Rational Lawvere Logic

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Abstract

We study Rational Lawvere logic ($\mathbb{RL}$). This logic is defined over the extended positive reals with an algebraic structure combining the Lawvere quantale (with the reversed order on the extended reals and a sum as tensor) and a multiplicative quantale (with the usual order on the extended reals and a multiplication as tensor); together they provide a semiring structure. The logic is designed for complex quantitative reasoning, including sequents expressing inequalities between rational functions over the extended positive reals. We give a deduction system and demonstrate its expressiveness by deriving a classical result from probability theory relating the Kantorovich and total variation distances. Our deductive system is complete for finitely axiomatizable theories. The proof of completeness relies on the Krivine-Stengle Positivstellensatz.

We additionally provide complexity results for both $\mathbb{RL}$ and its affine fragment $\mathbb{AL}$. We consider two decision problems: the satisfiability of a set of sequents and whether a sequent follows from a finite set of sequent. We show that both problems lie in PSPACE for $\mathbb{RL}$, and we give sharper complexity bounds for $\mathbb{AL}$: the first problem is NP-complete, while the second is co-NP-complete.

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1 Introduction

Some recent developments in theoretical computer science have questioned the usefulness of equality in semantics, advocating more nuanced, quantitative approaches to equivalence. For instance, exact equality is often too rigid for probabilistic systems where small changes can disrupt equivalence between processes. To address this, researchers used metrics to measure differences, thus shifting the focus from strict equivalence to quantitative comparisons. Metric-based reasoning has also been applied to other areas, such as privacy, security [19, 53], computational resource analysis [37, 38], and symbolic computation [26].

As a result, theories of semantic equality have evolved into quantitative frameworks, focusing on measuring differences rather than asserting equality. Notable examples include theories for program analysis [3, 15, 16, 39, 36, 38], distances for processes [20, 21, 24, 25, 5, 6, 10], and quantitative equational logics over algebras of terms [43, 44, 7, 45, 8, 48, 49, 1, 2].

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The latter, in particular, focuses on providing foundations for quantitative reasoning. The basic idea is to replace traditional equations $s = t$ between terms s, t of an algebra with *quantitative equations* of the form $s =_\varepsilon t$, expressing that s and t are at most ε apart, for a real $\varepsilon \geq 0$. Thus, quantitative algebraic theories are used to reason about the *distances* between elements of an algebra. However, equational logic is only one of many forms of logic and a question arises: how can extensions of classical logic be used to provide foundations for quantitative reasoning?

In his seminal work [40], Lawvere views the extended non-negative reals $[0, \infty]$ as the objects of a complete monoidal-closed category with $\geq$ as the sets of morphisms and an extended sum as tensor. A $[0, \infty]$-enriched category is then a generalised metric space. Further, in the introduction to [41], he regards the extended non-negative reals as a kind of truth-value, with 0 and ∞ as "true " and "false ", and speaks of $[0, \infty]$-valued relations. Further, all sups (= $[0, \infty]$-limits) are preserved by tensoring, and so $[0, \infty]$ is a quantale, which we call the *Lawvere quantale*. We argue that logical reasoning on the Lawvere quantale of truth values is a natural choice for studying metric spaces. Lawvere's generalized metric spaces are $[0, \infty]$-valued preorders in it. A quantitative equation $s =_\varepsilon t$ is expressed as a sequent $\varepsilon \vdash s = t$, which corresponds to the inequality $\varepsilon \geq$ "$s = t$".

From a logical point of view, $[0, \infty]$-valued propositional logic is then a natural place to start. Bacci et al. [9] began exploring a class of such quantitative logics, referred to as *Lawvere logics*[1]. Among them, Affine Lawvere propositional logic ($\mathbb{AL}$) was the most expressive. This logic features a tensor operation, interpreted as addition in the Lawvere quantale, a linear implication, interpreted as the adjoint residuum of addition, constants for all non-negative real numbers, and scalar multiplication by non-negative reals. So all affine functions on $[0, \infty]$ can be expressed in $\mathbb{AL}$. Logical conjunction and disjunction are derived operators. Sequents in $\mathbb{AL}$ are interpreted as affine inequalities on $[0, \infty]$.

A key innovation of [9] was the use of theorems from linear algebra, specifically Farkas' Lemma [23] and Motzkin's transposition theorem [50], to help establish completeness: consequence relations between finite sets of sequents and sequents were reduced to consequence relations between finite sets of linear inequalities and linear inequalities. This established a strong link between logic and classical arithmetic. However, many real-world quantitative phenomena involve non-linear interactions, making it desirable to express polynomial inequalities.

In this paper, we take on the challenge of developing *Rational Lawvere Logic* ($\mathbb{RL}$). This logic extends $\mathbb{AL}$ by adding multiplication and division as logical connectives, enabling sequents to represent rational inequalities. Our approach builds on Lawvere's idea by giving logical status to both sum and multiplication, with the key innovation being that the truth values come from a semiring structure involving two quantales over $[0, \infty]$: the *additive* Lawvere one (with reverse order and sum as tensor), and the *multiplicative* one (with the natural order and multiplication as tensor).

Our *main contributions* are:

1. We give a deduction system for $\mathbb{RL}$ and demonstrate its expressiveness by (a) deriving a classical result from probability theory relating the Kantorovich and the total variation distances and (b) giving an embedding of quantitative equational logic in it (Section 5).
2. We prove completeness for finitely axiomatizable theories (Theorem 9). (There is no finitary complete consistent proof system for general theories (Theorem 15) as compactness fails.) The core of the completeness proof differs significantly from that in [9]. Rather than

[1] The logics are named in honor of Lawvere.

reducing to formally proving relations between linear inequalities, when we can use Farkas' Lemma or Motzkin's transposition theorem, we reduce to formally proving relations between polynomial inequalities, when we can use Krivine-Stengle's Positivstellensatz [35, 58, 13], a real analogue of Hilbert's Nullstellensatz. As all such polynomial relations can be directly expressed in the logic, this indicates *a prima facie* need for the Positivstellensatz.

3. Unlike $\mathbb{AL}$, $\mathbb{RL}$ allows formulas and sequents to be "booleanized ." We use this to prove a deduction theorem (Theorem 8) that is not available in $\mathbb{AL}$.
4. The completeness proof employs a linear-time non-deterministic reduction that translates any $\mathbb{RL}$ inference to a set of inferences in polynomial form. Notably, when applied to $\mathbb{AL}$ inferences, it significantly simplifies the normalisation algorithm proposed in [9]. Perhaps this technique can be helpful to obtain, and/or simplify, other completeness proofs.
5. Relying on the reduction discussed above, we establish complexity results for two fundamental decision problems (for both $\mathbb{RL}$ and $\mathbb{AL}$): the semantical consequence of a sequent from a finite set of sequents, and the satisfiability of a finite set of sequents. We show that semantical consequence is in PSPACE for $\mathbb{RL}$ and is co-NP-complete for $\mathbb{AL}$ (Theorem 17), and obtain as a corollary that satisfiability is in PSPACE for $\mathbb{RL}$ and is NP-complete for $\mathbb{AL}$ (Corollary 18).

Related Work. Connections between arithmetic and logical reasoning are well known. A completeness interpretation of Farkas' Lemma appears already in the literature (*e.g.*, in [46]). In algebraic complexity there is the Nullstellensatz proof system which uses a simple reduction of propositional satisfaction to polynomial equation solvability (*e.g.*, [11, 52]) and the Positivstellensatz calculus [29] which considers polynomial inequalities.

Parallel to Lawvere's real-valued approach we must mention the vast development of fuzzy logic, for example [51, 12, 31]. Fuzzy logic generally employs (if not explicitly) quantales on the real interval $[0, 1]$. The most relevant for us is product logic [32, 30, 56, 22], defined over the multiplicative quantale on $[0, 1]$. Through the quantale isomorphism e^{-x}, $\mathbb{AL}$ corresponds to product logic extended with constants in $[0, 1]$, and $\mathbb{RL}$ corresponds to a further extension with an operation $e^{-\ln x \ln y}$. Neither of these extensions seems to be in the literature. Moreover, this interpretation of the logical connectives seems unnatural for quantitative reasoning over $[0, \infty]$, and impedes direct access to results we use, *e.g.*, in linear algebra (such as Khachiyan's ellipsoid method, used for complexity), and in real algebraic geometry (such as the Krivine-Stengle Positivstellensatz, used for completeness).

We must also mention the extensive works on graded (or weighted) structures, such as linear logic's exponentials, comonads, types, or categories (*e.g.*, [33, 28, 4, 17, 18, 42]). The gradings usually employ general semirings of some kind. However $[0, \infty]$ in particular is also discussed, for example in [28, 4, 33, 18]. Various possibilities for multiplication are considered: two commutative ones (ours is one) and a non-commutative one. In Section 2, we discuss all the possible monotonic, commutative, and associative addition and multiplication operations on $[0, \infty]$ that extend the usual ones on $(0, \infty)$. They are all definable in our logic (as are the non-commutative ones, as a straightforward extension of our discussion shows).

Synopsis. Section 2 gives preliminary definitions and notation. Section 3 gives the syntax and semantics of $\mathbb{RL}$, and Section 4 presents a deduction system for it. Section 5 presents some nontrivial applications. Section 6 develops the completeness result. Section 7 gives the complexity results for $\mathbb{RL}$ and its affine fragment $\mathbb{AL}$. Section 8 gives concluding remarks and discusses future work.

Table 1 Three variants of sum ($+_1$, $+_2$, $+_3$); truncated subtraction ($\dot{-}$) ; two variants of multiplication ($\times_1$, $\times_2$); and extended division ($\div$) (the first column lists numerators, the first row denominators). Note that $r, s \in (0, \infty)$.

$+_1$	0	s	∞
0	0	s	∞
r	r	$r+s$	∞
∞	∞	∞	∞

$+_2$	0	s	∞
0	0	0	0
r	0	$r+s$	∞
∞	0	∞	∞

$+_3$	0	s	∞
0	0	0	∞
r	0	$r+s$	∞
∞	∞	∞	∞

$\dot{-}$	0	s	∞
0	0	0	0
r	r	$\max\{r-s, 0\}$	0
∞	∞	∞	0

$\times_1$	0	s	∞
0	0	0	0
r	0	rs	∞
∞	0	∞	∞

$\times_2$	0	s	∞
0	0	0	∞
r	0	rs	∞
∞	∞	∞	∞

$\div$	0	s	∞
0	∞	0	0
r	∞	$\frac{r}{s}$	0
∞	∞	∞	∞

2 Preliminaries and Notation

A *quantale* [55] is a complete lattice with a binary, associative operation $\otimes$ (*tensor*) that distributes over joins in each argument; distributivity and completeness entail that the tensor has both right adjoints. A quantale is *commutative* if its tensor is; and *unital* if there is an element u (*unit*) s.t. $u \otimes a = a = a \otimes u$, for all a; if the unit is the top element, the quantale is *integral*. For commutative quantales, the right adjoints of $- \otimes a$ and $a \otimes -$ coincide.

As mentioned in the introduction, our interest concerns the extended non-negative reals $[0, \infty]$. We next discuss ways of extending sum and multiplication from the positive reals $(0, \infty)$ to $[0, \infty]$ and analyse the choices of quantales that one obtains from these extensions. To avoid confusion, in what follows we always use sup and inf on $[0, \infty]$ with respect to the natural order $\leq$, even when we speak of structures using different orders.

Addition. We would like to extend sum from the positive reals $(0, \infty)$ to $[0, \infty]$ so that we still get a sum that is associative, commutative, and monotonic w.r.t $\leq$ (equivalently w.r.t. $\leq^{op}$). One can show there are three choices for defining such a sum, as given in Table 1, with $+_1$ being the addition of the Lawvere quantale.

▶ **Lemma 1.**
1. *$([0, \infty], +_1, \leq^{op})$ is a commutative, unital, integral quantale; $([0, \infty], +_1, \leq)$ is not a quantale.*
2. *$([0, \infty], +_2, \leq)$ is a commutative quantale; $([0, \infty], +_2, \leq^{op})$ is not a quantale.*
3. *Neither $([0, \infty], +_3, \leq)$ nor $([0, \infty], +_3, \leq^{op})$ are quantales.*

Thus, for an additive quantale on $[0, \infty]$, if we use the natural order $\leq$, the correct choice for sum is $+_2$; if we use the reverse order $\leq^{op}$, the correct choice is $+_1$. The first is not unital, since $0 +_2 \infty = 0$; the Lawvere quantale, is both unital and integral. We chose $+_1$, as this enables us to directly encode examples from quantitative equational logic (Section 5). The right adjoint to $- +_1 a$, can be explicitly formulated in terms of truncated substruction $\dot{-}$, appropriately extended to $[0, \infty]$ as shown in Table 1. Indeed, it holds that $b \dot{-} a = \inf\{c \mid c +_1 a \geq b\}$.

Multiplication. One can show there are two associative, commutative, and monotonic extensions of multiplication from $[0, \infty)$ to $[0, \infty]$: $\times_1$ and $\times_2$, as given in Table 1.

▶ **Lemma 2.**
1. $([0,\infty], \times_1, \leq)$ *is a commutative, unital quantale;* $([0,\infty], \times_1, \leq^{op})$ *is not a quantale.*
2. $([0,\infty], \times_2, \leq^{op})$ *is a commutative, unital quantale;* $([0,\infty], \times_2, \leq)$ *is not a quantale.*

Thus, for a multiplicative quantale on $[0,\infty]$, if we use the natural order $\leq$, it is $\times_1$; if we use the reverse order $\leq^{op}$, it is $\times_2$. We discuss our choice of multiplication in relation to the Lawvere quantale. On the one hand, if the choice were dictated by the quantale order, $\times_2$ would seem the natural candidate. On the other hand, unlike $\times_2$, choosing $\times_1$ yields a semiring (both multiplications distribute over $+$, but the unit of $+_1$ is not the null element for $\times_2$, as $\infty \times_2 0 = \infty$). Ultimately, we choose $\times_1$. While no choice is perfect, having a semiring enables us to directly encode examples from measure theory (Section 5) and to obtain a deduction theorem (Theorem 8).

Although the logic will use the order of the Lawvere quantale, we will still exploit the quantalic structure associated with $\times_1$ by adding as a logical connective the right adjoint to $- \times_1 a$, which can be explicitly formulated in terms of division $\div$, appropriately extended to $[0,\infty]$ as given in Table 1. Indeed, it holds that $b \div a = \sup\{c \mid c \times_1 a \leq b\}$.

We conclude by showing that the other operations, namely $+_2$, $+_3$, and $\times_2$, can be expressed in terms of $+_1$, $\times_1$, $\dot{-}$, and ∞ (and so, eventually, in $\mathbb{RL}$). First, binary sups and infs can be:

▶ **Lemma 3.** *For* $a, b \in [0,\infty]$ *we have:*
1. $a \vee b = a + (b \dot{-} a)$
2. $a \wedge b = (a \dot{-} (a \dot{-} b)) \vee (b \dot{-} (b \dot{-} a))$

Next, we define functions $N, Z\colon [0,\infty] \to [0,\infty]$ by $N(a) = \infty \dot{-} a$ and $Z(a) = a \times_1 \infty$. These are "boolean functions" returning either 0 or ∞ (*i.e.*, $\top$ and $\bot$ in the Lawvere quantale), as:

$$N(a) = \begin{cases} 0 & \text{if } a = \infty \\ \infty & \text{otherwise}, \end{cases} \qquad Z(a) = \begin{cases} 0 & \text{if } a = 0 \\ \infty & \text{otherwise}. \end{cases}$$

Hence, N is a test for ∞, while Z is a test for 0. We can next define a conditional using $\vee$ and $\wedge$:

$$\textit{if } a \textit{ then } b \textit{ else } c = [N(Z(a)) \vee b] \wedge [Z(a) \vee c] = \begin{cases} b & \text{if } a = 0 \\ c & \text{otherwise}. \end{cases}$$

and finally obtain:

▶ **Lemma 4.** *For* $a, b \in [0,\infty]$ *we have:*
1. $a +_2 b = \textit{if } (Z(a) \vee Z(b)) \textit{ then } 0 \textit{ else } (a +_1 b)$
2. $a +_3 b = (a +_2 b) +_1 [\textit{if } (N(a) \vee N(b)) \textit{ then } \infty \textit{ else } 0]$
3. $a \times_2 b = \textit{if } [(Z(a) \wedge N(b)) \vee (Z(b) \wedge N(a))] \textit{ then } \infty \textit{ else } (a \times_1 b)$

Hereafter, when working on $[0,\infty]$, we simply write $+$ for the sum instead of $+_1$ and $\times$ for the multiplication instead of $\times_1$. The other operations, namely $\dot{-}$ and $\div$ (written as a fraction), are those from Table 1. We continue writing $\leq$ for the natural order on $[0,\infty]$ and $\leq^{op}$ for Lawvere's order.

3 Rational Lawvere Logic

In this section, we introduce *Rational Lawvere logic* ($\mathbb{RL}$), a propositional logic interpreted over our semiring on $[0,\infty]$. It extends Affine Lawvere logic ($\mathbb{AL}$) of [9], enabling one to reason with inequalities between rational functions over the non-negative extended reals.

Syntax. Let $\mathbb{P}$ be a set of *propositional letters*, ranged over by $P, Q, R, \ldots$. The formulas of $\mathbb{RL}$ are freely generated by the following grammar, for arbitrary $P \in \mathbb{P}$ and $r \in [0, \infty)$.

$$\phi, \psi ::= \bot \mid P \mid r \mid \phi \oplus \psi \mid \phi \multimap \psi \mid \phi\psi \mid \phi/\psi$$

We define expected logical connectives as derived operators:

$$\top := 0\,, \quad \neg\phi := \phi \multimap \bot\,, \quad \phi \wedge \psi := \phi \oplus (\phi \multimap \psi)\,,$$
$$\phi \vee \psi := ((\psi \multimap \phi) \multimap \phi) \wedge ((\phi \multimap \psi) \multimap \psi)\,, \quad \phi \multimapboth \psi := (\phi \multimap \psi) \wedge (\psi \multimap \phi)\,.$$

We assume the following precedence rule: multiplication and division have highest precedence, followed by $\neg$, then $\oplus$, next $\wedge$ and $\vee$, and finally $\multimap$ and $\multimapboth$ have lowest precedence. Thus, $\theta\phi \oplus \psi \wedge \neg\theta\psi \multimap \theta$ is interpreted as the formula $(((\theta\phi) \oplus \psi) \wedge (\neg(\theta\psi))) \multimap \theta$.

Semantics. Interpretations are maps $\mathcal{I} \colon \mathbb{P} \to [0, \infty]$, extended to all formulas as follows:

$$\mathcal{I}(\bot) := \infty\,, \quad \mathcal{I}(r) := r\,, \quad \mathcal{I}(\phi \oplus \psi) := \mathcal{I}(\phi) + \mathcal{I}(\psi)\,, \quad \mathcal{I}(\phi \multimap \psi) := \mathcal{I}(\psi) \dot{-} \mathcal{I}(\phi)\,,$$
$$\mathcal{I}(\phi\psi) := \mathcal{I}(\phi) \times \mathcal{I}(\psi)\,, \quad \mathcal{I}(\phi/\psi) := \frac{\mathcal{I}(\phi)}{\mathcal{I}(\psi)}\,.$$

Consequently, the derived connectives are interpreted as follows (recall Lemma 3):

$$\mathcal{I}(\top) = 0\,, \quad \mathcal{I}(\neg\phi) = \infty \dot{-} \mathcal{I}(\phi)\,, \quad \mathcal{I}(\phi \wedge \psi) = \max\{\mathcal{I}(\psi), \mathcal{I}(\phi)\}\,,$$
$$\mathcal{I}(\phi \vee \psi) = \min\{\mathcal{I}(\psi), \mathcal{I}(\phi)\}\,, \quad \mathcal{I}(\phi \multimapboth \psi) = \max\{\mathcal{I}(\phi) \dot{-} \mathcal{I}(\psi), \mathcal{I}(\psi) \dot{-} \mathcal{I}(\phi)\}\,.$$

Affine Lawvere Logic. Affine Lawvere Logic ($\mathbb{AL}$), introduced in [9], is the sublogic of $\mathbb{RL}$ defined for $P \in \mathbb{P}$ and $r \in [0, \infty)$, by the following grammar:[2]

$$\mathbb{AL}: \quad \phi, \psi ::= \bot \mid P \mid r \mid \phi \oplus \psi \mid \phi \multimap \psi \mid r\psi$$

Boolean formulas. While, in $\mathbb{RL}$, an interpretation evaluates a formula to a value in $[0, \infty]$, formulas such as $\neg\phi$ or $\phi\bot$ evaluate either to 0 ("true") or to ∞ ("false"). For example:

$$\mathcal{I}(\neg\phi) = \begin{cases} 0 & \text{if } \mathcal{I}(\phi) \text{ is infinite} \\ \infty & \text{otherwise}\,, \end{cases} \qquad \mathcal{I}(\phi\bot) = \begin{cases} 0 & \text{if } \mathcal{I}(\phi) = 0 \\ \infty & \text{otherwise}\,. \end{cases}$$

We call such formulas *boolean.* They yield derived operators, such as:

$$\begin{array}{lll} |\phi| := \neg\neg\phi & \phi = \psi := \mathrm{Z}(\phi \multimapboth \psi)\,, & \phi \geq \psi := \mathrm{Z}(\phi \multimap \psi)\,, \\ \mathrm{Z}\phi := \phi\bot & \phi \neq \psi := \neg\mathrm{Z}(\phi \multimapboth \psi)\,, & \phi > \psi := \neg\mathrm{Z}(\psi \multimap \phi)\,, \end{array} \quad |\phi|^+ := |\phi| \wedge \neg\mathrm{Z}(\phi)\,.$$

These have useful "boolean" meanings. For example:

$$\mathcal{I}(|\phi|) = \begin{cases} 0 & \text{if } \mathcal{I}(\phi) \text{ is finite} \\ \infty & \text{otherwise}\,, \end{cases} \quad \mathcal{I}(\mathrm{Z}\phi) = \begin{cases} 0 & \text{if } \mathcal{I}(\phi) = 0 \\ \infty & \text{otherwise}\,, \end{cases} \quad \mathcal{I}(|\phi|^+) = \begin{cases} 0 & \text{if } 0 < \mathcal{I}(\phi) < \infty \\ \infty & \text{otherwise}\,, \end{cases}$$

Using them, we can express useful facts, *e.g.*, $|\phi|$ says that "ϕ *is finite*" and $Z\phi$ that "ϕ *is strictly positive*". We use $\phi \leq \psi$ and $\phi < \psi$ as synonyms for $\psi \geq \phi$ and $\psi > \phi$.

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[2] In [9] $\wedge$ and $\vee$ belong to the syntax, but they can be obtained as derived operators, as in $\mathbb{RL}$.

Sequents. A *sequent* in $\mathbb{RL}$ is a syntactic construct of the form

$$\Gamma \vdash \phi\,, \qquad \text{(Sequent)}$$

where $\Gamma = \phi_1, \dots, \phi_n$ is a finite ordered list of formulas, possibly with repetitions, the *antecedents* of the sequent, and the formula ϕ is its *consequent*. For Γ and Δ such lists of formulas, Γ, Δ denotes their concatenation; and $\vdash \phi$ is a sequent with no antecedents.

A sequent $\phi_1, \dots, \phi_n \vdash \psi$ *is satisfied by* an interpretation $\mathcal{I}$ (alternatively, $\mathcal{I}$ is a *model* for the sequent), denoted $\mathcal{I} \models (\phi_1, \dots, \phi_n \vdash \psi)$, whenever

$$\mathcal{I}(\phi_1) + \cdots + \mathcal{I}(\phi_n) \geq \mathcal{I}(\psi)\,. \qquad \text{(Semantics of sequents)}$$

In particular, $\mathcal{I} \models (\vdash \psi)$ means that $\mathcal{I}(\psi) = 0$. We write $\mathcal{I} \models S$ and say that $\mathcal{I}$ is a model for S if $\mathcal{I}$ satisfies all sequents in S. A sequent is *satisfiable* if it has a model; it is *unsatisfiable* if it has no models; it is a *tautology* if it is satisfied by all interpretations. In particular, $\vdash \phi \multimap \phi$, $\vdash \top$, and $\vdash \neg\neg\phi \mathrel{\circ\!\!-\!\!\circ} (\bot > \phi)$ are tautologies, while $\vdash \phi \mathrel{\circ\!\!-\!\!\circ} (\neg\neg\phi)$ is not.

Note the distinction between $\phi \multimap \psi$ and the boolean formula $\phi \geq \psi$: while for all interpretations $\mathcal{I}$, we have $\mathcal{I} \models (\vdash \phi \multimap \psi)$ iff $\mathcal{I} \models (\vdash \phi \geq \psi)$, it may not hold that $\mathcal{I}(\phi \multimap \psi) = \mathcal{I}(\phi \geq \psi)$, as $\mathcal{I}(\phi \multimap \psi)$ could be a non-zero finite number.

▶ **Definition 5** (Semantic Consequence). *A sequent γ is a* semantic consequence *of a set S of sequents, in symbols $S \models \gamma$, if every model of S is also a model of γ.*

4 Deduction System for $\mathbb{RL}$

An *inference rule* is a syntactic construct of the form $\frac{S}{\gamma}$ with S a finite set of sequents, the *hypotheses*, and γ a sequent, the *conclusion*. As usual with inference rules, we may write $\frac{S \quad S'}{\gamma}$ for $\frac{S \cup S'}{\gamma}$ (so repetitions do not matter for hypotheses, only for sequents); γ for $\{\gamma\}$ in hypotheses; and $\frac{\gamma'}{\overline{\gamma}}$ for both $\frac{\gamma'}{\gamma}$ and $\frac{\gamma}{\gamma'}$.

▶ **Definition 6** (Disjunctive Consequence). *Given a set $\mathcal{D}$ of inference rules for a sublogic of $\mathbb{RL}$ containing $\vee$, the disjunctive consequence relation it induces is the smallest relation $S \Longrightarrow_{\mathcal{D}} \gamma$ between finite sets of sequents S and sequents γ such that:*

1. *If $\dfrac{S}{\gamma} \in \mathcal{D}$ then $S \Longrightarrow_{\mathcal{D}} \gamma$.*
2. *$S, \gamma \Longrightarrow_{\mathcal{D}} \gamma$.*
3. *If $S \Longrightarrow_{\mathcal{D}} \gamma_1, \dots, S \Longrightarrow_{\mathcal{D}} \gamma_n$ and $\gamma_1, \dots, \gamma_n \Longrightarrow_{\mathcal{D}} \gamma$ then $S \Longrightarrow_{\mathcal{D}} \gamma$ (for $n \geq 0$).*
4. *If $S, \Gamma \vdash \phi \Longrightarrow_{\mathcal{D}} \gamma$ and $S, \Gamma \vdash \psi \Longrightarrow_{\mathcal{D}} \gamma$ then $S, \Gamma \vdash \phi \vee \psi \Longrightarrow_{\mathcal{D}} \gamma$.*

Note that weakening holds: if $S \Longrightarrow_{\mathcal{D}} \gamma$ then $S, S' \Longrightarrow_{\mathcal{D}} \gamma$. We say γ is *$\mathcal{D}$-provable* from S if $S \Longrightarrow_{\mathcal{D}} \gamma$, omitting $\mathcal{D}$ if understood from the context, when we may write $\frac{S}{\gamma}$ for $S \Longrightarrow_{\mathcal{D}} \gamma$.

Disjunctive consequence for $\mathbb{RL}$ is that induced by the inference rules in Table 2; we use the same name $\mathbb{RL}$ to refer to it. It contains basic inference rules of logical deduction: (CUT), weakening (WEAK) and permutation (PERM) (note that contraction is not sound). The rule (BOT) behaves as expected. (ZERO) guarantees that the additive quantale is integral and (ONE) that one is finite. We also have weak-excluded-middle (WEM), stating that any formula is either finite or infinite, a prelinearity rule (LIN) that ensures the strong connectivity of the quantale order. (PREM) is a double inference that allows us to merge premises using $\oplus$; and (QUANT) is the double inference representing the (right) quantale implication rule. The cancellation (CANC) and subtraction (SUB) rules encode standard properties of addition and truncated subtraction, adjusted to allow for infinity. (PREM) and (ZERO), together with

Table 2 Inference rules for rational Lawvere logic $\mathbb{RL}$.

$$\frac{}{\phi \vdash \phi}\ (\textsc{id}) \qquad \frac{\Gamma \vdash \phi \quad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}\ (\textsc{cut}) \qquad \frac{\Gamma \vdash \phi}{\Gamma, \psi \vdash \phi}\ (\textsc{weak}) \qquad \frac{\Gamma, \phi, \psi, \Delta \vdash \theta}{\Gamma, \psi, \phi, \Delta \vdash \theta}\ (\textsc{perm})$$

$$\frac{}{\bot \vdash \phi}\ (\textsc{bot}) \qquad \frac{}{\vdash 0}\ (\textsc{zero}) \qquad \frac{}{\vdash |1|}\ (\textsc{one})$$

$$\frac{}{\vdash (\neg\phi) \vee (\neg\neg\phi)}\ (\textsc{wem}) \qquad \frac{}{\vdash (\phi \multimap \psi) \vee (\psi \multimap \phi)}\ (\textsc{lin})$$

$$\frac{\Gamma, \phi, \psi \vdash \theta}{\Gamma, \phi \oplus \psi \vdash \theta}\ (\textsc{prem}) \qquad \frac{\phi \oplus \psi \vdash \theta}{\phi \vdash \psi \multimap \theta}\ (\textsc{quant})$$

$$\frac{\theta \oplus \phi \vdash \psi \oplus \phi \quad \vdash |\phi|}{\theta \vdash \psi}\ (\textsc{canc}) \qquad \frac{\theta \vdash \phi}{\theta \vdash (\phi \multimap \theta) \oplus \phi}\ (\textsc{sub})$$

$$\frac{}{\vdash 0\phi \multimap\!\!\circ\, 0}\ (\textsc{null}) \qquad \frac{}{\vdash 1\phi \multimap\!\!\circ\, \phi}\ (\textsc{unit}) \qquad \frac{\phi \vdash \psi}{\theta\phi \vdash \theta\psi}\ (\textsc{comp}) \qquad \frac{\vdash \phi\psi}{\vdash \phi \vee \psi}\ (\textsc{zm})$$

$$\frac{}{\vdash (\phi\psi)\theta \multimap\!\!\circ\, \phi(\psi\theta)}\ (\textsc{assoc}) \qquad \frac{}{\vdash \phi\psi \multimap\!\!\circ\, \psi\phi}\ (\textsc{comm}) \qquad \frac{}{\vdash \theta(\phi \oplus \psi) \multimap\!\!\circ\, \theta\phi \oplus \theta\psi}\ (\textsc{distr})$$

$$\frac{}{\vdash (r \oplus s) \multimap\!\!\circ\, (r + s)}\ (\textsc{sum}) \qquad \frac{}{\vdash (rs) \multimap\!\!\circ\, (r \times s)}\ (\textsc{mult})$$

$$\frac{\phi/\theta \vdash \psi}{\phi \vdash \theta\psi}\ (\textsc{adj}) \qquad \frac{\vdash |\theta|^+}{\vdash \psi \multimap\!\!\circ\, \theta(\psi/\theta)}\ (\textsc{div}) \qquad \frac{}{\vdash 1/\bot}\ (\textsc{null})$$

the basic inference rules and (TOP), entail that $\oplus$ forms an ordered commutative monoid with a zero. (COMP), (ASSOC), (UNIT) and (COMM) express that multiplication is an ordered commutative monoid with a unit. Together with (DISTRIB) and (NULL) we then see that we have an ordered commutative semiring. Next, (ZM) states that if a product is zero, then one of its factors must also be zero. (SUM) and (MULT) ensure that $\oplus$ and logical multiplication correspond to $+$ and $\times$ respectively when applied to real constants. Finally, (ADJ) states the adjunction in the multiplicative quantale and (DIV) is a cancellation rule for multiplication.

▶ **Theorem 7** (Soundness). *If a sequent γ is provable from S in $\mathbb{RL}$, then γ is a semantic consequence of S. In symbols: $\frac{S}{\gamma}$ implies $S \models \gamma$.*

In $\mathbb{RL}$, $\phi_1, \ldots, \phi_n \vdash \psi$ is provably equivalent to $\phi_1 \oplus \ldots \oplus \phi_n \vdash \psi$; moreover $\phi \vdash \psi$ is provably equivalent to $\vdash \phi \multimap \psi$. Hence, without loss of generality, we may assume that arbitrary sequents are of the form $\vdash \theta$.

▶ **Theorem 8** (Deduction Theorem). *For arbitrary formulas ϕ, ψ in $\mathbb{RL}$, we have*

$$\frac{\vdash \phi}{\vdash \psi} \quad \textit{iff} \quad \frac{}{\vdash (0 \geq \phi) \multimap (0 \geq \psi)}$$

In [9] it is shown that $\mathbb{AL}$ does not enjoy a deduction theorem, not even in the weak form that holds for fuzzy logics, such as Łukasiewicz, Gödel, or product logics [31]. This is because we have proven that is not possible to "internalize" provability in the language of $\mathbb{AL}$. However, in $\mathbb{RL}$, the expressivity provided by multiplication allows us to "booleanize" the sequents.

5 Applications: Proving Properties of Distances

In this section, we show how the deductive system of $\mathbb{RL}$ can be used to reason about distances on probability distributions, namely, the total variation, the Kantorovich and the p-Wasserstein distances, and we discuss embedding quantitative equational logic in $\mathbb{RL}$.

Let $X = \{x_1, \dots, x_n\}$ be a finite (extended) metric space with distances d_{ij} between x_i and x_j possibly taking ∞ as value. Denote by $\mu, \nu, \rho, \dots$ generic discrete probabilities on X and by $\mu_i, \nu_i, \rho_i, \dots$ their probabilities at $x_i \in X$.

Total Variation. The total variation distance $d_{TV}(\mu,\nu) = \max_{A \subseteq X} |\mu(A) - \nu(A)|$, is encoded in $\mathbb{RL}$ by the formula $t_{\mu,\nu} := \bigwedge_{A \subseteq \{1..n\}} (\bigoplus_{i \in A} \mu_i \multimapboth \bigoplus_{i \in A} \nu_i)$. A simple example to start with is to demonstrate that the total variation is a pseudo-metric, *i.e.*, satisfies the axioms of reflexivity, symmetry, and triangle inequality, which can be expressed in $\mathbb{PL}$:

$$(\textsc{refl})\ \vdash t_{\mu,\mu} \qquad (\textsc{symm})\ t_{\mu,\nu} \vdash t_{\nu,\mu}\,, \qquad (\textsc{triang})\ t_{\mu,\nu}, t_{\nu,\rho} \vdash t_{\mu,\rho}\,.$$

The first two are trivial to derive. The derivation of the third is shown below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mu_i \multimap \nu_i \vdash \mu_i \multimap \nu_i}\,(\textsc{id})}{\mu_i, \mu_i \multimap \nu_i \vdash \nu_i}\,(\textsc{quant,prem}) \qquad \dfrac{\dfrac{}{\nu_i \multimap \rho_i \vdash \nu_i \multimap \rho_i}\,(\textsc{id})}{\nu_i \multimap \rho_i, \nu_i \vdash \rho_i}\,(\textsc{quant,prem})}{\mu_i \oplus (\mu_i \multimap \nu_i) \oplus (\nu_i \multimap \rho_i) \vdash \rho_i}\,(\textsc{cut,prem})}{(\mu_i \multimap \nu_i) \oplus (\nu_i \multimap \rho_i) \vdash \mu_i \multimap \rho_i}\,(\textsc{quant})}{(\mu_i \multimapboth \nu_i) \oplus (\nu_i \multimapboth \rho_i) \vdash \mu_i \multimap \rho_i}\,(\textsc{prem},\wedge_1) \qquad \dfrac{\dfrac{\dfrac{\text{similarly}\dots}{\rho_i \oplus (\nu_i \multimap \mu_i) \oplus (\rho_i \multimap \nu_i) \vdash \mu_i}}{(\nu_i \multimap \mu_i) \oplus (\rho_i \multimap \nu_i) \vdash \rho_i \multimap \mu_i}\,(\textsc{quant})}{(\mu_i \multimapboth \nu_i) \oplus (\nu_i \multimapboth \rho_i) \vdash \rho_i \multimap \mu_i}\,(\textsc{prem},\wedge_1)}{(\mu_i \multimapboth \nu_i) \oplus (\nu_i \multimapboth \rho_i) \vdash \mu_i \multimapboth \rho_i}\,(\wedge_2)}{\dfrac{\bigwedge_{A \subseteq \{1..n\}} (\bigoplus_{i \in A} \mu_i \multimapboth \bigoplus_{i \in A} \nu_i) \oplus \bigwedge_{A \subseteq \{1..n\}} (\bigoplus_{i \in A} \nu_i \multimapboth \bigoplus_{i \in A} \rho_i) \vdash \bigwedge_{A \subseteq \{1..n\}} (\bigoplus_{i \in A} \mu_i \multimapboth \bigoplus_{i \in A} \rho_i)}{t_{\mu,\nu}, t_{\nu,\rho} \vdash t_{\mu,\rho}}\,(\textsc{def,prem})}\,(\textsc{prem},\wedge_1,\wedge_2)$$

Note that $(\textsc{perm})$ is used implicitly and some steps of the derivation use meta-rules which are derivable from the rules in Table 2, such as $(\wedge_1)$ and $(\wedge_2)$.

The total variation is not just a pseudo-metric, but a proper metric satisfying the Fréchet positivity axiom, which can be expressed in $\mathbb{RL}$ by the sequent

$$(\textsc{positivity})\ \bigwedge (\mu_i \neq \nu_i) \vdash (t_{\mu,\nu} > 0)\,.$$

The above uses the boolean formulas of $\mathbb{RL}$, which can be expressed using multiplication by $\bot$. In fact, this is a non-linear property that cannot be captured by $\mathbb{AL}$ as it allows only affine formulas.

Kantorovich distance. The Kantorovich distance[3] between μ and ν can be defined using the following two equivalent (dual) formulations

$$d_K(\mu,\nu) = \inf_{\omega} \sum_{i,j} \omega_{ij} d_{ij} = \sup_{f} \Big| \sum_i f_i \mu_i - \sum_i f_i \nu_i \Big| \qquad \text{(K-R duality)}$$

where ω ranges over joint probability distributions with μ as left-marginal (*i.e.*, $\sum_j \omega_{ij} = \mu_i$, for all i) and ν as right-marginal (*i.e.*, $\sum_i \omega_{ij} = \nu_j$, for all j); and f over non-expanding $[0,\infty)$-valued maps on X, *i.e.*, $|f_i - f_j| \leq d_{ij}$, for all i, j.

As its definitions involve inf (infimum) on one hand, and sup (supremum) on the other hand, we cannot express the Kantorovich distance as a single formula in $\mathbb{RL}$. However, we

[3] Also known as the Wasserstein distance or Earth mover's distance.

can still reason about it if we can find a finite set of sequents that uniquely characterises its value. The set we propose, hereafter denoted by $\mathcal{K}$, contains the following sequents:

$$\vdash \bigwedge_i (\bigoplus_j W_{ij} \multimapboth \mu_i) \wedge \bigwedge_j (\bigoplus_i W_{ij} \multimapboth \nu_j)\,, \qquad \bigoplus_i F_i\mu_i \multimapboth \bigoplus_i F_i\nu_i \vdash K_{\mu,\nu}\,,$$

$$\vdash \bigwedge_{i,j} \big(d_{ij} \multimap (F_j \multimap F_i)\big) \wedge \bigwedge_i |F_i|\,, \qquad K_{\mu,\nu} \vdash \bigoplus_{i,j} W_{ij} d_{ij}\,,$$

where W_{ij}, F_i, and $K_{\mu,\nu}$ are propositional atoms. This set is derived by following the steps of the proof of (strong) duality in linear programs [57], tailored to the K-R duality presented above. The sequents to the left represent the conjunction of the constraints from both the primal and dual linear programs (*i.e.*, the marginal conditions on ω and the non-expanding condition on f). Those to the right imply $\bigoplus_i F_i\mu_i \multimapboth \bigoplus_i F_i\nu_i \vdash \bigoplus_{i,j} W_{ij}d_{ij}$, corresponding to the optimality condition for the feasible solutions. The atom $K_{\mu,\nu}$ is a convenience.

This encoding is such that all the models of $\mathcal{K}$ assign the atom $K_{\mu,\nu}$ value $d_K(\mu,\nu)$, *i.e.*, the Kantorovich distance between μ and ν. Indeed, next we show that from $\mathcal{K}$ we can deduce

$$\vdash K_{\mu,\nu} \multimapboth \Big(\bigoplus_i F_i\mu_i \multimapboth \bigoplus_i F_i\nu_i\Big) \qquad \text{and} \qquad \vdash K_{\mu,\nu} \multimapboth \bigoplus_{i,j} W_{ij}d_{ij}\,. \tag{1}$$

The above follows by deriving the following two sequents from $\mathcal{K}$

$$\bigoplus_{i,j} W_{ij}d_{ij} \oplus \bigoplus_i F_i\mu_i \vdash \bigoplus_j F_j\nu_j\,, \qquad \bigoplus_{i,j} W_{ij}d_{ij} \oplus \bigoplus_j F_j\nu_j \vdash \bigoplus_i F_i\mu_i$$

as they imply $\bigoplus_{i,j} W_{ij}d_{ij} \vdash \bigoplus_i F_i\mu_i \multimapboth \bigoplus_i F_i\nu_i$. Note that this corresponds to the steps of the proof of weak duality in linear programs. We show only the derivation of the first one as the other is similar. Below we provide only the schematic steps of the derivation, which would otherwise take too much space

$$\begin{aligned}\bigoplus_{i,j} W_{ij}d_{ij} \oplus \bigoplus_i F_i\mu_i &\vdash \bigoplus_{i,j} W_{ij}d_{ij} \oplus \bigoplus_i F_i(\bigoplus_j W_{ij})) && \text{(left-marginal)}\\ &\vdash \bigoplus_{i,j} F_jW_{ij} && (\text{DISTR, PREM, PERM, non-expanding})\\ &\vdash \bigoplus_j F_j\nu_j && (\text{DISTR, right-marginal})\end{aligned}$$

In the above a concatenation of the form $\phi \vdash \psi \vdash \vartheta$ means that both $\phi \vdash \psi$ and $\psi \vdash \vartheta$ are derivable; the desired result follows by repeated applications of (CUT).

Now that we have established a way to encode the Kantorovich distance, we can prove some of its properties. A well-known result from [27] relating the Kantorovich distance with the total variation is $d_K(\mu,\nu) \geq d_{\min} \cdot d_{TV}(\mu,\nu)$, where $d_{\min} = \min_{i\neq j} d_{ij}$. According to our encoding, such a statement is equivalent to establishing the provability of $K_{\mu,\nu} \vdash (\bigvee_{i\neq j} d_{ij})t_{\mu,\nu}$ from $\mathcal{K}$.

Due to a lack of space, below, we provide only the sketch of the proof. The key steps of it are to show that the sequents below follow from $\mathcal{K}$ for all $A \subseteq \{1,\ldots,n\}$

$$\bigoplus_{i\neq j} W_{ij} \oplus \bigoplus_{i\in A} \mu_i \vdash \bigoplus_{i\in A} \nu_i \qquad \bigoplus_{i\neq j} W_{ij} \oplus \bigoplus_{i\in A} \nu_i \vdash \bigoplus_{i\in A} \mu_i$$

from which, by using (QUANT), ($\vee_2$), one gets $\bigoplus_{i\neq j} W_{ij} \vdash t_{\mu,\nu}$. Thus, by applying the inference rules of $\mathbb{RL}$, (1), and the fact that $d_{ii} = 0$ for all i, we get

$$K_{\mu,\nu} \vdash \bigoplus_{i,j} W_{ij}d_{ij} \vdash \bigoplus_{i\neq j} W_{ij}d_{ij} \vdash \bigoplus_{i\neq j} W_{ij}(\bigvee_{i\neq j} d_{ij}) \vdash (\bigvee_{i\neq j} d_{ij})t_{\mu,\nu}\,.$$

The desired inference follows from the above by repeated applications of (CUT).

Quantitative Equational Logic (QEL). We showed in in [9] how one can embed the finitary part of QEL in $\mathbb{AL}$ (*i.e.*, the axioms and rules of QEL other than its infinitary rule). To do so, we add, as propositional letters in our logic, all the equalities of the form $\ulcorner s = t \urcorner$ for all terms s, t of a chosen quantitative algebra. A quantitative equation such as $\vdash s =_\varepsilon t$ is then encoded in Lawvere logic as the sequent $\varepsilon \vdash \ulcorner s = t \urcorner$, or equivalently as $\vdash \ulcorner s = t \urcorner \leq \varepsilon$.

Next, a quantitative judgement such as the triangle inequality, which in QA has the form

$$s =_\varepsilon t, \quad t =_\delta u \vdash s =_{\varepsilon+\delta} u$$

can be encoded in Lawvere logic as follows, if we want to emphasize ε and δ,

$$(\ulcorner s = t \urcorner \leq \varepsilon) \wedge (\ulcorner t = u \urcorner \leq \delta) \vdash \ulcorner s = u \urcorner \leq (\varepsilon \oplus \delta)$$

or if ε and δ are generic, we can use an even more compact encoding that emphasize the relation between triangle inequality and transitivity

$$\ulcorner s = t \urcorner, \quad \ulcorner t = u \urcorner \vdash \ulcorner s = u \urcorner.$$

The logic $\mathbb{AL}$ of [9] lacks a deduction theorem, and the embedding of QEL in $\mathbb{AL}$ therefore relied on extending $\mathbb{AL}$ with certain inference rules. However, in $\mathbb{RL}$ this problem disappears, as the inferences in [9, Table 2] can be formalized as proper sequents using the deduction theorem (Theorem 8), exactly as we have done above for the triangle inequality.

Additionally, while $\mathbb{AL}$ can handle only affine functions, $\mathbb{RL}$ can encode more complex examples, including polynomials and rational functions and even rational powers.

For instance, interpolative barycentric algebras (IBAs) were introduced in [43] as a quantitative generalization of Stone's barycentric algebras [59]. Barycentric algebras, sometimes called convex algebras, have binary operators $+_e$ for $e \in [0,1]$, where the intended interpretation of $s +_e t$ on reals or distributions is the e-convex combination of s and t. To characterize the p-Wasserstein metric on the space of distributions for a strictly positive integer p, IBAs must satisfy the following axiom:

$$(I_p): \qquad s =_{\varepsilon_1} t, \quad s' =_{\varepsilon_2} t' \vdash s +_e s' =_\delta t +_e t', \qquad \text{where } \delta = (e\varepsilon_1^p + (1-e)\varepsilon_2^p)^{\frac{1}{p}}$$

This can be encoded in $\mathbb{RL}$ using a couple of judgements. Let d be a fresh propositional letter; then (I_p) can be represented in $\mathbb{RL}$ by:

$$(I_p): \qquad \begin{cases} (\ulcorner s = t \urcorner \leq \varepsilon_1) \wedge (\ulcorner s' = t' \urcorner \leq \varepsilon_2) \vdash \ulcorner s +_e s' = t +_e t' \urcorner \leq d \\ \vdash d^p \multimapboth e\varepsilon_1^p \oplus (e \multimap 1)\varepsilon_2^p. \end{cases}$$

The compact quantitative algebraic theories of [47] have the property (in the case of QEL) that if a sequent is provable, then it is provable without the infinitary rule. So our finitary encodings of theories in $\mathbb{RL}$ are *complete* for compact theories in that any QEL consequence of such a theory is also, via the encoding, an $\mathbb{RL}$ consequence. As shown in [47], the theories of rational Wasserstein metrics are compact, as is the theory of quantitative semilattices [9].

6 Completeness and Incompleteness

We first prove that $\mathbb{RL}$ is complete for finite theories.

▶ **Theorem 9** (Finite Completeness). *Let S be a finite set of sequents in $\mathbb{RL}$. If a sequent γ is a semantic consequence of S, then γ is provable from S. That is, $S \models \gamma$ implies $\frac{S}{\gamma}$.*

The proof plan is to reduce the statement above to a restricted form of completeness, which applies only to sequents in a certain polynomial form and allows us to appeal to Krivine-Stengle's Positivstellensatz to obtain the desired result.

▶ **Definition 10.** *A formula in* $\mathbb{RL}$ *is in* polynomial form *if it is built up from propositional letters and constants using addition and multiplication (equivalently, if it has no occurrences of* $\bot$*,* $\multimap$*, or* $/$*).*

Formulas ϕ in polynomial form evidently correspond to polynomials $\widetilde{\phi}$ with positive coefficients over the propositional letters of ϕ, and we have $\widetilde{\phi} = \widetilde{\psi}$ iff $\vdash \phi \multimapboth \psi$ is provable. Further, every polynomial with positive coefficients is obtained in this way, and we may identify polynomials with positive coefficients with corresponding formulas in polynomial form (chosen in some standard manner). Note that $|P|$, which by definition is $P \multimap (P \multimap \bot)$, is not in polynomial form. We extend the definition of polynomial form to sequents and sets of sequents in the obvious way: $\phi_1, \dots \phi_n \vdash \psi$ is in polynomial form if all ϕ_i and ψ are; a set of sequents is in polynomial form if all its elements are. We say that a sequent is *finitising* if it is of the form $\vdash |P|$, and that a set $\mathfrak{F}$ of finitising sequents *restricts* a set of sequents S if it contains $\vdash |P|$ for every propositional letter P occurring in S.

▶ **Theorem 11** (Polynomial Completeness). *Let* γ *be a sequent and* S *a finite set of sequents, all in polynomial form, and let* $\mathfrak{F}$ *be a set of finitising sequents restricting* $S \cup \{\gamma\}$*. Then,* $S \cup \mathfrak{F} \models \gamma$ *implies* $\frac{S \quad \mathfrak{F}}{\gamma}$*.*

Note that $S \cup \mathfrak{F} \models \gamma$ represents a restricted form of semantical consequence where the models are assumed to be $[0, \infty)$-valued.

Before delving into the proof of Theorem 11 – which constitutes the core of the completeness result – we describe our non-deterministic linear reduction to it. The reduction is specified by *rules*, being finite sets

$$(S, \gamma) \longrightarrow (S_i, \gamma_i) \qquad \text{for } i = 1, \dots, k$$

of *moves* between *configurations* of the form (S, γ), where S is a finite set of sequents and γ is a sequent. To be sound, a rule must satisfy the following two properties:

Reliability: $S \models \gamma$ implies $\forall i.\, S_i \models \gamma_i$ (*i.e.*, if γ is a semantical consequence of S, then each γ_i is semantical consequence of S_i).

Faithfulness: $\forall i.\, \frac{S_i}{\gamma_i}$ implies $\frac{S}{\gamma}$ (*i.e.*, if γ_i is provable from the S_i, then γ is provable from S).

We present the reduction by means of rule schemas and divide it into five phases, performed in the following order: (1) reduction to PCF, (2) elimination of $\multimap$, (3) elimination of $/$, (4) choice of domain; and (5) reduction to polynomial form. For ease of presentation, without loss of generality, we assume that all sequents are either of the form $\phi \vdash \psi$ (with exactly one antecedent) or $\vdash |P|$ (finitising).

Phase 1 (Reduction to PCF). The first reduction comprises the following nine one-move rule schemas. The intent is to reduce the judgments in both the premises and the conclusions of configurations to a simplified canonical form, *propositional canonical form* (PCF), where logical connectives are applied only to propositional letters.

$$(S, \phi \vdash \psi) \longrightarrow (S \cup \{P \vdash \phi, \psi \vdash Q\}, P \vdash Q) \tag{C}$$
$$(S \cup \{\phi \oplus \psi \vdash \theta\}, \gamma) \longrightarrow (S \cup \{P \oplus Q \vdash \theta, \phi \vdash P, \psi \vdash Q\}, \gamma) \tag{$\oplus$-L}$$
$$(S \cup \{\theta \vdash \phi \oplus \psi\}, \gamma) \longrightarrow (S \cup \{\theta \vdash P \oplus Q, P \vdash \phi, Q \vdash \psi\}, \gamma) \tag{$\oplus$-R}$$
$$(S \cup \{\phi\psi \vdash \theta\}, \gamma) \longrightarrow (S \cup \{PQ \vdash \theta, \phi \vdash P, \psi \vdash Q\}, \gamma) \tag{$\times$-L}$$
$$(S \cup \{\theta \vdash \phi\psi\}, \gamma) \longrightarrow (S \cup \{\theta \vdash PQ, P \vdash \phi, Q \vdash \psi\}, \gamma) \tag{$\times$-R}$$
$$(S \cup \{\phi \multimap \psi \vdash \theta\}, \gamma) \longrightarrow (S \cup \{P \multimap Q \vdash \theta, P \vdash \phi, \psi \vdash Q\}, \gamma) \tag{$\multimap$-L}$$
$$(S \cup \{\theta \vdash \phi \multimap \psi\}, \gamma) \longrightarrow (S \cup \{\theta \vdash P \multimap Q, \phi \vdash P, Q \vdash \psi\}, \gamma) \tag{$\multimap$-R}$$
$$(S \cup \{\phi/\psi \vdash \theta\}, \gamma) \longrightarrow (S \cup \{P/Q \vdash \theta, \phi \vdash P, Q \vdash \psi\}, \gamma) \tag{/-L}$$
$$(S \cup \{\theta \vdash \phi/\psi\}, \gamma) \longrightarrow (S \cup \{\theta \vdash P/Q, P \vdash \phi, \psi \vdash Q\}, \gamma) \tag{/-R}$$

where $P, Q \in \mathbb{P}$ are fresh propositional letters not occurring in the source configurations of the moves (chosen in a standard way) and at least one among ϕ or ψ is not a propositional letter.

The correctness of the rules follows from the monotonicity properties of the connectives: $\oplus$ and $\times$ are monotone in both arguments; $\multimap$ is antimonotone in its first argument and monotone in its second; and $/$ is monotone in its first argument and antimonotone its second.

Observe that, since the rules bring subformulas to the top level, their repeated application ensures that every sequent is eventually brought into PCF. The next phases will keep sequents in this form, except for finitising ones.

Phase 2 (Elimination of $\multimap$). The following two rule schemas (the first with three moves) are designed to eliminate all occurrences of $\multimap$:

$$(S \cup \{P \multimap Q \vdash \phi\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, 0 \vdash \phi\}, \gamma) \tag{$\multimap$-EL1}$$
$$(S \cup \{P \multimap Q \vdash \phi\}, \gamma) \longrightarrow (S \cup \{\vdash |P|, P \vdash Q, 0 \vdash \phi\}, \gamma) \tag{$\multimap$-EL2}$$
$$(S \cup \{P \multimap Q \vdash \phi\}, \gamma) \longrightarrow (S \cup \{\vdash |P|, Q \vdash P, Q \vdash P \oplus R, R \vdash \phi\}, \gamma) \tag{$\multimap$-EL3}$$
$$(S \cup \{\phi \vdash P \multimap Q\}, \gamma) \longrightarrow (S \cup \{\phi \vdash R, R \oplus P \vdash Q\}, \gamma) \tag{$\multimap$-ER}$$

where P, Q, R are propositional letters and R, chosen in a standard way, is fresh in the move source configurations. The rule ($\multimap$-EL) removes occurrences of $P \multimap Q$ on the left-hand side of sequents; its correctness relies on the axioms (LIN) and (WEM). Dually, the rule ($\multimap$-R) removes occurrences of $P \multimap Q$ on the right-hand side of sequents; its correctness follows from (QUANT). The fresh propositional letter R is used to maintain the sequents in PCF. As for the previous phase, repeated applications of these rules ensure the elimination of $\multimap$ from all sequents except the finitising ones.

Phase 3 (Elimination of $/$). The two rule schemas below (the second one comprising four moves) remove all occurrences of $/$:

$$(S \cup \{P/Q \vdash \phi\}, \gamma) \longrightarrow (S \cup \{R \vdash \phi, P \vdash QR\}, \gamma) \tag{/-EL}$$
$$(S \cup \{\phi \vdash P/Q\}, \gamma) \longrightarrow (S \cup \{0 \vdash Q, \phi \vdash \bot\}, \gamma) \tag{/-ER1}$$
$$(S \cup \{\phi \vdash P/Q\}, \gamma) \longrightarrow (S \cup \{\vdash |Q|, R \vdash \bot, QR \vdash \bot, \phi \vdash T, TQ \vdash P\}, \gamma) \tag{/-ER2}$$
$$(S \cup \{\phi \vdash P/Q\}, \gamma) \longrightarrow (S \cup \{Q \vdash \bot, \vdash |P|, \phi \vdash 0\}, \gamma) \tag{/-ER3}$$
$$(S \cup \{\phi \vdash P/Q\}, \gamma) \longrightarrow (S \cup \{Q \vdash \bot, P \vdash \bot, \phi \vdash \bot\}, \gamma) \tag{/-ER4}$$

where $P, Q, R, T \in \mathbb{P}$ are propositional letters and R, T are fresh in the source configurations (chosen in a standard way). The rule (/-EL) removes occurrences of P/Q on the left-hand side of sequents; its correctness follows from (ADJ). Dually, the rule (/-ER) removes P/Q from the right-hand side of sequents; its soundness follows from (LIN). The propositional letter R is used to encode that Q is non-zero using the combinations of the sequents $R \vdash \bot$ and $QR \vdash \bot$; the propositional letter T to maintain the sequent in PCF.

Phase 4 (Choice of domain). This is a rule schema comprising two moves:

$$(S, \gamma) \longrightarrow (S \cup \{\vdash |P|\}, \gamma) \tag{F}$$

$$(S, \gamma) \longrightarrow (S \cup \{P \vdash \bot\}, \gamma) \tag{$\bot$}$$

where P is a propositional letter occurring in S such that neither $\vdash |P|$ nor $P \vdash \bot$ are in S.

The moves (F) and ($\bot$) correspond, respectively, to non-deterministically choosing whether P is finite or infinite. This phase is completed when all propositional letters in S have been "tagged" in one of the two ways above. Note that the applicability conditions ensure that the rules are never applied vacuously or repeated twice on the same propositional letter.

Phase 5 (Reduction to Polynomial Form). Recall that a formula is in polynomial form if it has no occurrences of $\bot$, $\multimap$, or $/$. The last two requirements have been taken care of by the previous phases. This phase, which we split into two stages, concerns the first requirement.

Stage 1. It removes the occurrences of infinitary propositional letters ($P \vdash \bot$) by means of the following seven rule schemas (the last two comprising two moves each)

$$(S \cup \{P \vdash \bot\}, \gamma) \longrightarrow (S, \gamma) \qquad \text{when } P \text{ does not occur in } (S, \gamma) \tag{$\bot$-E}$$

$$(S \cup \{P \vdash \bot\}, P \vdash \phi) \longrightarrow (S \cup \{P \vdash \bot\}, \bot \vdash \phi) \tag{$\bot$-CL}$$

$$(S \cup \{P \vdash \bot\}, \phi \vdash P) \longrightarrow (S \cup \{P \vdash \bot\}, \phi \vdash \bot) \tag{$\bot$-CR}$$

$$(S \cup \{P \vdash \bot, P \vdash \phi\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \bot \vdash \phi\}, \gamma) \qquad \text{when } \phi \neq \bot \tag{$\bot$-PL}$$

$$(S \cup \{P \vdash \bot, \phi \vdash P\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \phi \vdash \bot\}, \gamma) \tag{$\bot$-PR}$$

$$(S \cup \{P \vdash \bot, P \oplus Q \vdash \phi\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \bot \vdash \phi\}, \gamma) \tag{$\bot$-SL}$$

$$(S \cup \{P \vdash \bot, \phi \vdash P \oplus Q\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \phi \vdash \bot\}, \gamma) \tag{$\bot$-SR}$$

$$(S \cup \{P \vdash \bot, PQ \vdash \phi\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \vdash Q, \vdash \phi\}, \gamma) \tag{$\bot$-ML1}$$

$$(S \cup \{P \vdash \bot, PQ \vdash \phi\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \vdash |R|, QR \vdash 1, \bot \vdash \phi\}, \gamma) \tag{$\bot$-ML2}$$

$$(S \cup \{P \vdash \bot, \phi \vdash PQ\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \vdash Q, \phi \vdash 0\}, \gamma) \tag{$\bot$-MR1}$$

$$(S \cup \{P \vdash \bot, \phi \vdash PQ\}, \gamma) \longrightarrow (S \cup \{P \vdash \bot, \vdash |R|, QR \vdash 1, \phi \vdash \bot\}, \gamma) \tag{$\bot$-MR2}$$

where $P, Q, R \in \mathbb{P}$ are propositional letters and R is fresh. The rules ($\bot$-PL), ($\bot$-PR), ($\bot$-SL), ($\bot$-SR), ($\bot$-ML), and ($\bot$-MR) remove an occurrence of the infinitary propositional letter P when it appears atomically or in logical connectives — we assume they apply up to commutativity of $\oplus$ and $\times$. The rules ($\bot$-CL) and ($\bot$-CR) remove the occurrence of P from the conclusion. Once these rules can no longer apply, the rule ($\bot$-E) removes $P \vdash \bot$.

As the rule schemas apply for arbitrary infinitary propositional letters P, their repeated application will eventually eliminate all the occurrences of such propositional letters.

Stage 2. After the previous phases, the only sequents that are not in polynomial form apart from the finitising ones are either trivially valid ($\bot \vdash \phi$) or finitarily unsatisfiable ($\phi \vdash \bot$). The following two one-move rule schemas eliminate the last occurrences of $\bot$:

$$(S, \gamma) \longrightarrow \mathcal{V}(S, \gamma) \qquad \text{(Valid)}$$

$$(S, \gamma) \longrightarrow \mathcal{U}(S, \gamma) \qquad \text{(Unsat)}$$

where, $\mathcal{V}(S, \gamma)$ and $\mathcal{U}(S, \gamma)$ are obtained from (S, γ) by replacing sequents of the form $\bot \vdash \phi$ with $0 \vdash 1$ (which is still valid), and sequents of the form $\phi \vdash \bot$, where $\phi \neq \bot$, with $1 \vdash 0$ (which is still unsatisfiable), respectively. Note that $0 \vdash 1$ and $1 \vdash 0$ are in polynomial form.

▶ **Proposition 12.**
1. *The rules of the reduction are reliable and faithful.*
2. *The non-deterministic tree of moves is finite and the leaves are configurations of the form $(S \cup \mathfrak{F}, \gamma)$ where S and γ are in polynomial form, and $\mathfrak{F}$ is a finitising set of sequents restricting $S \cup \{\gamma\}$.*
3. *If the formulas of the initial configuration (S, γ) contain only rational constants, then so do all the configurations of the tree, and the height of the tree is linear in the size of (S, γ), as is the maximum size of the configurations in the tree.*

In the above, the size of a formula is intended as the total number of logical connectives and propositional atoms it contains, plus the number of bits required for the binary representation of the constants[4]. The size of a set of judgments is the sum of the sizes of its formulas, and similarly for configurations.

Now we are ready to prove polynomial completeness:

Proof of Theorem 11. Let $\gamma = \theta \vdash \vartheta$ be a sequent and $S = \{\theta_1 \vdash \vartheta_1, \dots, \theta_n \vdash \vartheta_n\}$ be a finite set of sequents, all in polynomial form, and let $\mathfrak{F}$ be a set of finitising sequents restricting $S \cup \{\gamma\}$. Assume that $S \cup \mathfrak{F} \models \gamma$ (thus, any $[0, \infty)$-valued model of S is also a model for γ).

Identifying polynomial formulas ϕ with their corresponding polynomials $\widetilde{\phi}$, the $[0, \infty)$-valued models of S are the solutions of the following system of polynomial inequalities

$$\theta_i - \vartheta_i \geq 0 \quad (\text{for } i = 1, \dots, n) \qquad\qquad P_j \geq 0 \quad (\text{for } j = 1, \dots, m)$$

where $P_1, \dots, P_m$ are the propositional letters occurring in $S \cup \{\gamma\}$. We recall one form of Krivine-Stengle's Positivstellensatz [35, 58] (see also [13, Corollary 4.4.3]).

▶ **Theorem 13** (Positivstellensatz)**.** *Let $f, f_1, \dots, f_r \in \mathbb{R}[X_1, \dots, X_n]$ n-variate polynomials over the reals and denote by $W = \{x \in \mathbb{R}^n \mid \forall i.\, f_i(x) \geq 0\}$ their semialgebraic set and by C the cone generated by them (i.e., the subsemiring generated by $f_1, \dots, f_r$ and squares of polynomials). Then,*

$$\forall x \in W.\, f(x) \geq 0 \iff \exists s \in \mathbb{N}.\, \exists h_1, h_2 \in C.\, h_1 f = f^{2s} + h_2 \,.$$

By the Positivstellensatz, there are polynomials $h_1, h_2 \in \mathbb{R}[P_1, \dots, P_m]$ (obtained using sums and multiplications from $(\theta_i - \vartheta_i)$, P_j, and squares of arbitrary polynomials) and integer $s \geq 0$ such that

$$h_1\theta = h_1\vartheta + (\theta - \vartheta)^{2s} + h_2$$

[4] For a rational $\frac{m}{n}$, we assume the common encoding format $\mathsf{bin}(m)\#\mathsf{bin}(n)$, where bin denotes binary encoding and # is a separator symbol not in the binary alphabet $\{0, 1\}$.

The first step is to find formulas ρ_1, ρ_2 such that:

$$\frac{S \quad \mathfrak{F}}{\vdash \rho_1\theta \multimapboth \rho_1\vartheta \oplus (\vartheta \multimapboth \theta)^{2s} \oplus \rho_2}\,. \tag{2}$$

To this end, for any polynomial f, write f^+ and f^- for its positive and negative parts, such that $f = f^+ - f^-$ and both f^+ and f^- have positive coefficients. For any set of judgements S, formula ϕ, and polynomial f, define

$$\phi =_S f \quad \text{iff} \quad \frac{S \quad \mathfrak{F}_{\text{tot}}}{\vdash \phi \oplus f^- \multimapboth f^+}$$

where $\mathfrak{F}_{\text{tot}} =_{\text{def}} \{\vdash |P| \mid P \in \mathbb{P}\}$. The next lemma allows us to turn equalities between not-necessarily positive polynomials into provable equalities between $\mathbb{RL}$ formulas in polynomial form.

▶ **Lemma 14.** *Let f, g be polynomials and ϕ, ψ be formulas in $\mathbb{RL}$. Then*

1. *If $\phi =_S f$ and $\psi =_S f$, then $\vdash \phi \multimapboth \psi$ is provable from S and $\mathfrak{F}_{\text{tot}}$.*
2. *If $\phi =_S f$ and $\psi =_S g$, then $\phi \oplus \psi =_S f + g$ and $\phi\psi =_S fg$.*
3. *If f has only positive coefficients, then $f =_S f$.*
4. *$(f^+ \multimapboth f^-)^2 =_S f^2$.*
5. *If f, g have only positive coefficients and $f \vdash g$ is provable from S and $\mathfrak{F}_{\text{tot}}$, then $g \multimap f =_S f - g$.*

Now, using Lemma 14.(2–5) we get formulas ρ_1 and ρ_2 such that $\rho_i =_S h_i$ (for $i = 1, 2$). By Lemma 14.(2–4), we further obtain $(\vartheta \multimapboth \theta)^{2s} =_S (\vartheta - \theta)^{2s}$. By combining the above with Lemma 14.(2) we finally get $\rho_1\theta =_S h_1\theta$ and $\rho_1\vartheta \oplus (\vartheta \multimapboth \theta)^{2s} \oplus \rho_2 =_S h_1\vartheta + (\theta - \vartheta)^{2s} + h_2$. Then, Lemma 14.(1) gives us $\rho_1\theta \multimapboth \rho_1\vartheta \oplus (\vartheta \multimapboth \theta)^{2s} \oplus \rho_2$, obtaining (2), as required.

We next show that $\frac{S \quad \mathfrak{F}}{\gamma}$. There are two cases. (Case $\vdash \rho_1 \neq 0$) From the conclusion of (2) we obtain $\vdash \rho_1\theta \multimap \rho_1\vartheta$ and so $\rho_1\theta \vdash \rho_1\vartheta$. Then $\theta \vdash \vartheta$, as required. (Case $\vdash \rho_1 = 0$) From the conclusion of (2) we get $\vdash 0 \multimapboth (\vartheta \multimapboth \theta)^{2s} \oplus \rho_2$. If $s = 0$, this is $\vdash 0 \multimapboth 1 \oplus \rho_2$, which is a contradiction. Otherwise, we get $\vdash (\theta \multimapboth \vartheta)^{2s}$ with $s > 0$, and so $\vdash \vartheta \multimapboth \theta$. From this, we derive $\vdash \theta \multimap \vartheta$ (recall that, $\vartheta \multimapboth \theta = (\vartheta \multimap \theta) \wedge (\theta \multimap \vartheta)$) and thus $\theta \vdash \vartheta$, as required. ◀

With that we can prove our main completeness theorem, Theorem 9. The root node of the reduction tree is (S, γ) where $S \models \gamma$. By Proposition 12, the leaf nodes have the form $(S' \cup \mathfrak{F}, \gamma')$ where S' and γ' are in polynomial form, and $\mathfrak{F}$ is a finitising set of sequents restricting $S' \cup \{\gamma'\}$. As the rules are reliable we have $S' \cup \mathfrak{F} \models \gamma'$ for all leaf nodes. Then, by polynomial completeness, we have $\frac{S' \quad \mathfrak{F}}{\gamma'}$ for them, and, finally, as the rules are faithful, we have $\frac{S}{\gamma}$, as required.

Turning to incompleteness, define *consequential compactness* to be that if $\frac{S}{\gamma}$ is valid for a set of sequents S, then $\frac{S_0}{\gamma}$ is valid for some finite $S_0 \subseteq S$ (using the evident extension of validity to allow an infinite set of hypotheses). This fails as the counterexample with $S = \{1/2^n \vdash P \mid \ n \in \mathbb{N}\}$ and $\gamma \ = \vdash P$ shows. As only a propositional letter and rational numbers are used here, we have:

▶ **Theorem 15** (Incompleteness). *There can be no finitary complete consistent proof system for any sublogic of $\mathbb{RL}$ containing a propositional letter and the rational numbers.*

The more usual compactness notion is that if every finite subset of a set S of sequents has a model, then so does S. The two are equivalent: if compactness fails with S then consequential compactness fails with $\frac{S}{\vdash \bot}$; and if consequential compactness fails with $\frac{S}{\phi \vdash \psi}$ then compactness fails with $S \cup \{\vdash \phi < \psi\}$.

7 Complexity Results

In this section, we give complexity bounds for two fundamental decision problems:

▶ **Definition 16** (Decision problems).

- *The* satisfiability problem *asks, given a finite set of sequents S, whether S has a model, i.e., whether $\mathcal{I} \models S$, for some $\mathcal{I}$.*
- *The* semantical consequence problem *asks, given a finite set of sequents S and a sequent γ, whether every model of S is also a model of γ, i.e., whether $S \models \gamma$.*

We restrict our attention to the case where formulas only have rational constants. The sizes of formulas, sequents, and sets of sequents are defined as discussed after Proposition 12.

▶ **Theorem 17.** *Semantical consequence is in PSPACE for $\mathbb{RL}$ and co-NP complete for $\mathbb{AL}$.*

Using faithfulness, reliability, and polynomial completeness, we see that the root node (S, γ) of the reduction tree is valid, in the sense that $S \models \gamma$, iff all the leaf nodes are. Membership of $\mathbb{RL}$-consequence in PSPACE follows by considering a nondeterministic exploration of the tree making use at the leafs of the fact that satisfiability in the existential theory of the reals [14, 54] is in PSPACE. For $\mathbb{AL}$, membership in co-NP is proved similarly, but now via reduction to the infeasibility of linear programs [34]; co-NP hardness follows by a linear-time reduction from boolean propositional logic.

Observe that S has a model if and only if $\bot$ is not a semantical consequence of S, in symbols, $S \not\models \bot$. We therefore obtain:

▶ **Corollary 18.** *Satisfiability is in PSPACE for $\mathbb{RL}$ and is NP-complete for $\mathbb{AL}$.*

Moreover, as $\mathbb{AL}$ is a sublanguage of $\mathbb{RL}$, satisfiability in $\mathbb{RL}$ is at least NP-hard.

8 Conclusions

We have developed and studied Rational Lawvere logic ($\mathbb{RL}$), a logic based on two quantales on $[0, \infty]$: one additive and one multiplicative, forming a semiring. We presented a deduction system for $\mathbb{RL}$ and showed the logic is complete for finitely axiomatized theories (but necessarily incomplete for general theories). The core of the completeness proof draws on a result from real algebraic geometry, the Krivine-Stengle Positivstellensatz, providing evidence of a deep connection between arithmetic and logical reasoning. We additionally presented new complexity results for both $\mathbb{RL}$ and its affine fragment ($\mathbb{AL}$). The satisfiability of a finite set of sequents is NP-complete in $\mathbb{AL}$ and in PSPACE for $\mathbb{RL}$; and semantical consequence from a finite set of sequents is co-NP-complete in $\mathbb{AL}$ and in PSPACE for $\mathbb{RL}$.

There are several possibilities for further work. One can ask if general $\mathbb{PL}$ theories have complete infinitary proof systems. One might seek a finitary approximation theory, building on the Weierstrass approximation theorem that continuous real-valued functions on compact subsets can be approximated arbitrarily well by polynomials, Natural extensions of $\mathbb{PL}$ beckon: predicate logics, modal logics, and μ-calculi.

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