to give another concrete example for the merits of the symmetric circuit framework, we show that a complexity dichotomy for the *immanant* families due to [5], whose hard cases are conditional on certain complexity-theoretic assumptions, holds unconditionally for symmetric circuits. The immanants interpolate between the permanent and determinant polynomials, which are in fact the extreme cases. Thus, the symmetric dichotomy for the immanants completes the picture begun in [15].



■ **Figure 1** Implications between properties of families of matrix-symmetric polynomials  $(p_{n,m})$ . The dashed implications hold under additional assumptions.

**Related work.** **Symmetric circuits** are an emerging area of research in recent years. We have already mentioned the lower bound for symmetric circuits for the permanent from [15]. In a similar vein, it has been shown that the determinant admits no small  $\text{Alt}_n \times \text{Alt}_n$ -symmetric circuits, even though it does have polynomial size  $\text{Sym}_n$ -symmetric ones [17]. Other known lower bounds are for example for the symmetric formula complexity of computing the product of permutation matrices [24], and a lower bound for certain types of symmetric  $\text{AC}^0$  circuits for the parity function [37]. More recent work establishes lower bounds for symmetric bounded-depth circuits with modular counting gates [26]. In [1], a precise characterisation of the power of uniform polynomial-size symmetric Boolean circuits with threshold gates was given: Such circuit families are equally powerful as fixed-point logic with counting, a formalism that is also related to the Weisfeiler-Leman algorithm. In particular, the graph properties computable with these circuits have bounded counting width, too. Similarly, in [16], rank logic is characterised via a class of symmetric circuits with more involved types of gates.

Another direction that has been explored are symmetric algebraic circuits in the context of algebraic proof complexity, more specifically in the circuit-based *Ideal Proof System* [11]. We hope that this direction will benefit from the framework we develop here, possibly leading to new lower bounds in proof complexity.

Another aspect of our work are the **homomorphism polynomials**. Interestingly, their complexity has been studied before with respect to a different restricted model, namely *monotone* circuits. In [28, 3] it is shown that the monotone circuit complexity of homomorphism polynomials is controlled by the treewidth of the pattern graphs in a similar way as we show this for symmetric circuits (and moreover, the circuit depth relates to the depth of the tree decompositions). It should be mentioned, though, that our setting is more general in two ways: We consider *linear combinations* of homomorphism polynomials, rather than single ones, and our set-up is non-uniform, while in [28, 3], the pattern graph is taken to be the same fixed graph for all sizes of host graphs. Certain types of homomorphism polynomials (defined differently from the ones we study) have also been shown to be  $\text{VP}$ -complete [19].

Our results also compare nicely to what is known on the *computational* complexity of **homomorphism** and **subgraph counting**. As already mentioned, it was shown in [6] that the computational complexity of graph parameters expressible as linear combinations of homomorphism counts (so-called *graph motif parameters*) depends only on the treewidth of the pattern graphs. The problem is in  $\text{FPT}$  if the treewidth is bounded, and  $\#\text{W}[1]$ -hard, otherwise. Our Theorem 1 yields the same tractability criterion for the symmetric circuit complexity of the polynomials expressing these parameters but our proof techniques are very

different. Moreover, as already mentioned, [6] concerns only uniform graph motif parameters, whereas we also cover the non-uniform case where we are counting different patterns for each host graph size.

The relationship between the complexity of subgraph counting and the *vertex cover number* of the pattern graphs that we obtain in Theorem 10 also appears in [6]. In [33, Theorem 1.3], a tight connection between the counting width of subgraph counts (for a fixed constant-size pattern) and the vertex cover number of the pattern is established. Our Theorem 10 generalises this result to the case of sublinearly growing patterns.

## 2 Preliminaries

**Permutation groups.** Let  $\Gamma$  be a group acting on a set  $X$ . For  $S \subseteq X$ , let  $\mathbf{Stab}_\Gamma(S) := \{\pi \in \Gamma \mid \pi(S) = S\}$  be the *stabiliser* of  $S$  and let  $\mathbf{Stab}_\Gamma^\bullet(S) := \{\pi \in \Gamma \mid \pi(s) = s \text{ for every } s \in S\}$  be the *pointwise stabiliser* of  $S$ . Let  $\Delta \leq \Gamma$  be a subgroup. A set  $S \subseteq X$  is a *support* of  $\Delta$  if  $\mathbf{Stab}_\Gamma^\bullet(S) \leq \Delta$ . For  $\Gamma = \mathbf{Sym}_n$  or  $\Gamma = \mathbf{Sym}_n \times \mathbf{Sym}_m$  it is known that many subgroups  $\Delta \leq \Gamma$  have a *unique* minimal support, which we denote  $\text{sup}(\Delta)$ , see [12, Lemma 4.1] and [10].

**Symmetric circuits.** An *algebraic circuit* over a variable set  $\mathcal{X}$  and a field  $\mathbb{F}$  is a connected DAG with multiedges such that each vertex is labelled with an element of  $\mathcal{X} \cup \mathbb{F} \cup \{+, \times\}$ . The vertices of a circuit are called *gates*, the edges *wires*. Gates labelled with elements from  $\mathcal{X} \cup \mathbb{F}$  are called *input gates* and they are not allowed to have incoming edges. Every element of  $\mathcal{X} \cup \mathbb{F}$  is allowed to appear at most once as the label of a gate – this is no restriction because one can simply identify the respective gates. We assume that a circuit contains exactly one gate without outgoing wires, and this is called the *output gate*. An algebraic circuit represents (or computes) a polynomial in  $\mathbb{F}[\mathcal{X}]$  in the obvious way: the gates  $+$  and  $\times$  are simply interpreted as addition and multiplication on polynomials, and we care about the polynomial at the output gate. Arrows are directed from a computation result towards the next gate where that result is used as an input. Multiedges are allowed so that for example the polynomial  $x^2$  can be represented with just one input and one multiplication gate. The *size* of a circuit  $C$ , is defined as the number of gates plus the number of wires, counted with multiplicities. It is denoted  $\|C\|$ .

Let  $\Gamma$  be a group that acts on  $\mathcal{X}$ . Then a circuit  $C$  is  $\Gamma$ -*symmetric* if the action of every  $\pi \in \Gamma$  on the input gates  $\mathcal{X}$  extends to an automorphism of the circuit: For every input gate  $g$ , let  $\ell(g)$  denote its label. This is either a variable or a field element. Field elements are fixed by every  $\pi \in \Gamma$ . We say that a permutation  $\pi \in \Gamma$  *extends to a circuit automorphism* of the circuit  $C$  if there exists a  $\sigma \in \mathbf{Sym}(V(C))$  such that  $\ell(\sigma(g)) = \pi(\ell(g))$  for every input gate, and such that  $\sigma$  is an automorphism of the multigraph structure of  $C$  (i.e. it maps edges to edges, non-edges to non-edges, and preserves the operation types of the gates). For more details, see [14].

A  $\Gamma$ -symmetric circuit is called *rigid* if it has no non-trivial circuit automorphism that fixes every input gate. This is equivalent to saying that for every  $\pi \in \Gamma$ , there is a *unique* circuit automorphism  $\sigma$  that extends  $\pi$ . In that case, we also write  $\pi(g)$  for internal gates  $g$ , to denote the application of the unique circuit automorphism that extends  $\pi$ . As shown in [12, Lemma 4.3], symmetric circuits can always be assumed to be rigid. An advantage of working with rigid circuits is that we obtain a well-defined notion of *support* of gates, and that there is a strong link between the size of these supports and the orbit sizes of the gates. In the following, we always assume that  $\Gamma$  is  $\mathbf{Sym}_n$  or  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ . In the former case, our variable set is  $\mathcal{X}_n = \{x_{ij} \mid i, j \in [n]\}$ , and  $\pi \in \mathbf{Sym}_n$  acts simultaneously on both indices, so  $\pi(x_{ij}) = x_{\pi(i)\pi(j)}$ . In the latter case, the variable set is  $\mathcal{X}_{n,m} = \{x_{ij} \mid i \in [n], j \in [m]\}$ . Then a pair of permutations  $(\pi, \pi') \in \mathbf{Sym}_n \times \mathbf{Sym}_m$  applied to  $x_{ij}$  yields  $x_{\pi(i)\pi'(j)}$ .

**Complexity measures for symmetric circuits.** An important measure for  $\Gamma$ -symmetric circuits is  $\max\text{Orb}(C)$ , the *maximum orbit size* of any gate in  $C$ , formally

$$\max\text{Orb}(C) := \max_{g \in V(C)} |\{\sigma(g) \mid \sigma \in \Gamma\}|.$$

If  $C$  is rigid, and  $\Gamma$  and  $C$  are such that every stabiliser group of a gate admits a unique minimal support, then we can also speak about the *support size* of  $C$ . In particular, this is possible if  $\Gamma \in \{\mathbf{Sym}_n, \mathbf{Sym}_n \times \mathbf{Sym}_m\}$  and  $\max\text{Orb}(C)$  is subexponential. We write  $\text{sup}(g) \subseteq [n] \uplus [m]$  to denote the *minimal support* of the group  $\mathbf{Stab}_{\mathbf{Sym}_n \times \mathbf{Sym}_m}(g)$  in  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ . The *support size* of a circuit  $C$  is then

$$\max\text{Sup}(C) := \max_{g \in V(C)} |\text{sup}(g)|.$$

We have the following relations between  $\max\text{Orb}$  and  $\max\text{Sup}$ . Lemma 3 establishes the converse of Lemma 2, part 1. Proofs of both lemmas are given in [12, Section 4.2].

► **Lemma 2.** *Let  $(\Gamma_{n,m})_{n,m \in \mathbb{N}}$  be either the sequence  $(\mathbf{Sym}_n)_{n \in \mathbb{N}}$  or  $(\mathbf{Sym}_n \times \mathbf{Sym}_m)_{n,m \in \mathbb{N}}$ . Let  $(C_{n,m})_{n,m \in \mathbb{N}}$  be a sequence of  $\Gamma_{n,m}$ -symmetric rigid circuits.*

1. *If  $\max\text{Orb}(C_{n,m})$  is polynomially bounded in  $n + m$ , then there exists a constant  $k \in \mathbb{N}$  such that for all large enough  $n, m \in \mathbb{N}$ , it holds  $\max\text{Sup}(C_{n,m}) \leq k$ .*
2. *If  $\max\text{Orb}(C_{n,m})$  is at most  $2^{o(n+m)}$ , then  $\max\text{Sup}(C_{n,m}) \leq o(n + m)$ .*

► **Lemma 3.** *Let  $(C_{n,m})_{n \in \mathbb{N}}$  be a sequence of rigid  $\Gamma_{n,m}$ -symmetric circuits, for  $\Gamma \in \{\mathbf{Sym}_n, \mathbf{Sym}_n \times \mathbf{Sym}_m\}$ . Assume that there is a constant  $k \in \mathbb{N}$  such that  $\max\text{Sup}(C_{n,m}) \leq k$ , for all  $n, m \in \mathbb{N}$ . Then  $\max\text{Orb}(C_{n,m}) \leq (n + m)^k$ .*

**Counting logic.** Counting logic is first-order logic augmented by counting quantifiers of the form  $\exists^{\geq i} x$ , for every  $i \in \mathbb{N}$ . A graph  $G$  satisfies a formula  $\exists^{\geq i} x \varphi(x)$ , written as  $G \models \exists^{\geq i} x \varphi(x)$ , if there exist at least  $i$  many distinct  $v \in V(G)$  such that  $G \models \varphi(v)$ . The  $k$ -variable fragment of counting logic is denoted  $\mathcal{C}^k$ . More details on counting logic and references may be found in [7]. For our purposes, the main interest is the equivalence relation that this induces on graphs. Two structures  $G, H$  are called  $\mathcal{C}^k$ -equivalent, denoted  $G \equiv_{\mathcal{C}^k} H$ , if  $G$  and  $H$  satisfy exactly the same sentences of  $\mathcal{C}^k$ . This family of equivalence relations has many characterisations, e.g. in combinatorics [20, 18], machine learning [40, 32], and optimisation [23, 31, 2]. In particular, it is known that  $G \equiv_{\mathcal{C}^{k+1}} H$  if, and only if,  $G$  and  $H$  are not distinguished by the  $k$ -dimensional Weisfeiler-Leman test for graph isomorphism [4].

### 3 Overview of results

#### 3.1 Characterisation of symmetric circuit complexity by homomorphism polynomials

We start by observing that any symmetric polynomial is expressible as a linear combination of *homomorphism* or *subgraph polynomials*. Let  $F$  be a bipartite multigraph with bipartition  $A \uplus B$ . For integers  $n, m \in \mathbb{N}$ , the *homomorphism polynomial*  $\text{hom}_{F,n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$  and the *subgraph polynomial*  $\text{sub}_{F,n,m} \in \mathbb{Q}[\mathcal{X}_{n,m}]$  of  $F$  are defined as

$$\begin{aligned} \text{hom}_{F,n,m} &:= \sum_{h: A \uplus B \rightarrow [n] \uplus [m]} \prod_{ab \in E(F)} x_{h(a)h(b)}, \\ \text{sub}_{F,n,m} &:= \frac{1}{|\mathbf{Aut}(F)|} \sum_{h: A \uplus B \hookrightarrow [n] \uplus [m]} \prod_{ab \in E(F)} x_{h(a)h(b)}. \end{aligned}$$

Here, we think of  $h$  as mapping  $A$  to  $[n]$  and  $B$  to  $[m]$ . For example,  $\text{hom}_{K_{2,n,m}} = \sum_{v \in [n]} \sum_{w \in [m]} x_{vw}$  and  $\text{sub}_{K_{n,m,n,m}} = \prod_{v \in [n]} \prod_{w \in [m]} x_{vw}$ . If  $A_G \in \{0, 1\}^{[n] \times [m]}$  is the bi-adjacency matrix of an undirected bipartite graph  $G$  with bipartition  $V(G) = [n] \uplus [m]$ , then  $\text{hom}_{F,n,m}(A_G)$  evaluates to the number of bipartition-respecting homomorphisms from  $F$  to  $G$ , and  $\text{sub}_{F,n,m}(A_G)$  is the number of occurrences of  $F$  as a subgraph in  $G$ . We also write  $\text{hom}_{F,n,m}(G)$  and  $\text{sub}_{F,n,m}(G)$  for the polynomials evaluated in the bi-adjacency matrix of  $G$ . Let

$$\mathfrak{G}_{n,m} := \left\{ \sum \alpha_i \text{hom}_{F_i,n,m} \mid \text{bipartite multigraphs } F_i, \alpha_i \in \mathbb{Q} \right\} \subseteq \mathbb{Q}[\mathcal{X}_{n,m}].$$

► **Lemma 4.** *For  $n, m \in \mathbb{N}$  and  $p \in \mathbb{Q}[\mathcal{X}_{n,m}]$ , the following are equivalent:*

1. *the polynomial  $p$  is  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric,*
2.  *$p \in \mathfrak{G}_{n,m}$ ,*
3.  *$p = \sum \alpha_i \text{sub}_{F_i,n,m}$  for some bipartite multigraphs  $F_i$  and  $\alpha_i \in \mathbb{Q}$ .*

For the proof, Item 3 can be concluded from Item 1 by partitioning the monomials in  $p$  into their orbits, and showing that each of these orbits is a subgraph polynomial for some pattern  $F$ . Via Möbius inversion (see [12, Corollary A.3]), subgraph polynomials can be expressed as linear combinations of homomorphism polynomials.

Thus, both subgraph and homomorphism polynomials can be seen as spanning sets for the vector space of matrix-symmetric polynomials. We are now interested in the ones that admit polynomial size symmetric circuits. For  $k, n, m \in \mathbb{N}$ , write

$$\mathfrak{T}_{n,m}^k := \left\{ \sum \alpha_i \text{hom}_{F_i,n,m} \mid \text{bipartite multigraphs } F_i \text{ s.t. } \text{tw}(F_i) < k, \alpha_i \in \mathbb{Q} \right\} \subseteq \mathfrak{G}_{n,m}.$$

for the set of finite  $\mathbb{Q}$ -linear combinations of homomorphism polynomials of bipartite multigraphs of treewidth less than  $k$ .

Our main result, Theorem 1, states that the families of polynomials in  $\mathfrak{T}_{n,m}^k$ , for constant  $k$ , are precisely the ones that admit symmetric circuits of polynomial orbit size. The full proof can be found in [12, Section 5] but the idea is the following:

**Proof sketch of Theorem 1.** The easier direction is to show that if  $p_{n,m} \in \mathfrak{T}_{n,m}^k$ , then it has circuits of polynomial orbit size. It turns out that each  $\text{hom}_{F,n,m}$  can be represented by a symmetric circuit with support size at most  $\text{tw}(F) < k$ . The circuit computes  $\text{hom}_{F,n,m}$  inductively along a tree decomposition of  $F$ , using ideas as in e.g. [20]. Then any polynomial in  $\mathfrak{T}_{n,m}^k$  is just a linear combination of such circuits. By Lemma 3, a circuit with support size  $\leq k$  has orbit size at most  $(n+m)^k$ , as desired. Note that a polynomial in  $\mathfrak{T}_{n,m}^k$  may be a linear combination involving a super-polynomial number of terms. Then only the orbit size of the circuit will be polynomial, but the total number of gates will not be.

The other direction of Theorem 1 requires substantial technical effort. It is proved using so-called *labelled* homomorphism polynomials: Imagine that a pattern  $F$  has  $k$  distinctly labelled vertices  $a_1, \dots, a_k$ . Then for any  $k$ -tuple of vertices  $b_1, \dots, b_k$  in the host graph  $G$ , we can ask how many homomorphisms  $h$  exist from  $F$  to  $G$  such that  $h(a_i) = b_i$ , for each  $i \in [k]$ . A labelled homomorphism polynomial is one that counts only these label-preserving homomorphisms.

We want to prove that every polynomial represented by a symmetric circuit of polynomial orbit size is in  $\mathfrak{T}_{n,m}^k$ , for some  $k \in \mathbb{N}$ . Lemma 2 tells us that the support size of every gate in such a circuit must be bounded by some constant  $k$ . We prove by induction on the circuit structure that at every gate  $g$  in the circuit, the polynomial computed at  $g$  is a linear combination of *labelled* homomorphism polynomials of patterns of treewidth  $\leq k$ . The crucial insight is that the supports of the gates always correspond exactly to the images of

the labels. That is, semantically, the gate  $g$  counts homomorphisms that map the labels  $a_1, \dots, a_k$  of the pattern graphs to the vertices  $b_1, \dots, b_k$  enumerating  $\text{sup}(g)$ . The root gate of a symmetric circuit always has empty support (as it is fixed by every permutation in  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ ), and so, the polynomial computed at the root is a linear combination of homomorphism polynomials for patterns  $F$  with  $\text{tw}(F) \leq k$  and without labels. Thus, the circuit computes a polynomial in  $\mathfrak{T}_{n,m}^k$ . ◀

We also prove that under the common assumption  $\text{VP} \neq \text{VNP}$ , Theorem 1 is best-possible in the sense that the characterisation cannot be improved to capture total circuit size rather than orbit size: There exist polynomials in  $\mathfrak{T}_{n,m}^k$ , for constant  $k$ , that do not admit polynomial size circuits (neither symmetric nor asymmetric), unless  $\text{VP} = \text{VNP}$ .

► **Theorem 5.** *There is a VNP-hard family of polynomials  $(p_n)_{n \in \mathbb{N}}$  such that  $p_n \in \mathfrak{T}_{n,n}^2$  for all  $n \in \mathbb{N}$ .*

### 3.2 Symmetric complexity of homomorphism polynomials

In order to make the characterisation in Theorem 1 more usable, we study  $\mathfrak{T}_{n,m}^k$  in relation with the class of polynomials of bounded counting width.

► **Definition 6** ([1]). *Let  $p$  be a graph parameter on  $(n, m)$ -vertex bipartite graphs. The counting width of  $p$  on simple graphs is the least integer  $k$  such that, for all  $(n, m)$ -vertex simple bipartite graphs  $G$  and  $H$ , if  $G$  and  $H$  are  $\mathcal{C}^k$ -equivalent, then  $p(G) = p(H)$ .*

It follows with known arguments that whenever a family  $(p_{n,m})_{n,m \in \mathbb{N}}$  admits circuits of polynomial orbit size, then its counting width is bounded (see [12, Theorem 6.2]). We ask in which cases the converse is true. As a start, we identify where the difficulty lies in this question. Namely, if we only care about the semantics of the polynomials on simple graphs, that is  $\{0, 1\}$ -valued rather than  $\mathbb{Q}$ -valued matrices, then we do have a conclusive answer. On  $\{0, 1\}$ -assignments, any polynomial  $p$  is equivalent to its *multilinearisation*. This is the polynomial obtained from  $p$  by replacing every exponent greater than 1 with 1. Up to multilinearisation, bounded counting width is indeed sufficient for the existence of polynomial orbit size symmetric circuits:

► **Theorem 7.** *Let  $k, n, m \in \mathbb{N}$ . For  $p \in \mathbb{Q}[\mathcal{X}_{n,m}]$ , the following are equivalent:*

1. *there exists  $q \in \mathfrak{T}_{n,m}^k$  such that  $p$  and  $q$  are equal up to multilinearisation,*
2. *the counting width of  $p$  on  $(n, m)$ -vertex simple bipartite graphs is at most  $k$ .*

Thus, if we only care about evaluating the polynomial  $p$  on  $\{0, 1\}$  (equivalently, we regard it as a parameter on simple, unweighted, bipartite graphs) the symmetric tractability question is completely settled. But this does not settle the general case, where we wish to consider it on all  $\mathbb{Q}$ -valued assignments (see, e.g. [12, Example 8.10]). However, we can characterise it in more specific cases. In particular, we obtain a precise characterisation of the bounded counting width case for linear combinations of homomorphism polynomials with sublinear-size patterns:

► **Theorem 8** (see [12, Theorem 7.9]). *For  $n, m \in \mathbb{N}$ , let  $F_{n,m,i}$  be bipartite multigraphs and  $\alpha_{n,m,i} \in \mathbb{Q} \setminus \{0\}$  for  $i \in [s_{n,m}]$  for some  $s_{n,m} \in \mathbb{N}$ . Let  $p_{n,m} := \sum_i \alpha_{n,m,i} \text{hom}_{F_{n,m,i}, n, m}$ . Suppose that, for all  $\nu, \mu \in \mathbb{N}$ , there are  $n', m' \in \mathbb{N}$  such that  $\max_i |F_{\nu, \mu, i}| \leq \min\{n, m\}$  for all  $n > n'$  and  $m > m'$ . Then the following are equivalent:*

1.  *$\max_i \text{tw}(F_{n,m,i})$  is bounded,*
2. *the counting width of  $p_{n,m}$  on  $(n, m)$ -vertex simple graphs is bounded,*
3. *the  $p_{n,m}$  admit  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of orbit size polynomial in  $n + m$ .*

If the linear expansions of the  $p_{n,m}$  are comprised of polynomially many homomorphism polynomials, then the above conditions are equivalent to the  $p_{n,m}$  admitting  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of total size polynomial in  $n + m$ . In the case where  $p_{n,m} = \text{hom}_{F_{n,m},n,m}$  for all  $n, m \in \mathbb{N}$ , i.e. we just have a single homomorphism polynomial for each  $n, m$ , we obtain the above result even if  $|F_{n,m}|$  is not in  $o(n + m)$ :

► **Theorem 9** (see [12, Theorem 7.1]). *For  $n, m \in \mathbb{N}$ , let  $F_{n,m}$  be bipartite multigraphs. Then the following are equivalent:*

1.  $\text{tw}(F_{n,m})$  is bounded,
2. the counting width of  $p_{n,m}$  on  $(n, m)$ -vertex simple graphs is bounded,
3. the  $p_{n,m}$  admit  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of orbit size polynomial in  $n + m$ .

We prove [12, Theorem 7.9] by generalising notions introduced by Robertson [36] for describing the distinguishing power of graph isomorphism relaxations via homomorphism counts. By [20, 18], two graphs  $G$  and  $H$  are  $\mathcal{C}^k$ -equivalent if, and only if, they are *homomorphism indistinguishable* over the graphs of treewidth less than  $k$ , i.e.  $\text{hom}(F, G) = \text{hom}(F, H)$  for all  $F$  such that  $\text{tw}(F) < k$ . Hence, a graph parameter  $p$  has bounded counting width if, and only if, it is determined by homomorphism counts from bounded-treewidth graphs. For uniform graph motif parameters  $\sum \alpha_F \text{hom}(F, -)$ , it was known [39, 33, 29] that this happens if and only if  $\max_F \text{tw}(F)$  is bounded. We generalise this result to non-uniform graph motif parameters, whose patterns and coefficients may depend on the size of the host graph.

### 3.3 Symmetric complexity of subgraph polynomials

By Lemma 4, linear combinations of subgraph and homomorphism polynomials can be transformed into one another. Nevertheless, we also study subgraph polynomials on their own because it seems likely that in this case we find another natural parameter of the pattern graphs that fully controls the counting width and symmetric complexity. Indeed, we show that for patterns of sublinear size, this parameter is  $\text{vc}(F)$ , the *vertex cover number*.

► **Theorem 10** (see [12, Theorem 8.8]). *Let  $(F_{n,m})_{n,m \in \mathbb{N}}$  be a family of simple bipartite graphs such that, for all  $\nu, \mu \in \mathbb{N}$ , there exist  $n', m' \in \mathbb{N}$ , such that  $|F_{\nu n, \mu m}| \leq \min\{n, m\}$  for all  $n > n'$  and  $m > m'$ . The following are equivalent:*

1.  $\text{vc}(F_{n,m})$  is bounded,
2. the counting width of  $\text{sub}_{F_{n,m},n,m}$  on  $(n, m)$ -vertex simple graphs is bounded,
3. the  $\text{sub}_{F_{n,m},n,m}$  admit  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of orbit size polynomial in  $n + m$ ,
4. the  $\text{sub}_{F_{n,m},n,m}$  admit  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of size polynomial in  $n + m$ .

The size bound in this theorem is necessary as it is easy to find examples of subgraph polynomials (of size  $\Theta(n + m)$ ) whose complexity is not governed by the pattern's vertex cover number. Consider e.g.  $p_{n,m} = \text{sub}_{K_{n,m},n,m} = \prod_{v \in [n]} \prod_{w \in [m]} x_{vw}$ . The natural symmetric circuit for this linear-size pattern has just one product gate that ranges over all variables, so this family  $(p_{n,m})_{n,m \in \mathbb{N}}$  has low symmetric circuit complexity (see also [12, Example 8.2]). But,  $\text{vc}(K_{n,m})$  is clearly unbounded.

More generally, for any function  $f: \mathbb{N} \rightarrow \mathbb{N}$  with  $f(n) \leq n$ , and the complete bipartite graphs  $K_{f(n),f(n)}$  as patterns, we show that the only tractable cases for  $\text{sub}_{K_{f(n),f(n)},n,n}$  are when  $\min\{f(n), n - f(n)\}$  is constant [12, Theorem 8.3]. This suggests that it might be the minimum vertex cover number of the pattern graph and its complement that determines the

symmetric complexity of  $\text{sub}_{F_{n,m,n,m}}$ . We call this parameter the *logical vertex cover number*  $\text{lvc}_{n,m}(F_{n,m})$ . In particular, we can show for all  $F$  that the counting width of  $\text{sub}_{F_{n,m}}$  and  $\text{sub}_{\overline{F}_{n,m}}$  is the same [12, Theorem 8.9], giving further evidence for this hypothesis.

► **Conjecture 11.** *For every family  $(F_{n,n})_{n,m \in \mathbb{N}}$  of simple bipartite graphs, the following are equivalent:*

1.  $\text{lvc}_{n,m}(F_{n,m})$  is bounded,
2. the counting width of  $\text{sub}_{F_{n,m,n,m}}$  on  $(n,m)$ -vertex  $\mathbb{Q}$ -edge-weighted graphs is bounded,
3. the  $\text{sub}_{F_{n,m,n,m}}$  admit  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of orbit size polynomial in  $n + m$ ,
4. the  $\text{sub}_{F_{n,m,n,m}}$  admit  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric circuits of size polynomial in  $n + m$ .

Theorem 10 proves this conjecture for pattern graphs of sublinear size.

### 3.4 Symmetric complexity of the immanants

So far, we have studied  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric families. There are also polynomials in the variables  $\mathcal{X}_{n,n}$  which are symmetric under the simultaneous action of  $\mathbf{Sym}_n$  on rows and columns but not under  $\mathbf{Sym}_n \times \mathbf{Sym}_n$ . One such example is the *determinant*, and more generally, other *immanants*. We have chosen to develop our theory for  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ -symmetric families but we believe that most of it can be adapted to the  $\mathbf{Sym}_n$ -symmetric case. Then, the relevant homomorphism or subgraph polynomials involve *directed* and not necessarily bipartite graphs. Rather than doing that here, we consider the immanants as a concrete  $\mathbf{Sym}_n$ -symmetric example and show that their symmetric circuit complexity is subject to a dichotomy similar to what is known for their computational complexity [5].

*Immanants* are families of  $\mathbf{Sym}_n$ -symmetric polynomials  $(p_n)_{n \in \mathbb{N}}$  that are defined via so-called *irreducible characters* of the symmetric group. An irreducible character of  $\mathbf{Sym}_n$  is a homomorphism  $f$  from  $\mathbf{Sym}_n$  to the multiplicative group  $\mathbb{C} \setminus \{0\}$  which is constant on the conjugacy classes of  $\mathbf{Sym}_n$ . Given such an  $f$ , the corresponding immanant is defined as  $\text{imm}_f = \sum_{\pi \in \mathbf{Sym}_n} f(\pi) \prod_{i \in [n]} x_{i,\pi(i)}$ . If  $f$  is constantly 1, then  $\text{imm}_f$  is the permanent. The determinant is obtained by letting  $f = \text{sgn}$ . While all immanants are  $\mathbf{Sym}_n$ -symmetric, the symmetry groups of particular members of this family may be larger. In particular, for the permanent it is, in fact,  $\mathbf{Sym}_n \times \mathbf{Sym}_n$ , while the determinant is symmetric under a subgroup of this of index 2 consisting of those pairs of permutations  $(\pi, \sigma)$  for which  $\text{sgn}(\pi) = \text{sgn}(\sigma)$ . The  $\mathbf{Sym}_n$  symmetric circuit complexity of the permanent and the determinant was studied in [15] and for the larger symmetry groups in [17].

In [5], Curticapean shows a complexity dichotomy for the immanants: In the tractable case,  $\text{imm}_f$  is in VP (i.e. admits polynomial size algebraic circuits) and is computable in polynomial time. In the intractable case, it is not in VP, unless  $\text{VFPT} = \text{VW}[1]$ , and also, it is not computable in polynomial time, unless  $\text{FPT} = \#\text{W}[1]$ . The complexity is controlled by a certain parameter of  $f$ .

We show that the same parameter of  $f$  produces the analogous dichotomy with regards to the complexity of  $\mathbf{Sym}_n$ -symmetric algebraic circuits for the immanants, but without the need for complexity-theoretic assumptions for the hardness part. This generalizes the results of [15]. The irreducible characters  $f$  correspond naturally to partitions of  $[n]$  (see, for example, [38, Prop. 1.10.1]). If  $\lambda$  is a partition of  $[n]$ , we denote by  $\text{imm}_\lambda$  the immanant defined by the irreducible character corresponding to  $\lambda$ . Let  $b(\lambda) := n - s$ , where  $s$  denotes the number of parts and  $n$  the size of the set that is partitioned by  $\lambda$ . This is the parameter that controls the complexity of  $\text{imm}_\lambda$ .

A family  $\Lambda$  of partitions is said to *support growth*  $g: \mathbb{N} \rightarrow \mathbb{N}$  if for every  $n \in \mathbb{N}$  there is a partition  $\lambda^{(n)}$  in  $\Lambda$  with  $b(\lambda^{(n)}) \geq g(n)$  and size  $\Theta(n)$ . According to the dichotomy from [5],  $(\text{imm}_\lambda)_{\lambda \in \Lambda}$  is tractable if there is a constant that bounds  $b(\lambda)$  for all  $\lambda \in \Lambda$ . Otherwise, if  $\Lambda$  supports growth  $\omega(1)$ , then the immanant family is (conditionally) intractable. If  $\Lambda$  even supports growth  $\omega(n^k)$ , then  $(\text{imm}_\lambda)_{\lambda \in \Lambda}$  is VNP-complete. Our analogous result for symmetric circuits is the following.

► **Theorem 12.** *Let  $\Lambda$  be a family of partitions.*

1. *If there exists a  $k \in \mathbb{N}$  such that  $b(\lambda) \leq k$  for all  $\lambda \in \Lambda$ , then the  $(\text{imm}_\lambda)_{\lambda \in \Lambda}$  admit  $\mathbf{Sym}_n$ -symmetric circuits of polynomial size in  $n$ .*
2. *If  $\Lambda$  supports growth  $g \in \omega(1)$ , then the counting width of  $(\text{imm}_\lambda)_{\lambda \in \Lambda}$  on directed  $\mathbb{Q}$ -edge-weighted  $n$ -vertex graphs is unbounded, where  $n$  is the integer that is partitioned by the respective  $\lambda$ . Thus, all  $\mathbf{Sym}_n$ -symmetric circuits for the immanant family have super-polynomial size in  $n$ .*
3. *If  $\Lambda$  supports growth  $g \in \omega(n^k)$  for a constant  $k$ , then the counting width of  $(\text{imm}_\lambda)_{\lambda \in \Lambda}$  on directed  $\mathbb{Q}$ -edge-weighted  $n$ -vertex graphs is  $\Omega(n)$ , so all  $\mathbf{Sym}_n$ -symmetric circuits have size  $2^{\Omega(n)}$ .*

## 4 Conclusion

In recent years, a rich theory of symmetric computation has been emerging which has established a tight and surprising relationship between logical definability, circuit complexity and also computation in other models such as linear programming (see [8, 9] for pointers). A central plank of this is the use of variations of the Cai-Fürer-Immerman construction to establish unconditional lower bounds in the context of models of computation with natural symmetries. The work in [15] is a significant example of this, showing an unconditional exponential separation between the complexity of the determinant and the permanent polynomials in the context of symmetric algebraic circuits.

The present work gives a sweeping generalisation of these results in two different directions. First, considering matrix polynomials symmetric under arbitrary permutations of the rows and columns (i.e. the  $\mathbf{Sym}_n \times \mathbf{Sym}_m$  action), we give a complete characterisation of the tractable cases in terms of homomorphism polynomials of bounded treewidth, linking this study to other complexity classifications such as [6]. For polynomials on *square* matrices symmetric under the simultaneous application of a permutation to the rows and columns (i.e. the  $\mathbf{Sym}_n$  action), we generalise the separation of the determinant and permanent by giving a complete classification of the symmetric complexity of all immanants.

The work raises a number of directions for future research. It suggests that there is a close connection between the counting width of polynomials, which we define, and their symmetric algebraic complexity. We have not established this in full generality and thus the first important direction is to prove or disprove Conjecture 11. A step in this direction would be to understand the complexity of linear combinations of homomorphism polynomials where neither Theorem 8 nor Theorem 9 applies. That is to say, there is more than one homomorphism polynomial involved and the size of the pattern graphs is not sublinearly bounded.

Our most general results are for matrix polynomials symmetric under  $\mathbf{Sym}_n \times \mathbf{Sym}_m$ , where we have a complete characterisation of tractable families. The polynomials on square matrices symmetric under the  $\mathbf{Sym}_n$  action include many that are not  $\mathbf{Sym}_n \times \mathbf{Sym}_n$ -symmetric (for instance, the determinant and many other immanants). While we give a complete classification of the immanant polynomials, there are many other such polynomial

families. It seems plausible that a complete classification in terms of homomorphism polynomials from directed graphs might be achievable in this case, and we leave it for future work.

It would be interesting to relate this to other methods for establishing lower bounds in algebraic complexity and to natural algorithmic methods in that field. Indeed, one of the striking features of the study of symmetric computation models is that a surprising range of algorithms that are used (for instance in combinatorial optimisation) are indeed symmetric and thus the unconditional lower bounds show the limitations of standard techniques.

Finally, this work may provide the means to prove lower bounds for symmetric circuits for interesting polynomial families beyond the determinant and the permanent. One potential application would be in proof complexity. There has been considerable recent interest in the *Ideal Proof System* [22], in which refutations of unsatisfiable Boolean formulas or systems of polynomial equations take the form of algebraic circuits. One could conceivably use the methods developed here to show that certain families of formulas with natural symmetries do not admit succinct symmetric refutations.

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