

Mean-Payoff and Energy Discrete-Bidding Games

Guy Avni 

Department of Computer Science, University of Haifa, Israel

Suman Sadhukhan 

Department of Computer Science, University of Haifa, Israel

Abstract

A *bidding* game is played on a graph as follows. A token is placed on an initial vertex and both players are allocated budgets. In each turn, the players simultaneously submit bids that do not exceed their available budgets, the higher bidder moves the token, and pays the bid to the lower bidder. We focus on *discrete*-bidding, which are motivated by practical applications and restrict the granularity of the players' bids, e.g., bids must be given in cents. We study, for the first time, discrete-bidding games with *mean-payoff* and *energy* objectives. In contrast, mean-payoff *continuous*-bidding games (i.e., no granularity restrictions) are understood and exhibit a rich mathematical structure. The *threshold* budget is a necessary and sufficient initial budget for winning an energy game or guaranteeing a target payoff in a mean-payoff game. We first establish existence of threshold budgets; a non-trivial property due to the concurrent moves of the players. Moreover, we identify the structure of the thresholds, which is key in obtaining compact strategies, and in turn, showing that finding threshold is in NP and coNP even in succinctly-represented games.

2012 ACM Subject Classification Theory of computation → Algorithmic game theory and mechanism design; Theory of computation → Formal languages and automata theory

Keywords and phrases Bidding games, Discrete-bidding, Mean-payoff games, energy games

Digital Object Identifier 10.4230/LIPIcs.CSL.2026.32

Related Version *Full Version*: <https://arxiv.org/abs/2509.00506> [13]

Funding This research was supported in part by ISF grant no. 1679/21.

1 Introduction

Two-player *graph games* constitute a fundamental model with applications in *reactive synthesis* [27] and multi-agent systems [2], and a deep connection to foundations of logic [28]. A game is played on a graph as follows. A token is placed on a vertex and the players move the token throughout the graph to generate an infinite path (*play*). Two orthogonal characterizations for graph games are (1) the *mode* by which the players move the token, e.g., in *turn-based* games, the players alternate turns in moving the token, and (2) the players' objectives, which determine the winner or utilities in a play.

We study *bidding games* [22, 23] in which an auction (bidding) determines which player acts in each turn: both players are allocated initial budgets, and in each turn, they simultaneously submit bids that do not exceed their budgets, the higher bidder moves the token, and pays their bid to the opponent. *Discrete bidding* [20], which is the focus of this paper, impose granularity restrictions: budgets are given in “cents” and the smallest positive bid is a “cent”. In contrast, *continuous bidding* allows arbitrarily small bids. We study, for the first time, discrete-bidding games with *mean-payoff* and *energy* objectives (formally defined in Sec. 2).

The motivation for discrete bidding is practical; every practical application requires some granularity restriction. We describe examples of applications of bidding games.

Auction-based scheduling [11] applies bidding games in a “decoupled” synthesis procedure: given two objectives ψ_1 and ψ_2 , the idea is to independently construct two policies f_1 and f_2 , where policy f_i only aims to satisfy ψ_i , for $i \in \{1, 2\}$, and to compose f_1 with f_2



© Guy Avni and Suman Sadhukhan;

licensed under Creative Commons License CC-BY 4.0

34th EACSL Annual Conference on Computer Science Logic (CSL 2026).

Editors: Stefano Guerrini and Barbara König; Article No. 32; pp. 32:1–32:18

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

at runtime using a bidding for who chooses the action in each turn. For example, consider the task of finding a plan for a robot waiter, where ψ_1 specifies delivering food and ψ_2 specifies collecting dishes. The challenge in [11] is to ensure that the composition of f_1 and f_2 satisfies $\psi_1 \wedge \psi_2$ even though they are constructed independently. Our work enables a combination of discrete bidding, which, again, is necessary in practice, with quantitative specifications. For example, consider the task of finding a plan for a patrolling robot, where ψ_i specifies maximizing the time spent at location t_i , for $i \in \{1, 2\}$.

Fair allocation of resources is timely (e.g., [3, 14]). The goal is to allocate a collection of items to agents in a *fair* manner. A mechanism based on bidding games is both natural and useful [24, 15]: each agent is allocated an initial budget, and the items are auctioned sequentially. Repeated applications of this mechanism are used for ongoing allocation of resources [21], e.g., daily allocation of GPU time to users. Mean-payoff objectives naturally specify a strong notion of fairness. For example, “in the long-run, the users are scheduled for the same duration of time”. As another example, online advertisement platforms hold auctions for allocation of ad slots [25], then an advertiser might aim to maximize the long-run average daily exposure, again, a mean-payoff objective.

Previous results

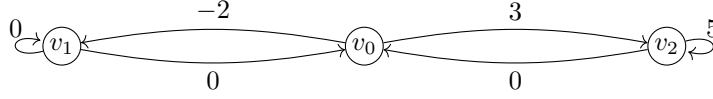
Continuous-bidding games. We briefly survey relevant literature. Continuous-bidding games with *reachability* objectives were studied in [23, 22] and *parity* objectives in [5]. The central quantity in these games is the *threshold budget*, which is roughly a necessary and sufficient initial budget for winning the game. Thresholds satisfy the *average property*: the threshold in a vertex is the average of two of its neighbors. This leads to an equivalence between bidding games and a class of *stochastic games* [19] called *random-turn games* [26].

Mean-payoff continuous-bidding games have been extensively studied. A generalized equivalence with random-turn games was shown in [5]. It implies that, somewhat surprisingly, in a strongly-connected game, the optimal payoff depends only on the structure of the game; that is, a player *cannot* guarantee a higher payoff, given a higher initial budget. Moreover, intricate equivalences between mean-payoff continuous-bidding games and random-turn games were shown for various bidding mechanisms [6, 7, 8]; including mechanisms for which equivalences in finite-duration games are not known and unlikely to exist.

Discrete-bidding games. Discrete-bidding games are far less understood than their continuous-bidding counterparts. So far, only qualitative objectives have been studied. Reachability discrete-bidding games were studied in [20]. It was shown that threshold budgets exist and satisfy a discrete version of the average property. Existence of thresholds in infinite-duration games was established in [1], but the question of whether thresholds satisfy the average property was left open. Moreover, in both papers, the algorithms for finding thresholds are exponential when the budgets are given succinctly¹; in practice, a succinct representation is appealing since large budgets imply reduced granularity constraints and high precision. Recently, both problems were solved in [12]: thresholds in parity discrete-bidding games were shown to satisfy the average property and based on this, finding thresholds was shown to be in NP and coNP.

Previous results leave a gap in our understanding of mean-payoff bidding games: under continuous-bidding, the literature is a rich, whereas under discrete-bidding, even the basic properties were not known.

¹ More formally, the total budget, later denoted k , is part of the representation of a discrete-bidding game, and we assume throughout the paper that k is represented in binary.



■ **Figure 1** A mean-payoff discrete bidding game where optimal payoff depends on the initial budget.

Our results

The central quantity that we study in mean-payoff games is threshold budgets, which we define as follows: for a target payoff c for Max, the threshold budget is necessary and sufficient for guaranteeing payoff c . Before elaborating on our results, we point to an inherent distinction between mean-payoff discrete- and continuous-bidding games: in strongly-connected games, under continuous-bidding, Max can guarantee the same payoff for every initial budget, whereas the following example shows that this is not the case under discrete bidding.

► **Example 1.** Consider the game that is depicted in Fig. 1. A *configuration*² $\langle v, B_{\text{Max}}, B_{\text{Min}}^* \rangle$ means that the token is placed on v , Max and Min's budgets are respectively $B_{\text{Max}}, B_{\text{Min}} \in \mathbb{N}$. Tie breaking is resolved as follows, based on [20]. One of the players (in this case Min) holds the *tie-breaking advantage*, marked with *. Min chooses a bid $b \leq B_{\text{Min}}$ and chooses whether she uses the advantage: (i) she uses it by bidding b^* , then if Max bids $b' \leq b$, she wins and pays Max b^* , and (ii) she does not use it by bidding b , then Max wins if he bids $b' \geq b$.

We describe *optimal plays* that arise from optimal play of both players. Note that upon winning a bidding, it is optimal for Min and Max to proceed left and right, respectively. We write $c \xrightarrow{b_{\text{Max}}, b_{\text{Min}}} c'$ to indicate that c' results from configuration c when the players respectively bid b_{Max} and b_{Min} . First, the optimal play from $\langle v_0, 1^*, 0 \rangle$ is $\langle v_0, 1^*, 0 \rangle \xrightarrow{0^*, 0} \langle v_2, 1, 0^* \rangle \xrightarrow{0, 0^*} \langle v_0, 1^*, 0 \rangle \xrightarrow{0^*, 0} \langle v_2, 1, 0^* \rangle \dots$ with corresponding path $(v_0, v_2)^\omega$ whose payoff is $\frac{3}{2}$. Second, consider the configuration $\langle v_0, 0^*, 1 \rangle$, in which Max has less budget. The optimal play is $\langle v_0, 0^*, 1 \rangle \xrightarrow{0^*, 1} \langle v_1, 1^*, 0 \rangle \xrightarrow{0^*, 0} \langle v_0, 1, 0^* \rangle \xrightarrow{1, 0^*} \langle v_2, 0, 1^* \rangle \xrightarrow{0, 0^*} \langle v_0, 0^*, 1 \rangle \dots$ with corresponding path $(v_0, v_1, v_0, v_2)^\omega$ with payoff $\frac{1}{4}$. This is a key distinction from continuous-bidding. There, the optimal payoff that Max can guarantee does not depend on the initial budget; it is roughly 1 in this game, for *every* positive initial budget.

Technically, we study energy games, and the results directly apply to mean-payoff bidding games, similar to turn-based games (e.g., [17]). An energy bidding game is played between *Consumer* (Cons) and *Preserver* (Pres) on a weighted graph. A play π corresponds to an infinite sequence of weights. Fix an initial energy $M \in \mathbb{N}$. Cons's goal is to “consume” the energy; formally, Cons wins iff there is a prefix of length m such that $M + \text{sum}(\pi^{\leq m}) < 0$.

We study two types of thresholds in energy games. First, we show existence of *energy thresholds*; for every initial vertex v , for every initial budget B , we show that there exists an initial energy, denoted $\text{Energy}(v, B)$ that is both necessary and sufficient for Pres to guarantee winning. We point out that existence of energy thresholds implies *determinacy*, namely from each initial configuration, one of the players has a (pure) winning strategy. Bidding games are formally a subclass of *concurrent games* [2], the latter are not determined; e.g., neither player has a winning strategy in “matching pennies”. Still, we show that energy bidding games are a determined subclass of concurrent games (see also [1, 16]). Second, we define

² Later we will specify only one of the players' budgets in a configuration and the other budget is implicit.

threshold budgets in energy bidding games. Before describing the definition, note that at v , there could be a budget B for which Pres loses with every initial energy. A simple example is a sink with a negative self loop. In such cases, $\text{Energy}(v, B) = \infty$. Further note that $\text{Energy}(v, B)$ increases as B decreases. We define the threshold in a vertex v , denoted $Th(v)$, as the minimal budget B such that $\text{Energy}(v, B)$ is finite.

A key result in the paper shows that Th satisfies the average property. This is an important ingredient in constructing concise *budget agnostic* strategies, which ignore “excess” budget; more formally, at vertex v , for budget $B \geq Th(v)$, a budget agnostic strategy acts in B and B^* as if the budget is $Th(v)$ and $Th(v)^*$, respectively. Existence of winning budget agnostic strategies is key in proving that finding threshold budgets³ is in NP and coNP.

Comparison with previous works. We establish existence of thresholds via a value-iteration algorithm, similar to the approach in previous works. However, previously, establishing the average property and constructing budget-agnostic strategies was a simple byproduct [20, 12] of the corresponding value-iteration algorithm, and in our case, it is significantly more challenging. In fact, we show in Ex. 24, that our algorithm produces strategies that are *not* budget agnostic. We circumvent this challenge by developing a novel proof structure. We first establish the average property directly (Thm. 19), from which we obtain “budget agnostic bids”. For both Cons and Pres separately (Sections 5 and 6), we identify certain scenarios in which the value-iteration algorithm’s winning strategies match the budget agnostic bids. This is particularly challenging for Cons, for which we do not have an explicit strategy. These observations are used to show that eventually, our strategies maintain energy invariants like the value-iteration strategies.

Second, interestingly, our constructions are conceptually very different from strategies in mean-payoff continuous-bidding games. There, strategies are *not* budget agnostic: Max maintains an invariant between accumulated energy and budget so that when the energy increases, both Max’s budget and his bids decrease, thus the strategy is quite the opposite of being budget agnostic. Our strategies are conceptually closer to constructions in turn-based games (e.g., [17]) in that they guarantee that eventually the play avoids “bad cycles”, namely cycles with average weight lower than the target payoff.

2 Preliminaries

We denote $\mathbb{N}$ as the set of natural numbers including 0 and $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$.

Concurrent games

A bidding game is formally, a succinctly represented *concurrent game*. We define concurrent games, and then describe the concurrent game that a bidding game corresponds to.

Intuitively, a concurrent game is a two-player game that is played on a graph, where each vertex is associated with a set of *allowed actions* for each player. The game proceeds as follows. A token is initially placed on a vertex. In each turn, the players simultaneously choose an allowed action, and their joint actions determine the next vertex the token moves to. This generates an infinite path, which determines the winner of the game.

³ Formally, given a game, a vertex v , and a value t , decide whether $Th(v) \geq t$.

Formally a concurrent game is played on an arena $\langle A, Q, \lambda, \delta \rangle$, where A is a set of actions, Q is a set of states, $\lambda : Q \times \{1, 2\} \rightarrow 2^A \setminus \emptyset$ specifies the allowed actions for each player at a state, and the transition function is $\delta : Q \times A \times A \rightarrow Q$. The *neighbors* of $q \in Q$ are $N(q) = \{q' \in Q : \exists a_1, a_2 \in A \text{ such that } q' \in \delta(q, a_1, a_2)\}$. For an infinite path $\pi = q_0, q_1, \dots$, we denote the prefix of length $m \in \mathbb{N}$ by $\pi^{\leq m} = q_0, \dots, q_m$.

A *strategy* is intuitively a *recipe* for playing the game. For $i \in \{1, 2\}$, a strategy for *Player* i is a function $\sigma_i : Q^* \rightarrow A$, which prescribes which action to take given a *history* of the game. We restrict to strategies that choose only allowed actions, that is for a history $h \in Q^*$, a *Player* i strategy chooses an action $a_i \in \lambda(q, i)$. The *play* that two strategies σ_1 and σ_2 and an initial vertex q_0 give rise to, denoted $\text{play}(q_0, \sigma_1, \sigma_2)$, is defined inductively as follows. The play starts from q_0 . Suppose that the prefix $\pi^{\leq m}$ of length $m \geq 1$ of $\text{play}(q_0, \sigma_1, \sigma_2)$ is defined, *Player* i takes action $a_i^j = \sigma_i(\pi^{\leq m} \cdot q_j)$, for $i \in \{1, 2\}$, then the next state is $q_{j+1} = \delta(q_j, a_1^j, a_2^j)$. We say that a play π is *consistent* with σ_1 from q_0 if there is σ_2 such that $\pi = \text{play}(q_0, \sigma_1, \sigma_2)$, and similarly for σ_2 .

For $i \in \{1, 2\}$, we say *Player* i *controls* a state $q \in Q$, if intuitively, the next state is determined solely based on their choice of action. Formally, *Player 1* controls state q if for any $a_1 \in \lambda(q, 1)$ and $a_2, a'_2 \in \lambda(q, 2)$, we have $\lambda(q, a_1, a_2) = \lambda(q, a_1, a'_2)$. The definition is dual for *Player 2*. *Turn-based* games are a special case of concurrent games, where each state q is controlled by one of the players. Note that a concurrent game which is not turn-based may still have some states which are controlled by one of the players.

Bidding games

A discrete bidding game is played on an arena $\langle V, E, k \rangle$, where V is the set of vertices, E is the set of edges, and $k \in \mathbb{N}$ is the total budget in the game. The neighbors of a vertex v , denoted $N(v)$, are $N(v) = \{u : (v, u) \in E\}$.

We introduce notation to formalize the tie-breaking mechanism, called *advantage-based* tie-breaking [20]. We denote the advantage with $*$. Thus, when a player's budget is B^* , this means that the player has a budget of $B \in \mathbb{N}$ and holds the advantage. Similarly, when we say that a player bids b^* , we mean that they bid $b \in \mathbb{N}$, and in case a tie occurs, they will use the advantage. Denote $\mathbb{N}^* = \{0, 0^*, 1, 1^*, \dots\}$ and $[k] = \{0, 0^*, 1, 1^*, \dots, k, k^*\}$. The integral part of $B \in \mathbb{N}^*$ is denoted $|B|$. We define two operators $\oplus$ and $\ominus$ over $\mathbb{N}^*$. We describe how the operators are used. Suppose that *Player 1*'s budget is m^* and the players bid b_1 and b_2 , respectively. Recall that the higher bidder pays the lower bidder. Thus, when $b_1 > b_2$, *Player 1*'s budget is updated to $m^* \ominus b_1$, and when $b_2 > b_1$, *Player 1*'s budget is updated to $m^* \oplus b_2$. Note that $x^* \oplus y^*$ and $x \ominus y^*$, for $x, y \in \mathbb{N}$, will not occur in the setting above, still it is useful to define both for convenience and completion. Formally,

► **Definition 2** ($\oplus$ and $\ominus$ operators). For $x, y \in \mathbb{N}$, we define $x^* \oplus y = x \oplus y^* = (x + y)^*$, $x \oplus y = x + y$ and $x^* \oplus y^* = x + y + 1$. For $x, y \in \mathbb{N}$, we define $x^* \ominus y = (x - y)^*$, $x^* \ominus y^* = x - y$ and $x \ominus y = x - y$. Finally, $x \ominus y^* = (x - y - 1)^*$.

Consider the natural order $\prec$ over $\mathbb{N}^*$ as $0 \prec 0^* \prec 1 \prec 1^* \prec \dots$. We will frequently use the *successor* and *predecessor* according to this order: for $B \in \mathbb{N}^*$, the successor of B is $B \oplus 0^*$ and its predecessor is $B \ominus 0^*$.

Bidding games as concurrent games

Consider an arena $\mathcal{A} = \langle V, E, k \rangle$ of a bidding game. The *configurations* of $\mathcal{A}$ are $\mathcal{C} = \{\langle v, B \rangle : v \in V, B \in [k] \cup \{k + 1\}\}$, where $\langle v, B \rangle \in \mathcal{C}$ means that the token is placed at vertex v and *Player 1*'s current budget is B . Implicitly, *Player 2*'s budget is $k^* \ominus B$. The arena of the

corresponding concurrent game is $\langle [k] \times V, \mathcal{C}, \lambda, \delta \rangle$, where we define the allowed actions λ and transitions δ next. Choosing an action $\langle b, v \rangle \in [k] \times V$ corresponds to bidding b and moving to v upon winning the bidding. Consider a configuration $\langle v, B \rangle$. Define $\lambda(\langle v, B \rangle, 1) = \{0, \dots, B\} \times N(v)$, that is *Player 1* must choose a bid within his budget and must move to a neighbor of v upon winning the bidding. Similarly, $\lambda(\langle v, B \rangle, 2) = \{0, \dots, k^* \ominus B\} \times N(v)$. We define δ next. Suppose that the token is placed on $c = \langle v, B \rangle$, and *Player i* chooses $\langle b_i, v_i \rangle$, for $i = \{1, 2\}$. If $b_1 > b_2$, then the token moves to $\langle v_1, B \ominus b_1 \rangle$; that is, *Player 1* wins the bidding, pays *Player 2*, and moves the token. Dually, if $b_2 > b_1$, then the token moves to $\langle v_2, B \oplus b_2 \rangle$. The remaining case is $b_1 = b_2$. Note that this occurs only when the player with the advantage does not use it. In this case, the other player wins the bidding, and the token moves as in the above.

As above, two strategies σ_1 and σ_2 and an initial configuration $c_0 = \langle v_0, B_0 \rangle$ give rise to an infinite play $\text{play}(c_0, \sigma_1, \sigma_2) = c_0, c_1, \dots \in \mathcal{C}^\omega$. We use $\text{path}(c_0, \sigma_1, \sigma_2) = v_0, v_1, \dots$ to refer to the path in $\langle V, E \rangle$ that corresponds to the play, assuming $c_j = \langle v_j, B_j \rangle$, for $j \geq 0$.

► **Remark 3 (Representation size).** Consider an arena $\mathcal{A} = \langle V, E, k \rangle$ of a bidding game. We assume that k is encoded in binary. Thus, the size of $\mathcal{A}$ is $O(|V| + |E| + \log k)$. Note the size (number of configurations) of the explicit concurrent game that corresponds to $\mathcal{A}$ is $k \cdot |V|$, thus exponentially larger than $\mathcal{A}$.

Mean-payoff and energy bidding games

Both mean-payoff and an energy bidding games are played on an arena $\langle V, E, k, w \rangle$, where $\langle V, E, k \rangle$ is as above, and $w : E \rightarrow \{-W, \dots, W\}$ is a function that assigns integer weights to edges, W being the largest absolute weight. Consider an infinite path $\pi = v_0, v_1, \dots$. We call the sum of weights traversed by the prefix $\pi^{\leq m}$ as its *energy*, $\text{sum}(\pi^{\leq m}) = \sum_{0 \leq j < m} w(\langle v_j, v_{j+1} \rangle)$. We define below which player wins in π under the two objectives, and later, in Thm. 15, we will show an equivalence between the two objectives.

Energy objective. We call the players in an energy game *preserver* (Pres) and *consumer* (Cons). Intuitively, the game starts with an initial energy $M \in \mathbb{N}$, Cons's objective is to drop the energy below 0, and Pres wins otherwise, namely if the energy stays non-negative throughout the whole play. Formally, for an initial energy $M \in \mathbb{N}$, Cons *M-wins* π if there is $m \in \mathbb{N}$ such that $M + \text{sum}(\pi^{\leq m}) < 0$ and Pres *M-wins* π if for all $m \in \mathbb{N}$, we have $M + \text{sum}(\pi^{\leq m}) \geq 0$. We say that Pres *M-wins* from a configuration $\langle v, B \rangle$ if she has a strategy σ such that for every Cons strategy τ , Pres *M-wins* $\text{path}(\langle v, B \rangle, \sigma, \tau)$, and the definition for Cons is dual.

Mean-payoff objective. We call the players in a mean-payoff game *maximizer* (Max) and *minimizer* (Min). The *payoff* of an infinite path, which is Max's reward and Min's cost, is the long-run average of the weights traversed. Formally, we define $\text{Mean-Payoff}(\pi) = \liminf_{m \rightarrow \infty} \frac{1}{m} \text{sum}(\pi^{\leq m})$. Max wins in π if $\text{Mean-Payoff}(\pi) \geq 0$ and Min wins π if $\text{Mean-Payoff}(\pi) < 0$. Max wins from a configuration $\langle v, B \rangle$ if he has a strategy σ such that for every Min strategy τ , Max wins $\text{path}(\langle v, B \rangle, \sigma, \tau)$, and the definition for Min is dual.

3 Existence of Energy Thresholds in Energy Bidding Games

In this section, we show existence of an *energy threshold*, which is a necessary and sufficient energy required for Pres to win from an initial configuration. Importantly, when Pres loses with every initial energy, we call the energy threshold ∞ . Formally,

► **Definition 4** (Energy threshold). Consider an energy bidding game $\mathcal{G} = \langle V, E, k, w \rangle$. The energy threshold is $\text{Energy} : V \times [k] \rightarrow \mathbb{N}^\infty$ such that for configuration $\langle v, B \rangle \in \mathcal{C}$:

- If $\text{Energy}(v, B) = M \in \mathbb{N}$, then (1) Pres M -wins from $\langle v, B \rangle$ and (2) Cons $(M - 1)$ -wins from $\langle v, B \rangle$.
- If $\text{Energy}(v, B) = \infty$, for every $M \in \mathbb{N}$, Cons M -wins from $\langle v, B \rangle$.

► **Remark 5** (Energy thresholds and determinacy). We point out that existence of Energy is not trivial. Indeed, as seen Sec. 2, bidding games are succinctly represented concurrent games, and even simple concurrent games are not *determined*, namely neither player can guarantee winning.⁴ Existence of Energy implies determinacy of energy bidding games. Indeed, assume that Energy exists, consider an initial configuration $c = \langle v, B \rangle$, and an initial energy level M . Then, if $M \geq \text{Energy}(v, B)$, Pres has a winning strategy and if $M < \text{Energy}(v, B)$, Cons has a winning strategy. The game is thus determined.

3.1 Energy thresholds exist in finite-duration games

Fix an energy game $\mathcal{G} = \langle V, E, k, w \rangle$ for the remainder of this section. Let $n \in \mathbb{N}$. The truncated game $\mathcal{G}_n$ intuitively favors Pres: she needs to keep the energy non-negative *only* in the first n turns. We will show existence of energy thresholds in every truncated game, which is still not trivial since $\mathcal{G}_n$ is a concurrent game. In the next section we extend to $\mathcal{G}$.

Formally, for $M \in \mathbb{N}$, Pres M -wins a path $\pi = v_0, v_1 \dots$ in $\mathcal{G}_n$ if for every $m \leq n$, we have $M + \text{sum}(\pi^{\leq m}) \geq 0$ and Cons wins otherwise. We define threshold energies in $\mathcal{G}_n$, denoted $\text{Energy}_n : V \times [k] \rightarrow \mathbb{N}$, by plugging in the definition of M -wins in $\mathcal{G}_n$ in Def. 4. Recall that the minimal possible weight is $-W$, thus an initial energy of nW suffices for Pres to win in $\mathcal{G}_n$. It follows that energy thresholds in $\mathcal{G}_n$ are finite.

We show existence of Energy_n via an algorithm to compute it. We recursively define $\mu_i : V \times [k] \rightarrow \mathbb{N}$ and show that $\text{Energy}_n \equiv \mu_n$. For the base case, Pres always wins in $\mathcal{G}_0$, thus $\mu_0 \equiv \text{Energy}_0 \equiv 0$. For the inductive step, consider a configuration $\langle v, B \rangle$. Intuitively, we define the initial energy $\mu_n(v, B)$ to suffice for winning even when Pres reveals her bid first, Cons responds adversarially, and the game proceeds to a configuration $\langle v', B' \rangle$, which requires an energy of $\mu_{n-1}(v', B')$.

We define μ_n formally. We first define $\text{trump} : [k] \times [k] \rightarrow [k]$ as the minimal bid that lets Cons “trump” a Pres bid and win the bidding. The definition depends on the tie-breaking status: if $B \in \mathbb{N}^* \setminus \mathbb{N}$ and $b \in \mathbb{N}$ (Pres has and does not use the advantage), then $\text{trump}(B, b) = b$ and otherwise $\text{trump}(B, b) = b \oplus 0^*$. Consider a configuration $\langle v, B \rangle$ and a Pres bid of $b \in [0, B]$. We consider two bidding outcomes: (1) Pres wins the bidding: the next configuration is $\langle v_{\text{win}}, B \ominus b \rangle$, where Pres chooses $v_{\text{win}} \in N(v)$, and (2) Cons wins the bidding: the next configuration is $\langle v_{\text{lose}}, B \oplus \text{trump}(B, b) \rangle$, where Cons chooses $v_{\text{lose}} \in N(v)$. Note that, Cons can win the bidding only if $B \oplus \text{trump}(B, b) \leq k^*$. The path accumulates energy $w(v, v_{\text{win}})$ or $w(v, v_{\text{lose}})$, respectively. The minimum required energy for Pres to win in the respective cases is:

$$e_{\text{win}}^n(v, B, b) = \min_{v' \in N(v)} \max\{\mu_{n-1}(v', B \ominus b) - w(v, v'), 0\} \quad (1)$$

$$e_{\text{lose}}^n(v, B, b) = \max_{v' \in N(v)} \max\{\mu_{n-1}(v', B \oplus \text{trump}(B, b)) - w(v, v'), 0\} \quad (2)$$

⁴ Note that we restrict to *pure* strategies as opposed to *mixed* strategies that allow choosing a probability distribution over actions.

We stress that both $e_{\text{lose}}^n(v, B, b) \geq 0$ and $e_{\text{win}}^n(v, B, b) \geq 0$. Moreover, $e_{\text{lose}}^n(v, B, b)$ is defined only when $B \oplus \text{trump}(B, b) \leq k^*$. Define $e_{\text{next}}^n(v, B, b) = \max\{e_{\text{win}}^n(v, B, b), e_{\text{lose}}^n(v, B, b)\}$ if $B \oplus \text{trump}(B, b) \leq k^*$, and $e_{\text{next}}^n(v, B, b) = e_{\text{win}}^n(v, B, b)$ otherwise. Then,

$$\mu_n(v, B) = \min_{b \leq B} e_{\text{next}}^n(v, B, b) \quad (3)$$

A Pres strategy that maintains the energy above μ_n follows from the construction above, thus we obtain the following.

► **Lemma 6.** *For every $\langle v, B \rangle$ and $n \in \mathbb{N}$, Pres $\mu_n(v, B)$ -wins from $\langle v, B \rangle$ in $\mathcal{G}_n$.*

The following lemma, whose proof can be found in the full version [13], shows that an energy of $\mu_n(v, B)$ is necessary for Pres to win. The proof proceeds by showing that for every Pres strategy, Cons has a winning response. Existence of a winning strategy for Cons follows from determinacy of reachability discrete-bidding games [20, 1].

► **Lemma 7.** *For every $\langle v, B \rangle$ and $n \in \mathbb{N}$, Cons $(\mu_n(v, B) - 1)$ -wins from $\langle v, B \rangle$ in $\mathcal{G}_n$.*

Combining Lem. 6 and 7, we obtain the following.

► **Theorem 8.** *For every $n \geq 0$, Energy_n exists. Moreover, we have $\text{Energy}_n \equiv \mu_n$.*

3.2 Extending to un-bounded energy games

In this section, we show existence of energy thresholds in unbounded energy games. A first attempt to define μ in unbounded games would be to simply consider the fixed point of the sequence $\{\mu_n : n \geq 0\}$, however the sequence might not reach a fixed point; indeed, when $\text{Energy}(v, B) = \infty$, every $\mu_n(v, B)$ is finite. Instead, we define a sequence of trimmed functions $\tilde{\mu}_n : V \times [k] \rightarrow \mathbb{N}^\infty$ (see details in the full version [13]).

$$\tilde{\mu}_n(v, B) := \begin{cases} \mu_n(v, B) & \text{if } \mu_n(v, B) \leq |V|kW \\ \infty & \text{otherwise} \end{cases} \quad (4)$$

Recall that $|V|kW$ is the largest absolute weight appearing on the arena. In the full version [13], we established monotonicity. Since there are only finitely many different $\tilde{\mu}_n$ functions, monotonicity means that the sequence $\{\tilde{\mu}_n : n \geq 0\}$ reaches a fixed point.

► **Lemma 9 (Monotonicity).** *For all $n \geq 0$, $\tilde{\mu}_n \leq \tilde{\mu}_{n+1}$. Moreover, for any vertex v , and two budgets $B_1, B_2 \in [k]$ with $B_1 \geq B_2$, $\tilde{\mu}_n(v, B_1) \leq \tilde{\mu}_n(v, B_2)$.*

We denote the fixed point by μ and define the strategy that it gives rise to as follows.

► **Definition 10.** *We define a strategy $\sigma_{\text{VI}} : V \times [k] \rightarrow [k] \times V$. At configuration $\langle v, B \rangle$, we define $\langle b', u' \rangle = \sigma_{\text{VI}}(v, B)$ as follows. First, replace μ_{n-1} with μ in Eq. 1 and Eq. 2, which intuitively consider the case that Pres bids b , and identify the necessary energy needed for winning following a bidding win $e_{\text{win}}(v, B, b)$ and a bidding lose $e_{\text{lose}}(v, B, b)$. To define the necessary energy, we distinguish between the case that overbidding b is a possible Cons action, i.e., $B \oplus \text{trump}(B, b) < k + 1$, in which case $e_{\text{next}}(v, B, b) = \max\{e_{\text{win}}(v, B, b), e_{\text{lose}}(v, B, b)\}$, otherwise $e_{\text{next}}(v, B, b) = e_{\text{win}}(v, B, b)$. Then, $b' = \arg \min_{b \leq B} e_{\text{next}}(v, B, b)$, and $u' = \arg \min_{u \in N(v)} \max\{\mu(u, B \ominus b') - w(v, u), 0\}$.*

We show how Pres wins by maintaining an *energy invariant*. This is reused in Sec. 5.

► **Lemma 11.** *If $\mu(v, B) < \infty$, then σ_{VI} is a Pres' strategy by which she $\mu(v, B)$ -wins from configuration $\langle v, B \rangle$ in $\mathcal{G}$.*

Proof. When $\mu(v, B) < \infty$, we have $\mu(v, B) = \min_{b \leq B} \max\{e_{\text{win}}(v, B, b), e_{\text{lose}}(v, B, b)\}$.

Consider an initial configuration $\langle v_0, B_0 \rangle$ and an initial energy $e_0 \geq \mu(v, B)$. We describe a Pres winning strategy inductively. Suppose that the game reaches $\langle v, B \rangle$ with an accumulated energy of e . Pres maintains that $e_0 + e \geq \mu(v, B)$. Since $\mu \geq 0$, it follows that a non-negative energy is preserved throughout the play, and Pres wins. Pres chooses $\langle b', u' \rangle$ such that b' attains the minimum in the definition of μ and v' attains the minimum in the definition of e_{win} . It is not hard to verify that the invariant is maintained: no matter what the next configuration $\langle v'', B'' \rangle$ is, the accumulated energy $e' = e + w(v, v'')$ satisfies $e' \geq \mu(v'', B'')$. ◀

We show that Cons wins $\mathcal{G}$ by simulating a winning strategy in a finite game $\mathcal{G}_n$, for a large enough n . This idea is reused in Sec. 6.

► **Lemma 12.** *If $\mu(v, B) < \infty$, then Cons $(\mu(v, B) - 1)$ -wins from $\langle v, B \rangle$ in $\mathcal{G}$. If $\mu(v, B) = \infty$, then for every $e \in \mathbb{N}$, Cons e -wins from $\langle v, B \rangle$ in $\mathcal{G}$.*

Proof sketch. We describe the proof idea and the details can be found in the full version [13]. Assume $\mu(v, B) = \infty$, thus $\mu_n(v, B) > |V|kW$, for some n . Consider an initial energy e . We construct a Cons strategy τ as follows. Let τ_n be a $(\mu_n(v, B) - 1)$ -winning strategy in $\mathcal{G}_n$. Intuitively, τ “simulates” τ_n and follows its actions. Consider a play π_n that is consistent with τ_n . Note that π_n coincides with a prefix of a play π in $\mathcal{G}$ that is consistent with τ . Since τ_n is winning, π_n consumes at least $|V|kW$ units of energy. Recall that the minimal weight in $\mathcal{G}$ is $-W$, thus the length of π_n is at least $|V| \cdot k$, which implies that π_n must contain a negative configuration cycle. Formally, there is an index m , such that $\pi_n^{\leq m} = \pi'_n \cdot \chi_n$, where χ_n is cycle of configurations with $\text{sum}(\chi_n) < 0$. We define τ to intuitively omit χ_n and restart the simulation of τ_n at π'_n . That is, the next action it chooses is $\tau_n(\pi'_n)$. By repeating this process, we obtain that a play π that is consistent with τ consists of only negative cycles and at most n additional configuration, thus e is eventually gets consumed and τ is winning. ◀

Combining Lem. 11 and Lem. 12, we obtain the following.

► **Theorem 13.** *Consider an energy bidding game $\mathcal{G} = \langle V, E, k, w \rangle$. The energy threshold function $\text{Energy} : V \times [k] \rightarrow \mathbb{N}^\infty$ exists and satisfies $\mu \equiv \text{Energy}$.*

We observe the following about Energy :

► **Corollary 14.** *Energy inherits the monotonicity of $\tilde{\mu}_n$ for budgets: for every vertex v , and two budgets $B_1 \geq B_2$, we have $\text{Energy}(v, B_1) \leq \text{Energy}(v, B_2)$.*

We close this section by an equivalence between energy and mean-payoff games, proof of which can be found in the full version [13].

► **Theorem 15.** *Consider a game $\mathcal{G} = \langle V, E, k, w \rangle$ and a configuration $\langle v, B \rangle$. If $\text{Energy}(v, B)$ is finite, then Max can guarantee non-negative payoff from $\langle v, B \rangle$. If $\text{Energy}(v, B) = \infty$, Min can guarantee negative payoff from $\langle v, B \rangle$.*

4 On Threshold Budgets

Our goal is to develop succinct winning strategies. Towards this goal, in this section, we define threshold budgets, identify their mathematical structure, and deduce succinct *budget agnostic* strategies from this structure.

Recall that $\text{Energy}(v, B)$ is the necessary and sufficient initial energy for Pres to win from a configuration $\langle v, B \rangle$. Further recall that $\text{Energy}(v, B)$ is (weakly) monotonically increasing as B decreases (a lower initial budget, requires a higher initial energy). The *threshold budget* is the lowest budget B such that $\text{Energy}(v, B)$ is finite.

► **Definition 16** (Threshold budgets). Define $Th : V \rightarrow [k] \cup \{k+1\}$ such that (1) if $\text{Energy}(v, B) = \infty$ for all $B \in [k]$, then $Th(v) = k+1$ and (2) $Th(v) = \min\{B < k+1 : \mu(v, B) < \infty\}$ otherwise.

► **Remark 17** (Thresholds in mean-payoff games). Thm. 15 implies that thresholds directly apply to mean-payoff games; $Th(v)$ is a necessary and sufficient budget for Max to guarantee a non-negative payoff.

In reachability continuous-bidding games [23], the threshold of a vertex v is the average of two of its neighbors, v^+ and v^- , which respectively denote the neighbor with the maximal and minimal threshold. Below, we describe a discrete version of this *average property* [20].

► **Definition 18** (Average property). Consider a graph $\langle V, E \rangle$ and $k \in \mathbb{N}$. A function $T : V \rightarrow [k] \cup \{k+1\}$ satisfies the average property if

$$T(v) = \left\lfloor \frac{|T(v_T^+)| + |T(v_T^-)|}{2} \right\rfloor + \varepsilon$$

where $v_T^+ = \arg \max_{v' \in N(v)} T(v')$, $v_T^- = \arg \min_{v' \in N(v)} T(v')$, and (1) if $|T(v_T^+)| + |T(v_T^-)|$ is even and $T(v_T^-) \in \mathbb{N}$, then $\varepsilon = 0$, (2) if $|T(v_T^+)| + |T(v_T^-)|$ is odd and $T(v_T^-) \in \mathbb{N}^*$, then $\varepsilon = 1$, and (3) otherwise $\varepsilon = *$. We often drop T from the notation of v_T^+ and v_T^- .

The main result in this section states that thresholds in energy games satisfy the average property. Our proof technique is very different from previous works; both for reachability [20] and parity games [12], the proof that thresholds satisfy the average property is a byproduct of a value-iteration algorithm. Our value-iteration algorithm (Sec. 3) focuses on the energy threshold and does not immediately imply the average property for the budget thresholds. Instead, in the full version [13] we proceed as follows. For each $v \in V$, we define $f(v) := \left\lfloor \frac{|Th(v_{Th}^+)| + |Th(v_{Th}^-)|}{2} \right\rfloor + \varepsilon$ as in Def. 18. We show that $f \equiv Th$ by showing that for every vertex v , we have $\text{Energy}(v, f(v)) < \infty$ and $\text{Energy}(v, f(v) \ominus 0^*) = \infty$. The proof proceeds by a careful case-by-case analysis.

► **Theorem 19.** Consider an energy game $\mathcal{G} = \langle V, E, k, w, \text{energy} \rangle$. The threshold budget $Th : V \rightarrow [k] \cup \{k+1\}$ satisfies the average property.

A budget-agnostic partial strategy

We seek succinct winning strategies, which intuitively choose bids ignoring excess budget.

► **Definition 20** (Budget agnostic strategy). Define $Trim : V \times [k] \rightarrow [k]$ that “trims” excess budget: for $\langle v, B \rangle$ with $B \geq Th(v)$, define $Trim(v, B)$ to be whichever of $Th(v)$ or $Th(v) \oplus 0^*$ agrees with B on the tie-breaking advantage. A winning strategy f is budget agnostic if for every $v \in V$ and $B \geq Th(v)$, and every two histories h_1, h_2 that end in $\langle v, B \rangle$, we have $\langle b, u_1 \rangle = f(h_1)$ and $\langle b, u_2 \rangle = f(h_2)$.

In fact, in Rem. 40, we will show existence of winning *positional* budget agnostic strategies.

We proceed as follows. A function T that satisfies the average property gives rise to a *partial* budget-agnostic strategy f_T ; namely it assigns to each configuration $\langle v, B \rangle$ a pair $\langle b, S \rangle$, where b is a bid and $S \subseteq V$ is a set of *allowed vertices* to move to upon winning

the bidding. Intuitively, a strategy that *agrees* with f_T maintains a *budget invariant* (see Lem. 23 below). In subsequent sections, we will construct winning budget-agnostic strategies σ_{agn} and τ_{agn} for Pres and Cons, respectively, that agree with a partial strategy f_T , namely the strategies choose the same bid as f_T and choose one of the allowed vertex, thus they maintain a budget invariant. We define the bids and allowed vertices below.

► **Definition 21** (Bid choice). *Consider a function $T : V \rightarrow [k]$ that satisfies the average property. We define a bid $\text{bid}^T(v, B)$ in two steps. First, let*

$$b^T(v) = \begin{cases} T(v) \ominus T(v^-) & \text{if } T(v^-) \in \mathbb{N} \\ T(v) \ominus (|T(v^-)| + 1) & \text{otherwise} \end{cases}$$

Second, we define $\text{bid}^T(v, B) = b^T(v)$ when both $b^T(v)$ and B belong to either $\mathbb{N}$ or $\mathbb{N}^ \setminus \mathbb{N}$, and $b^T(v) \oplus 0^*$ otherwise.*

Intuitively, Pres “attempts” to bid $b^T(v)$ at $\langle v, B \rangle$. If $b^T(v)$ requires the advantage and B does not have it, Pres bids $|b^T(v)| + 1 \in \mathbb{N}$, which does not require the advantage.

Next, we define the allowed vertices as the neighbors of v that minimize T .

► **Definition 22** (Allowed vertices). *For a function T that satisfies the average property, the set of allowed vertices at vertex v are: if $T(v^-) \in \mathbb{N}$, then $A^T(v) = \{u \in N(v) : T(u) = T(v^-)\}$, and otherwise $A^T(v) = \{u \in N(v) : T(u) \leq T(v^-) \oplus 0^*\}$.*

Consider a configuration $\langle v, B \rangle$ with $B \geq T(v)$. The following lemma shows that choosing an action $\langle b, u \rangle$ with $b = \text{bid}^T(v, B)$ and $u \in A^T(v)$, maintains a budget invariant: no matter how the opponent acts, in the next configuration $\langle v', B' \rangle$, we have $B' \geq T(v')$. In particular, this implies that a $\text{bid}^T(v, B)$ is a legal bid, i.e., $\text{bid}^T(v, B) \leq B$.

► **Lemma 23.** [12] *Let T be a function that satisfies the average property, and $v \in V$. Then, $T(v) \ominus b^T(v) \geq T(v^-)$ and $T(v) \oplus (b^T(v) \oplus 0^*) \geq T(v^+)$.*

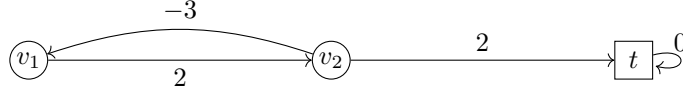
5 Constructing a Budget Agnostic Winning Strategy for Pres

In this section we construct a budget-agnostic strategy σ_{agn} for Pres. Recall that σ_{V1} is the strategy that is constructed by the value-iteration algorithm (see Def. 10). In qualitative games [20, 12], the value-iteration algorithm outputs a budget-agnostic strategy. However, the following example shows that σ_{V1} is not budget agnostic.

► **Example 24.** Consider the energy discrete-bidding game depicted in Fig. 2. Set $k = 5$. Suppose that the game starts from $\langle v_1, 1 \rangle$ with initial energy 2, and we describe a play that is consistent with σ_{V1} when Cons responds optimally: $\langle v_1, 1 \rangle \xrightarrow{0,0} \langle v_2, 1 \rangle \xrightarrow{1,1^*} \langle v_1, 2^* \rangle \xrightarrow{0,0} \langle v_2, 2^* \rangle \xrightarrow{0^*,1} \langle v_1, 3^* \rangle \xrightarrow{0,0} \langle v_2, 3^* \rangle \xrightarrow{3^*,2} \langle t, 0 \rangle$. Intuitively, observe that traversing the cycle v_1, v_2, v_1 causes both a decrease in energy and an increase to Pres’s budget. Note that at v_2 , the bids are 0^* until the last visit in which σ_{V1} bids 3^* . Since Cons budget is 2, this forces the game to t , where Pres wins.

The budget agnostic strategy σ_{agn} that we construct will always bid 0^* at v_2 . It too is winning, but requires traversing the cycle twice more. Indeed, after two more traversals, we reach configuration $\langle v_2, 5^* \rangle$ in which Cons’s budget is 0, and Pres’s bid of 0^* is winning. Finally, we note that while an initial energy of 2 suffices for σ_{V1} to win from $\langle v_1, 1 \rangle$, σ_{agn} requires an initial energy of 5.

32:12 Mean-Payoff and Energy Discrete-Bidding Games



■ **Figure 2** A mean-payoff discrete bidding game where σ_{v_I} and σ_{agn} sometimes act differently.

In the full version [13], we prove the following key lemma. It identifies configurations in which the bids chosen by σ_{v_I} coincide with the bids that are deduced from the average property (Def. 21).

► **Lemma 25.** *Let $\langle v, B \rangle$ with $B \in \{Th(v), Th(v) \oplus 0^*\}$. Then, $\sigma_{v_I}(v, B) = \langle bid^{Th}(v, B), u \rangle$ for some $u \in A^{Th}(v)$.*

We define σ_{agn} as follows and note that it is budget agnostic by construction. Recall that $Trim(v, B)$ is whichever of $Th(v)$ or $Th(v) \oplus 0^*$ agrees with B on the tie-breaking advantage.

► **Definition 26.** *For a configuration $\langle v, B \rangle$ with B greater than or equal to $Th(v)$, we define $\sigma_{agn}(v, B) = \sigma_{v_I}(v, Trim(v, B))$.*

Intuitively, σ_{agn} enjoys two features. First, since it follows σ_{v_I} , it maintains an energy invariant as in the proof of Lem. 11. Second, Lem. 25 means that it maintains a budget invariant as in Lem. 23. But Lem. 25 does not apply in all configurations. In the proof of the following theorem, we will show that both features are guaranteed to *eventually* hold.

► **Theorem 27.** *Consider an energy game $\mathcal{G} = \langle V, E, k, w \rangle$ and a configuration $\langle v, B \rangle$ with $B \geq Th(v)$. Then, there exists a finite energy $M = (k + 1) \cdot (W + \max_{v \in V} Energy(v, Th(v)))$ such that σ_{agn} M -wins from $\langle v, B \rangle$.*

Proof. Consider a play $\pi = \langle v_0, B_0 \rangle \langle v_1, B_1 \rangle, \dots$ consistent with σ_{agn} from $\langle v, B \rangle$. We define a function $\nabla_\pi : \mathbb{N} \rightarrow [k]$ that intuitively assigns to each turn $i \in \mathbb{N}$, Pres's *spare change*, which is roughly the difference between B_i and the required threshold budget $Th(v_i)$. Formally, $\nabla_\pi(i) = |B_i \ominus Th(v_i)|$. We first observe that $\nabla_\pi(i)$ is monotonically increasing. Indeed, σ_{agn} bids at turn i as if the budget is B_i , which suffices to ensure that $B_{i+1} \ominus \nabla_\pi(i+1) \geq Th(v_{i+1})$, thus the spare change is unused and can only increase. Clearly, ∇_π is bounded by k , thus it eventually stabilizes; let $N \geq 0$ and $r \in \mathbb{N}$ such that $\nabla_\pi(m) = r$, for every $m \geq N$.

In the full version [13], we show that when ∇_π is stable, the energy invariant is maintained. Formally,

▷ **Claim 28.** If $\nabla_\pi(m) = \nabla_\pi(m+1)$ for some $m \geq 0$, then $Energy(v_m, Trim(v_m, B_m)) + w(v_m, v_{m+1}) \geq Energy(v_{m+1}, Trim(v_{m+1}, B_{m+1}))$.

Suppose that ∇_π stabilizes at turn N . Using Claim 28, in the full version [13] we show: (1) Pres wins the suffix from turn N , if the energy is at least $Energy(v_N, Trim(v_N, B_N))$, and (2) we bound the initial energy that σ_{agn} requires to guarantee that at turn N , the energy is at least $Energy(v_N, Trim(v_N, B_N))$. For (2), intuitively, we partition $\pi^{\leq N}$ into “patches” such that ∇_π is stable within each patch and changes between patches. We show that within each patch, the energy is preserved, and between patches, the energy increases by at most W . Since there are at most k patches, σ_{agn} requires factor $k \cdot W$ more initial energy than σ_{v_I} . ◀

6 Constructing a Budget Agnostic Winning Strategy for Cons

In this section, we show existence of a Cons budget-agnostic winning strategy. There are two challenges w.r.t. the construction for Pres. First, the proof that Cons has a winning strategy in a finite energy game (Lem. 7) is existential and, unlike the case of Pres, does not provide an explicit construction. Second, while Pres needs to maintain an energy invariant, Cons needs to “make progress” and consume energy.

Throughout this section, in order to avoid clutter, we use primed notation to refer to Cons’s perspective: we use B' to refer to Cons budget, thus configuration $\langle v, B' \rangle$ (from Cons perspective) refers to $\langle v, k^* \ominus B' \rangle$ (from Pres perspective), we denote by Th' the threshold from Cons perspective, formally, $Th'(v) = (k + 1) \ominus Th(v)$, and so on.

Consider a function $T : V \rightarrow [k]$ that satisfies the average property. The *complement* of T , denoted by T' , intuitively represents Cons’s budget when Pres’s budget is $T(v) \ominus 0^*$.

► **Lemma 29** ([12]). *Consider a function $T : V \rightarrow [k] \cup \{k + 1\}$ that satisfies the average property. The complement of T , denoted $T' : V \rightarrow [k] \cup \{k + 1\}$, is $T'(v) = (k + 1) \ominus T(v)$. Then, T' satisfies the average property.*

Since Th satisfies the average property (Thm. 19), its complement Th' also satisfies it.

On winning strategies in finite-duration energy bidding games

Intuitively, our budget-agnostic strategy has two features. First, its bids match the bids derived from Th' (Def. 21), thus it maintains a *budget invariant* as in Lem. 23. Second, its actions follow some winning strategy $\tau_{VI,n}$ in a truncated game $\mathcal{G}_n$. This means that it maintains an energy invariant, which implies energy consumption. In this section, we establish properties of $\tau_{VI,n}$ that enable these features.

We denote by Energy'_n the energy threshold in $\mathcal{G}_n$ from Cons perspective, namely from $\langle v, B' \rangle$, Cons can consume $\text{Energy}'_n(v, B')$ units of energy within n turns, but cannot consume $\text{Energy}'_n(v, B') + 1$ units. Formally, $\text{Energy}'_n(v, B') = \text{Energy}_n(v, B) - 1$ where $B = k^* \ominus B'$.

Recall that Th' is defined such that $\text{Energy}'(v, Th'(v)) = \infty$ but $\text{Energy}'_n(v, Th'(v))$ is finite, for every $n \in \mathbb{N}$. Intuitively, the following lemma (see the proof in the full version [13]) states that for every energy e , there is a truncated game G_N in which Cons e -wins from $\langle v, Th'(v) \rangle$.

► **Lemma 30.** *For every $e < \infty$ and $v \in V$, there exists N such that $\text{Energy}'_N(v, Th'(v)) \leq e$.*

Let $n \in \mathbb{N}$. We denote by $\tau_{VI,n}$, a Cons strategy that $\text{Energy}'_n(v, Th'(v))$ -wins from $\langle v, Th'(v) \rangle$ in $\mathcal{G}$. In Lem. 7, we show that $\tau_{VI,n}$ exists, moreover, it operates on an arena in which each vertex is (v, e, m) , where v is a vertex, the current energy is e , and $m \leq n$ is a counter that marks the remaining turns. In the full version [13], we establish the following properties.

► **Lemma 31.** *Consider $x = (v, e, m)$, a Cons budget $B' \geq Th'(v)$ such that, $e \leq \text{Energy}'_m(v, B')$, and let $\langle b, u \rangle = \tau_{VI,n}(x, B')$. The following hold:*

- *There is $\tau_{VI,n}$ such that $\tau_{VI,n}(x, B') = \tau_{VI,n}((v, \text{Energy}'_m(v, B'), m), B')$*
- *$\tau_{VI,n}$ maintains an energy invariant, ensuring that the updated energy at the next configuration remains less than or equal to the energy required from that configuration: (i) Cons wins the bidding: $e + w(v, u) \leq \text{Energy}'_{m-1}(u, B' \ominus b)$, and (ii) Cons loses the bidding: assuming $B' \oplus (b \oplus 0^*) < k + 1$, then $e + w(v, v') \leq \text{Energy}'_{m-1}(v', B' \oplus (b \oplus 0^*))$, for every $v' \in N(v)$.*
- *If $B' \in \{Th'(v), Th'(v) \oplus 0^*\}$ and $\text{Energy}'_m(v, B') > |V|kW$, then $b = \text{bid}^{Th'}(v, B')$ and $u \in A^{Th'}(v)$.*

Intuitively, in the first item, note that $\langle v, \text{Energy}'_{m-1}(v, B'), m \rangle$ is a worse configuration than x for Cons, thus Cons wins in the former by following a winning strategy of the latter. The third item is a key property that is analogous to Lem. 25 for Pres. It identifies inputs such that $\tau_{\text{VI},n}$ matches the bids derived from Th' .

A budget-agnostic Cons winning strategy

We proceed as follows. We start by defining a budget agnostic strategy τ'_{agn} , which drops energy from $2|V|kW + 1$ to $|V|kW + 1$. The definition of τ'_{agn} is based on $\tau_{\text{VI},n}$, and since it operates at high energy levels, we obtain the properties in Lem. 31. Then, we define τ_{agn} to simulate τ'_{agn} and repeatedly omit negative configuration cycles, as in Lem. 12.

We define $\text{Trim}' : V \times [k] \rightarrow [k]$: for $\langle v, B' \rangle$ with $B' \geq Th'(v)$, define $\text{Trim}'(v, B')$ to be $Th'(v)$ or $Th'(v) \oplus 0^*$ that agrees with B' on the tie-breaking advantage.

► **Definition 32.** Let $\langle v, B' \rangle$ with $B' \geq Th'(v)$, energy $e > |V|kW$, and $P(e) \in \mathbb{N}$ be the minimal integer such that $\text{Energy}'_{P(e)}(v, \text{Trim}'(v, B')) \geq e$. We define $\tau'_{\text{agn}}(v, e, B') = \tau_{\text{VI},P(e)}((v, e, P(e)), \text{Trim}'(v, B'))$.

Note that the third item in Lem. 31 implies that τ'_{agn} is budget agnostic by construction. Indeed, the bid chosen by $\tau_{\text{VI},P(e)}$ is $\text{bid}^{Th'}(v, B')$.

► **Lemma 33.** From a configuration $\langle v, B' \rangle$ with $B \geq Th'(v)$, and initial energy level $e_0 \geq 2|V|kW$, τ'_{agn} ensures that the energy drops to $|V|kW$.

Proof Sketch. Consider a play π' consistent with τ'_{agn} . Similar to the proof of Thm. 27, we define the *spare change* of Cons $\nabla_{\pi'}$ and show that it eventually stabilizes due to the budget invariant that τ'_{agn} maintains. When $\nabla_{\pi'}$ is stable, we show that τ'_{agn} maintains an energy invariant, which implies that energy is consumed. See details in the full version [13]. ◀

We proceed to prove the main result in this section.

► **Theorem 34.** Consider an energy game $G = \langle V, E, k, w \rangle$. There exists a Cons budget-agnostic strategy τ_{agn} that M -wins from every configuration $\langle v, B' \rangle$ with $B' \geq Th'(v)$, for every energy $M \in \mathbb{N}$.

Proof. We proceed similar to the proof of Lem. 12. We observe that any play π' consistent with τ'_{agn} from $\langle v, B' \rangle$ and initial energy $e_0 = 2|V|kW$ must contain a negative configuration cycle before reaching energy $|V|kW$. We define τ_{agn} to simulate τ'_{agn} from (v, e_0, B') until such a negative cycle is closed, omit it, and restart the simulation. Since τ'_{agn} is budget agnostic, so is τ_{agn} . By repeating this process, we obtain that a play $\hat{\pi}$ that is consistent with τ_{agn} , consists of only negative cycles and a finitely many additional configurations (see the full version [13] for an upper bound). Thus, τ_{agn} is M -winning from $\langle v, B' \rangle$, for every energy M . ◀

7 Finding Threshold Budgets is in NP and coNP

We formalize the problem of finding threshold budgets as a decision problem:

► **Problem 35** (Finding Threshold Budgets). Given an energy bidding game $\mathcal{G} = \langle V, E, k, w \rangle$, a vertex v , and a budget $\ell \in [k]$, decide whether $Th(v) \geq \ell$.

We show that Prob. 35 is in NP and coNP. Our approach is similar to [12]. The core of the algorithm is to decide, given a function $T : V \rightarrow [k] \cup \{k+1\}$ that satisfies the average property whether $T \equiv Th$. Note that T can be guessed since its size is $\mathcal{O}(|V| \cdot \log(k))$.

From bidding games to turn-based games. We show how to decide $T \geq Th$ and showing that $T \leq Th$ is dual. Given an energy bidding game $\mathcal{G}$ and T that satisfies the average property, we construct and solve a turn-based energy game $G_{T,\mathcal{G}}$. We describe the idea and the details can be found in the full version [13]. Intuitively, a Pres winning strategy in $G_{T,\mathcal{G}}$ corresponds to a winning budget-agnostic strategy in $\mathcal{G}$. For each vertex $v \in V$, there are two Pres vertices $\langle v, B \rangle$ for $B \in \{T(v), T(v) \oplus 0^*\}$, which are configurations in $\mathcal{G}$, and a third copy $\langle v, \top \rangle$, which is a winning sink for Pres. Vertex $\langle v, B \rangle$ in $G_{T,\mathcal{G}}$ simulates configuration $\langle v, \tilde{B} \rangle$ in $\mathcal{G}$ with $B = \text{Trim}(v, \tilde{B})$. Pres can choose any $v' \in A^T(v)$. This corresponds to choosing action $\langle \text{bid}^T(v, B), v' \rangle$ in $\mathcal{G}$. Then, Cons responds by either: (1) letting Pres win the bidding and proceeding to $\langle v', B \ominus \text{bid}^T(v, B) \rangle$ or (2) win the bidding and choosing the successor vertex u , then the next vertex is $\langle u, \tilde{B} \rangle$, where $\tilde{B} = B \oplus (\text{bid}^T(v, B) \oplus 0^*)$ if $\tilde{B} \in \{T(u), T(u) \oplus 0^*\}$, otherwise the budget is trimmed to $\top$. The weights in $G_{T,\mathcal{G}}$ are derived from $\mathcal{G}$ such that a play in $G_{T,\mathcal{G}}$ that does not end in a sink, corresponds to a play in $\mathcal{G}$ that traverses the same sequence of weights.

The following lemma shows soundness of the procedure. Energy in turn-based games is similar to Def. 4; a necessary and sufficient initial energy for Pres to win, see also [18].

► **Lemma 36.** *If $\text{Energy}(\langle v, B \rangle) < \infty$ in $G_{T,\mathcal{G}}$, for every v with $T(v) < k + 1$, then $T \geq Th$.*

Proof sketch. We construct a Pres budget-agnostic strategy σ in $\mathcal{G}$ that simulates the operation of a winning strategy σ' in $G_{T,\mathcal{G}}$ as follows. See details in the full version [13]. Suppose that $\mathcal{G}$ is in $\langle v, \tilde{B} \rangle$ with $\tilde{B} \geq T(v)$, then the simulation of $G_{T,\mathcal{G}}$ is in $\langle v, \text{Trim}(v, \tilde{B}) \rangle$. We define $\langle \text{bid}^T(v, B), v' \rangle = \sigma(v, \tilde{B})$ such that σ chooses v' in $G_{T,\mathcal{G}}$. Pres simulates in $G_{T,\mathcal{G}}$, Cons's response in $\mathcal{G}$, which we assume wlog, is either $\langle 0, u \rangle$ or $\langle \text{bid}^T(v, B) \oplus 0^*, u \rangle$, for some $u \in N(v)$. If the next vertex in $G_{T,\mathcal{G}}$ is not a sink, we repeat. A sink is reached only when Cons wins the bidding and Pres's updated budget is $\tilde{B}' = B \oplus (\text{bid}^T(v, B) \oplus 0^*)$ with $\tilde{B}' > T(u) \oplus 0^*$. We restart the simulation of $G_{T,\mathcal{G}}$ in $\langle u, \text{Trim}(u, \tilde{B}') \rangle$. Note that Pres's spare change has increased, thus a play can end in a sink only finitely often. ◀

We proceed to prove completeness.

► **Lemma 37.** *If $T \equiv Th$, then $\text{Energy}(\langle v, B \rangle) < \infty$ in $G_{T,\mathcal{G}}$, for every v with $T(v) < k + 1$.*

Proof sketch. Suppose towards contradiction that $T \equiv Th$ and for configuration $\langle v, B \rangle$, we have $\text{Energy}(v, B) = \infty$ in $G_{T,\mathcal{G}}$ but $\text{Energy}(v, B) < \infty$ in $\mathcal{G}$. Let σ_{agn} be a M -winning from $\langle v, B \rangle$ in $\mathcal{G}$ (see Thm. 27) and let τ' be a Cons M -winning strategy in $G_{T,\mathcal{G}}$. We simulate σ_{agn} against τ' in both games. Crucially, the simulation in $G_{T,\mathcal{G}}$ is possible only since the actions that σ_{agn} chooses imply that the configuration updates in both games are the same. We obtain two plays, one in $\mathcal{G}$ and the other in $G_{T,\mathcal{G}}$ with the same sequence of weights, which is a contradiction since Cons M -wins in $G_{T,\mathcal{G}}$ while Pres M -wins in $\mathcal{G}$. See details in the full version [13]. ◀

Finally, we verify $T' \geq Th'$. We proceed as before to construct $G_{T',\mathcal{G}}$, except that Pres responds to Cons and sink vertices are winning for Cons, i.e. $\langle v, \top \rangle$ has a (-1) -valued self-loop. Dually it follows:

► **Lemma 38.** *If $\text{Energy}(\langle v, B \rangle) = \infty$ in $G_{T',\mathcal{G}}$, for every v with $Th'(v) < k + 1$, then $T' \geq Th'$. If $T' \equiv Th'$, then $\text{Energy}(\langle v, B \rangle) = \infty$ in $G_{T,\mathcal{G}}$, for every v with $T'(v) < k + 1$.*

Since solving mean-payoff turn-based games is in NP and coNP [29], we obtain:

► **Theorem 39.** *Finding threshold budgets in energy bidding game is in NP and coNP.*

► **Remark 40.** Since there exist optimal strategies in turn-based energy games that are memoryless, the strategies in $\mathcal{G}$ constructed in Lem. 36 and Lem. 38 strengthen Thm. 27 and Thm. 34: there exists a winning *positional* budget agnostic strategy $\hat{\sigma}_{\text{agn}} : V \times [k] \rightarrow [k] \times V$ such that $\hat{\sigma}_{\text{agn}}(v, B) = \hat{\sigma}_{\text{agn}}(v, \text{Trim}(v, B))$, for every $B \geq \text{Th}(v)$, and similarly for Cons.

8 Discussion

We study, for the first time, a combination of discrete-bidding with mean-payoff and energy objectives. We define threshold budgets, establish existence, and construct concise budget-agnostic winning strategies, which serve as the basis for showing that finding thresholds is in NP and coNP, even when the budgets in the game are represented in binary.

We believe that our work opens the door to extensions and generalizations, which are technically challenging to study in combination with continuous bidding. One example is bidding games in which the players are partially-informed of their opponent's budgets [9], which is common in practice, but results under continuous bidding are limited due to the demanding technicalities. Other examples that have not yet been considered for mean-payoff objectives include bidding games with charging [4], stochastic transitions [10], and non-zero sum games, which require refined solution concepts [11].

Finally, we point out that our result positions mean-payoff discrete-bidding games in a peculiar state of affairs: on the one hand, solving turn-based mean-payoff games (in NP and coNP but not known to be in P) easily reduces to solving mean-payoff discrete-bidding games with total budget 0, and on the other hand, the result applies to a seemingly exponentially harder problem of finding thresholds in bidding games with budgets given in binary.

References

- 1 M. Aghajohari, G. Avni, and T. A. Henzinger. Determinacy in discrete-bidding infinite-duration games. *Log. Methods Comput. Sci.*, 17(1), 2021. URL: <https://lmcs.episciences.org/7148>.
- 2 R. Alur, T. A. Henzinger, and O. Kupferman. Alternating-time temporal logic. *J. ACM*, 49(5):672–713, 2002. doi:10.1145/585265.585270.
- 3 G. Amanatidis, G. Birmpas, A. Filos-Ratsikas, and A. A. Voudouris. Fair division of indivisible goods: A survey. In Luc De Raedt, editor, *Proceedings of the Thirty-First International Joint Conference on Artificial Intelligence, IJCAI 2022, Vienna, Austria, 23-29 July 2022*, pages 5385–5393. ijcai.org, 2022. doi:10.24963/ijcai.2022/756.
- 4 G. Avni, E. Kafshdar Goharshady, T. A. Henzinger, and K. Mallik. Bidding games with charging. In Rupak Majumdar and Alexandra Silva, editors, *35th International Conference on Concurrency Theory, CONCUR 2024, Calgary, Canada, September 9-13, 2024*, volume 311 of *LIPICs*, pages 8:1–8:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024. doi:10.4230/LIPICs.CONCUR.2024.8.
- 5 G. Avni, T. A. Henzinger, and V. Chonev. Infinite-duration bidding games. *J. ACM*, 66(4):31:1–31:29, 2019. doi:10.1145/3340295.
- 6 G. Avni, T. A. Henzinger, and R. Ibsen-Jensen. Infinite-duration poorman-bidding games. In George Christodoulou and Tobias Harks, editors, *Web and Internet Economics - 14th International Conference, WINE 2018, Oxford, UK, December 15-17, 2018, Proceedings*, volume 11316 of *Lecture Notes in Computer Science*, pages 21–36. Springer, 2018. doi:10.1007/978-3-030-04612-5_2.
- 7 G. Avni, T. A. Henzinger, and D. Zikelic. Bidding mechanisms in graph games. *J. Comput. Syst. Sci.*, 119:133–144, 2021. doi:10.1016/j.jcss.2021.02.008.
- 8 G. Avni, I. Jecker, and Đ. Žikelić. Infinite-duration all-pay bidding games. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms, SODA 2021, Virtual Conference, January 10 - 13, 2021*, pages 617–636. SIAM, 2021. doi:10.1137/1.9781611976465.38.

- 9 G. Avni, I. Jecker, and D. Zikelic. Bidding graph games with partially-observable budgets. In Brian Williams, Yiling Chen, and Jennifer Neville, editors, *Thirty-Seventh AAAI Conference on Artificial Intelligence, AAAI 2023, Thirty-Fifth Conference on Innovative Applications of Artificial Intelligence, IAAI 2023, Thirteenth Symposium on Educational Advances in Artificial Intelligence, EAAI 2023, Washington, DC, USA, February 7-14, 2023*, pages 5464–5471. AAAI Press, 2023. doi:10.1609/aaai.v37i5.25679.
- 10 G. Avni, M. Kurecka, K. Mallik, P. Novotný, and S. Sadhukhan. Bidding games on markov decision processes with quantitative reachability objectives. In *Proceedings of the 24th International Conference on Autonomous Agents and Multiagent Systems, AAMAS 2025, Detroit, MI, USA, May 19-23, 2025*, pages 161–169. International Foundation for Autonomous Agents and Multiagent Systems / ACM, 2025. doi:10.5555/3709347.3743528.
- 11 G. Avni, K. Mallik, and S. Sadhukhan. Auction-based scheduling. In Bernd Finkbeiner and Laura Kovács, editors, *Tools and Algorithms for the Construction and Analysis of Systems - 30th International Conference, TACAS 2024, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2024, Luxembourg City, Luxembourg, April 6-11, 2024, Proceedings, Part III*, volume 14572 of *Lecture Notes in Computer Science*, pages 153–172. Springer, 2024. doi:10.1007/978-3-031-57256-2_8.
- 12 G. Avni and S. Sadhukhan. Computing threshold budgets in discrete-bidding games. *Theoret-iCS*, 4, 2025. doi:10.46298/theoretics.25.5.
- 13 Guy Avni and Suman Sadhukhan. Mean-payoff and energy discrete bidding games. *CoRR*, abs/2509.00506, 2025. doi:10.48550/arXiv.2509.00506.
- 14 H. Aziz, B. Li, He. Moulin, and X. Wu. Algorithmic fair allocation of indivisible items: a survey and new questions. *SIGecom Exch.*, 20(1):24–40, 2022. doi:10.1145/3572885.3572887.
- 15 M. Babaioff, T. Ezra, and U. Feige. Fair-share allocations for agents with arbitrary entitlements. In Péter Biró, Shuchi Chawla, and Federico Echenique, editors, *EC '21: The 22nd ACM Conference on Economics and Computation, Budapest, Hungary, July 18-23, 2021*, page 127. ACM, 2021. doi:10.1145/3465456.3467559.
- 16 B. Bordais, P. Bouyer, and S. Le Roux. From local to global determinacy in concurrent graph games. In *41st IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2021, Virtual Conference, December 15-17, 2021*, volume 213 of *LIPIcs*, pages 41:1–41:14. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.FSTTCS.2021.41.
- 17 P. Bouyer, U. Fahrenberg, K. Guldstrand Larsen, N. Markey, and J. Srba. Infinite runs in weighted timed automata with energy constraints. In Franck Cassez and Claude Jard, editors, *Formal Modeling and Analysis of Timed Systems, 6th International Conference, FORMATS 2008, Saint Malo, France, September 15-17, 2008. Proceedings*, volume 5215 of *Lecture Notes in Computer Science*, pages 33–47. Springer, 2008. doi:10.1007/978-3-540-85778-5_4.
- 18 L. Brim, J. Chaloupka, L. Doyen, R. Gentilini, and J.-F. Raskin. Faster algorithms for mean-payoff games. *Formal Methods Syst. Des.*, 38(2):97–118, 2011. doi:10.1007/s10703-010-0105-x.
- 19 A. Condon. The complexity of stochastic games. *Inf. Comput.*, 96(2):203–224, 1992. doi:10.1016/0890-5401(92)90048-K.
- 20 M. Develin and S. Payne. Discrete bidding games. *Electron. J. Comb.*, 17(1):R85, 2010. doi:10.37236/357.
- 21 A. Gorokh, S. Banerjee, and K. Iyer. The remarkable robustness of the repeated fisher market. In *EC '21: The 22nd ACM Conference on Economics and Computation, Budapest, Hungary, July 18-23, 2021*, page 562. ACM, 2021. doi:10.1145/3465456.3467560.
- 22 A. J. Lazarus, D. E. Loeb, J. G. Propp, W. R. Stromquist, and D. H. Ullman. Combinatorial games under auction play. *Games and Economic Behavior*, 27(2):229–264, 1999. doi:10.1006/game.1998.0676.
- 23 A. J. Lazarus, D. E. Loeb, J. G. Propp, and D. Ullman. Richman games. *Games of No Chance*, 29:439–449, 1996. doi:10.1017/9781009701839.037.

- 24 R. Meir, G. Kalai, and M. Tennenholtz. Bidding games and efficient allocations. *Games Econ. Behav.*, 112:166–193, 2018. doi:10.1016/j.geb.2018.08.005.
- 25 S. Muthukrishnan. Ad exchanges: Research issues. In Stefano Leonardi, editor, *Internet and Network Economics, 5th International Workshop, WINE 2009, Rome, Italy, December 14-18, 2009. Proceedings*, volume 5929 of *Lecture Notes in Computer Science*, pages 1–12. Springer, 2009. doi:10.1007/978-3-642-10841-9_1.
- 26 Y. Peres, O. Schramm, S. Sheffield, and D. B. Wilson. Tug-of-war and the infinity laplacian. *J. Amer. Math. Soc.*, 22:167–210, 2009. URL: <https://www.jstor.org/stable/40587228>.
- 27 A. Pnueli and R. Rosner. On the synthesis of a reactive module. In *Conference Record of the Sixteenth Annual ACM Symposium on Principles of Programming Languages, Austin, Texas, USA, January 11-13, 1989*, pages 179–190. ACM Press, 1989. doi:10.1145/75277.75293.
- 28 M.O. Rabin. Decidability of second order theories and automata on infinite trees. *Transaction of the AMS*, 141:1–35, 1969. doi:10.2307/1995086.
- 29 U. Zwick and M. Paterson. The complexity of mean payoff games on graphs. *Theor. Comput. Sci.*, 158(1&2):343–359, 1996. doi:10.1016/0304-3975(95)00188-3.