

Satisfiability in Łukasiewicz Logic and Its Unbounded Relative

Zuzana Haniková 

Institute of Computer Science of the Czech Academy of Sciences, Prague, Czech Republic

Filip Jankovec 

Institute of Computer Science of the Czech Academy of Sciences, Prague, Czech Republic

Abstract

Unbounded Łukasiewicz logic is a substructural logic that combines features of infinite-valued Łukasiewicz logic with those of abelian logic. The logic is finitely strongly complete w.r.t. the additive ℓ -group on the reals expanded with a distinguished element -1 . We show that the existential theory of this structure is NP-complete. This provides a complexity upper bound for the set of theorems and the finite consequence relation of unbounded Łukasiewicz logic. The result is obtained by reducing the problem to the existential theory of the MV-algebra on the reals, the standard semantics of Łukasiewicz logic. This provides a new connection between both logics. The result entails a translation of the existential theory of the standard MV-algebra into itself.

2012 ACM Subject Classification Theory of computation $\rightarrow$ Logic

Keywords and phrases unbounded Łukasiewicz Logic, Łukasiewicz Logic, Abelian Logic, existential theory, computational complexity, NP-completeness

Digital Object Identifier 10.4230/LIPIcs.CSL.2026.14

Funding Both authors were supported by ERDF-Project *Knowledge in the Age of Distrust*, No. CZ.02.01.01/00/23_025/0008711.

Filip Jankovec: SVV-2025-260837.

Acknowledgements Three anonymous reviews helped to improve the text.

1 Introduction

In the realm of nonclassical logics, and particularly in its subarea known as many-valued logics, the infinite-valued Łukasiewicz logic $\mathbb{L}$ has long occupied a prominent position. Its evolution (discussed in detail in, e.g., [8, 36, 22]) spanned the introduction of its three-valued antecedent by Łukasiewicz in his famous analysis of future contingents [29], as well as the infinite-valued calculus also introduced by Łukasiewicz and elaborated by himself and Tarski in the paper [30], which first discussed, but did not prove, completeness w.r.t. the intended semantics. A completeness theorem was given much later by Rose and Rosser [40]. MV-algebras, the equivalent algebraic semantics, were introduced and analyzed by Chang, and completeness was proved [6, 7]. Numerous monographs deal exclusively or substantially in the topic, e.g., [8, 22, 33, 36, 11]. Moreover, Łukasiewicz logic and its extensions can fruitfully be investigated in the framework of substructural logics [39, 19].

Increasingly then, the well-established area of Łukasiewicz logic and MV-algebras has been taken as a vantage point for various theoretical or applied problems, be it expansions of its language with constants in the pioneering works of Pavelka [37], modal expansions [16, 17, 23, 42], or reasoning about probabilities [18], to name a few.

Mundici initiated the study of computational complexity problems raised by Łukasiewicz logic in his famous work [35] which showed satisfiability and positive satisfiability of terms in the MV-algebra on the real unit interval with the natural order, $[0, 1]_{\mathbb{L}}$, to be NP-complete. An earlier important work, relevant to many subsequent contributions to the



© Zuzana Haniková and Filip Jankovec;

licensed under Creative Commons License CC-BY 4.0

34th EACSL Annual Conference on Computer Science Logic (CSL 2026).

Editors: Stefano Guerrini and Barbara König; Article No. 14; pp. 14:1–14:18

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

area of algorithmic problems in Łukasiewicz logic, came in McNaughton's work [32] that characterized term-definable functions in $[0, 1]_{\mathbb{L}}$ as continuous, piecewise linear functions with integer coefficients. Algebraic methods are the mainstay of studying computational problems in the area, see especially [2, 4, 5, 3]. Komori's classification of subvarieties of MV [27, 28] yielded the coNP-completeness result for axiomatic extensions of $\mathbb{L}$ [10], and Mundici's NP-completeness result for term satisfiability generalizes to the existential theory of $[0, 1]_{\mathbb{L}}$ [24]. While Mundici's results place the satisfiability and the tautology problems in $\mathbb{L}$ on a par with their counterparts in classical propositional logic (either being NP-complete and coNP-complete respectively), this is not a given in the infinite-valued setting. More generally, Łukasiewicz logic is dissimilar from classical logic in many respects and a list to showcase this phenomenon ought not to omit $\mathbb{L}$ not being structurally complete [14], the admissible rules being PSPACE complete [26], the first-order standard tautologies not being recursively enumerable [41] and in fact Π_2 -complete in the arithmetical hierarchy [38], or the complex structure of the lattice of extensions of $\mathbb{L}$, which is dually isomorphic to the lattice of subquasivarieties of the variety of MV-algebras. This lattice is Q-universal [1]. Consequently, the extensions cannot all be decidable for cardinality reasons; this is also the case already for term satisfiability problems for various MV-algebras on the rational domain [25].

Unbounded Łukasiewicz logic $\mathbb{L}_u$ was considered recently in the papers [9, 12]. Formally, it is obtained by expanding the language of lattice-ordered abelian groups with a propositional constant -1 with suitable axioms ensuring that in any algebra within its equivalent algebraic semantics, the upset of the interpretation of 0 is the set of designated elements, representing truth, whereas the downset of -1 represents falsity. By design, then, the logic bears resemblance to both Łukasiewicz logic and to abelian logic, as considered in Meyer and Slaney's paper [34]. Unlike $\mathbb{L}$, and like abelian logic, $\mathbb{L}_u$ is inconsistent with classical logic.

Weispfenning [43] proved that the existential theory of the variety of ℓ -groups is NP-complete. So the universal theory of ℓ -groups is coNP-complete and *a fortiori*, the quasiequational theory and the equational theory are in coNP. It is also known that ℓ -groups are generated as a quasivariety by the totally ordered members [13], in fact, by any nontrivial totally ordered member, since the universal theories of each two nontrivial totally ordered abelian groups coincide [20]. One easily shows that the equational theory of the totally ordered additive group on the integers is coNP-hard, thus any of the above universal fragments is coNP-complete.

The purpose of this paper is to provide a complexity estimate for $\mathbb{L}_u$ by exhibiting a polynomial-time reduction of the existential theory of the intended semantics of $\mathbb{L}_u$ – the algebra $\mathbf{R}_{-1}$, the pointed abelian totally ordered additive group on the reals where -1 is interpreted by itself – to the existential theory of the intended semantics of $\mathbb{L}$, namely $[0, 1]_{\mathbb{L}}$, the standard MV-algebra on the reals. In both cases, the logics are finitely strongly complete w.r.t. the intended semantics. In our opinion the method surpasses the result in terms of interest, since the result might be anticipated. The method consists in taking a linear map of a sufficiently large neighbourhood of the element 0 to the interval $[0, 1]$, with the coefficient of the linear map determined by the instance (i.e., a given existential sentence).

In combination with the extant method of [12] that reduces in polynomial time the set of valid universal formulas (valid identities) of $[0, 1]_{\mathbb{L}}$ to the set of valid universal formulas (valid identities respectively) of the algebra $\mathbf{R}_{-1}$ one obtains firstly, a coNP-completeness of each of the universal fragments (i.a., the logic $\mathbb{L}_u$ is coNP-complete) and secondly, a non-trivial interpretation of the existential theory of the standard MV-algebra in itself.

The result invites a comparison with the work of Marchioni and Montagna [31], which concerns the logic $\mathbb{L}\Pi^{1/2}$ and its relation to the universal theory of real closed fields (RCF). The authors show that the universal fragment of RCF has a polynomial time reduction to

the (theorems of) the logic $\mathbb{L}\Pi^{1/2}$. The relevance consists in, on the one hand, the logic $\mathbb{L}\Pi^{1/2}$ being a conservative expansion of the logic $\mathbb{L}$ (the theorems of $\mathbb{L}$ are exactly those theorems of $\mathbb{L}\Pi^{1/2}$ that only use the language $\mathcal{L}_{\text{MV}}$) and on the other hand, the universal theory of pointed abelian totally ordered group being term-equivalent to the multiplication- and division-free fragment of the universal theory of RCF. Their reduction starts with fixing a bijection between $\mathbf{R}$ and the interval $(0, 1)$ and then defining the operations of the real closed field on $\mathbf{R}$ in the $\mathbb{L}\Pi^{1/2}$ -algebra on the reals. This approach cannot be used in the present case because it relies on the multiplicative function symbols of the logic Π even when defining the real addition; so it does not scale well to language fragments.

This paper is structured as follows. Key notions are presented in section 2. Section 3 establishes that assignments that make an open formula of the language of pointed ℓ -groups true, if any, can be found in a suitably bounded interval. Section 4 presents the main result, the polynomial-time reduction. Section 5 then composes two reductions in order to obtain a nontrivial reduction of the existential theory of $[0, 1]_{\mathbb{L}}$ to itself.

2 Preliminaries

Let $\text{Var} = \{x_1, x_2, \dots\}$ be a denumerable set of variables. A first-order language $\mathcal{L}$ uniquely determines the set $\text{Term}_{\mathcal{L}}$ of terms of $\mathcal{L}$. The logical symbol $=$ is the only predicate symbol of any algebraic language; however, we may additionally use binary predicate symbols $\leq, <$ (in Section 3). If φ and ψ are terms of $\mathcal{L}$, then $\varphi \diamond \psi$ is an atomic formula (or “atom”) of $\mathcal{L}$ for $\diamond \in \{=, \leq, <\}$. If there are no predicate symbols in $\mathcal{L}$ then $\mathcal{L}$ -atoms are just $\mathcal{L}$ -identities $\varphi = \psi$. We use lowercase Greek for terms and uppercase Greek for first-order formulas. For boolean symbols occurring in first-order formulas (boolean combinations of atoms of $\mathcal{L}$) we use $\neg$ for boolean negation, $\wedge$ and $\vee$ for boolean conjunction and disjunction respectively, $\bigwedge$ and $\bigvee$ for finite boolean conjunctions and disjunctions respectively. An open formula of $\mathcal{L}$ is an arbitrary finite boolean combination (i.e., a $\{\neg, \wedge, \vee\}$ -combination) of $\mathcal{L}$ -atoms; we do not in general assume a normal form for first-order boolean formulas. Moreover, if $\Rightarrow$ denotes boolean implication, quasiidentities of $\mathcal{L}$ are formulas of the form $\bigwedge_{i=1}^n (\varphi_i = \psi_i) \Rightarrow (\varphi_0 = \psi_0)$ where $\{\varphi_i, \psi_i\}_{0 \leq i \leq n}$ is a set of $\mathcal{L}$ -terms. Since quasiequational theories are not of particular interest in this work, $\Rightarrow$ is not taken as primitive and a formula $\Phi \Rightarrow \Psi$ is understood as $\neg \Phi \vee \Psi$ with only an insubstantial increase in length. An existential (universal) sentence of a language $\mathcal{L}$ is an existential (universal respectively) closure of an open formula of $\mathcal{L}$.

If φ is an $\mathcal{L}$ -term, $\text{Var}(\varphi)$ denotes the set of variables that occur in φ , and analogously for $\text{Var}(\Phi)$ if Φ is an $\mathcal{L}$ -formula. We use $\bar{x}$ for $\langle x_1, \dots, x_n \rangle$ if n is of no consequence.

The following languages will be used:¹

- $\mathcal{L}_{\text{Ab}} = \langle +, -, 0, \wedge, \vee \rangle$ is the language of abelian lattice-ordered groups (ℓ -groups) and Term_{Ab} is the corresponding set of well-formed terms;
- $\mathcal{L}_{\text{pAb}} = \langle +, -, 0, -1, \wedge, \vee \rangle$ is the language of pointed ℓ -groups and Term_{pAb} the set of its well-formed terms;
- $\mathcal{L}_{\text{MV}} = \langle \oplus, \otimes, \neg, \rightarrow, 0, 1, \wedge, \vee \rangle$ is the language of MV-algebras and Term_{MV} the set of its well-formed terms;
- $\mathcal{L}_{\text{MV}[\frac{1}{2}]} = \langle \oplus, \otimes, \neg, \rightarrow, 0, 1, \frac{1}{2}, \wedge, \vee \rangle$ expands the language of MV-algebras with a new constant $\frac{1}{2}$ and $\text{Term}_{\text{MV}[\frac{1}{2}]}$ is the set of its well-formed terms.

¹ We write Term_{Ab} instead of $\text{Term}_{\mathcal{L}_{\text{Ab}}}$. We use the same notation for a function symbol of a language and for its interpretation in a particular algebra where possible, to avoid cluttering.

$\mathbf{R} = \langle \mathbb{R}, +, -, \wedge, \vee, 0 \rangle$ is an abelian lattice-ordered group (ℓ -group) on the real numbers with the usual order, (the usual) addition, subtraction and 0. The algebra $\mathbf{R}$ provides a semantics to the language $\mathcal{L}_{\text{Ab}}$.

$\mathbf{R}_{-1} = \langle \mathbb{R}, +, -, \wedge, \vee, 0, -1 \rangle$ is a *pointed* abelian ℓ -group whose $\mathcal{L}_{\text{Ab}}$ -reduct is $\mathbf{R}$ and the element -1 interprets the constant -1 .²

In the *standard MV-algebra* $[0, 1]_{\mathbf{L}}$ on the reals, which provides an interpretation to the language $\mathcal{L}_{\text{MV}}$, the domain is $[0, 1]$, as for the basic operations, $\neg a$ is $1 - a$ and $a \otimes b$ is $\max(0, a + b - 1)$ for $a, b \in [0, 1]$. The remaining function symbols are definable.

- $a \oplus b$ is $\neg(\neg a \otimes \neg b)$
- $a \rightarrow b$ is $\neg a \oplus b$
- $a \wedge b$ is $a \otimes (a \rightarrow b)$
- $a \vee b$ is $(a \rightarrow b) \rightarrow b$
- 0 is $a \otimes \neg a$ for some a
- 1 is $\neg 0$

In any MV-algebra, $\neg\neg a = a$ holds (involutive law), analogously in any abelian ℓ -group, we have $--a = a$. In view of this, we assume that terms are given in a reduced form within their respective languages, containing no subterms of the form $\neg\neg\varphi$ or $--\varphi$, respectively.

The two element boolean algebra is a subalgebra of $[0, 1]_{\mathbf{L}}$. The algebra $[0, 1]_{\mathbf{L}}^{1/2}$ interprets the language $\mathcal{L}_{\text{MV}[1/2]}$. Its $\mathcal{L}_{\text{MV}}$ -reduct is the algebra $[0, 1]_{\mathbf{L}}$ and the element $1/2$ interprets the constant $1/2$.

It is assumed throughout that no well-formed term of the languages under consideration contains two consecutive occurrences of the symbol $-$ or the symbol $\neg$. This can be assumed without a loss of generality, relying on the valid involutive laws.

One can introduce the (infinite-valued) Łukasiewicz logic $\mathbf{L}$ and the unbounded Łukasiewicz logic $\mathbf{Lu}$ axiomatically and then obtain that $\mathbf{L}$ is finitely strongly complete w.r.t. $[0, 1]_{\mathbf{L}}$ and $\mathbf{Lu}$ is finitely strongly complete w.r.t. $\mathbf{R}_{-1}$. We take these results for granted, referring the reader to the works [8, 22, 12]. Moreover, without further ado we use Mundici's result on the NP-completeness of the term satisfiability problem in $[0, 1]_{\mathbf{L}}$ and the coNP-completeness of term validity therein. Recall that the designated element of $[0, 1]_{\mathbf{L}}$ is 1; accordingly, a term $\varphi(\bar{x})$ holds under an assignment $\bar{a}$ provided that $\varphi(\bar{a}) = 1$, and $\varphi(\bar{x})$ is *satisfiable* in $[0, 1]_{\mathbf{L}}$ provided that an assignment exists that makes it true there. We retain this terminology also for first-order open formulas: such a formula $\Phi(\bar{x})$ is satisfiable in $[0, 1]_{\mathbf{L}}$ provided an assignment to its variables exists that makes it true therein. A term φ of $\mathcal{L}_{\text{MV}}$ is *valid* in $[0, 1]_{\mathbf{L}}$ provided it is true under any assignment therein. The set of designated elements of $\mathbf{R}_{-1}$ is the set of its nonnegative elements (the upset of 0), and accordingly a term is satisfiable in $\mathbf{R}_{-1}$ provided some assignment sends it to a nonnegative element; the rest is analogous.

If $\mathcal{L}$ is a language and $\mathbf{A}$ is an $\mathcal{L}$ -algebra, the existential theory ($\exists$ -theory) of $\mathbf{A}$ is the set of all existential $\mathcal{L}$ -sentences valid in $\mathbf{A}$; dually for the universal theory ($\forall$ -theory) and universal $\mathcal{L}$ -sentences. The (quasi)equational theory of $\mathbf{A}$ is the set of universal closures of $\mathcal{L}$ -(quasi)identities valid in $\mathbf{A}$. Both the equational theory and the quasiequational theory of an algebra $\mathbf{A}$ are fragments of the universal theory of $\mathbf{A}$.

Each term φ of $\mathcal{L}_{\text{Ab}}$ in n variables $x_1, \dots, x_n$ in Var uniquely determines a function $f_{\varphi}: \mathbb{R}^n \rightarrow \mathbb{R}$ (on the domain of the algebra $\mathbf{R}$) using the following inductive steps:

² While the algebra $\mathbf{R}_{-1}$ seems very similar to $\mathbf{R}$, the additional constant in the language grants much more expressivity and brings the logic $\mathbf{Lu}$ closer to the logic $\mathbf{L}$.

- $f_x = x$ for each variable x in Var ;
- $f_{\varphi+\psi} = f_\varphi + f_\psi$;
- $f_{-\varphi} = -f_\varphi$;
- $f_{\varphi\wedge\psi} = \min(f_\varphi, f_\psi)$;
- $f_{\varphi\vee\psi} = \max(f_\varphi, f_\psi)$;
- $f_0 = 0$.

Analogously for a term φ of $\mathcal{L}_{\text{pAb}}$. Furthermore, a term φ of $\mathcal{L}_{\text{MV}}$ uniquely determines a function $f_\varphi: [0, 1]^n \rightarrow [0, 1]$ (on the domain of the standard MV-algebra), i.e.,

- $f_{\varphi\otimes\psi} = \max(0, f_\varphi + f_\psi - 1)$,
- $f_{\neg\varphi} = 1 - f_\varphi$,

etc., and analogously for a term of the language $\mathcal{L}_{\text{MV}[\frac{1}{2}]}$.

For a term φ of $\mathcal{L}_{\text{Ab}}$, the *depth* of φ is defined by induction: $\text{depth}(0) = 0$; $\text{depth}(x) = 0$ if $x \in \text{Var}$; $\text{depth}(-\varphi) = \text{depth}(\varphi) + 1$; and $\text{depth}(\varphi \circ \psi) = \max(\text{depth}(\varphi), \text{depth}(\psi)) + 1$ if $\circ$ is one of $\{+, \wedge, \vee\}$. Analogously for the other languages, with depth of constants set to 0.

For a term φ and a first-order formula Φ of $\mathcal{L}_{\text{Ab}}$, $\sharp(\varphi)$ and $\sharp(\Phi)$ will denote the *length* (or *size*) of φ and Φ respectively. We define $\sharp(\varphi)$ as the total number of occurrences of variables and constants in φ ; should one view φ as a tree, $\sharp(\varphi)$ is the number of leaves. Recalling that in φ no two consecutive negations “ $-$ ” can occur, the number of occurrences of function symbols in φ (i.e., of the inner nodes of the tree) is bounded by $2\sharp(\varphi)$. The sizes of representations of each variable and each constant are neglected and representation in unit length is considered.

For an open Φ , we use $\text{At}^\sharp(\Phi)$ for the number of occurrences of atomic formulas. We define $\sharp(\Phi)$ to be the sum of the sizes of all the occurrences of terms in the atomic identities; we have $\sharp(\Phi) \leq \text{At}^\sharp(\Phi) \cdot 2 \max\{\sharp(\varphi) \mid \varphi \text{ atom in } \Phi\}$. Again any two consecutive boolean negations are omitted. Analogous considerations apply to the remaining languages.

Let $\mathcal{L}$ be a language and Φ an open $\mathcal{L}$ -formula. A *Tseitin variant* of Φ is an open formula Φ' with some desirable properties: $\sharp(\Phi')$ is of polynomial size in $\sharp(\Phi)$; every atom of Φ' contains at most one function symbol (in particular, terms of Φ' have depth at most 1); and Φ' is equisatisfiable with Φ in any $\mathcal{L}$ -structure $\mathbf{M}$ (i.e., the existential closure of Φ holds in $\mathbf{M}$ iff so does the existential closure of Φ'). We proceed to define Tseitin variants formally.

Let T be the set of all subterms of $\mathcal{L}$ -terms that occur in $\Phi(x_1, \dots, x_n)$. Let $|T|$ be the cardinality of T . To each term φ in T that is not a variable among $x_1, \dots, x_n$, assign a new variable x_φ from Var , not occurring in Φ , with distinct variables assigned to distinct terms. We have $|T| \leq 3\sharp(\Phi)$. Define a new open formula Ψ from Φ as follows. For every atom $\varphi \diamond \psi$ occurring in Φ (here $\diamond$ is among $\{=, \leq, <\}$), if either of φ or ψ is not a variable among $x_1, \dots, x_n$, replace this occurrence of φ or ψ with the new variable x_φ or x_ψ respectively. For convenience, if the term φ is a variable, x_φ denotes the original variable φ . Then the Tseitin variant of Φ is

$$\Phi' := \Psi \wedge \bigwedge_{\substack{\varphi \text{ occurs in } \Phi \\ \varphi \text{ is } f(\varphi_1, \dots, \varphi_n) \text{ for some } n\text{-ary function symbol } f \in \mathcal{L}}} x_\varphi = f(x_{\varphi_1}, \dots, x_{\varphi_n}),$$

where $x_\varphi, x_{\varphi_1}, \dots, x_{\varphi_n}$ are metavariables for elements of Var . Observe Φ' has no compound terms. Furthermore, $\text{At}^\sharp(\Phi') \leq \text{At}^\sharp(\Phi) + |T|$. Lastly, Φ' is indeed equisatisfiable with Φ in any $\mathcal{L}$ -structure $\mathbf{M}$: for any assignment $v_{\mathbf{M}}$ in $\mathbf{M}$ one has that $v_{\mathbf{M}}$ satisfies Φ in $\mathbf{M}$ if and only if Φ' is satisfied by the assignment $v'_{\mathbf{M}}$, where $v'_{\mathbf{M}}$ extends $v_{\mathbf{M}}$ by sending each added variable x_φ to $v_{\mathbf{M}}(\varphi)$.

3 Bounded models

This section establishes a technical result to be used below: if an existential sentence $\exists x_1 \exists x_2 \dots \exists x_n \Phi$ in the language $\mathcal{L}_{\text{pAb}}$ holds in $\mathbf{R}_{-1}$, then there is a satisfying (rational) assignment whose binary representation has size polynomial in $\sharp(\Phi)$. Our exposition follows (in some detail) the one in the paper [15] as to the general method.

► **Lemma 1.** *Let k, m, n be positive natural numbers. Let $Ax \leq b$ be a system of linear inequalities with A an $n \times m$ integer matrix, b a rational n -tuple, and $a_{i,j}, b_i \in [-k, k]$ for $i \leq n, j \leq m$. Assume the system has a solution in $\mathbb{R}$. Then there is a (rational) solution in the interval $[-(mk)^m, (mk)^m]$.*

Proof. The polytope $P = \{x \mid Ax \leq b\}$ is nonempty. Fix a non-empty face of P minimal under inclusion. Such a minimal face is given by a system of equations $\hat{A}x = \hat{b}$, a subsystem of $Ax = b$. To solve this system, it is enough to solve a system $A^0 y = b^0$, where A^0 and b^0 are obtained from $\hat{A}$ and $\hat{b}$ by fixing d columns that contain pivots after Gauss elimination (permuting columns if necessary, assume the first d columns of A) and fixing the corresponding d linearly independent rows. The matrix A^0 is regular of rank d and b^0 is a d -tuple of rationals. A solution of $A^0 y = b^0$ also yields a solution of $\hat{A}x = \hat{b}$ by assigning the value 0 to all variables $x_{d+1}, \dots, x_m$. Cramer's rule yields the solution

$$\langle y_i \rangle_{i \leq d} = \left\langle \frac{\det(A_i^0)}{\det(A^0)} \right\rangle_{i \leq d}.$$

Since A has integer entries, $|\det(A^0)| \geq 1$ and thus $|y_i| \leq |\det(A_i^0)|$ for each $i \leq d$. We have $|\det(A_i^0)| \leq d! \cdot k^d \leq d^d \cdot k^d = (dk)^d$. So $y_i \in [-(dk)^d, (dk)^d]$ for each $i \leq d$, hence for each $i \leq m$. Since $d \leq \min(m, n)$ we have $x_i \in [-(mk)^m, (mk)^m]$ for $i \leq m$. ◀

► **Lemma 2 (Bounded assignment).** *Let $\Phi(x_1, \dots, x_n)$ be an open formula of the language $\mathcal{L}_{\text{pAb}}$ with n variables and let $N = \sharp(\Phi)$. Assume that $\mathbf{R}_{-1} \models \exists x_1 \dots \exists x_n \Phi$. Then there are integers c and M and real numbers $\langle a_1, \dots, a_n \rangle$ such that $\Phi(a_1, \dots, a_n)$ holds in $\mathbf{R}_{-1}$, $-2^M \leq a_i \leq 2^M$ for each $1 \leq i \leq n$, and the binary representation of M has at most $cN \log_2 N$ bits.*

Proof. For convenience, the symbols $\leq$ and $<$ are added to the language $\mathcal{L}_{\text{pAb}}$. Let $\Phi(x_1, \dots, x_n)$ be given. Applying de Morgan rules and removing any double boolean negations that occur, Φ can be rewritten so that any boolean negation is applied to an atom. Using trichotomy, any atom $\neg(\varphi = \psi)$ can be replaced with $(\varphi < \psi) \vee (\psi < \varphi)$. This yields an equivalent negation-free formula $\Phi'(x_1, \dots, x_n)$ whose atoms are identities and strict inequalities. Moreover $\text{At}^\sharp(\Phi') \leq 2\text{At}^\sharp(\Phi)$.

Φ' can be brought to a DNF without negations, obtaining a

$$\Phi'' = \Psi_1 \vee \Psi_2 \vee \dots \vee \Psi_m$$

for some m , where each Ψ_j for $1 \leq j \leq m$ is a conjunction of atoms of the form $(\varphi = \psi)$ or $(\varphi < \psi)$. The value of m need not be polynomial in $\text{At}^\sharp(\Phi')$. The remaining part of the proof establishes that a bounded satisfying assignment to Φ exists on the basis of the existence of the formula $\Phi'' = \text{DNF}(\Phi')$ and the fact, which we proceed to show, of an existence of a bounded satisfying assignment to at least one among the Ψ_j . Both Φ' and Φ'' are equivalent to Φ ; namely, a tuple $\langle a_1, \dots, a_n \rangle$ makes Φ true in $\mathbf{R}_{-1}$ if and only if it makes Φ' true, and the latter is the case if and only if it makes some Ψ_j true for some $j \leq m$. Trivially $\text{At}^\sharp(\Psi_j) \leq \text{At}^\sharp(\Phi')$ for $1 \leq j \leq m$.

Next, we remove any complex terms from Φ'' using a Tseitin transformation. Let $\Psi'_1, \dots, \Psi'_m$ be Tseitin variants of $\Psi_1, \dots, \Psi_m$. Without loss of generality, we assume that the Tseitin transformation assigns the same variable x_t to identical terms t , regardless of the disjunct in which they appear. Consequently, $|\bigcup_{j=1}^m \text{Var}(\Psi'_j)| = |T|$, where $|T|$ is the cardinality of the set T of all $\mathcal{L}_{\text{pAb}}(\text{sub})$ terms that occur in Φ' .

Let us set

$$(\Phi''') = (\Psi'_1 \wp \dots \wp \Psi'_m).$$

Since Ψ_j is equisatisfiable with Ψ'_j for each $j \leq m$, we obtain that Φ'' is equisatisfiable with Φ''' . Let k denote the total number of variables in Φ''' . Clearly, we have $k = |T| \leq 3\sharp(\Phi')$.

The formula Φ''' is still in DNF and we have $\text{At}^\sharp(\Psi'_j) \leq |T| + \sharp(\Psi_j) \leq 4\sharp(\Phi')$ for each $j \leq m$.

By the assumption, there is an assignment v in $\mathbf{R}_{-1}$ with $v(x_i) = a_i$ for $i \leq k$ such that the vector $\langle a_1, \dots, a_k \rangle$ makes both Φ and Φ''' true in $\mathbf{R}_{-1}$. Fix such an assignment v . Then there is a total ordering given by a permutation σ of $\{1, \dots, k\}$ such that, in $\mathbf{R}$, $a_{\sigma(i)} \leq a_{\sigma(i+1)}$ for each $i < k$. Let

$$\rho(x_1, \dots, x_k) := \bigwedge_{\{i < k\}; a_{\sigma(i)} < a_{\sigma(i+1)}} (x_{\sigma(i)} < x_{\sigma(i+1)}) \wp \bigwedge_{\{i < k\}; a_{\sigma(i)} = a_{\sigma(i+1)}} (x_{\sigma(i)} = x_{\sigma(i+1)}).$$

We have $\text{At}^\sharp(\rho) \leq k$. For some $j \leq m$, $(\rho \wp \Psi'_j)(a_1, \dots, a_k)$ is true in $\mathbf{R}_{-1}$. Fix such $j \leq m$ and let us show that there exists an assignment that makes the formula $(\rho \wp \Psi'_j)$ true in $\mathbf{R}_{-1}$ and whose values are bounded as required. We refer to the formula Ψ'_j simply as Ψ .

Using ρ , one can omit atoms with the function symbols $\wedge$ and $\vee$. Indeed, since $\rho \wp \Psi$ is true under v , it must also be the case that if, e.g., $x_\varphi \wedge x_\psi = x_\chi$ occurs as an atom in Ψ (which means either that the term χ is the term $\varphi \wedge \psi$ in Φ , or that $\chi = \varphi \wedge \psi$ is an identity in Φ), then the identity $\varphi(a) \wedge \psi(a) = \chi(a)$ holds and hence, this is captured by ρ , namely $x_\chi =_\rho \min_\rho(x_\varphi, x_\psi)$. The other cases featuring $\vee$ and $<$ are analogous. Thus ρ already captures all the information that is provided by those atoms in $\rho \wp \Psi$ in which the lattice function symbols $\wedge$ and $\vee$ occur, whereby these atomic formulas are redundant and can be omitted from the conjunction.

Such an omission yields a new conjunction of identities and strict inequalities $\rho \wp \Psi^*$. This formula has k variables, and

$$l := \text{At}^\sharp(\rho \wp \Psi^*) = \text{At}^\sharp(\rho) + \text{At}^\sharp(\Psi'_j) \leq \sharp(\Phi') + 4\sharp(\Phi') = 5\sharp(\Phi') \leq 10\sharp(\Phi).$$

Now $\rho \wp \Psi^*$ defines a system S of linear equations and strict inequalities in $\mathbf{R}$ as follows. Any atomic formula in $\rho \wp \Psi^*$ has at most three distinct variables and at most one function symbol. Subtracting the expression on right-hand side of each atomic formula from both sides, we get a system consisting of a subsystem $Ax = c$ with l_1 rows, and a subsystem $Bx < d$ with l_2 rows, both in the variables $x_1, \dots, x_k$ and with $l_1 + l_2 = l$, thus $l_1, l_2 \leq 10\sharp(\Phi)$. Since $\langle a_1, \dots, a_k \rangle$ is a solution to the system, there has to exist a rational vector $d' = \langle d'_1, \dots, d'_{l_2} \rangle$ where $d_i - 1 \leq d'_i < d_i$, such that a modified system S' consisting of $Ax = c$ and $Bx \leq d'$ still has a solution. Clearly, any solution to S' is also a solution to S .

Rewriting each of the equations in $Ax = c$ as two inequalities and adding the rows of $Bx \leq d'$ yields a system $A'x \leq b'$, s.t. there is an integer $c \leq 20$ with the number of both rows and columns in $A'x \leq b'$ bounded by $c \cdot \sharp(\Phi)$. A' is an integer matrix, b' is a rational vector, and the absolute values of entries in either are bounded by 3. Applying Lemma 1 yields a solution whose absolute value is bounded by $(3c\sharp(\Phi))^{c\sharp(\Phi)}$. It is then enough to pick $M = \lceil c\sharp(\Phi) \log_2(3c\sharp(\Phi)) \rceil$. $\blacktriangleleft$

Using the tools of this section, one might proceed to show that the existential theory of $\mathbf{R}_{-1}$ is in NP. Indeed, one gets almost for free from the above, for any open $\mathcal{L}_{\text{pAb}}$ -formula Φ with a real assignment that satisfies it, also a satisfying assignment that is rational with both the numerators and the denominators of polynomial size in $\sharp(\Phi)$. This assignment is then a witness of the fact that the existential closure of Φ is true, whence the existential theory of $\mathbf{R}_{-1}$ is in NP. However, our intent is to use the results of this section to obtain suitable maps on the reals in the next section, and hence a polynomial-time reduction of the existential theory of $\mathbf{R}_{-1}$ to the existential theory of $[0, 1]_{\mathbb{L}}$. In other words, we answer the satisfiability question, the instances of which are open formulas, indirectly. It is worth remarking that, if one established NP-completeness of the existential theory of $\mathbf{R}_{-1}$ anyhow, then one would still get a poly-time reduction between this problem and the existential theory of $[0, 1]_{\mathbb{L}}$, because the latter problem is known to be NP-complete (using, e.g., [21, 24]). Our reduction has the added value of being explicit.

4 Reducing the $\exists$ -theory of $\mathbf{R}_{-1}$ to the $\exists$ -theory of $[0, 1]_{\mathbb{L}}$

Consider the mapping $\tau: \mathcal{T}m_{\text{Ab}} \rightarrow \mathcal{T}m_{\text{MV}[\frac{1}{2}]}$, assigning to each term φ of $\mathcal{L}_{\text{Ab}}$ a term of $\mathcal{L}_{\text{MV}[\frac{1}{2}]}$ as follows:

1. $\tau(x) = x$ for $x \in \text{Var}(\varphi)$;
2. $\tau(0) = \frac{1}{2}$;
3. $\tau(-\varphi) = \neg(\tau(\varphi))$;
4. $\tau(\varphi + \psi) = (\tau(\varphi) \oplus \tau(\psi)) \otimes (\frac{1}{2} \oplus (\tau(\varphi) \otimes \tau(\psi)))$;
5. $\tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi)$;
6. $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$.

For each pair of nonnegative integers M and k , let $r_{M,k}: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$r_{M,k}(a) = \frac{a}{2^{M+k+1}} + \frac{1}{2}. \quad (1)$$

► **Lemma 3.** *Let M and k be fixed nonnegative integers. Let $\varphi(x_1, \dots, x_n)$ be a term in $\mathcal{T}m_{\text{Ab}}$ with n variables, let $\text{depth}(\varphi) = l$ with $l \leq k$. Assume $a_1, \dots, a_n \in [-2^M, 2^M]$. Then*

$$f_{\tau(\varphi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = r_{M,k}(f_{\varphi}(a_1, \dots, a_n)).$$

Moreover, $r_{M,k}(f_{\varphi}(a_1, \dots, a_n)) \in [\frac{1}{2} - \frac{1}{2^{k-l+1}}, \frac{1}{2} + \frac{1}{2^{k-l+1}}]$.

Proof. First let us note that for each $a \in [-2^M, 2^M]$ it holds that $r_{M,k}(a) \in [\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$. We will proceed by induction on term depth.

■ Let φ be a variable x . By definition we have

$$f_{\tau(\varphi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = r_{M,k}(f_{\varphi}(a_1, \dots, a_n)) \in [\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}].$$

■ Let $\varphi = 0$. We have

$$f_{\tau(\varphi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = \frac{1}{2} = r_{M,k}(f_{\varphi}(a_1, \dots, a_n)).$$

Now assume $\text{depth}(\varphi) = l$ and the statement holds for any $l' < l$.

- Let φ be $-\psi$. We assume $f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = r_{M,k}(f_\psi(a_1, \dots, a_n))$. We have

$$\begin{aligned} f_{\tau(-\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= f_{-\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = \\ 1 - f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= 1 - r_{M,k}(f_\psi(a_1, \dots, a_n)) = \\ 1 - \left(\frac{f_\psi(a_1, \dots, a_n)}{2^{M+k+1}} + \frac{1}{2} \right) &= \frac{1}{2} - \frac{f_\psi(a_1, \dots, a_n)}{2^{M+k+1}} = \frac{1}{2} + \frac{f_{-\psi}(a_1, \dots, a_n)}{2^{M+k+1}} = \\ &= r_{M,k}(f_{-\psi}(a_1, \dots, a_n)). \end{aligned}$$

Moreover, since by (the induction) hypothesis

$$f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) \in \left[\frac{1}{2} - \frac{1}{2^{k-(l-1)+1}}, \frac{1}{2} + \frac{1}{2^{k-(l-1)+1}} \right],$$

we obtain

$$\begin{aligned} f_{\tau(-\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= \\ 1 - f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &\in \left[\frac{1}{2} - \frac{1}{2^{k-l+2}}, \frac{1}{2} + \frac{1}{2^{k-l+2}} \right] \subseteq \left[\frac{1}{2} - \frac{1}{2^{k-l+1}}, \frac{1}{2} + \frac{1}{2^{k-l+1}} \right]. \end{aligned}$$

- Let $\varphi = \psi \wedge \gamma$. We assume

$$\begin{aligned} f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= r_{M,k}(f_\psi(a_1, \dots, a_n)), \\ f_{\tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= r_{M,k}(f_\gamma(a_1, \dots, a_n)). \end{aligned}$$

We obtain

$$\begin{aligned} f_{\tau(\psi \wedge \gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= f_{\tau(\psi) \wedge \tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = \\ (f_{\tau(\psi)} \wedge f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= ((r_{M,k} \circ f_\psi) \wedge (r_{M,k} \circ f_\gamma))(a_1, \dots, a_n). \end{aligned}$$

Since $r_{M,k}$ is order preserving we obtain

$$((r_{M,k} \circ f_\psi) \wedge (r_{M,k} \circ f_\gamma))(a_1, \dots, a_n) = r_{M,k}((f_\psi \wedge f_\gamma)(a_1, \dots, a_n)) = r_{M,k}((f_{\psi \wedge \gamma})(a_1, \dots, a_n)).$$

Since

$$f_{\tau(\psi \wedge \gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) \in \{f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)), f_{\tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n))\},$$

then necessarily

$$f_{\tau(\psi \wedge \gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) \in \left[\frac{1}{2} - \frac{1}{2^{k-(l-1)+1}}, \frac{1}{2} + \frac{1}{2^{k-(l-1)+1}} \right] \subseteq \left[\frac{1}{2} - \frac{1}{2^{k-l+1}}, \frac{1}{2} + \frac{1}{2^{k-l+1}} \right].$$

- The case $\varphi = \psi \vee \gamma$ is analogous.

- Now let φ be $\psi + \gamma$. We assume

$$\begin{aligned} f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= r_{M,k}(f_\psi(a_1, \dots, a_n)), \\ f_{\tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= r_{M,k}(f_\gamma(a_1, \dots, a_n)) \end{aligned}$$

and since $l = \text{depth}(\varphi) = \max(\text{depth}(\psi), \text{depth}(\gamma)) + 1$ we get

$$f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)), f_{\tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) \in \left[\frac{1}{2} - \frac{1}{2^{k-l+2}}, \frac{1}{2} + \frac{1}{2^{k-l+2}} \right].$$

We distinguish two cases:

1. In case

$$(f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) \geq 1$$

we have $f_{\tau(\psi) \oplus \tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = 1$ and

$$f_{\tau(\psi) \otimes \tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) - 1.$$

Thus, we have

$$\begin{aligned} f_{\tau(\psi+\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= \\ f_{(\tau(\psi) \oplus \tau(\gamma)) \otimes (\frac{1}{2} \oplus (\tau(\psi) \otimes \tau(\gamma)))}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= \\ f_{1 \otimes (\frac{1}{2} \oplus (\tau(\psi) \otimes \tau(\gamma)))}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= \\ f_{\frac{1}{2} \oplus (\tau(\psi) \otimes \tau(\gamma))}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= \\ \min(1, f_{\tau(\psi) \otimes \tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) + \frac{1}{2}) &= \\ \min(1, (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) - 1 + \frac{1}{2}) &= \\ \min(1, (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) - \frac{1}{2}) &= \\ \min(1, (r_{M,k} \circ f_{\psi} + r_{M,k} \circ f_{\gamma})(a_1, \dots, a_n) - \frac{1}{2}). \end{aligned}$$

Moreover, by the induction hypothesis

$$(r_{M,k} \circ f_{\psi} + r_{M,k} \circ f_{\gamma})(a_1, \dots, a_n) - \frac{1}{2} \in [\frac{1}{2} - \frac{1}{2^{k-l+1}}, \frac{1}{2} + \frac{1}{2^{k-l+1}}]$$

and thus, we can conclude

$$\min(1, (r_{M,k}(f_{\psi} + f_{\gamma}))(a_1, \dots, a_n)) - \frac{1}{2} = (r_{M,k} \circ f_{\psi} + r_{M,k} \circ f_{\gamma})(a_1, \dots, a_n) - \frac{1}{2}.$$

Therefore,

$$\begin{aligned} f_{\tau(\psi+\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= (r_{M,k} \circ f_{\psi} + r_{M,k} \circ f_{\gamma})(a_1, \dots, a_n) - \frac{1}{2} = \\ \frac{(f_{\psi} + f_{\gamma})(a_1, \dots, a_n)}{2^{M+k+1}} + \frac{1}{2} &= r_{M,k}(f_{\psi+\gamma}(a_1, \dots, a_n)). \end{aligned}$$

This completes the proof of the first case.

2. In case

$$(f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) \leq 1,$$

we have $f_{\tau(\psi) \otimes \tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = 0$ and

$$f_{\tau(\psi) \oplus \tau(\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)).$$

Thus, we have

$$\begin{aligned}
& f_{\tau(\psi+\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = \\
& f_{(\tau(\psi) \oplus \tau(\gamma)) \otimes (\frac{1}{2} \oplus (\tau(\psi) \otimes \tau(\gamma)))}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = \\
& f_{(\tau(\psi) \oplus \tau(\gamma)) \otimes (\frac{1}{2} \oplus 0)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = \\
& f_{(\tau(\psi) \oplus \tau(\gamma)) \otimes \frac{1}{2}}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = \\
& \max(0, (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) + \frac{1}{2} - 1) = \\
& \max(0, (f_{\tau(\psi)} + f_{\tau(\gamma)})(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) - \frac{1}{2}) = \\
& \max(0, (r_{M,k} \circ f_{\psi} + r_{M,k} \circ f_{\gamma})(a_1, \dots, a_n) - \frac{1}{2}).
\end{aligned}$$

Since by the induction hypothesis

$$(r_{M,k} \circ f_{\psi} + r_{M,k} \circ f_{\gamma})(a_1, \dots, a_n) - \frac{1}{2} \in [\frac{1}{2} - \frac{1}{2^{k-l+1}}, \frac{1}{2} + \frac{1}{2^{k-l+1}}]$$

we can conclude that

$$\begin{aligned}
f_{\tau(\psi+\gamma)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) &= (r_{M,k} \circ f_{\psi} + r_{M,k} \circ f_{\gamma})(a_1, \dots, a_n) - \frac{1}{2} = \\
& \frac{(f_{\psi} + f_{\gamma})(a_1, \dots, a_n)}{2^{M+k+1}} + \frac{1}{2} = r_{M,k}(f_{\psi+\gamma}(a_1, \dots, a_n)).
\end{aligned}$$

This completes the second case and thus concludes the proof. $\blacktriangleleft$

We will extend τ to open $\mathcal{L}_{\text{Ab}}$ -formulas. For Φ of $\mathcal{L}_{\text{Ab}}$ define $\tau'(\Phi)$ by replacing each occurrence of an atom $\varphi = \psi$ with the atom $\tau(\varphi) = \tau(\psi)$. By definition then, τ' does not interfere with the boolean structure of Φ .

► **Lemma 4.** *Let M and k be fixed nonnegative integers, let Φ be an open formula in $\mathcal{L}_{\text{Ab}}$ with n variables such that k is an upper bound on the depth of terms in Φ . Then for any $a_1, \dots, a_n \in [-2^M, 2^M]$*

$$\mathbf{R} \models \Phi(a_1, \dots, a_n) \text{ if and only if } [0, 1]_{\mathbb{L}}^{1/2} \models \tau'(\Phi)(r_{M,k}(a_1), \dots, r_{M,k}(a_n)). \quad (2)$$

Equivalently, for any $b_1, \dots, b_n \in [\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$

$$[0, 1]_{\mathbb{L}}^{1/2} \models \tau'(\Phi)(b_1, \dots, b_n) \text{ if and only if } \mathbf{R} \models \Phi(r_{M,k}^{-1}(b_1), \dots, r_{M,k}^{-1}(b_n)). \quad (3)$$

Proof. We only prove the first equivalence, the second following from the first one using the fact that $r_{M,k}$ is a bijection between $[-2^M, 2^M]$ and $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$. The proof is by induction on boolean structure of Φ . For an atom $\varphi = \psi$ (where φ, ψ are terms), if $f_{\varphi}(a_1, \dots, a_n) = f_{\psi}(a_1, \dots, a_n)$ then $r_{M,k}(f_{\varphi}(a_1, \dots, a_n)) = r_{M,k}(f_{\psi}(a_1, \dots, a_n))$ and using Lemma 3,

$$f_{\tau(\varphi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) = f_{\tau(\psi)}(r_{M,k}(a_1), \dots, r_{M,k}(a_n)),$$

which implies $[0, 1]_{\mathbb{L}}^{1/2} \models \tau'(\varphi = \psi)(r_{M,k}(a_1), \dots, r_{M,k}(a_n))$. The other implication is obtained analogously using the fact that $r_{M,k}$ is one-to-one.

Now assume the induction hypothesis for open formulas Ψ and Δ , that is

$$\mathbf{R} \models \Psi(a_1, \dots, a_n) \text{ iff } [0, 1]_{\mathbb{L}}^{1/2} \models \tau'(\Psi)(r_{M,k}(a_1), \dots, r_{M,k}(a_n)) \text{ and} \quad (4)$$

$$\mathbf{R} \models \Delta(a_1, \dots, a_n) \text{ iff } [0, 1]_{\mathbb{L}}^{1/2} \models \tau'(\Delta)(r_{M,k}(a_1), \dots, r_{M,k}(a_n)). \quad (5)$$

14:12 Satisfiability in Łukasiewicz Logic and Its Unbounded Relative

First, since

$$\mathbf{R} \not\models \Psi(a_1, \dots, a_n) \text{ iff } [0, 1]_{\mathbb{L}}^{1/2} \not\models \tau'(\Psi)(r_{M,k}(a_1), \dots, r_{M,k}(a_n)),$$

and $\tau'(\neg\Psi) = \neg\tau'(\Psi)$, it follows that

$$\mathbf{R} \models \neg\Psi(a_1, \dots, a_n) \text{ iff } [0, 1]_{\mathbb{L}}^{1/2} \models \tau'(\neg\Psi)(r_{M,k}(a_1), \dots, r_{M,k}(a_n)).$$

Also, $\tau'(\Psi \mathbin{\&\&} \Delta) = \tau'(\Psi) \mathbin{\&\&} \tau'(\Delta)$ and thus clearly

$$\mathbf{R} \models (\Psi \mathbin{\&\&} \Delta)(a_1, \dots, a_n) \text{ iff } [0, 1]_{\mathbb{L}}^{1/2} \models \tau'(\Psi \mathbin{\&\&} \Delta)(r_{M,k}(a_1), \dots, r_{M,k}(a_n)).$$

The induction step for the connective $\mathbb{W}$ can be checked analogously. $\blacktriangleleft$

Let us recap what has been established. Let M and k be fixed nonnegative integers. Let Φ be an open formula with n variables in $\mathcal{L}_{\text{Ab}}$ and term depth bounded by k be given. Using the bijection $r_{M,k}$ of $[-2^M, 2^M]$ onto $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$ one can map satisfying assignments of (variables in) Φ in $\mathbf{R}_{-1}$ to those of $\tau'(\Phi)$ in $[0, 1]_{\mathbb{L}}$ (which fall into $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$), and conversely, any satisfying assignment to $\tau'(\Phi)$ in the interval $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$ of $[0, 1]_{\mathbb{L}}$ can be mapped to a satisfying assignment to Φ (in $[-2^M, 2^M]$).

The translation τ' , as presented, can lead to an exponential increase in term size. For example $\tau(\dots(x_1 + x_2) + x_3) + \dots + x_n$ has 2^{n-1} occurrences of variable x_1 . To attain a polynomial-size translation of open $\mathcal{L}_{\text{pAb}}$ -formulas to open $\mathcal{L}_{\text{MV}}$ -formulas, complex terms need to be removed from Φ while preserving satisfying assignments, before applying τ' . This can be done relying (again) on Tseitin transformations.

► **Lemma 5.** *Let Φ be a $\mathcal{L}_{\text{Ab}}$ formula. Let Φ' be the result of a Tseitin transformation of Φ . Then for any assignment v in $[0, 1]_{\mathbb{L}}^{1/2}$ one has that v satisfies $\tau'(\Phi)$ if and only if $\tau'(\Phi')$ is satisfied by the assignment v' , where v' uniquely extends v by sending each added variable x_φ to $v(\tau(\varphi))$. Moreover, $\sharp(\tau'(\Phi'))$ is polynomial in $\sharp(\Phi)$.*

► **Lemma 6.** *Let M, k be fixed nonnegative integers. Consider the following set of equations in $\mathcal{L}_{\text{MV}}$:*

1. $z_1 = \neg z_1$.
2. $(z_{i+1} \oplus z_{i+1} = z_i)$ for each $i \leq M + k$.
3. $q = \neg(z_1 \oplus z_{M+k+1})$ and $r = \neg(z_1 \oplus z_{k+1})$.

Any assignment v in $[0, 1]_{\mathbb{L}}$ that satisfies all these equations has $v(z_i) = \frac{1}{2^i}$, $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$ and $v(r) = \frac{1}{2} - \frac{1}{2^{k+1}}$.

Notice that in the following lemma, the translation S operates on $\mathcal{L}_{\text{pAb}}$ -formulas and brings them to $\mathcal{L}_{\text{MV}}$ -formulas.

► **Lemma 7.** *Let $\Phi(x_1, \dots, x_n)$ be an open formula in the language $\mathcal{L}_{\text{pAb}}$. Let Φ' be a Tseitin variant of Φ . Assume that the variables $\{z_1, \dots, z_{M+k+1}, r, q\}$ do not occur in $\text{Var}(\Phi')$. Let $\zeta(\Phi')$ be obtained from Φ' by replacing each occurrence of the constant -1 in Φ' with the variable q , applying τ' to the resulting formula and then replacing all occurrences of the constant $1/2$ with the variable z_1 . Let k be the maximal depth of terms in Φ and, denoting $N := \sharp(\Phi)$, set $M := \lceil cN \log_2 N \rceil$ for some large integer c . Let $S(\Phi, M, k)$ be the following open formula in $\mathcal{L}_{\text{MV}}$:*

$$\begin{aligned} &\zeta(\Phi') \mathbin{\&\&} (z_1 = \neg z_1) \mathbin{\&\&} (r = \neg(z_1 \oplus z_{k+1})) \mathbin{\&\&} \\ &(q = \neg(z_1 \oplus z_{M+k+1})) \mathbin{\&\&} \bigwedge_{i \leq M+k} (z_{i+1} \oplus z_{i+1} = z_i) \mathbin{\&\&} \bigwedge_{i \leq n} (r \leq x_i) \mathbin{\&\&} \bigwedge_{i \leq n} (x_i \leq \neg r). \end{aligned}$$

Then Φ has a satisfying assignment in $\mathbf{R}_{-1}$ if and only if $S(\Phi, M, k)$ has a satisfying assignment in $[0, 1]_{\mathbb{L}}$. Moreover, $\sharp(S(\Phi, M, k))$ is polynomial in $\sharp(\Phi)$.

Proof. Assume first that $\Phi(x_1, \dots, x_n)$ is satisfied in $\mathbf{R}_{-1}$. By Lemma 2, there is a satisfying assignment $v(x_i) = a_i$ with $a_i \in [-2^M, 2^M]$. Then Φ' in variables $x_1, \dots, x_m$ (with $m \geq n$) is satisfied by any assignment that extends v on $x_1, \dots, x_n$ by sending the extra variables $x_{n+1}, \dots, x_m$ to the values $a_{n+1}, \dots, a_m$ of subterms in Φ in the interval $[-2^{M+k+1}, 2^{M+k+1}]$. W.l.o.g. we assume that v itself is this extended, satisfying assignment. Apply Lemma 4, sending -1 to $\frac{1}{2} - \frac{1}{2^{M+k+1}}$ via the mapping $r_{M,k}$. Then $\zeta(\Phi')$ has a satisfying assignment $r_{M,k}(a_1), \dots, r_{M,k}(a_m)$ in $[0, 1]_{\mathbb{L}}$ and moreover, by construction, one that sends q to $\frac{1}{2} - \frac{1}{2^{M+k+1}}$ and z_1 to $\frac{1}{2}$. By Lemma 6, the formula

$$(z_1 = \neg z_1) \mathbin{\&\&} (q = \neg(z_1 \oplus z_{M+k+1})) \mathbin{\&\&} (r = \neg(z_1 \oplus z_{k+1})) \mathbin{\&\&} \bigwedge_{i \leq M+k} (z_{i+1} \oplus z_{i+1} = z_i) \quad (6)$$

has the unique satisfying assignment in $[0, 1]_{\mathbb{L}}$, which sends z_i to $1/2^i$ for each $i \leq M+k$, q to $\frac{1}{2} - \frac{1}{2^{M+k+1}}$ and r to $\frac{1}{2} - \frac{1}{2^{k+1}}$. By construction, we have that $r_{M,k}(a_i)$ for $i \leq n$ belongs to $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$, so $\bigwedge_{i \leq n} (r \leq x_i) \mathbin{\&\&} \bigwedge_{i \leq n} (x_i \leq \neg r)$ is satisfied too.

For the converse implication assume $S(\Phi, M, k)$ is satisfied in $[0, 1]_{\mathbb{L}}$ by an assignment v . Let $v(x_i) = b_i$ for $i \leq n$. In particular, then, Equation 6 is satisfied and hence by Lemma 6, $v(z_1) = \frac{1}{2}$, $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$ and $v(r) = \frac{1}{2} - \frac{1}{2^{k+1}}$. This implies that $v(x_i)$ for $i \leq n$ are in $[\frac{1}{2} - \frac{1}{2^{k+1}}, \frac{1}{2} + \frac{1}{2^{k+1}}]$. Using Lemma 5 one can easily show that any evaluation satisfying $\zeta(\Phi')$ will also satisfy $\zeta(\Phi)$. Thus v satisfies $\zeta(\Phi)$ with $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$ and $v(z_1) = \frac{1}{2}$. By Lemma 4 the assignment $v'(x_i) = r_{M,k}^{-1}(b_i)$ for $i \leq n$ satisfies Φ , which completes the proof. $\blacktriangleleft$

► **Theorem 8.** *The existential theory of $\mathbf{R}_{-1}$ reduces in polynomial time to the existential theory of $[0, 1]_{\mathbb{L}}$. Therefore, the former theory is in NP.*

It follows that the universal theory of $\mathbf{R}_{-1}$ is in coNP. It was already established in the paper [12] that the equational theory of $\mathbf{R}_{-1}$ is coNP-hard. Hence the universal theory and the quasiequational theory of $\mathbf{R}_{-1}$ are coNP-complete and the existential theory is NP-complete.

5 Reducing the $\exists$ -theory of $[0, 1]_{\mathbb{L}}$ to itself

This section investigates the composition of the reduction from $[0, 1]_{\mathbb{L}}$ to $\mathbf{R}_{-1}$ and that from $\mathbf{R}_{-1}$ to $[0, 1]_{\mathbb{L}}^{1/2}$. First let us introduce a variant of the translation used in [12] which provided a faithful interpretation of the universal theory of $[0, 1]_{\mathbb{L}}$ (in the language $\mathcal{L}_{MV}$) in the universal theory of $\mathbf{R}_{-1}$ (in the language $\mathcal{L}_{pAb}$), mentioned in Section 1. Such a mapping δ from Tm_{MV} to Tm_{pAb} is defined recursively as follows:

1. $\delta(x) = (x \vee -1) \wedge 0$ for $x \in \text{Var}$;
2. $\delta(\varphi \rightarrow \xi) = (\delta(\xi) - \delta(\varphi)) \wedge 0$;
3. $\delta(\varphi \otimes \psi) = (\delta(\varphi) + \delta(\psi)) \vee -1$;
4. $\delta(0) = -1$;
5. $\delta(1) = 0$;
6. $\delta(\varphi \circ \xi) = \delta(\varphi) \circ \delta(\xi)$ for $\circ \in \{\wedge, \vee\}$.

► **Lemma 9.** *Let $\varphi(x_1, \dots, x_n)$ be a term in $\mathcal{L}_{MV}$ with n variables. Assume $a_1, \dots, a_n$ are elements of the domain of $[0, 1]_{\mathbb{L}}$. Then $f_{\delta(\varphi)}((a_1 - 1), \dots, (a_n - 1)) + 1 = f_{\varphi}(a_1, \dots, a_n)$.*

Proof. As in the proof of [12, Theorem 5.8]. $\blacktriangleleft$

14:14 Satisfiability in Łukasiewicz Logic and Its Unbounded Relative

As in the case of τ defined in Section 4, the mapping δ extends to a mapping on open formulas of $\mathcal{L}_{MV}$: for Φ of $\mathcal{L}_{MV}$ define $\delta'(\Phi)$ by replacing each occurrence of an atom $\varphi = \psi$ with the atom $\delta(\varphi) = \delta(\psi)$.

► **Lemma 10.** *Let Φ be an open formula in $\mathcal{L}_{MV}$ with n variables and let $a_1, \dots, a_n \in [0, 1]$. Then*

$$[0, 1]_{\mathbb{L}} \models \Phi(a_1, \dots, a_n) \text{ if and only if } \mathbf{R}_{-1} \models \delta'(\Phi)(a_1 - 1, \dots, a_n - 1).$$

Proof. As in the proof of Lemma 4. ◀

Our aim is to compose δ with τ . This is not immediately possible, since τ is defined only on Tm_{Ab} (lacking the constant -1). First, therefore, the definition of τ will be extended to its variant τ_q . Let $\varphi \in Tm_{pAb}$ and let q be a variable not used in φ . The function $\tau_q(\varphi)$ is defined recursively as follows:

1. $\tau_q(x) = (x \vee q) \wedge \frac{1}{2}$ for $x \in \text{Var}(\varphi)$;
2. $\tau_q(0) = \frac{1}{2}$;
3. $\tau_q(-1) = q$;
4. $\tau_q(-\varphi) = \neg(\tau_q(\varphi))$;
5. $\tau_q(\varphi \circ \psi) = \tau_q(\varphi) \circ \tau_q(\psi)$ for $\circ \in \{\vee, \wedge\}$;
6. $\tau_q(\varphi + \psi) = (\tau_q(\varphi) \oplus \tau_q(\psi)) \otimes (\frac{1}{2} \oplus (\tau_q(\varphi) \otimes \tau_q(\psi)))$.

We now look into the composition $\tau_q \circ \delta$ which maps Tm_{MV} into $Tm_{MV[\frac{1}{2}]}$. The following lemma shows that, as far as the semantics provided by $[0, 1]_{\mathbb{L}}$ is concerned, the composition of these two translations can be expressed by a function that increases the length of a term polynomially, rather than exponentially as one might expect, given the behaviour of τ discussed in Section 4.

► **Lemma 11.** *Let $\varphi \in Tm_{MV}$ and let q be a variable not among $\text{Var}(\varphi)$. The functions $f_{\tau_q(\delta(\varphi))}$ and $f_{\sigma_q(\varphi)}$ coincide on the domain of $[0, 1]_{\mathbb{L}}$, where the mapping σ_q on Tm_{MV} is defined recursively as follows:*

1. $\sigma_q(x) = (x \vee q) \wedge \frac{1}{2}$ for $x \in \text{Var}(\varphi)$;
2. $\sigma_q(1) = \frac{1}{2}$;
3. $\sigma_q(0) = q$;
4. $\sigma_q(a \rightarrow b) = ((\sigma_q(a) \rightarrow \sigma_q(b)) \otimes \frac{1}{2}) \wedge \frac{1}{2}$;
5. $\sigma_q(x \otimes y) = ((\sigma_q(x) \oplus \sigma_q(y)) \otimes \frac{1}{2}) \vee q$;
6. $\sigma_q(\varphi \circ \xi) = \sigma_q(\varphi) \circ \sigma_q(\xi)$ for $\circ \in \{\wedge, \vee\}$.

Moreover, for any term ψ in $\mathcal{L}_{MV}$, $\sharp(\sigma_q(\psi))$ is polynomial in $\sharp(\psi)$.

Proof. By induction on term depth.

- Base case (depth 0): By definition $\sigma_q(x) = (x \vee q) \wedge \frac{1}{2} = \tau_q(\delta(x))$, $\sigma_q(1) = \frac{1}{2} = \tau_q(\delta(1))$ and $\sigma_q(0) = q = \tau_q(\delta(0))$.

Now assume that $f_{\sigma_q(\psi)} = f_{\tau_q(\delta(\psi))}$ and $f_{\sigma_q(\gamma)} = f_{\tau_q(\delta(\gamma))}$.

- Clearly, for $\varphi = \psi \circ \gamma$ we have $f_{\tau_q(\delta(\psi \circ \gamma))} = f_{\sigma_q(\psi \circ \gamma)}$ for $\circ \in \{\vee, \wedge\}$.
- For $\varphi = \psi \rightarrow \gamma$ we have

$$f_{\tau_q(\delta(\psi \rightarrow \gamma))} = f_{\tau_q((\delta(\gamma) - \delta(\psi)) \wedge 0)} = f_{((\neg \tau_q(\delta(\psi)) \oplus \tau_q(\delta(\gamma))) \otimes (\frac{1}{2} \oplus (\neg \tau_q(\delta(\psi)) \otimes \tau_q(\delta(\gamma)))) \wedge \frac{1}{2}}.$$

By the induction hypothesis this is equivalent to

$$f_{((\neg \sigma_q(\psi) \oplus \sigma_q(\gamma)) \otimes (\frac{1}{2} \oplus (\neg \sigma_q(\psi) \otimes \sigma_q(\gamma)))) \wedge \frac{1}{2}} = \max(0, f_{\neg \sigma_q(\psi) \oplus \sigma_q(\gamma)} + f_{\frac{1}{2} \oplus (\neg \sigma_q(\psi) \otimes \sigma_q(\gamma))} - 1) \wedge \frac{1}{2}.$$

Let us distinguish two cases.

1. First assume $(f_{\neg\sigma_q(\psi)} + f_{\sigma_q(\gamma)})(a_1, \dots, a_n) \geq 1$. Thus, $f_{\neg\sigma_q(\psi) \oplus \sigma_q(\gamma)}(a_1, \dots, a_n) = 1$ and consequently

$$\begin{aligned} & \max(0, (f_{\neg\sigma_q(\psi) \oplus \sigma_q(\gamma)} + f_{\frac{1}{2} \oplus (\neg\sigma_q(\psi) \otimes \sigma_q(\gamma))})(a_1, \dots, a_n) - 1) \wedge \frac{1}{2} = \\ & \max(0, f_{\frac{1}{2} \oplus (\neg\sigma_q(\psi) \otimes \sigma_q(\gamma))}(a_1, \dots, a_n)) \wedge \frac{1}{2} = \frac{1}{2} = \\ & f_{((\sigma_q(\psi) \rightarrow \sigma_q(\gamma)) \otimes \frac{1}{2}) \wedge \frac{1}{2}}(a_1, \dots, a_n) = f_{\sigma_q(\psi \rightarrow \gamma)}(a_1, \dots, a_n). \end{aligned}$$

2. For the other case assume $(f_{\neg\sigma_q(\psi)} + f_{\sigma_q(\gamma)})(a_1, \dots, a_n) \leq 1$. Then we have

$$f_{((\neg\sigma_q(\psi) \oplus \sigma_q(\gamma)) \otimes (\frac{1}{2} \oplus (\neg\sigma_q(\psi) \otimes \sigma_q(\gamma))) \wedge \frac{1}{2})} = f_{((\neg\sigma_q(\psi) \oplus \sigma_q(\gamma)) \otimes \frac{1}{2}) \wedge \frac{1}{2}} = f_{\sigma_q(\psi \rightarrow \gamma)}.$$

This completes the second case.

- For $\varphi = \psi \otimes \gamma$ we have

$$f_{\tau_q(\delta(\psi \otimes \gamma))} = f_{\tau_q((\delta(\psi) + \delta(\gamma)) \vee -1)} = f_{((\tau_q(\delta(\psi)) \oplus \tau_q(\delta(\gamma))) \otimes (\frac{1}{2} \oplus (\tau_q(\delta(\psi)) \otimes \tau_q(\delta(\gamma)))) \vee q)}.$$

By induction hypothesis this is equal to

$$f_{((\sigma_q(\psi) \oplus \sigma_q(\gamma)) \otimes (\frac{1}{2} \oplus (\sigma_q(\psi) \otimes \sigma_q(\gamma)))) \vee q}.$$

Since $f_{\sigma_q(\gamma)}(a_1, \dots, a_n) \leq \frac{1}{2}$ for each $a_1, \dots, a_n \in [0, 1]_{\mathbb{L}}$ and for each $\gamma \in \mathcal{T}m_{MV}$, we have for each $a_1, \dots, a_n \in [0, 1]_{\mathbb{L}}$ that $f_{\sigma_q(\psi) \otimes \sigma_q(\gamma)}(a_1, \dots, a_n) = 0$. Thus,

$$f_{\tau_q(\delta(\psi \otimes \gamma))} = f_{((\sigma_q(\psi) \oplus \sigma_q(\gamma)) \otimes \frac{1}{2}) \vee q}.$$

This completes the second case and also the proof. $\blacktriangleleft$

We extend definitions of τ_q and σ_q to open formulas. For Φ of $\mathcal{L}_{PAb}$ define $\tau'_q(\Phi)$ by replacing each occurrence of an atom $\varphi = \psi$ with the atom $\tau_q(\varphi) = \tau_q(\psi)$. Similarly for Φ of $\mathcal{L}_{MV}$ define $\sigma'_q(\Phi)$ by replacing each occurrence of an atom $\varphi = \psi$ with the atom $\sigma_q(\varphi) = \sigma_q(\psi)$.

► **Corollary 12.** *Let Ψ be open formula of $\mathcal{L}_{MV}$ with n variables and let $q \notin \text{Var}(\Psi)$. Let $a_1, \dots, a_n \in [0, 1]$. Then*

$$[0, 1]_{\mathbb{L}}^{1/2} \models \tau'_q(\delta'(\Psi))(a_1, \dots, a_n) \text{ if and only if } [0, 1]_{\mathbb{L}}^{1/2} \models \sigma'_q(\Psi)(a_1, \dots, a_n).$$

Notice that if φ is a $\mathcal{L}_{MV}$ -term and v an assignment in $\mathbf{R}_{-1}$ such that $v(x_i) \in [-1, 0]$ for each $x_i \in \text{Var}(\varphi)$, then for each subterm ψ of φ , $v(\delta(\psi)) \in [-1, 0]$. Using this observation, we obtain the following variant of Lemma 4, now featuring the translation τ'_q in the role of τ' , and moreover avoiding the assumption regarding the depth of terms in the open first-order formula. The proof of this lemma is a variant of those of Lemma 3 and Lemma 4 with only minor modifications.

► **Lemma 13.** *Let M and k be fixed nonnegative integers, $\Phi(x_1, \dots, x_n)$ an open formula in the language $\mathcal{L}_{MV}$ and $q \notin \text{Var}(\Phi)$. For any assignment v in $\mathbf{R}_{-1}$ where $v(x_i) = a_i \in [-1, 0]$, let v' be an assignment in $[0, 1]_{\mathbb{L}}^{1/2}$ with*

$$v'(x_i) = r_{M,k}(a_i) \text{ for each } i \leq n \text{ and } v'(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}.$$

Then v satisfies the formula $\delta'(\Phi)$ in the algebra $\mathbf{R}_{-1}$ if and only if v' satisfies the formula $\tau'_q(\delta'(\Phi))$ in $[0, 1]_{\mathbb{L}}^{1/2}$.

This is equivalent to the following. For any assignment v in $[0, 1]_{\mathbb{L}}^{1/2}$ where $v(x_i) = b_i \in [\frac{1}{2} - \frac{1}{2^{M+k+1}}, \frac{1}{2}]$ let v' be an assignment in $\mathbf{R}_{-1}$ with $v'(x_i) = r_{M,k}^{-1}(b_i)$. Assume that $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$. Then v satisfies the formula $\tau'_q(\delta'(\Phi))$ in the algebra $[0, 1]_{\mathbb{L}}^{1/2}$ if and only if v' satisfies the formula $\delta'(\Phi)$ in $\mathbf{R}_{-1}$.

► **Theorem 14** (A translation of the existential theory of $[0, 1]_{\mathbb{L}}$ to itself). *Let $\Phi(x_1, \dots, x_n)$ be an open formula of $\mathcal{L}_{MV}$. Assume that the variables $\{z_1, \dots, z_{M+k+1}, q\}$ do not occur in Φ . Let $M, k \in \mathbb{N}$. Let $\zeta(\Phi)$ be obtained from Φ by applying σ'_q on Φ and then replacing all occurrences of the constant $\frac{1}{2}$ with the variable z_1 . Let $S(\Phi, M, k)$ be the following open formula in $\mathcal{L}_{MV}$:*

$$\zeta(\Phi) \mathbin{\wedge} (z_1 = \neg z_1) \mathbin{\wedge} (q = \neg(z_1 \oplus z_{M+k+1})) \mathbin{\wedge} \bigwedge_{i \leq M+k} (z_{i+1} \oplus z_{i+1} = z_i).$$

Then Φ has a satisfying assignment in $[0, 1]_{\mathbb{L}}$ if and only if $S(\Phi, M, k)$ has a satisfying assignment in $[0, 1]_{\mathbb{L}}$. Moreover, $\sharp(S(\Phi, M, k))$ is polynomial in $\sharp(\Phi)$.

Proof. Assume first that $\Phi(x_1, \dots, x_n)$ has a satisfying assignment $v(x_i) = a_i$ with $a_i \in [0, 1]_{\mathbb{L}}$. By Lemma 10 $\delta'(\Phi)$ has a satisfying assignment $x_i \mapsto a_i - 1$ with $(a_i - 1) \in [-1, 0]$.

By Lemma 13 $\tau'_q(\delta'(\Phi))$ has a satisfying assignment $x_i \mapsto r_{M,k}(a_i - 1)$ and $q \mapsto \frac{1}{2} - \frac{1}{2^{M+k+1}}$ in $[0, 1]_{\mathbb{L}}^{1/2}$. By Lemma 12 $\sigma'_q(\Phi)$ is satisfied by the very same assignment. Consequently, $\zeta(\Phi)$ has satisfying assignment $x_i \mapsto r_{M,k}(a_i - 1)$, $q \mapsto \frac{1}{2} - \frac{1}{2^{M+k+1}}$ and $z_1 \mapsto \frac{1}{2}$ in $[0, 1]_{\mathbb{L}}$.

By Lemma 6, the formula $(z_1 = \neg z_1) \mathbin{\wedge} (q = \neg(z_1 \oplus z_{M+k+1})) \mathbin{\wedge} \bigwedge_{i \leq M+k} (z_{i+1} \oplus z_{i+1} = z_i)$ has the unique satisfying assignment in $[0, 1]_{\mathbb{L}}$, which sends z_i to $1/2^i$ for each $i \leq M+k$ and q to $\frac{1}{2} - \frac{1}{2^{M+k+1}}$.

Conversely, assume $S(\Phi, M, k)$ has a satisfying assignment v in $[0, 1]_{\mathbb{L}}$. The conjunct

$$(z_1 = \neg z_1) \mathbin{\wedge} (q = \neg(z_1 \oplus z_{M+k+1})) \mathbin{\wedge} \bigwedge_{i \leq M+k} (z_{i+1} \oplus z_{i+1} = z_i)$$

forces by Lemma 6 that $v(z_1) = \frac{1}{2}$ and $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$. Observe that by the definition of ζ , every occurrence of a variable x_i in $\zeta(\Phi)$ appears strictly within the term $(x_i \vee q) \wedge z_1$ for each $i \leq n$. Therefore, we can assume without loss of generality that $\frac{1}{2} - \frac{1}{2^{M+k+1}} = v(q) \leq v(x_i) \leq v(z_1) = \frac{1}{2}$ for all $i \leq n$. Since v satisfies $S(\Phi, M, k)$, it must satisfy $\zeta(\Phi)$. By definition of ζ , this means the formula $\sigma'_q(\Phi)$ is satisfied by the assignment v in $[0, 1]_{\mathbb{L}}^{1/2}$. By Lemma 12 the assignment v also satisfies $\tau'_q(\delta'(\Phi))$ in $[0, 1]_{\mathbb{L}}^{1/2}$. We can now apply Lemma 13.

Since $\frac{1}{2} - \frac{1}{2^{M+k+1}} \leq v(x_i) \leq \frac{1}{2}$ and $v(q) = \frac{1}{2} - \frac{1}{2^{M+k+1}}$, by Lemma 13 the assignment $v'(x_i) = r_{M,k}^{-1}(v(x_i))$ satisfies $\delta'(\Phi)$ in $\mathbf{R}_{-1}$. Finally, applying Lemma 10, the assignment $v''(x_i) = v'(x_i) + 1$ satisfies the original formula Φ in $[0, 1]_{\mathbb{L}}$. This completes the proof. ◀

The last theorem can be viewed as providing a way of testing the validity of existential sentences in $[0, 1]_{\mathbb{L}}$ on a subinterval of its domain (rather than the full domain), provided the value $1/2$ belongs to this interval. The statement relies on the well-known fact that rational values in $[0, 1]$ are implicitly definable by $\mathcal{L}_{MV}$ -terms, and moreover, the size of the defining term is polynomial in the length of the positional representation of the rational number in question.

References

- 1 M. E. Adams and W. Dziobiak. Q-universal Quasivarieties of Algebras. *Proceedings of the American Mathematical Society*, 120(4):1053–1059, 1994.
- 2 Stefano Aguzzoli. *Geometric and Proof-Theoretic Issues in Łukasiewicz Propositional Logics*. PhD thesis, University of Siena, Siena, 1998.
- 3 Stefano Aguzzoli. An asymptotically tight bound on countermodels for Łukasiewicz logic. *International Journal of Approximate Reasoning*, 43(1):76–89, 2006. doi:10.1016/J.IJAR.2006.02.003.

- 4 Stefano Aguzzoli and Agata Ciabattini. Finiteness in infinite-valued Łukasiewicz logic. *Journal of Logic, Language and Information*, 9(1):5–29, 2000. doi:10.1023/A:1008311022292.
- 5 Stefano Aguzzoli and Brunella Gerla. Finite-valued reductions of infinite-valued logics. *Archive for Mathematical Logic*, 41(4):361–399, 2002. doi:10.1007/S001530100118.
- 6 Chen Chung Chang. Algebraic analysis of many-valued logics. *Transactions of the American Mathematical Society*, 88(2):467–490, 1958.
- 7 Chen Chung Chang. A new proof of the completeness of the Łukasiewicz axioms. *Transactions of the American Mathematical Society*, 93(1):74–80, 1959.
- 8 Roberto Cignoli, Itala M.L. D'Ottaviano, and Daniele Mundici. *Algebraic Foundations of Many-Valued Reasoning*, volume 7 of *Trends in Logic*. Kluwer, Dordrecht, 1999.
- 9 Petr Cintula, Berta Grima, Carles Noguera, and Nicholas J. J. Smith. These degrees go to eleven: fuzzy logics and gradable predicates. *Synthese*, 200(6):Paper No. 445, 38, 2022. doi:10.1007/s11229-022-03909-2.
- 10 Petr Cintula and Petr Hájek. Complexity issues in axiomatic extensions of Łukasiewicz logic. *Journal of Logic and Computation*, 19(2):245–260, 2009. doi:10.1093/LOGCOM/EXN052.
- 11 Petr Cintula, Petr Hájek, and Carles Noguera, editors. *Handbook of Mathematical Fuzzy Logic*, volume 1&2 of *Studies in Logic*, vol. 37–38. College Publications, London, 2011.
- 12 Petr Cintula, Filip Jankovec, and Carles Noguera. Superabelian logics, 2024. submitted. arXiv:2409.20170.
- 13 A. H. Clifford. Partially ordered abelian groups. *Ann. of Math. (2)*, 41:465–473, 1940. doi:10.2307/1968728.
- 14 Wojciech Dzik. Unification in some substructural logics of BL-algebras and hoops. *Reports on Mathematical Logic*, 43:73–83, 2008. URL: <https://rml.tcs.uj.edu.pl/rml-43/a-dzi-43.htm>.
- 15 Ronald Fagin, Joseph Y. Halpern, and Nimrod Megiddo. A logic for reasoning about probabilities. *Information and Computation*, 87(1–2):78–128, 1990. doi:10.1016/0890-5401(90)90060-U.
- 16 Melvin C. Fitting. Many-valued modal logics. *Fundamenta Informaticae*, 15(3–4):235–254, 1991.
- 17 Melvin C. Fitting. Many-valued modal logics II. *Fundamenta Informaticae*, 17(1–2):55–73, 1992.
- 18 Tommaso Flaminio and Franco Montagna. A logical and algebraic treatment of conditional probability. *Archive for Mathematical Logic*, 44(2):245–262, 2005. doi:10.1007/S00153-004-0253-Z.
- 19 Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.
- 20 Yu. Sh. Gurevich and A. I. Kokorin. Universal equivalence of ordered abelian groups. *Algebra i Logika. Sem.*, 2(1):37–39, 1963.
- 21 Reiner Hähnle. Many-valued logic and mixed integer programming. *Annals of Mathematics and Artificial Intelligence*, 12(3–4):231–264, December 1994. doi:10.1007/BF01530787.
- 22 Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998. doi:10.1007/978-94-011-5300-3.
- 23 Petr Hájek and Dagmar Harmancová. A many-valued modal logic. In *Proceedings IPMU'96. Information Processing and Management of Uncertainty in Knowledge-Based Systems*, pages 1021–1024, 1996.
- 24 Zuzana Haniková. Computational complexity of propositional fuzzy logics. In Petr Cintula, Petr Hájek, and Carles Noguera, editors, *Handbook of Mathematical Fuzzy Logic*, volume 2, pages 793–851. College Publications, 2011.
- 25 Zuzana Haniková and Petr Savický. Term satisfiability in FL_{ew} -algebras. *Theoretical Computer Science*, 631:1–15, 2016. doi:10.1016/J.TCS.2016.03.009.

- 26 Emil Jeřábek. The complexity of admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 23(3):693–705, 2013. doi:10.1093/LOGCOM/EXS007.
- 27 Yuichi Komori. Super-Łukasiewicz implicational logics. *Nagoya Mathematical Journal*, 72:127–133, 1978.
- 28 Yuichi Komori. Super-Łukasiewicz propositional logics. *Nagoya Mathematical Journal*, 84:119–133, 1981.
- 29 Jan Łukasiewicz. On determinism. *The Polish Review*, 13(3):47–61, 1968.
- 30 Jan Łukasiewicz and Alfred Tarski. Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, cl. III*, 23(iii):30–50, 1930.
- 31 Enrico Marchioni and Franco Montagna. Complexity and definability issues in $\mathbb{L}\Pi_2^1$. *Journal of Logic and Computation*, 17(2):311–331, 2007.
- 32 Robert McNaughton. A theorem about infinite-valued sentential logic. *Journal of Symbolic Logic*, 16(1):1–13, 1951. doi:10.2307/2268660.
- 33 George Metcalfe, Nicola Olivetti, and Dov M. Gabbay. *Proof Theory for Fuzzy Logics*, volume 36 of *Applied Logic Series*. Springer, 2008.
- 34 Robert K. Meyer and John K. Slaney. Abelian logic from A to Z. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic: Essays on the Inconsistent*, pages 245–288. Philosophia Verlag, 1989.
- 35 Daniele Mundici. Satisfiability in many-valued sentential logic is NP-complete. *Theoretical Computer Science*, 52(1–2):145–153, 1987. doi:10.1016/0304-3975(87)90083-1.
- 36 Daniele Mundici. *Advanced Łukasiewicz Calculus and MV-Algebras*, volume 35 of *Trends in Logics*. Springer, 2011.
- 37 Jan Pavelka. On fuzzy logic I, II, III. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 25:45–52, 119–134, 447–464, 1979.
- 38 Matthias Emil Ragaz. *Arithmetische Klassifikation von Formelmengen der unendlichwertigen Logik*. PhD thesis, Swiss Federal Institute of Technology, Zürich, 1981.
- 39 Greg Restall. *An Introduction to Substructural Logics*. Routledge, New York, 2000.
- 40 Alan Rose and J.Barkley Rosser. Fragments of many-valued statement calculi. *Transactions of the American Mathematical Society*, 87(1):1–53, 1958.
- 41 Bruno Scarpellini. Die Nichtaxiomatisierbarkeit des unendlichwertigen Prädikatenkalküls von Łukasiewicz. *Journal of Symbolic Logic*, 27(2):159–170, 1962.
- 42 Amanda Vidal. Undecidability and non-axiomatizability of modal many-valued logics. *The Journal of Symbolic Logic*, 87(4):1576–1605, 2022. doi:10.1017/JSL.2022.32.
- 43 Volker Weispfenning. The complexity of the word problem for Abelian ℓ -groups. *Theoretical Computer Science*, 48(1):127–132, 1986. doi:10.1016/0304-3975(86)90089-7.