Boolean Basis and Succinctness of Modal Logic via Hella-Vilander Games

Sebastian Pfau
Technische Universität Ilmenau, Germany

Abstract

The Hella-Vilander game for modal logic is a model comparison game that captures the formula size necessary to separate sets of pointed Kripke structures. We introduce the $\mathcal{M}$-ON game as a modification of this game. Our game captures the necessary number of modal operators, i.e., $\Diamond$ and $\Box$ instead of formula size. We use our game to show that the bi-implication $\leftrightarrow$, sometimes also called equivalence, enables us to write modal logic formula with significantly fewer modal operators. With this we show, that with bi-implications we can also write significantly shorter modal logic formulas. This result holds even if only special classes of Kripke structures are considered. To be more precise we show that there is an exponential succinctness gap between modal logic and its extension with bi-implication on the class of structures with a transitive and reflexive accessibility relation, as well as on the class of structures with a symmetrical and reflexive accessibility relation. Lastly we show that for the class of structures with a transitive and symmetrical accessibility relation this succinctness gap disappears.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics

Keywords and phrases succinctness, modal logic, model comparison games

Digital Object Identifier 10.4230/LIPIcs.CSL.2026.35

1 Introduction

Model comparison games, like the Ehrenfeucht-Fraïssé game [4, 6], are a type of logic games. They can be used to prove that it is difficult to express certain properties in a logic. For example the game defined by Adler and Immerman in [1] has been used to prove lower bounds for formula size in $\mathrm{FO}^2(\mathrm{TC})$ and the logic CTL. Extended syntax trees [8] can be seen as an equivalent method to this game, even though they drop the game setting. The game and the syntax tree method can also be modified to work on other logics, see [7, 12, 13] for examples. In [9] Hella and Vilander introduce their own game, which builds on the Adler-Immerman game. This Hella-Vilander game is defined for modal logic and the μ-calculus, but it can also be adapted for other logics. The main idea of this game is as follows: players S (*Spoiler, Samson*) and D (*Duplicator, Delilah*) play on two sets of structures $\mathbb{A}$ and $\mathbb{B}$ with a resource parameter k. Player S spends the parameter k in order to make his moves, while D answers. S has to play a winning move before k runs out. In such a play S has a winning strategy if, and only if, there is a formula φ of size at most k of the corresponding logic, that separates $\mathbb{A}$ and $\mathbb{B}$.

There are also model comparison games that only count the number of occurrences of specific characters, instead of the more general formula size. See [3, 5] for examples. As part of their conclusion Hella and Vilander suggest that their game can be modified to count a more specific parameter as well. One such possible parameter is the number of modal operators used, i.e., $\Diamond$ and $\Box$. This can be done, because every move of the Hella-Vilander game corresponds to some character of a separating formula. So by simply adjusting the way the parameter k has to be spend, we can create a game for the number of modal operators instead of formula size. The main problem with this approach is that some moves do not reduce the parameter at all. Firstly this opens up the possibility of endless games, where S cannot win, but he can ensure he does not lose either. Secondly it makes the game inefficient, since moves that do not change the parameter are inherently less interesting.

In order to deal with these problems we look at another model-comparison game used by Vinall-Smeeth in [14]. He calls this game the k-QVT game. However in this paper we use the parameter k with a different meaning. So in order to avoid confusion we will use the name l-QVT game when referring to this game instead. The l-QVT counts the number of quantifiers in formulas of the l-variable fragment of first-order logic. It can be seen as a modification of the Adler-Immerman game, where after every move all structures without a partially isomorphic counterpart in the respective other set are removed. This is possible, because these are exactly the structures that can be removed in the Adler-Immerman game without playing any quantifier-moves. With this idea we can create our own game, which counts the number of modal-operators of separating modal logic formulas. We call this game the $\mathcal{M}$-*Operator-Number* game.

Afterwards we use our new game in order to look at how different sets of Boolean connectives can be used to write more succinct formulas. This topic was already researched by Berkholz, Kuske and Schwarz in [2]. They show that all complete operator sets, which only contain so-called *locally monotone* operators, have the same succinctness up to a polynomial. On the other hand all sets that contain some non locally monotone operator also coincide in succinctness. In this paper we use the De Morgan basis, consisting of $\wedge$, $\vee$ and $\neg$, as a candidate from the first class and the De Morgan basis extended by the bi-implication $\leftrightarrow$ as a candidate of the second. Berkholz et al. also show that for modal logic there is an exponential succinctness gap between these two classes. They prove this by working directly with formulas and using intricate and ad hoc arguments. This gap still exists if only reflexive Kripke structures are taken into consideration. However if the focus is tightened even further to only Kripke structures with an equivalence relation as accessibility relation, then this succinctness gap is closed.

From these findings some interesting follow-up questions emerge: Is it possible to remove $\leftrightarrow$ without significantly increasing the number of modal operators? This question is interesting because modal logic formulas with few modal operators are easier to work with. However we answer this question negatively by showing that there is an exponential operator-succinctness gap between modal logic and modal logic with bi-implication as well. Another interesting follow-up question is the following: What about other classes of Kripke-structures? We want to answer this question in this paper. In particular we show that for the class of Kripke structures with a reflexive and transitive accessibility relation, as well as the class of structures with a reflexive and symmetrical relation this exponential succinctness gap exists. This directly implies that the gap also exists for the class of transitive and the class of symmetrical structures. For all of these classes there is an exponential operator-succinctness gap as well. Only for the class of structures with a transitive and symmetrical relation this gap disappears. In this way our work offers a more complete and systematic examination of the connection between succinctness and the basis of boolean connectives in modal logic on special classes of Kripke structures.

2 Preliminaries

2.1 Modal logic with Bi-implication and Kripke structures

Let P be an infinite set of propositional variables, also called *labels*. We define ML[$\leftrightarrow$] as the set of all modal logic formulas over P using $\top$ and $\bot$ as constants, $\Diamond$ and $\Box$ as modal operators, as well as $\neg$, $\vee$, $\wedge$ and $\leftrightarrow$ as Boolean connectives. Formally ML[$\leftrightarrow$] is the set of all formulas generated by the following grammar:

$$\varphi ::= \bot \mid \top \mid p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid (\varphi \leftrightarrow \varphi) \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in P$ is a propositional variable. Now modal logic can be defined as a restriction of modal logic with bi-implication. So ML is defined as the set of all formulas from ML$[\leftrightarrow]$ that do not contain any occurrences of $\leftrightarrow$.

Let $\mathcal{M} = (W, R, V)$ be a connected, directed, labelled graph, where W is the set of possible worlds or nodes, $R \subseteq W \times W$ is a binary accessibility relation and $V : W \to \mathcal{P}(P)$ is a labelling function, assigning sets of propositional variables to each node. For a node $w \in W$ we call $V(w) \subseteq P$ the *colour* of w. If $V(W) := \bigcup_{w \in W} V(w)$, the set of all labels used in $\mathcal{M}$, is finite, we say $\mathcal{M}$ is a *finitely labelled graph*. The tuple $(\mathcal{M}, w)$ is called a *pointed Kripke structure* or just *Kripke structure*. It should be noted here, that our naming conventions might differ from the norm. In the literature the labelled graph $\mathcal{M}$ is sometimes referred to as a Kripke structure, while $(\mathcal{M}, w)$ is called a pointed Kripke structure. However in this paper we do not view $\mathcal{M}$ as a structure. Also our conventions allow us to use the terms pointed Kripke structure, Kripke structure or sometimes just structure interchangeably when referring to $(\mathcal{M}, w)$.

Let $A \subseteq W$ be a set of nodes. We write $(\mathcal{M}, A)$ to mean the class of structures $\{(\mathcal{M}, w) \mid w \in A\}$. We define $\Box w = \{w' \in W \mid (w, w') \in R\}$ as the set of all successors of a node w. This notation can also be extended to sets of nodes with $\Box A = \bigcup_{w \in A} \Box w$ and structures with $\Box(\mathcal{M}, w) = (\mathcal{M}, \Box w) = \{(\mathcal{M}, w') \mid w' \in \Box w\}$. The extension to classes of structures is defined in the obvious way. We can now start giving the definition of the semantics of modal logic with bi-implication on Kripke structures.

The satisfaction relation $\models$ is defined inductively as follows:

1. $(\mathcal{M}, w) \models \top$ and $(\mathcal{M}, w) \not\models \bot$ for all $(\mathcal{M}, w)$
2. $(\mathcal{M}, w) \models p$ iff $p \in V(w)$
3. $(\mathcal{M}, w) \models \neg\varphi$ iff $(\mathcal{M}, w) \not\models \varphi$
4. $(\mathcal{M}, w) \models (\varphi \vee \psi)$ iff $(\mathcal{M}, w) \models \varphi$ or $(\mathcal{M}, w) \models \psi$
5. $(\mathcal{M}, w) \models (\varphi \wedge \psi)$ iff $(\mathcal{M}, w) \models \varphi$ and $(\mathcal{M}, w) \models \psi$
6. $(\mathcal{M}, w) \models \varphi \leftrightarrow \psi$ iff $(\mathcal{M}, w) \models (\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$
7. $(\mathcal{M}, w) \models \Diamond\varphi$ iff $(\mathcal{M}, v) \models \varphi$ holds for some $v \in \Box w$
8. $(\mathcal{M}, w) \models \Box\varphi$ iff $(\mathcal{M}, v) \models \varphi$ holds for all $v \in \Box w$

Next we extend this satisfaction relation to classes of structures. Let $\mathfrak{A}$ be a class of pointed Kripke structures. Then $\mathfrak{A} \models \varphi$ if, and only if, $(\mathcal{M}, w) \models \varphi$ for all $(\mathcal{M}, w) \in \mathfrak{A}$. We say a formula φ separates two classes of pointed Kripke structures $\mathfrak{A}$ and $\mathfrak{B}$ if either $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$, or $\mathfrak{B} \models \varphi$ and $\mathfrak{A} \models \neg\varphi$. So φ must be true for all structures of one class and for no structure of the other class. In this case we will also call φ a *separating formula*. It should be noted here that our definition differs slightly from the definition used by Hella and Vilander. In their definition φ is a separating formula only if $\mathfrak{A} \models \varphi$ and $\mathfrak{B} \models \neg\varphi$. However it is easy to see that if φ is a separating formula by our definition then φ or $\neg\varphi$ is a separating formula by their definition. For $\mathfrak{A} = \{\mathcal{A}\}$ and $\mathfrak{B} = \{\mathcal{B}\}$ we might also say φ separates $\mathcal{A}$ and $\mathcal{B}$ instead of $\mathfrak{A}$ and $\mathfrak{B}$.

We say that two formulas φ and ψ are *equivalent*, in writing $\varphi \equiv \psi$, if every pointed Kripke structure $(\mathcal{M}, w)$ satisfies φ if, and only if, it satisfies ψ. Let $\mathcal{C}$ be some class of Kripke structures. Then two formulas φ and ψ are *equivalent on* $\mathcal{C}$, in writing $\varphi \equiv_{\mathcal{C}} \psi$, if every pointed Kripke structure $(\mathcal{M}, w) \in \mathcal{C}$ from $\mathcal{C}$ satisfies φ if, and only if, it satisfies ψ.

2.2 Formula size

The *size* of a formula φ is the number of symbols needed to write it, excluding brackets. This can be expressed quite easily using the following inductive definition:

1. $|\varphi| = 1$ if $\varphi \in \{\top, \bot\} \cup P$
2. $|\varphi| = |\psi| + 1$ if $\varphi = \neg\psi$, $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$
3. $|\varphi| = |\psi| + |\chi| + 1$ if $\varphi = \psi \wedge \chi$, $\varphi = \psi \vee \chi$ or $\varphi = \psi \leftrightarrow \chi$

Similarly the *operator-number* of φ is the number of modal operators, i.e. $\Diamond$ and $\Box$, used in the formula. The inductive definition is as follows:

1. $|\varphi|_O = 0$ if $\varphi \in \{\top, \bot\} \cup P$
2. $|\varphi|_O = |\psi|_O$ if $\varphi = \neg\psi$
3. $|\varphi|_O = |\psi|_O + 1$ if $\varphi = \Diamond\psi$ or $\varphi = \Box\psi$
4. $|\varphi|_O = |\psi|_O + |\chi|_O$ if $\varphi = \psi \wedge \chi$, $\varphi = \psi \vee \chi$, or $\varphi = \psi \leftrightarrow \chi$

The term succinctness is used to compare logics based on the length of formulas needed to express the same property. Given two logics L_1, L_2 with the same expressiveness and a function $f : \mathbb{N} \to \mathbb{N}$, we say L_1 is *f-times more succinct* than L_2 if, and only if, for every $n \in \mathbb{N}$ there is a formula $\varphi \in L_1$ with $|\varphi| \geq n$, so that $|\psi| \geq f(|\varphi|)$ for every L_2-formula ψ with $\psi \equiv \varphi$. If f is an exponential function, we also say L_1 is exponentially more succinct than L_2. Let $\mathcal{C}$ be a class of structures, then succinctness on $\mathcal{C}$ is defined by taking the definition of succinctness and replacing $\equiv$ with $\equiv_{\mathcal{C}}$.

Succinctness is based on formula size, but we can define a corresponding measurement for operator-number. We say that L_1 is *f-times more operator-succinct* than L_2 if, and only if, for every $n \in \mathbb{N}$ there is a formula $\varphi \in L_1$ with $|\varphi|_O \geq n$, so that $|\psi|_O \geq f(|\varphi|_O)$ for every L_2-formula ψ with $\psi \equiv \varphi$. Exponential operator-succinctness and operator-succinctness on special classes of structures are defined in the obvious way.

Counter intuitively it is theoretically possible that there are two logics L_1 and L_2 where L_1 is exponentially more succinct than L_2 and L_2 is exponentially more succinct than L_1. This is because there might be some properties that can be expressed more succinctly in L_1, while some other properties can be expressed more succinctly in L_2. However this cannot happen for the logics examined in this paper. Since every ML formula is also an ML[$\leftrightarrow$] formula, ML cannot be more succinct than ML[$\leftrightarrow$].

3 The operator-number game

Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph and let $A, B \subseteq W$ be two sets of nodes of the graph. Then $\{A, B\}$ is an *unordered pair*. From this point on, we will use just the term pair to refer to unordered pairs. We say such a pair $\{A, B\}$ is *tidy*, if for every $a \in A$ there is a $b \in B$ with $V(a) = V(b)$ and vice versa. For any sets A and B, $\{A, B\}$ can be tidied up by removing all nodes $a \in A$ without a fitting node in B and all $b \in B$ without a fitting node in A. Let $t(\{A, B\}) = \{A', B'\}$, where $A' \subseteq A$ and $B' \subseteq B$ are the largest subsets of A and B respectively, so that $\{A', B'\}$ is tidy.

▶ **Definition 1.** *Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph, $A_0, B_0 \subseteq W$ two sets of nodes and $k_0 \in \mathbb{N}$. The $\mathcal{M}$-*Operator-Number *game, or $\mathcal{M}$-ON game for short, played from A_0 and B_0 with parameter k_0 has two players S and D. An S-*position *$(\{A, B\}, k)$ of the game consists of a tidy pair of sets of nodes $A, B \subseteq W$, as well as a parameter $k \in \mathbb{N}$. The starting position is $(t(\{A_0, B_0\}), k_0)$. A *turn *for S starts from an S-position and proceeds in one of the following two ways:*

1. ***Split-move:** S may only play this type of move if $k \geq 2$. He picks a set $X \in \{A, B\}$ and splits it into two disjoint and non-empty sets X_1 and X_2. Then he chooses two parameters $k_1, k_2 \geq 1$ with $k_1 + k_2 = k$. Let Y be the set that fulfils $\{X, Y\} = \{A, B\}$. This play by S creates the D-position (X_1, X_2, Y, k_1, k_2). Afterwards D plays by taking the set Y and picking a subset $Y' \subseteq Y$ of it. Then she chooses $i \in \{1, 2\}$ and the game continues from the S-position $(t(\{X_i, Y'\}), k_i)$.*

2. ***Operator-move:*** *S may only play this type of move if $k \geq 1$. He picks a set $X \in \{A, B\}$ and then picks a function $f : X \to \Box X$, where $f(x) \in \Box x$ for every $x \in X$. Let $X' = \{f(x) \mid x \in X\}$ be the image of X under f and let Y be the set with $\{X, Y\} = \{A, B\}$. Then the D-position after this move by S is $(X', Y, k-1)$. Afterwards D plays by taking the set Y and then determining a set $Y' \subseteq \Box Y$ of successors of the nodes from Y. The game continues from the position $(t(\{X', Y'\}), k-1)$.*

Player S wins a play of the game, if the play arrives at a position $(\{A, B\}, k)$ with $A = B = \emptyset$. Otherwise if S cannot make another move, player D wins.

When looking at an operator-move we might say *S picks x' as a successor of x* instead of *S picks a function f with $f(x) = x'$*. We usually make D play in such a way that no nodes are removed during tidy-up. For this reason we will only mention the tidy-up process if nodes are getting removed. The composition of a D-position depends on the type of move played by S. However in contrast to S-positions we mainly view D-positions as intermediate positions. So it is not necessary to give a full definition for them.

The parameter k is reduced in every move and the play cannot continue, once the parameter is depleted. So a play with starting parameter k can go on for k rounds at most. In particular every play has to be finite. This implies in every play either player S or player D wins. So our game is a finite, chance-less two-player game with complete information, that cannot end in a draw. Then by Zermelo's Theorem for every position of the $\mathcal{M}$-ON game one of the players, S or D, has a positional winning strategy.

Such a positional strategy for a player, S or D, is a partial map f_S or f_D. For S a strategy f_S takes as input an S-position and outputs a D-position, that can be reached from the input position via a single move by S. It is a winning strategy, if every play where S plays according to f_S is won by him. Similarly a strategy f_D for D takes a D-position as input and gives an S-position, reachable in one move by D, as output. It is a winning strategy, if D wins every play in which she plays f_D. If player S has a winning strategy for some position $(\{A, B\}, k)$, we will say this position is a winning position for player S. The definition of winning positions for D is done in the obvious way.

As previously stated, the main influence for our game is the Hella-Vilander game from [9], which in turn builds on the Adler-Immerman game from [1]. Before we show that our game can be used to prove lower bounds for operator-number, we want to discuss how our game differs from these games.

The $\mathcal{M}$-ON game is played by two players, S (Samson) and D (Delilah), just like the game defined by Hella and Vilander. However instead of playing on two sets of structures, in our game the players play on unordered pairs of sets of nodes. This shows two significant differences between our game and their game. Firstly we are using nodes, where they used structures. This change however is only notational, because we will still implicitly talk about pointed Kripke structures. We will use the node w as a representation of the structure $(\mathcal{M}, w)$. In order to do so, we have to fix a single graph $\mathcal{M}$ for the entire game. So every pointed Kripke structure that appears in a play must be based on the same labelled graph. While this appears like a real restriction, it is possible to merge multiple labelled graphs into one without changing the validity of formulas at any world.

The second significant difference is that we use unordered pairs instead of ordered pairs or tuples. This change allows us to switch around the elements of our pairs at will. In the game by Adler and Immerman, such a switch is used as a not-move. So basically in our game, we can simulate a not-move without actually making a move. This is possible, because we are only interested in the number of operators and not in the number of negations. We also add an additional step to every move, where we tidy up the pair of our position. By doing so we remove all elements, that could have been removed in the Hella-Vilander game by S playing a number of turns without playing an operator-move. A similar approach is used in the l-QVT game from [14]. In this game structures without a partially isomorphic counterpart in the other set are removed after every step. A problem with this method occurs when tidying up would skip an infinite number of moves. Vinall-Smeeth solves this problem by only considering finite structures. However for our purposes a weaker condition is sufficient. We only demand that our graphs are finitely labelled.

As a last major difference: In the Hella-Villander game, player D is forced to play following the so-called *oblivious* strategy. This means she always has to answer with the largest possible set. This works, because this oblivious strategy is always optimal. So if D can win the game, she can do so by playing obliviously. While this simplifies the definition of the game, it would also complicate our proofs. Simply put we only want to prove that D has a winning strategy. Whether this strategy is also optimal is of no importance to us. Sometimes D playing in a suboptimal way might lead to smaller positions, which can simplify our proofs.

Our aim is to prove that the $\mathcal{M}$-ON game can be used to characterize the number of operators needed in order to separate sets of nodes. As a stepping stone towards this goal, we state the following two lemmas.

▶ **Lemma 2.** *Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph and $A, B, A', B' \subseteq W$ be sets of nodes with $t(\{A, B\}) = \{A', B'\}$. If there is a separating formula φ' for $(\mathcal{M}, A')$ and $(\mathcal{M}, B')$, then there is also a separating formula φ for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with $|\varphi|_O = |\varphi'|_O$.*

▶ **Lemma 3.** *Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph, $A, B \subseteq W$ be sets of nodes that form a tidy pair and let φ be an ML-formula without any modal operators. Then $(\mathcal{M}, A) \models \varphi$ if, and only if, $(\mathcal{M}, B) \models \varphi$.*

Now we want to prove the following Correctness- and Completeness-Theorem for our game. We do so by proving both directions separately, beginning with the proof of correctness.

▶ **Lemma 4.** *Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph, $A, B \subseteq W$ two sets of nodes and $k \in \mathbb{N}$. If S has a winning strategy in the $\mathcal{M}$-ON game played from A and B with parameter k, then there is an ML-formula φ separating $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with $|\varphi|_O \leq k$.*

Proof. For this proof we assume that S has a winning strategy for the $\mathcal{M}$-ON game played from A and B with parameter k. Because of Lemma 2 we may assume that A and B already form a tidy pair. Then the starting position of this game is $(\{A, B\}, k)$. We construct the separating formula φ by induction over k. First we look at the base case $k = 0$. Since the parameter is already used up, S cannot make a move at all. So the only possible way for him to win is, if the current position already fulfils $\{A, B\} = \{\emptyset, \emptyset\}$. Then $\varphi = \top$ is a trivial separating formula.

We now assume, the implication holds for all parameters $k \leq i$ and we look at some winning strategy f_S for the $\mathcal{M}$-ON game starting from $(\{A, B\}, i+1)$. We have to differentiate between the types of moves f_S prescribes as the next move from this position.

We begin by assuming that f_S prescribes a split-move. In this case S picks a set from $\{A, B\}$, without loss of generality he picks A. Then S splits it into two sets A_1 and A_2 and he splits $i + 1$ into k_1 and k_2, both of which have to be at most i. Since f_S is a winning strategy for S, it has to win against every possible strategy of D, in particular against the following ones: D answers by picking $B' = B$ and then picks either 1 or 2. The game continues from $(t(\{A_1, B\}), k_1)$ or $(t(\{A_2, B\}), k_2)$ respectively. From both positions f_S is still a winning strategy. So thanks to our induction hypothesis and Lemma 2, for $j \in \{1, 2\}$

we have separating formulas φ_j for $(\mathcal{M}, A_j)$ and $(\mathcal{M}, B)$ with $|\varphi_j|_O \leq k_j$. We may assume $(\mathcal{M}, A_j) \models \varphi_j$, because otherwise we could use $\neg\varphi_j$ instead. Then $\varphi = \varphi_1 \vee \varphi_2$ is a separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with $|\varphi|_O = |\varphi_1|_O + |\varphi_2|_O \leq k_1 + k_2 = i + 1$.

Next we look into f_S prescribing an operator-move. Once again we may assume that S plays on A. For every node in A he picks a successor and forms A'. The strategy f_S has to win against a strategy of D that simply picks $B' = \Box B$. So S can also win from the position $(t(\{A', \Box B\}), i)$. This means there has to be a formula φ' with $(\mathcal{M}, A') \models \varphi'$, $(\mathcal{M}, \Box B) \models \neg\varphi'$ and $|\varphi'|_O \leq i$. So every node in A has at least one successor, namely those picked by S during the construction of A', that satisfies φ'. On the other hand no node in B can have such a successor, since all of them are part of $\Box B$. So $\varphi = \Diamond\varphi'$ is a separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with at most $i + 1$ operators. This finishes our prove of the first implication. ◀

Next we prove the completeness of our game.

▶ **Lemma 5.** *Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph, φ an ML-formula, $A, B \subseteq W$ two sets of nodes and $k \in \mathbb{N}$, so that φ is a separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with $|\varphi|_O \leq k$. Then S has a winning strategy in the $\mathcal{M}$-ON game played from A and B with parameter k.*

Proof. To begin with we may assume that the separating formula φ only contains the connectives $\vee$ and $\neg$, as well as the operator $\Diamond$, since otherwise we can use the well known equivalences $\psi \wedge \chi \equiv \neg(\neg\psi \vee \neg\chi)$ and $\Box\psi \equiv \neg\Diamond\neg\psi$ in order to remove all occurrences of $\wedge$ and $\Box$. None of these changes increase the number of modal operators, so $|\varphi|_O \leq k$ still holds. The $\mathcal{M}$-ON game played from A and B with parameter k has the starting position $(t(\{A, B\}), k)$. If φ is a separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$, it is also a separating formula for the tidied up subsets of A and B. So we may assume that $\{A, B\}$ is already tidy, meaning $t(\{A, B\}) = \{A, B\}$. We construct f_S, the winning strategy of S, by playing according to φ. This construction is done by induction over the formula size of φ.

For the base case, we look at all atomic formulas, so $\varphi \in P \cup \{\top, \bot\}$. Because of Lemma 3, $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ agree on φ. So the only way for φ to be a separating formula is $A = B = \emptyset$. In this case $(\{A, B\}, k)$ is already a winning S-position and S does not have to make a move in order to win the play.

We now assume the implication holds for all formulas ψ with $|\psi| \leq i$ and we look at a separating formula φ for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with $|\varphi| = i + 1$ and $|\varphi|_O \leq k$. We have to differentiate based on the outermost operator of φ.

We begin with $\varphi = \neg\psi$. Because of our definition of separating formula, ψ also is a separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with $|\psi| = i$ and $|\psi|_O \leq k$. So thanks to our induction hypothesis, S already has a winning strategy for the $\mathcal{M}$-ON game played on A and B with parameter k.

Next we look at $\varphi = \Diamond\psi$ with $|\psi| = i$ and $|\psi|_O \leq k - 1$. This also implies $k \geq 1$. So every node in one of the sets, w.l.o.g. we assume A, has a successor that satisfies ψ, while no node in the other set, B, has such a successor. Then f_S prescribes an operator-move, where S picks for every $a \in A$ a successor a' with $(\mathcal{M}, a') \models \psi$ and collects them into the set A'. Then D answers by gathering some nodes $b' \in \Box B$ into a set B'. However none of the nodes picked by D can satisfy ψ. The game continues from the position $(t(\{A', B'\}), k - 1)$, where ψ is a separating formula for A' and B' and thus also for the sets after tidying. So because of our induction hypothesis, for every choice made by D during this turn, S has a winning strategy from the resulting position. So f_S simply copies the fitting strategy for D's choice of B'.

For $\varphi = \psi \vee \chi$ we have to make an additional differentiation. It is possible that one of the sub-formulas, w.l.o.g. we assume χ, does not contain any modal operators. Then one of the sets, we assume A, satisfies $\psi \vee \chi$, while the other satisfies its negation, so $(\mathcal{M}, B) \models \neg\psi \wedge \neg\chi$. This implies $(\mathcal{M}, B) \models \neg\psi$ and $(\mathcal{M}, B) \models \neg\chi$. So because of Lemma 3, we also know $(\mathcal{M}, A) \models \neg\chi$, which in turn implies $(\mathcal{M}, A) \models \psi$. This means ψ is also a separating formula for A and B with $|\psi| \leq i$ and $|\psi|_O \leq k$. So our hypothesis offers a winning strategy for S in $(\{A, B\}, k)$.

Lastly we look at $\varphi = \psi \vee \chi$, where both ψ and χ contain at least one modal operator. This implies $k \geq 2$. Once again we assume $(\mathcal{M}, A) \models \psi \vee \chi$ and thus $(\mathcal{M}, B) \models \neg\psi \wedge \neg\chi$. Let A_1 and A_2 be the disjoint subsets of A with $A_1 := \{a \in A \mid (\mathcal{M}, a) \models \psi\}$ and $A_2 := \{a \in A \mid (\mathcal{M}, a) \models \neg\psi \wedge \chi\} = A \setminus A_1$. If one of these sets, w.l.o.g. we assume A_2, is empty, then $A = A_1$ and thus $(\mathcal{M}, A) \models \psi$ holds. So ψ is a shorter separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ and according to our induction hypothesis S has a winning strategy for the position $(\{A, B\}, k)$. Otherwise we know $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. So S may play a split-move where he splits A into A_1 and A_2. Then he picks the new parameters as $k_1 = |\psi|_O$ and $k_2 = k - k_1$, which implies $k_2 \geq |\chi|_O$. Player D answers by picking a set $B' \subseteq B$. Independent from her choice $(\mathcal{M}, B') \models \neg\psi \wedge \neg\chi$ holds. The game continues from either $(t(\{A_1, B'\}), k_1)$ or $(t(\{A_2, B'\}), k_2)$. For the first case, S can use ψ as a separating formula for A_1 and B'. Similarly χ is a separating formula for A_2 and B'. So by our induction hypothesis every possible follow-up position is a winning position for S and thus $(\{A, B\}, k)$ is as well. ◀

Lemma 4 and Lemma 5 can be combined into the following Correctness- and Completeness-Theorem.

▶ **Theorem 6.** *Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph, $A, B \subseteq W$ two sets of nodes and $k \in \mathbb{N}$. Then S has a winning strategy in the $\mathcal{M}$-ON game played from A and B with parameter k if, and only if, there is an ML-formula φ with $|\varphi|_O \leq k$, that is a separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$.*

4 Succinctness between ML[$\leftrightarrow$] and ML

We begin this section by proving some simple upper bounds for succinctness.

▶ **Lemma 7.** *The logic ML[$\leftrightarrow$] is at most exponentially more succinct and at most exponentially more operator-succinct than ML.*

Proof. It is quite easy to see that the succinctness gap between ML and ML[$\leftrightarrow$] is at most exponential. We can simply replace every sub-formula $\varphi \leftrightarrow \psi$ with $(\varphi \wedge \psi) \vee (\neg\varphi \wedge \neg\psi)$. Every time this is done, the formula size is doubled at most. Since the number of bi-implications cannot exceed the formula size, this procedure leads to an exponential size increase at worst.

However this method cannot provide an upper bound for the operator-succinctness. While every such substitution only doubles the numbers of operators as well, we cannot bound the number of operators in the resulting formula by some function of the number of operators in the original formula. E.g. for a formula φ with only one modal operator that appears in the scope of n nested bi-implications this method would result in a formula with 2^n modal operators.

However there is another method, that guarantees an at most exponential increase in operator-number. This method is based on the construction used by Spira in [11]. We can use the equivalence $\alpha(\Diamond\beta) \equiv (\alpha(\top) \wedge \Diamond\beta) \vee (\alpha(\bot) \wedge \neg\Diamond\beta)$, which holds as long as $\Diamond\beta$ does not appear in α in the scope of any operator. This allows us to inductively move all modal operators outside of the scope of any bi-implication. Every time this is done, the number of modal operators is doubled, which leads to an exponential increase. Then the bi-implications can be removed in the known way without increasing the number of modal operators any further. A more in-depth proof of this is given as part of the appendix. ◀

In the following Section 4.1 we show that there is an exponential succinctness gap between ML and ML[$\leftrightarrow$] on the class of Kripke structures with a transitive and reflexive accessibility relation. Afterwards in Section 4.2 we show that this exponential succinctness gap also exists on the class of Kripke structures with a symmetrical and reflexive accessibility relation. Since these two claims are proven in similar ways, we want to use this opportunity to give a general overview of these proofs.

To begin with we construct an infinite graph $\mathcal{M}$ as well as two series of nodes $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$. Then for every $n \in \mathbb{N}$ we give an ML[$\leftrightarrow$]-formula φ_n. This formula separates $(\mathcal{M}, a_n)$ and $(\mathcal{M}, b_n)$. The number of modal operators in φ_n is bound from above by a polynomial over n. Then we give some parameter k_n that is exponential in n and show that $(\{\{a_n\}, \{b_n\}\}, k_n)$ is a winning position for player D in the $\mathcal{M} - ON$ game. It follows that S cannot have a winning strategy for this position. So by Theorem 6, there cannot be an ML-formula ψ that separates $(\mathcal{M}, a_n)$ from $(\mathcal{M}, b_n)$ with $|\psi|_O \leq k_n$ implying that every ML-formula equivalent to φ_n must be of size exponential in n at least. This proves an exponential succinctness gap between ML and ML[$\leftrightarrow$] for all classes of Kripke structures $\mathcal{M}$ belongs to.

Now we construct the infinite labelled graph $\mathcal{M} = (W, R, V)$, which will be the basis for all of our other graphs. The nodes of this graph are formally defined as $W := \{a_0, b_0\} \cup \bigcup_{i\geq 1} \{a_i, b_i, 1_i, 2_i, 3_i, 4_i, 5_i, 6_i, 7_i, 8_i, 9_i\}$. The edges of this graph are as follows:

$$R = \bigcup_{i\geq 1} \left\{ \begin{array}{l} (a_i, 1_i), (a_i, 2_i), (a_i, 3_i), (b_i, 1_i), (b_i, 3_i), (1_i, 4_i), (1_i, 6_i), (2_i, 4_i), (2_i, 7_i), (3_i, 5_i), \\ (3_i, 7_i), (4_i, 8_i), (4_i, a_{i-1}), (5_i, 8_i), (5_i, b_{i-1}), (6_i, a_{i-1}), (6_i, 9_i), (7_i, b_{i-1}), (7_i, 9_i) \end{array} \right\}$$

The graph $\mathcal{M}$ can be separated into segments. Then segment i contains all nodes with index i, as well as a_{i-1} and b_{i-1}. So the latter two nodes are shared nodes between segment i and segment $i-1$. The nodes and edges of segment i are presented graphically in Figure 1. All edges of $\mathcal{M}$ connect nodes within a segment and all segments have equivalent edges. So this figure fully represents the nodes and edges of $\mathcal{M}$. The numbers on the left side of the figure count the layers of the graph. Segment i contains all nodes of layers $3i-1$, $3i-2$ and $3(i-1)$, as well as the nodes a_i and b_i from layer $3i$. Lastly we define the labelling function of $\mathcal{M}$. The labels we use are $\{p_0, p_1, q_0, q_1, q_2\} \subseteq P$. The only node labelled with p_0 is a_0. For every $i \geq 1$ the node 8_i is the only node of segment i labelled with p_1. Lastly we use the remaining labels q_0, q_1 and q_2 in order to label the layers of the graph in an alternating pattern. More precisely for every $i \in \mathbb{N}$ we label all nodes of layer $3i$ with q_0, all nodes of layer $3i-1$ with q_1 and all nodes of layer $3i-2$ with q_2. The formal definition of the labelling function is given by the following list:

- $V(9_i) = V(b_{i-1}) = \{q_0\}$ for all $i \geq 1$,
- $V(8_i) = \{p_1, q_0\}$ for all $i \geq 1$ and
- $V(a_{i-1}) = \begin{cases} \{p_0, q_0\} & \text{for } i = 1 \\ \{q_0\} & \text{for all } i > 1 \end{cases}$
- $V(1_i) = V(2_i) = V(3_i) = \{q_1\}$ for all $i \geq 1$,
- $V(4_i) = V(5_i) = V(6_i) = V(7_i) = \{q_2\}$ for all $i \geq 1$,

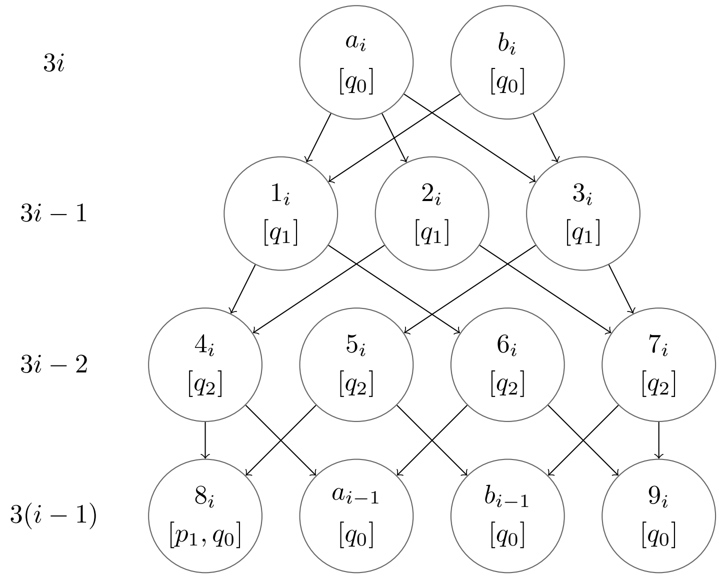

Figure 1 The ith segment of $\mathcal{M}$ including labels.

▷ Claim 8. Let φ_i be the ML formula inductively defined by $\varphi_0 := p_0$ and $\varphi_i := \Diamond\Box((\Diamond p_1) \leftrightarrow (\Diamond\varphi_{i-1}))$. Then φ_i is a separating formula for $(\mathcal{M}, a_i)$ and $(\mathcal{M}, b_i)$ with $(\mathcal{M}, a_i) \models \varphi_i$ for every $i \in \mathbb{N}$.

Proof. This proof is done via induction on i. For the base case $i = 0$ the formula $\varphi_i = p_0$ holds only in $(\mathcal{M}, a_0)$, because a_0 is the only node labelled with p_0. Next let $i \geq 1$ be some positive number. We assume that our claim holds for $i-1$. So $(\mathcal{M}, a_{i-1}) \models \varphi_{i-1}$ and $(\mathcal{M}, b_{i-1}) \models \neg\varphi_{i-1}$. Now we look at the other nodes of segment i. We know $(\mathcal{M}, 8_i) \models p_1$, while no other node of layer $3(i-1)$ is labelled with p_1. This implies $(\mathcal{M}, 4_i) \models (\Diamond p_1) \wedge (\Diamond\varphi_{i-1})$, $(\mathcal{M}, 5_i) \models (\Diamond p_1) \wedge \neg(\Diamond\varphi_{i-1})$, $(\mathcal{M}, 6_i) \models \neg(\Diamond p_1) \wedge (\Diamond\varphi_{i-1})$, as well as $(\mathcal{M}, 7_i) \models \neg(\Diamond p_1) \wedge \neg(\Diamond\varphi_{i-1})$. So $(\mathcal{M}, 4_i)$ and $(\mathcal{M}, 7_i)$ model $(\Diamond p_1) \leftrightarrow (\Diamond\varphi_{i-1})$, while $(\mathcal{M}, 5_i)$ and $(\mathcal{M}, 6_i)$ do not. Because 4_i and 7_i are the only successors of 2_i, this also means $(\mathcal{M}, 2_i) \models \Box((\Diamond p_1) \leftrightarrow (\Diamond\varphi_{i-1}))$. On the other hand, 1_i is a predecessor of 6_i and 3_i is a predecessor of 5_i. So neither $(\mathcal{M}, 1_i)$ nor $(\mathcal{M}, 3_i)$ satisfy this formula. Lastly 2_i is a successor of a_i, so $(\mathcal{M}, a_i) \models \Diamond\Box((\Diamond p_1) \leftrightarrow (\Diamond\varphi_{i-1})) = \varphi_i$. But 1_i and 3_i are the only successors of b_i, so $(\mathcal{M}, b_i) \models \neg\varphi_i$, which proves the claim. ◁

It should be noted here that the formula φ_i does not contain any occurrences of either q_0, q_1 or q_2. However we need those labels in order to adapt φ_i to the modified graphs we use in the following proofs.

4.1 Transitive and reflexive graphs

The graph $\mathcal{M}_{t,r} = (W, R_{t,r}, V)$ is defined as an extension of $\mathcal{M} = (W, R, V)$. The nodes and labels stay the same. First we add a self-loop for every node. So (x, x) is a new edge for every node x. Then we add a new edge from a node x to a node y, for all x and y where x is a node on layer i and y is a node on layer j with $i \geq j + 2$. So every node is now a predecessor to all nodes two or more layers below. So formally $R_{t,r} = R \cup \{(x, x) \mid x \in W\} \cup \{(x, y) \mid x \text{ belongs to layer } i, y \text{ belongs to layer } j, i \geq j + 2\}$. It is easy to see that $\mathcal{M}_{t,r}$ is a reflexive graph. On top of that it is also transitive. Let (x, y) and (y, z) be edges of the graph. If either of them is a self loop, then (x, z) is the same as the other edge. Otherwise y must be on a lower layer than x and z must be on a lower layer than y. So z is at least two layers below x and (x, z) is an edge in $\mathcal{M}_{t,r}$.

▶ **Lemma 9.** *There are two polynomial functions $g_1, g_2 \colon \mathbb{N} \to \mathbb{N}$, so that for every $n \in \mathbb{N}$ there is an ML[$\leftrightarrow$]-formula ψ_n, which separates $(\mathcal{M}_{t,r}, a_n)$ from $(\mathcal{M}_{t,r}, b_n)$, with $|\psi_n|_O = g_1(n) \geq n$ and $|\psi_n| = g_2(n) \geq n$.*

Proof. This is proven, by simply adapting the formula φ_n from Claim 8 to separate a_n and b_n in $\mathcal{M}_{t,r}$ instead of in $\mathcal{M}$. In $\mathcal{M}$ every edge goes down exactly one layer. But all edges added to R when constructing $R_{t,r}$ go down at least two layers or are self-loops. So we need a way to determine what layer a node belongs to. We look at the formula χ_i^+ defined by $\chi_0^+ := \top$ and $\chi_{i+1}^+ := \Diamond(q_j \land \chi_i^+)$ with $i \in \mathbb{N}$ and $j = i \mod 3$. This formula states, that there is a path of length i that starts at the current node and on this path the labels q_0, q_1 and q_2 appear in an alternating pattern. Such a path cannot use any self-loops, so every edge has to move down at least one layer. Then only nodes of layer i or above satisfy this formula. Similarly only nodes of layer i or below satisfy $\chi_i^- := \neg\chi_{i+1}^+$. Now we can use the formula $\chi_i := \chi_i^+ \land \chi_i^-$ in order to check whether or not a node belongs to layer i.

Then for any node w of layer $i \geq 1$ $(\mathcal{M}, w) \models \Diamond(\chi_{i-1} \land \varphi)$ holds if, and only if, w has a successor exactly one layer below that satisfies φ. Similarly $(\mathcal{M}, w) \models \Box(\neg\chi_{i-1} \lor \varphi)$ holds if, and only if, every successor of w that is exactly one layer below the current layer satisfies φ. So we can add conditions like these to φ_n in order to only consider edges that go down exactly one layer, which are all edges from R, while ignoring all edges from $R_{t,r} \setminus R$. We define ψ_n inductively with $\psi_0 := p_0$ and $\psi_i := \Diamond(\chi_{3i-1} \land \Box(\neg\chi_{3i-2} \lor ((\Diamond(p_1 \land \chi_{3(i-1)}) \leftrightarrow (\Diamond(\psi_{i-1} \land \chi_{3(i-1)})))))$. Then for all nodes w of layer $3n$, $(\mathcal{M}_{t,r}, w) \models \psi_n$ holds if, and only if $(\mathcal{M}, w) \models \varphi_n$. So by Claim 8, ψ_n is a separating formula for $(\mathcal{M}_{t,r}, a_n)$ and $(\mathcal{M}_{t,r}, b_n)$. For a rough estimation of the size of φ_n we can assert that the number of times a sub-formula χ_i occurs in φ_n is linear in n. Also the size of such a formula is linear in i, which is bounded from above by $3n - 1$. So the overall size of φ_n is quadratic in n, which is polynomial. A more in-depth examination leads to the findings $|\varphi_n|_O = 24 \cdot n^2 + 10 \cdot n =: g_1(n)$ and $|\varphi_n| = 36 \cdot n^2 + 17 \cdot n + 1 =: g_2(n)$. ◀

▶ **Lemma 10.** *There is an exponential function $h : \mathbb{N} \to \mathbb{N}$, so that for every $n \in \mathbb{N}$ and for every ML-formula χ_n which separates $(\mathcal{M}_{t,r}, a_n)$ from $(\mathcal{M}_{t,r}, b_n)$, we have $|\chi_n|_O > h(n)$.*

Proof. First we define $h(n) = 2^n - 2$. It is easy to see that $|\chi_0|_O > -1 = h(0)$ holds. So from now on we may assume $n \geq 1$. Then we show that there is no separating ML-formula for $\{(\mathcal{M}_{t,r}, a_n)\}$ and $\{(\mathcal{M}_{t,r}, b_n)\}$ with at most $h(n)$ modal operators. This is done by playing the $\mathcal{M}_{t,r}$-ON game on $\{a_n\}$ and $\{b_n\}$ with parameter $h(n)$ and showing that D has a winning strategy f_D for these games. In order to do so we define the following sets of positions of the game.

- $W_D^0 := \{(\{A, B\}, k) \mid A, B \subseteq W, A \cap B \neq \emptyset, k \in \mathbb{N}\}$
- $W_D^1 := \{(\{\{a_i\}, \{b_i\}\}, k) \mid i \geq 1, k \leq 2^i - 2\}$
- $W_D^2 := \{(\{\{2_i\}, \{1_i, 3_i\}\}, k) \mid i \geq 2, k \leq 2^i - 3\}$
- $W_D^3 := \{(\{\{2_i, 4_i\}, \{1_i, 5_i\}\}, k) \mid i \geq 2, k \leq 2^i - 4\}$
- $W_D^4 := \{(\{\{2_i, 7_i\}, \{3_i, 6_i\}\}, k) \mid i \geq 2, k \leq 2^i - 4\}$
- $W_D^5 := \{(\{\{4_i, 7_i\}, \{5_i, 6_i\}\}, k) \mid i \geq 2, k \leq 2^i - 4\}$
- $W_D^6 := \{(\{\{1_i\}, \{2_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1}\}$
- $W_D^7 := \{(\{\{2_i\}, \{3_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1}\}$
- $W_D^8 := \{(\{\{4_i\}, \{5_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1} - 1\}$
- $W_D^9 := \{(\{\{6_i\}, \{7_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1} - 1\}$

Then $W_D = \bigcup_{0 \leq j \leq 9} W_D^j$ is defined as the union of these sets. For $n \geq 1$ it is easy to see, that $(\{\{a_n\}, \{b_n\}\}, 2^n - 2) \in W_D^1$ and thus $(\{\{a_n\}, \{b_n\}\}, 2^n - 2) \in W_D$ holds. It is also easy to see that no W_D^j contains a position $(\{A, B\}, k)$ with $A = B = \emptyset$. So W_D does not contain any immediate winning position for S. Player D can also stop player S from leaving W_D. More precisely if S makes a move from any position in W_D, then D can answer in such a way that the follow-up position is also in W_D. We can prove this fact by going through all positions within W_D and checking every possible move that S can make. In this shortened proof we will showcase this for the set W_D^2. The remaining cases are given as part of the appendix.

So let $(\{\{2_i\}, \{1_i, 3_i\}\}, k)$ with $i \geq 2$ and $k \leq 2^i - 3$ be some position from W_D^2. In this position player S can play an operator-move, where he picks some successor of 2_i. If he picks a node that is also a successor of 1_i or 3_i, then D can answer by picking the same node. In this case the next position $(\{A, B\}, k - 1)$ satisfies $A \cap B \neq \emptyset$. So it belongs to W_D^0. Then S cannot leave W_D by picking a shared successor, meaning a node that is a successor of both sets of the position. So for $\mathcal{M}_{t,r}$ in particular we can ignore all moves where S picks a node two or more layers below any of the current nodes. Every successor of 2_i besides 2_i itself is also a successor of 1_i or 3_i. If S picks 2_i as its own successor then D can answer by also using the self-loops of 1_i and 3_i. So this move leads to the new position $(\{\{2_i\}, \{1_i, 3_i\}\}, k - 1)$, which is also a part of W_D^2. Alternatively S can also play an operator-move where he picks successors of 1_i and 3_i. Once again if S picks a node that can also be picked by D, then the next position is a member of W_D^0. The only other options of S are to pick 1_i or 6_i as a successor of 1_i and 3_i or 5_i as a successor of 3_i. If he uses the self-loop both times, D answers with her own self-loop and the follow-up position belongs to W_D^2. If S choose 1_i and 5_i then D answers $\{2_i, 4_i\}$. This moves the play to $(\{\{2_i, 4_i\}, \{1_i, 5_i\}\}, k - 1)$, a position from W_D^3. On the other hand if S picked 6_i and 3_i, then D answers with $\{2_i, 7_i\}$, which leads to $(\{\{2_i, 7_i\}, \{3_i, 6_i\}\}, k - 1)$, a member of W_D^4. Lastly if he choose 5_i and 6_i, D picks $\{4_i, 7_i\}$. The new position $(\{\{4_i, 7_i\}, \{5_i, 6_i\}\}, k - 1)$ belongs to W_D^5. So S cannot leave W_D from W_D^2 by playing an operator-move.

At $(\{\{2_i\}, \{1_i, 3_i\}\}, k)$ S can also play a split-move. In this case he has to split $\{1_i, 3_i\}$ into $\{1_i\}$ and $\{3_i\}$. He also splits the parameter k into k_1 and k_2. One of these two new parameters is at most half of the original parameter k. So D can pick j in such a way, that $k_j \leq 2^{i-1}$. Then she answers with $\{2_i\}$. The new position of the play is either $(\{\{1_i\}, \{2_i\}\}, k_j) \in W_D^6$ or $(\{\{2_i\}, \{3_i\}\}, k_j) \in W_D^7$. So we have shown that S cannot leave W_D by making a move in a position from W_D^2.

The same holds for all other positions from W_D. Since S cannot leave W_D and W_D does not contain any immediate winning position, S cannot win from any of these positions. Then all positions in W_D are winning positions for D. This includes $(\{\{a_n\}, \{b_n\}\}, 2^n - 2)$. So there cannot be a separating formula for $\{a_n\}$ and $\{b_n\}$ with $2^n - 2$ or less modal operators. ◀

▶ **Theorem 11.** *The logic ML[$\leftrightarrow$] is exponentially more operator-succinct and exponentially more succinct than ML on the class of Kripke structures with a transitive and reflexive accessibility relation.*

Proof. Let $n \in \mathbb{N}$ be some natural number. Then ψ_n is the ML[$\leftrightarrow$]-formula given in Lemma 9. We know $|\psi_n|_O = g_1(n) \geq n$ and $|\psi_n| = g_2(n) \geq n$ for some polynomial functions g_1 and g_2. We also know that ψ_n separates the two transitive and reflexive Kripke structures $(\mathcal{M}_{t,r}, a_n)$ and $(\mathcal{M}_{t,r}, b_n)$ from one another. Let χ_n be some ML-formula equivalent to ψ_n on the class of Kripke structures with a transitive and reflexive accessibility relation. Then χ_n has to be a separating formula for $(\mathcal{M}_{t,r}, a_n)$ and $(\mathcal{M}_{t,r}, b_n)$ as well. So, according to Lemma 10, $|\chi_n|_O \geq$

$h(n)$ for some exponential function h. This implies $|\chi_n| \geq h(n)$ as well. We define $f_1 \colon \mathbb{N} \to \mathbb{N}$ as $f_1(n) = h\left(\lfloor (g_1)^{-1}(n) \rfloor\right)$. Then $|\chi_n|_O \geq h(n) \geq h\left(\lfloor (g_1)^{-1}(|\psi_n|_O) \rfloor\right) = f_1\left(|\psi_n|_O\right)$ holds. So ML[$\leftrightarrow$] is f_1-times more operator-succinct than ML and thus exponentially more operator-succinct. We also define $f_2 \colon \mathbb{N} \to \mathbb{N}$ as $f_2(n) = h\left(\lfloor (g_2)^{-1}(n) \rfloor\right)$. We know $|\chi_n| \geq f_2\left(|\psi_n|\right)$ holds. This implies that ML[$\leftrightarrow$] is f_2-times more succinct than ML, so it is exponentially more succinct as well. ◀

From this theorem it is easy to infer, that the succinctness gap also exists on all classes of structures, which are supersets of the class of Kripke structures with a transitive and reflexive accessibility relation. We say an accessibility relation R is *serial*, if it satisfies the condition $\forall x \exists y : R(x, y)$. It is called *weakly dense*, if it satisfies $\forall x \forall y : (R(x, y) \to \exists z : (R(x, z) \wedge R(z, y))$. Both of these conditions are true for every reflexive relation.

▶ **Corollary 12.** *The logic ML[$\leftrightarrow$] is exponentially more succinct than ML on the following classes of pointed Kripke structures:*

1. *The class of all pointed Kripke structures.*
2. *The class of all pointed Kripke structures with a transitive accessibility relation.*
3. *The class of all pointed Kripke structures with a reflexive accessibility relation.*
4. *The class of all pointed Kripke structures with a serial accessibility relation.*
5. *The class of all pointed Kripke structures with a weakly dense accessibility relation.*

4.2 Symmetrical and reflexive graphs

The graph $\mathcal{M}_{s,r}$ is also constructed as an extension of $\mathcal{M} = (W, R, V)$. We do not change the nodes or their labels. We only add edges to $\mathcal{M}$. For every edge $(x, y) \in R$ we add its inverse (y, x) to our new graph. For nodes x and y, that belong to the same level, we also add the edges (x, y) and (y, x). This explicitly includes all possible self-loops. So formally $R_{s,r} = R \cup \{(y, x) \mid (x, y) \in R\} \cup \{(x, y) \mid \exists i \in \mathbb{N} \colon x \text{ and } y \text{ belong to layer } i\}$. It is easy to see that $\mathcal{M}_{s,r}$ is both symmetrical and reflexive.

▶ **Lemma 13.** *There are two polynomial functions $g_1, g_2 \colon \mathbb{N} \to \mathbb{N}$, so that for every $n \in \mathbb{N}$ there is an ML[$\leftrightarrow$]-formula ψ_n, which separates $(\mathcal{M}_{s,r}, a_n)$ from $(\mathcal{M}_{s,r}, b_n)$, with $|\psi_n|_O = g_1(n) \geq n$ and $|\psi_n| = g_2(n) \geq n$.*

Proof. Once again this is proven by adapting φ_n from Claim 8. All edges in $\mathcal{M}$ go down exactly one layer. But all of the edges added to R during the construction of $R_{s,r}$ either go up one layer or stay on the same layer. Since we used the labels q_0, q_1 and q_2 in an alternating pattern, the current layer, the layer one below and the layer one above are all coloured differently. This means we can use these labels to make sure that only the original edges from R are considered. We define ψ_n via $\psi_0 := p_0$ and $\psi_i := \Diamond(q_1 \wedge \Box(\neg q_2 \vee ((\Diamond(p_1 \wedge q_0) \leftrightarrow (\Diamond(\psi_{i-1} \wedge q_0)))))$. For all nodes w of layer $3n$ we know that $(\mathcal{M}_{s,r}, w) \models \psi_n$ holds if, and only if, $(\mathcal{M}, w) \models \varphi_n$. So, by Claim 8, ψ_n is a separating formula for $(\mathcal{M}_{s,r}, a_n)$ and $(\mathcal{M}_{s,r}, b_n)$ with $|\psi_n|_O = |\psi_{n-1}|_O + 4$ and $|\psi_n| = |\psi_{n-1}| + 15$. So $g_1(n) = 4 \cdot n$ and $g_2(n) = 15 \cdot n + 1$ fulfil all conditions of the lemma. ◀

▶ **Lemma 14.** *There is an exponential function $h : \mathbb{N} \to \mathbb{N}$, so that for every $n \in \mathbb{N}$ and for every ML-formula χ_n, which separates $\{(\mathcal{M}_{s,r}, a_n)\}$ from $\{(\mathcal{M}_{s,r}, b_n)\}$, we have $|\chi_n|_O > h(n)$.*

Proof. This proof is very similar to that of Lemma 10. We also use $h(n) = 2^n - 2$, so $|\chi_0|_O > -1 = h(0)$ holds. Then for $n \geq 1$ we show that $(\{\{a_n\}, \{b_n\}\}, h(n))$ is a winning position for player D in the $\mathcal{M}_{s,r}$-ON game. For this purpose we define the following sets of positions of the $\mathcal{M}_{s,r}$-ON game.

- $W_D^0 := \{(\{A, B\}, k) \mid A, B \subseteq W, A \cap B \neq \emptyset, k \in \mathbb{N}\}$
- $W_D^1 := \{(\{\{a_i\}, \{b_i\}\}, k) \mid i \geq 1, k \leq 2^i - 2\}$
- $W_D^2 := \{(\{\{2_i\}, \{1_i, 3_i\}\}, k) \mid i \geq 2, k \leq 2^i - 3\}$
- $W_D^3 := \{(\{\{4_i, 7_i\}, \{5_i, 6_i\}\}, k) \mid i \geq 2, k \leq 2^i - 4\}$
- $W_D^4 := \{(\{\{1_i\}, \{2_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1}\}$
- $W_D^5 := \{(\{\{1_i\}, \{3_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1}\}$
- $W_D^6 := \{(\{\{2_i\}, \{3_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1}\}$
- $W_D^7 := \{(\{\{4_i\}, \{5_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1} - 1\}$
- $W_D^8 := \{(\{\{6_i\}, \{7_i\}\}, k) \mid i \geq 2, k \leq 2^{i-1} - 1\}$

Then $W_D := \bigcup_{0 \leq j \leq 8} W_D^j$ is defined as the union of these sets. For $n \geq 1$ it is easy to see that $(\{\{a_n\}, \{b_n\}\}, 2^n - 2) \in W_D^1$ and thus $(\{\{a_n\}, \{b_n\}\}, 2^n - 2) \in W_D$ holds. It is also easy to see, that no W_D^j contains a position $(\{A, B\}, k)$ with $A = B = \emptyset$, so W_D does not contain any immediate winning position for S. Next we show that for every move of S from a position of W_D player D can defend in such a way that the next position is in W_D as well. In this shortened version of our proof we will only give a proper explanation for W_D^2. The description of the other cases is once again moved to the appendix.

So we have to look at positions $(\{\{2_i\}, \{1_i, 3_i\}\}, k)$ with $i \geq 2$ and $k \leq 2^i - 3$. Once again if S picks a shared successor, D can move the game to a position from W_D^0. For $\mathcal{M}_{s,r}$ this happens every time S picks a node of the same layer as any node in the other set. In particular if S uses the self-loop on any nodes. Since every successor of 2_i is also a successor of 1_i or 3_i, player S cannot leave W_D by playing an operator-move on $\{2_i\}$. Similarly when playing on $\{1_i, 3_i\}$ we only have to look at S picking b_i or 6_i as a successor of 1_i, as well as b_i or 5_i as a successor of 3_i. This is because every other successor is also a successor of 2_i. If S picked b_i at all, then D answers with $\{a_i\}$. If he choose 5_i or 6_i in addition to b_i, then these nodes will be removed when tidying up, because they are labelled differently from a_i and b_i. So in this case the game continues with $(\{\{a_i\}, \{b_i\}\}, k - 1)$, a position from W_D^1. On the other hand, if S picked both 5_i and 6_i as successors, D answers with $\{4_i, 7_i\}$. Then the follow-up position is $(\{\{4_i, 7_i\}, \{5_i, 6_i\}\}, k - 1)$ from W_D^3. So S cannot leave W_D by playing an operator-move at W_D^2. Lastly we look at what happens when D plays a split-move. He has to split $\{1_i, 3_i\}$ into $\{1_i\}$ and $\{3_i\}$ and k into k_1 and k_2. Then D can answer in such a way that $k_j \leq 2^{i-1}$ and the follow-up position is either $(\{\{1_i\}, \{2_i\}\}, k_j) \in W_D^4$ or $(\{\{2_i\}, \{3_i\}\}, k_j) \in W_D^6$. With this we have shown that S cannot leave W_D in one move from W_D^2.

The same goes for all other positions from W_D. So D can stop S from leaving W_D and there is no immediate winning position for S in W_D. Then all position in W_D, including $(\{\{a_n\}, \{b_n\}\}, 2^n - 2)$, have to be winning positions for player D. This is why there cannot be a separating formula for $\{a_n\}$ and $\{b_n\}$ with $2^n - 2$ or less modal operators. ◀

▶ **Theorem 15.** *The logic ML[$\leftrightarrow$] is exponentially more operator-succinct and exponentially more succinct than ML on the class of Kripke structures with a symmetrical and reflexive accessibility relation.*

Proof. The proof of this theorem can be done analogous to the proof of Theorem 11. Simply replace all mentions of transitivity with reflexivity and use Lemma 13 and Lemma 14 instead of Lemma 9 and Lemma 10 respectively. ◀

Since the succinctness gap exists when looking only at symmetrical and reflexive graphs, it has to exist for any class of graphs that contains all symmetrical and reflexive graphs.

▶ **Corollary 16.** *The logic ML[$\leftrightarrow$] is exponentially more succinct than ML on the class of Kripke structures with a symmetrical accessibility relation.*

4.3 Transitive and symmetrical graphs

So far in this paper we have shown that adding bi-implications allows us to write more succinct formulas. Even when only looking at special classes of Kripke structures, like those with a transitive accessibility relation and those with a symmetrical one. In this section we are going to show that the same does not hold if we combine these two conditions. More precisely we show that there is no exponential succinctness gap between ML and ML[$\leftrightarrow$] on the class of Kripke structures with a transitive and symmetrical accessibility relation.

We start with a few remarks about our notation in this section. Let $\mathcal{S}_{S5}$ be the class of all Kripke structures with a transitive, symmetrical and reflexive accessibility relation. We write $\varphi \equiv_{T,S} \psi$ if φ and ψ are equivalent on all pointed Kripke structures with a transitive and symmetrical accessibility relation. Similarly we use $\equiv_{\mathcal{S}_{S5}}$ when talking about structures from $\mathcal{S}_{S5}$ and $\equiv_E$ for graphs with an empty accessibility relation.

All connected, transitive and symmetrical graphs, that are not reflexive as well, contain a single node and no edges. So $\varphi \equiv_{T,S} \psi$ holds if, and only if, both $\varphi \equiv_{\mathcal{S}_{S5}} \psi$ and $\varphi \equiv_E \psi$ hold. This fact offers us a way to find formulas that are equivalent on transitive and symmetrical structures, which we will use to prove the following Theorem.

▶ **Theorem 17.** *For every ML[$\leftrightarrow$]-formula φ there is an ML-formula ψ with $\varphi \equiv_{T,S} \psi$ and the size of ψ is polynomial in the size of φ.*

Proof. In [2] it is shown, that ML[$\leftrightarrow$] has polynomial translations with regards to $\mathcal{S}_{S5}$ in ML. So for every ML[$\leftrightarrow$]-formula φ there is an ML-formula ψ_1 with $\varphi \equiv_{\mathcal{S}_{S5}} \psi_1$ and the size of ψ_1 is polynomial in the size of φ. We now look at the graph with only one node and no edges. We know $\Diamond\chi \equiv_E \bot$ as well as $\Box\chi \equiv_E \top$ for all formulas χ. We can use this to replace all operators in φ, obtaining a operator free formula ψ' with $\psi' \equiv_E \varphi$. Since this formula does not contain any operators, we can treat it as a formula of propositional logic. So as described by Pratt in [10], we can use Spira's construction from [11] in order to construct an ML-formula ψ_2 with $\psi_2 \equiv \psi' \equiv_E \varphi$. The size of ψ_2 is polynomial in the size of ψ', which in turn is bounded from above by the size of φ. Now we just need to combine these formulas to obtain ψ. In order to do so we use the formula $\Diamond\top$, which is true on all graphs from $\mathcal{S}_{S5}$ and false for all edgeless graphs, as well as its negation $\Box\bot$. We construct the wanted ML-formula as $\psi := (\psi_1 \wedge \Diamond\top) \vee (\psi_2 \wedge \Box\bot)$. This formula fulfils the following:

$$\psi \equiv_{\mathcal{S}_{S5}} (\psi_1 \wedge \Diamond\top) \vee (\psi_2 \wedge \Box\bot) \equiv_{\mathcal{S}_{S5}} (\psi_1 \wedge \top) \vee (\psi_2 \wedge \bot) \equiv_{\mathcal{S}_{S5}} \psi_1 \equiv_{\mathcal{S}_{S5}} \varphi$$

$$\psi \equiv_E (\psi_1 \wedge \Diamond\top) \vee (\psi_2 \wedge \Box\bot) \equiv_E (\psi_1 \wedge \bot) \vee (\psi_2 \wedge \top) \equiv_E \psi_2 \equiv_E \varphi$$

Which also implies $\psi \equiv_{T,S} \varphi$. The size of this formula is $|\psi_1| + |\psi_2| + 7$, so the sum of two polynomials over $|\varphi|$ and a constant, which is also polynomial in the size of φ. ◀

5 Conclusion

The main result of this paper is the development of a new model comparison game, the $\mathcal{M}$-Operator-Number game. This game can be used to prove lower bounds for the number of modal operators necessary to express certain properties in modal logic. An interesting follow-up question to this might be: For which logics and which parameters can a similar game be designed? If there is a version of the Hella-Vilander game for some logic, it should be possible to adjust that game to only count all occurrences of one specific type of operator. This should work for every type of operator of that logic. The more interesting part of that question is whether or not some process similar to our idea of 'tidying up' is possible. A positive answer is given for FO^l in [14]. However additional research is necessary to achieve a better understanding of this topic.

We also give a more complete overview of the succinctness of modal logic with bi-implication on different classes of Kripke structures. For every possible combination of the properties reflexive, transitive and symmetrical, we have either proven or disproven the existence of an exponential succinctness gap between ML and ML[$\leftrightarrow$] on the corresponding class of structures.

A possible future task could be to look into even more classes of Kripke structures, for example those with a function or a partial function as accessibility relation. Although these examples might require a quite different proof from the ones presented in this paper. Instead we could also attempt to adjust our proofs for a different logic. Our current goal is to do this for the logic CTL. This logic can be seen as an extension of ML, where more complex operators are added. Our hope is that we can show that bi-implications can be used to write exponentially shorter formulas in CTL as well.

References

1 Micah Adler and Neil Immerman. An n! lower bound on formula size. *ACM Transactions on Computational Logic*, 4(3):296–314, 2003. doi:10.1145/772062.772064.

2 Christoph Berkholz, Dietrich Kuske, and Christian Schwarz. Boolean basis, formula size, and number of modal operators. *Logical Methods in Computer Science*, Volume 21, Issue 3, July 2025. doi:10.46298/lmcs-21(3:10)2025.

3 Marco Carmosino, Ronald Fagin, Neil Immerman, Phokion Kolaitis, Jonathan Lenchner, and Rik Sengupta. Multi-structural games and beyond. *Logical Methods in Computer Scienc*, 20:27:1–27:40, 2024. doi:10.48550/arXiv.2301.13329.

4 Andrzej Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fundamenta Mathematicae*, 49:129–141, 1960. URL: https://eudml.org/doc/213582.

5 Ronald Fagin, Jonathan Lenchner, Kenneth W. Regan, and Nikhil Vyas. Multi-structural games and number of quantifiers. *Logical Methods in Computer Science*, 21(1), 2025. doi:10.46298/lmcs-21(1:10)2025.

6 Roland Fraïssé. On some classifications of relationship systems. *Publications scientifiques de l'Université d'Alger, Série A, Sciences mathématiques*, 1954.

7 Tim French, Wiebe van der Hoek, Petar Iliev, and Barteld Kooi. Succinctness of epistemic languages. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence (IJCAI-11)*, pages 881–886, 2011. doi:10.5591/978-1-57735-516-8/IJCAI11-153.

8 Martin Grohe and Nicole Schweikardt. The succinctness of first-order logic on linear orders. *Logical Methods in Computer Science*, 1(1), 2005. doi:10.2168/LMCS-1(1:6)2005.

9 Lauri T. Hella and Miikka S. Vilander. Formula size games for modal logic and μ-calculus. *Journal of Logic and Computation*, 29(8):1311–1344, 2019. doi:10.1093/logcom/exz025.

10 Vaughan R. Pratt. The effect of basis on size of boolean expressions. In *Proceedings - Annual IEEE Symposium on Foundations of Computer Science, FOCS*, pages 119–121, October 1975. doi:10.1109/SFCS.1975.29.

11 Philip Spira. On time-hardware complexity tradeoffs for boolean functions. In *Proceedings of the 4th Hawaii Symposium on System Sciences, 1971*, pages 525–527, 1971.

12 Wiebe van der Hoek, Petar Iliev, and Barteld Kooi. On the relative succinctness of two extensions by definitions of multimodal logic. *Lecture Notes in Computer Science*, 7318 LNCS:323–333, 2012. doi:10.1007/978-3-642-30870-3_33.

13 Hans van Ditmarsch, Jie Fan, Wiebe van der Hoek, and Petar Iliev. Some exponential lower bounds on formula-size in modal logic. In *Advances in Modal Logic*, volume 10, pages 139–157, August 2014. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-84924671231&partnerID=40&md5=3f43caf649380733487e57e7338bd34d.

14 Harry Vinall-Smeeth. From quantifier depth to quantifier number: Separating structures with k variables. In *Proceedings - Symposium on Logic in Computer Science*, 2024. doi:10.1145/3661814.3662125.

A Appendix

A.1 Proof of Lemma 2

▷ Claim of Lemma 2. Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph and $A, B, A', B' \subseteq W$ be sets of nodes with $t(\{A, B\}) = \{A', B'\}$. If there is a separating formula φ' for $(\mathcal{M}, A')$ and $(\mathcal{M}, B')$, then there is also a separating formula φ for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$ with $|\varphi|_O = |\varphi'|_O$.

Proof. Without loss of generality we assume $A' \subseteq A$ and $B' \subseteq B$. In order to prove this lemma, we look at the sets $A'' = A \setminus A'$ and $B'' = B \setminus B'$. We know that no node in A'' has the same colour as any node in B and no node in B'' has the same color as any node in A. It is easy to see that two pointed Kripke structures based on differently coloured nodes can be separated by a literal. In this case let $\psi_{a,b}$ be a literal with $(\mathcal{M}, a) \models \psi_{a,b}$ and $(\mathcal{M}, b) \not\models \psi_{a,b}$. Then for every $a \in A''$ we define $\Psi_a := \{\psi_{a,b} \mid b \in B\}$. All nodes in W agree on all labels that are not used in $\mathcal{M}$. Then every formula in Ψ_a must be either an atomic formula from $V(W)$ or the negation of such an atomic formula. So even though B might be infinite, Ψ_a will always be finite, since $V(W)$ is finite. So we can construct the formula $\psi_a := \bigwedge_{\psi_{a,b} \in \Psi_a} \psi_{a,b}$, which separates $(\mathcal{M}, a)$ from $(\mathcal{M}, B)$. Next we look at the set $\Psi := \{\psi_a \mid a \in A''\}$ and the formula $\psi := \bigvee_{\psi_a \in \Psi} \psi_a$. This formula ψ holds in $(\mathcal{M}, A'')$, while its negation holds in $(\mathcal{M}, B)$.

In a similar fashion we define $X_b := \{\psi_{a,b} \mid a \in A\}$ and $\chi_b := \bigvee_{\psi_{a,b} \in X_b} \psi_{a,b}$ for every $b \in B''$. Next we construct $X := \{\chi_b \mid b \in B''\}$ and $\chi := \bigwedge_{\chi_b \in X} \chi_b$. This formula holds in $(\mathcal{M}, A)$ and its negation holds in $(\mathcal{M}, B'')$. With this we can finally define the wanted formula φ. Without loss of generality we assume $(\mathcal{M}, A') \models \varphi'$. Then $\varphi = (\varphi' \wedge \chi) \vee \psi$ is a separating formula for $(\mathcal{M}, A)$ and $(\mathcal{M}, B)$. Since neither ψ nor χ contain any operators, $|\varphi|_O = |\varphi'|_O$ holds as well. ◁

A.2 Proof of Lemma 3

▷ Claim of Lemma 3. Let $\mathcal{M} = (W, R, V)$ be a finitely labelled graph, $A, B \subseteq W$ be sets of nodes that form a tidy pair and let φ be an ML-formula without any modal operators. Then $(\mathcal{M}, A) \models \varphi$ if, and only if, $(\mathcal{M}, B) \models \varphi$.

Proof. If $A = B = \emptyset$, then the Lemma is trivially true. Otherwise let $(\mathcal{M}, A) \models \varphi$. For every $b \in B$ there is a $a \in A$ with $V(a) = V(b)$. Then a and b agree on all atomic formulas and thus on all operator-free ML-formulas, so especially on φ. This means $(\mathcal{M}, b) \models \varphi$ for every $b \in B$ and thus $(\mathcal{M}, B) \models \varphi$. The converse implication follows from $\{A, B\} = \{B, A\}$. ◁

A.3 Proof of Lemma 7

During the proof of Lemma 7 we sketch a method that removes all $\leftrightarrow$ from an ML[$\leftrightarrow$]-formula, while guaranteeing an exponential increase at most in the number of modal operators. Here we give a proper formal definition of this construction. We also prove its correctness.

Proof. We recursively define a function $f\colon \text{ML}[\leftrightarrow] \to \text{ML}[\leftrightarrow]$. The base case consist of all formulas φ without any operators and in this case $f(\varphi) := \varphi$. For any other formula φ there must be a sub-formula χ so that $\Diamond\chi$ appears in φ outside of the scope of any other operator. Then φ can be written as $\psi(\Diamond\chi)$ for some formula ψ. We define $f(\varphi)$ as $f(\varphi) := (f(\psi(\top)) \wedge \Diamond f(\chi)) \vee (f(\psi(\bot)) \wedge \neg\Diamond f(\chi))$.

Next we want to show that $f(\varphi)$ fulfils the following three conditions for every ML[$\leftrightarrow$] formula φ: Firstly $f(\varphi) \equiv \varphi$, secondly no operator in $f(\varphi)$ appears in the scope of any bi-implication and thirdly $|f(\varphi)|_O \leq 2^{|\varphi|_O} \cdot |\varphi|_O$. We do this via induction on the number of operators in φ. As our base case we look at all ML[$\leftrightarrow$] formulas φ with no modal operators. Then we know $f(\varphi) = \varphi$, so the first condition is obviously fulfilled. Because $f(\varphi)$ does not contain any operators, the second condition is also trivially true and because of $|f(\varphi)|_O = |\varphi|_O = 0$, the third condition is also fulfilled.

For the induction step we look at all formulas φ with $|\varphi|_O = i$ for some $i \geq 1$. In this case let ψ and χ be the sub-formulas with $\varphi = \psi(\Diamond\chi)$, that were used when constructing $f(\varphi)$. Because of $|\psi|_O < |\varphi|_O$ and $|\chi|_O < |\varphi|_O$ we may assume as our induction hypothesis that both $f(\psi)$ and $f(\chi)$ fulfil all three conditions. Then we know $\varphi \equiv (\psi(\top) \wedge \Diamond\chi) \vee (\psi(\bot) \wedge \neg\Diamond\chi)$. Because of $f(\psi) \equiv \psi$ and $f(\chi) \equiv \chi$, this is equivalent to $(f(\psi(\top)) \wedge \Diamond f(\chi)) \vee (f(\psi(\bot)) \wedge \neg\Diamond f(\chi)) = f(\varphi)$. No operator in $f(\psi)$ or in $f(\chi)$ appears in the scope of any bi-implication. So the same must hold for $f(\varphi)$, since no new bi-implication is added during its construction. For the number of operators we know $|\varphi|_O = |\psi|_O + |\chi|_O + 1$ and $|f(\varphi)|_O = 2 \cdot |f(\psi)|_O + 2 \cdot |f(\chi)|_O + 2$. So by our induction hypothesis we know that $|f(\varphi)|_O \leq 2^{|\psi|_O+1} \cdot |\psi|_O + 2^{|\chi|_O+1} \cdot |\chi|_O + 2$. The value of the right hand side of this inequality becomes largest if all but one operators of φ belong to the same sub-formula. So it is bounded from above by $2^{|\varphi|_O} \cdot (|\varphi|_O - 1) + 2$. For $|\varphi|_O \geq 1$ this is in turn bounded from above by $2^{|\varphi|_O} \cdot |\varphi|_O$. So the third condition is fulfilled as well.

In total we have shown that for every ML[$\leftrightarrow$]-formula φ there exists an equivalent ML[$\leftrightarrow$]-formula $f(\varphi)$ where no operator appears in the scope of a bi-implication and the number of operators in φ' is at most exponential in the number of operators in φ. Lastly we can replace every sub-formula $\psi \leftrightarrow \chi$ in $f(\varphi)$ with $(\psi \wedge \chi) \vee (\neg\psi \wedge \neg\chi)$, in order to transform $f(\varphi)$ into an ML-formula, without increasing the number of operators any further. This construction proves, that the operator-succinctness gap between ML and ML[$\leftrightarrow$] is at most exponential. ◀

A.4 Proof of Lemma 10

During the proof of Lemma 10 we claimed that player D can ensure that a play of the game never leaves the set W_D. Here we want to give a complete proof of this fact.

Proof. We begin with some general facts. Since $\mathcal{M}_{t,r}$ is reflexive, player S can always play an operator-move, where he picks every node as its own successor. However D can answer by doing the same. So the follow up position consists of the same pair of sets as the current position only with the parameter reduced by one. So the new position belongs to the same set as the old position. Also S cannot play a split-move, as long as both sets of the current position contain only one element each.

We start with W_D^0. So let $(\{A, B\}, k) \in W_D^0$ be some position and let $w \in A \cap B$ be some node shared between A and B. For $k \geq 2$, S can play a split-move in this position. He picks $X \in \{A, B\}$ and then splits this set into X_1 and X_2. Then the shared node w has to be in one of these two sets, w.l.o.g. we can assume $w \in X_1$. Then D can ensure $w \in Y'$ for her answer and the game continues from the position $(t(\{X_1, Y'\}), k_1)$. The node w is also a shared node for X_1 and Y'. For this reason w cannot be removed by tidying up, because there will always be a node of the same colour in the respective other set, namely w itself. So in the position after the move, the two sets still have w as a shared node. Then this new position lies in W_D^0 as well and thus in W_D. On the other hand S might play an operator-move instead. In this case he picks $X \in \{A, B\}$ and then for every $x \in X$ he picks a successor $f(x)$. Because of $w \in Y$, we know $f(w) \in \Box Y$. So D can ensure $f(w) \in Y'$. Then $f(w)$ is a shared node for X' and Y' and thus it is not removed by tidying up. This means that the follow-up position $(t(\{X', Y'\}), k-1)$ is also a member of W_D^0 and of W_D. So D can ensure that the $\mathcal{M}_{s,r}$-ON game cannot move from a position inside of W_D^0 to a position outside of W_D.

During an operator-move, if S tries to pick some node x' as $f(x)$ with $x' \in \Box Y$, the D can ensure $x' \in Y'$. So x' is a shared node for the new position and thus this position is a part of W_D^0. Then S cannot leave W_D by playing like this. So from now on we may ignore all of his moves where he picks some shared successor. This is especially relevant, because in $\mathcal{M}_{t,r}$ all nodes that are two or more layers below the current node are successors of the current node. So every move where D picks some node that is two or more layers below any node from Y leads to a position from W_D^0. For the rest of this proof we may ignore every move of S with this property.

Next we look at W_D^1. Let $(\{\{a_i\}, \{b_i\}\}, k)$ be a position with $i \geq 1$ and $k \leq 2^i - 2$. For $i = 1$ we know $k = 0$. So S cannot play any moves, meaning he cannot leave W_D. So from now on we can assume $i > 1$. Then S has to play an operator-move in such a position. However every successor of b_i is either b_i itself or a successor of b_i as well. Also the only successor of a_i that is not a successor of b_i as well is 2_i. So S has to pick 2_i as a successor of a_i. To this D answers with $\{1_i, 3_i\}$ and the follow-up position $(\{\{2_i\}, \{1_i, 3_i\}\}, k-1)$ is a part of W_D^2.

We already gave a detailed explanation for W_D^2 during the main proof, so we will skip it here.

Now we look at W_D^3, which contains positions $(\{\{2_i, 4_i\}, \{1_i, 5_i\}\}, k)$ with $i \geq 2$ and $k \leq 2^i - 4$. Every successor of 4_i is also a successor of 1_i as well, including 4_i itself. So we can ignore all operator-moves, where S tries to pick successors for 2_i and 4_i. Similarly every proper successor of 5_i is also a successor of 2_i. This means we only need to look at S picking 5_i as a successor of itself and 6_i as successor of 1_i. Then D answers with $\{4_i, 7_i\}$ and the new position $(\{\{4_i, 7_i\}, \{5_i, 6_i\}\}, k-1)$ belongs to W_D^5. S can also play a split-move, where he either splits $\{2_i, 4_i\}$ into $\{2_i\}$ and $\{4_i\}$ or $\{1_i, 5_i\}$ into $\{1_i\}$ and $\{5_i\}$. Either way he also has to split k into k_1 and k_2. Because of $k \leq 2^i - 4$, D can always pick a $j \in \{1, 2\}$ so that $k_j \leq 2^{i-1} - 1$. The set D picks depends on the choice of X_j. She answers with $\{2_i\}$ to $\{1_i\}$

and the other way around. Then the follow-up position $(\{\{1_i\},\{2_i\}\},k_j)$ is a member of W_D^6. On the other hand D picks $\{5_i\}$ as an answer to $\{4_i\}$ and vice versa. In this case the play moves to $(\{\{4_i\},\{5_i\}\},k_j)$, which belongs to W_D^8. Either way S cannot leave W_D from W_D^3.

For the next set, W_D^4, we look at positions $(\{\{2_i,7_i\},\{3_i,6_i\}\},k)$ with $i \geq 2$ and $k \leq 2^i - 4$. The arguments for these positions are very similar to those used for W_D^3. Every successor of 7_i is also a successor of 3_i and every proper successor of 6_i is a successor of 2_i. So the only operator-move we need to look at is S picking 5_i as a successor of 3_i and 6_i as its own successor. To this D answers with $\{4_i,7_i\}$ The follow-up position is $(\{\{4,7\},\{5,6\}\},k-1)$ and belongs to W_D^5. S can also either split $\{2_i,7_i\}$ into $\{2_i\}$ and $\{7_i\}$ or he can split $\{3_i,6_i\}$ into $\{3_i\}$ and $\{6_i\}$. Either way D can always guarantee $k_j \leq 2^{i-1}-1$ for some $j \in \{1,2\}$. Then she can answer in such a way that the next position is either $(\{\{2_i\},\{3_i\}\},k_j)$ from W_D^7 or $(\{\{6_i\},\{7_i\}\},k_j)$ from W_D^9. So S cannot leave W_D from W_D^4 either.

W_D^5 contains all positions $(\{\{4_i,7_i\},\{5_i,6_i\}\},k)$ with $i \geq 2$ and $k \leq 2^i - 4$. Every proper successor of 4_i or 7_i is also a successor of 5_i or 6_i and vice versa. So we already know that every operator-move leads to a position from W_D. However S can also play a split move. He either splits $\{4_i,7_i\}$ into $\{4_i\}$ and $\{7_i\}$ or he splits $\{5_i,6_i\}$ into $\{5_i\}$ and $\{6_i\}$. He also splits the parameter k into k_1 and k_2. Then D can always pick $j \in \{1,2\}$ in such a way that $k_j \leq 2^{i-1}-1$. She can also choose Y' in such a way that the follow-up position is either $(\{\{4_i\},\{5_i\}\},k_j)$, which belongs to W_D^8, or $(\{\{6_i\},\{7_i\}\},k_j)$, a member of W_D^9.

The next set we look at is W_D^6, which contains positions $(\{\{1_i\},\{2_i\}\},k)$ with $i \geq 2$ and $k \leq 2^{i-1}$. Here S has to play an operator-move. Since we can ignore all shared successors, the only moves we need to look at are picking 6_i as a successor of 1_i and picking 7_i as a successor of 2_i. In any case D answers with the respective other node. Then the new position is $(\{\{6_i\},\{7_i\}\},k-1) \in W_D^9$.

In positions $(\{\{2_i\},\{3_i\}\},k)$ from W_D^7 S has to play an operator-move as well. The only moves we need to check are S picking 4_i as a successor of 2_i or 5_i as a successor of 3_i. In either case D can guarantee $(\{\{4_i\},\{5_i\}\},k-1)$, a position from W_D^8, as the follow-up position.

The set W_D^8 contains positions $(\{\{4_i\},\{5_i\}\},k)$ with $i \geq 2$ and $k \leq 2^{i-1}-1$. In such a position S has to play an operator-move. Aside from picking shared successors or using only self-loops, S can only pick a_{i-1} as a successor of $4-i$ or b_{i-1} as a successor of 5_i. Then D answers with whatever node S did not pick and the next position is $(\{\{a_{i-1}\},\{b_{i-1}\}\},k-1)$, a member of W_D^2.

Lastly we look at the set W_D^9, so positions $(\{\{6_i\},\{7_i\}\},k)$ with $i \geq 2$ and $k \leq 2^{i-1}-1$. We only need to check what happens when S picks either a_{i-1} as a successor of 6_i or b_{i-1} as a successor of 7_i. Independent of his choice, D can guarantee $(\{\{a_{i-1}\},\{b_{i-1}\}\},k-1) \in D_W^1$ as the new position. ◀

A.5 Proof of Lemma 14

During the proof of Lemma 14 we stated that player D can ensure that it is impossible to leave the set W_D by playing a move of the $\mathcal{M}_{s,r}$-ON game. Here we want to give a complete proof of this fact.

Proof. During the proof of Lemma 10 we showed that in $\mathcal{M}_{t,r}$ player S cannot leave W_D^0. The argumentation we used was completely independent from the actual edges of the graph. So we can use exactly the same argumentation here to show that S cannot leave W_D^0 in $\mathcal{M}_{s,r}$ either. This also means that S cannot leave W_D by picking some shared successor. For $\mathcal{M}_{s,r}$ in particular this includes all moves, where he tries to pick a node of the current layer.

The next set we want to look at is W_D^1. So let $(\{\{a_i\}, \{b_i\}\}, k)$ be a position with $i \geq 1$ and $k \leq 2^i - 2$. For $i = 1$ we know $k = 0$. As already mentioned S cannot make another move for $k = 0$, meaning he cannot leave W_D. For $i \geq 2$ he cannot play a split-move, since both sets have only one element each. So he has to play an operator-move instead. Recall that the nodes a_i and b_i belong segment $i + 1$ as well as to segment i and since $\mathcal{M}_{s,r}$ is a symmetrical graph, these nodes also have successors in the layer above, layer $3i + 1$. Then S can pick 4_{i+1}, 6_{i+1} or 2_i as a successor of a_i. Alternatively he may pick 5_{i+1} or 7_{i+1} as a successor of b_i. Every other successor of a_i or b_i is a shared successor. If he choose either 4_{i+1} or 5_{i+1}, then D answers with a set that contains only the respective other node. This leads to the follow-up position $(\{\{4_{i+1}\}, \{5_{i+1}\}\}, k - 1)$. Because of $k - 1 \leq 2^i - 3$ this position belongs to W_D^7. Alternatively if S picked 6_{i+1} or 7_{i+1}, then D also answers with the respective other node. Then the follow-up position is $(\{\{6_{i+1}\}, \{7_{i+1}\}\}, k - 1)$, which belongs to W_D^8. Lastly if S decided to move from a_i to 2_i, then D answers with the set $\{1_i, 3_i\}$ which progresses the play to the position $(\{\{2_i\}, \{1_i, 3_i\}\}, k - 1)$. This position belongs to W_D^2. So independent of the choices made by S, the play always continues from some position within W_D.

We already checked W_D^2 during the main proof, so we skip it here.

We continue by analysing W_D^3. Let $(\{\{4_i, 7_i\}, \{5_i, 6_i\}\}, k)$ be some position with $i \geq 2$ and $k \leq 2^i - 4$. If S plays an operator-move from this position, he has to pick 2_i as a successor of both 4_i and 7_i, since every other successor is a shared successor. To this, D answers with the set $\{1_i, 3_i\}$. Then the follow-up position $(\{\{2_i\}, \{1_i, 3_i\}\}, k - 1)$ belongs to W_D^2. However for $k \geq 2$, S can also play a split-move. In this case he has to split either $\{4_i, 7_i\}$ into $\{4_i\}$ and $\{7_i\}$ or $\{5_i, 6_i\}$ into $\{5_i\}$ and $\{6_i\}$. He also splits k into k_1 and k_2. Let k_j be the smaller of these two. Then we know $k_j \leq 2^{i-1} - 2$. Then D answers by picking the side with k_j. The set she picks depends on what choice was made by S for X_j. She answers with $\{5_i\}$ to $\{4_i\}$ and vice versa. Similarly she picks $\{6_i\}$ as an answer to $\{7_i\}$ and the other way around. So the follow-up position is either $(\{\{4_i\}, \{5_i\}\}, k_j)$, which is a member of W_D^7, or $(\{\{6_i\}, \{7_i\}\}, k_j)$, which is a member of W_D^8. So S cannot leave W_D from W_D^3 either.

The next set is W_D^4. So we look at a position $(\{\{1_i\}, \{2_i\}\}, k)$ with $i \geq 2$ and $k \leq 2^{i-1}$. In this position, S cannot play a split-move, since both sets contain only one node each. This will be the case for all sets from this point on, so we will stop mentioning it going forward. Now player S has to play an operator-move, where he picks b_i or 6_i as a successor of 1_i or 7_i as a successor of 2_i. If he picked b_i, D answers with $\{a_i\}$. This moves the game to $(\{\{a_i\}, \{b_i\}\}, k - 1)$. Because of $i \geq 2$, which implies $2^{i-1} \leq 2^i - 2$, this position belongs to W_D^1. On the other hand, if S choose 6_i or 7_i, then D answers with the set only containing the respective other node. Then the follow-up position is $(\{\{6_i\}, \{7_i\}\}, k - 1)$, which belongs to W_D^8. Then D can force any play from W_D^4 to stay in W_D.

We continue our proof with W_D^5. The positions within this set are the shape of $(\{1_i\}, \{3_i\}\}, k)$ with $i \geq 2$ and $k \leq 2^{i-1}$. When playing an operator-move, S may pick 4_i or 6_i as a successor of 1_i or he may pick 5_i or 7_i as a successor of 3_i. Then D answers with $\{5_i\}$ to 4_i, with $\{7_i\}$ to 6_i, with $\{4_i\}$ to 5_i and lastly with $\{6_i\}$ to 7_i. Then the play continues with $(\{\{4_i\}, \{5_i\}, k - 1)$, a position from W_D^7, or with $(\{\{6_i\}, \{7_i\}\}, k - 1)$, a position from W_D^8. This finishes the case of W_D^5.

Now we look at the set W_D^6, which contains all positions $(\{\{2_i\}, \{3_i\}\}, k)$ with $i \geq 2$ and $k \leq 2^{i-1}$. Here S may pick 4_i as a successor of 2_i or alternatively b_i or 5_i as a successor of 3_i. If he picked b_i then D answers with $\{a_i\}$. Then the follow-up position $(\{\{a_i\}, \{b_i\}\}, k - 1)$ is a member of W_D^1. If S choose 4_i or 5_i instead, then D answers by picking a singleton of the respective other node. In this case the game moves on to the position $(\{\{4_i\}, \{5_i\}\}, k - 1)$, which belongs to W_D^7.

The set W_D^7 consists of all positions $(\{\{4_i\},\{5_i\}\},k)$ with $i \geq 2$ and $k \leq 2^{i-1}-1$. Then S can play an operator-move by picking 1_i, 2_i or a_{i-1} as a successor of 4_i. Alternatively he may also pick 3_i or b_{i-1}. If S choose 2_i or 3_i, then D answers with $\{3_i\}$ or $\{2_i\}$ respectively. This creates the follow-up position $(\{\{2_i\},\{3_i\}\},k-1)$, which is a part of W_D^6. If S picked 1_i, then D answers with $\{3_i\}$, creating $(\{\{1_i\},\{3_i\}\},k-1)$ instead. This position belongs to W_D^5. On the other hand if S picked a_{i-1} or b_{i-1}, then D answers with a set only containing the respective other node. In this case the follow-up position $(\{\{a_{i-1}\},\{b_{i-1}\}\},k-1)$ belongs to W_D^1.

Lastly we look at the set W_D^8. This set contains all positions $(\{\{6_i\},\{7_i\}\},k)$ with $i \geq 2$ and $k \leq 2^{i-1}-1$. From such a position, S may play an operator-move where he picks either 1_i or a_{i-1} as a successor of 6_i or 2_i, 3_i or b_{i-1} as a successor of 7_i. If he choose 1_i or 2_i, then D answers with the respective other node as a singleton. The follow-up position $(\{\{1_i\},\{2_i\}\},k-1)$ is a member of W_D^4. If S choose 3_i instead, then D answers with $\{1_i\}$ and the new position is $(\{\{1_i\},\{3_i\}\},k-1)$, a member of W_D^5. On the other hand if S picked a_{i-1} or b_{i-1}, then D picks $\{b_{i-1}\}$ and $\{a_{i-1}\}$ respectively. Then the game continues with $(\{\{a_{i-1}\},\{b_{i-1}\}\},k-1)$, a position from W_D^1. So all follow-up positions belong to W_D. ◀