

The Biequivalence of Path Categories and Axiomatic Martin-Löf Type Theories

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Abstract

The semantics of extensional type theory has an elegant categorical description: models of extensional $=$ -types, 1 -types, and Σ -types are biequivalent to finitely complete categories, while adding Π -types yields locally Cartesian closed categories. We establish parallel results for axiomatic type theory, which includes systems like cubical type theory, where the computation rule of the $=$ -types only holds as a propositional axiom instead of a definitional reduction. In particular, we prove that models of axiomatic $=$ -types, and standard 1 - and Σ -types are biequivalent to certain path categories, while adding axiomatic Π -types yields dependent homotopy exponents.

This biequivalence simplifies axiomatic $=$ -types, which are more intricate than extensional ones since they permit higher dimensional structure. Specifically, path categories use a primitive notion of equivalence instead of a direct reproduction of the syntactic elimination rules and computation axioms. We apply our correspondence to prove a coherence theorem: we show that these weak homotopical models can be turned into equivalent strict models of axiomatic type theory. In addition, we introduce a more modular notion, that of a display map path category, which only models axiomatic $=$ -types by default, while leaving room to add other axiomatic type formers such as 1 -, Σ -, and Π -types.

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1 Introduction and motivation

Dependent type theory, and most notably *intensional type theory (ITT)*, has become a cornerstone of logic, with applications in proof assistants, programming languages, and the foundations of mathematics. Models of type theory form categories, and categorical formulations [15, 38, 29] provide a powerful tool to understand the behaviour of these systems. However, giving a satisfactory categorical formulation of dependent type theory is challenging.



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First of all, we want our requirements to be simple categorical properties instead of direct reproductions of the syntax; and secondly, categorical structure usually only satisfies stability conditions up to isomorphism instead of the strict equality imposed in the syntax. This means that it is significantly easier to produce *weak models*, which only satisfy stability up to equivalence. To deal with the second issue, strictification techniques [26, 5, 33, 12, 22, 31] have been developed to turn weakly stable structure into strictly stable structure. This paper focuses on the first issue: providing a categorical and homotopy theoretic presentation of the weak models of dependent type theory. We combine our results with existing techniques for the second issue to turn our weak models into (*strict*) *models*.

One difficulty in studying ITT is the fact that it has two notions of equality: *propositional equality* ($=$), which is an internal equality type, and *definitional equality* (\equiv), which is an external judgment. Distinguishing these notions provides ITT with its good computational properties: definitional equality is a decidable fragment of propositional equality and allows type checking to be algorithmically decidable. This fragment can be changed, such as by allowing the addition of rewrite rules [17]. The two extremes are: *extensional type theory (ETT)* [36], where every equality is definitional, and *axiomatic type theory (ATT)* [8, 44], where no equality is definitional.

For ETT the external and internal notions of equality match. This prevents it from being compatible with homotopy type theory: it proves uniqueness of identity proofs (UIP) which contradicts univalence [27, 47]. In addition, definitional equality is undecidable in ETT, which means that type checking is undecidable as well. By contrast, the semantics of ETT is easier to explore and allows for a simple categorical description of its type formers. Models can be formulated using locally Cartesian closed categories as seen by Seely [43]. However, Seely interprets substitutions using pullbacks, which are only specified up to isomorphism, leading to coherence problems. The precise statement is a biequivalence of 2-categories as shown by Clairambault and Dybjer [16] building on work of Curien, Hofmann and Garner [19, 26, 20], which has been extended by Maietti [35] and Van der Weide [48]:

| Syntax | Semantics |
|--|-------------------------------------|
| extensional $=, \mathbf{1}, \Sigma$ | finitely complete categories |
| extensional $=, \mathbf{1}, \Sigma, \Pi$ | locally Cartesian closed categories |

In this paper we generalise this to ATT. The main reason to study axiomatic type theory is that – while it only has one notion of equality like ETT, and is therefore easier to study – it is minimal and therefore compatible with ITT and its extensions, and provides a larger class of models. In addition, there are conservativity results for ATT which means that we can use it to study ITT and ETT: for example, ETT is conservative over ATT extended with function extensionality and UIP [49, 13, 45]. Moreover, from a mathematical perspective, axiomatic type formers enjoy the advantage of being *homotopy invariant*, as explained in [3, 4]: any type that is homotopy equivalent to the one given by the formation rule of a type former satisfies the same rules.

We generalise the known correspondence by considering the notion of a path category introduced by Van den Berg and Moerdijk [9, 6]. Path categories have equivalences as a primitive notion, which allows the omission of explicit interpretations of the elimination and computation rules. This is a considerable simplification, since it is exactly these rules that become unwieldy in ATT, especially in a setting without Π -types. We show a correspondence:

| Syntax | Semantics |
|---|---|
| extensional $\mathbf{1}, \Sigma$ and axiomatic $=$ | path categories |
| extensional $\mathbf{1}, \Sigma$ and axiomatic $=, \Pi$ | path cats with dependent homotopy exponents |

This extends the existing results since finitely complete categories can be seen as degenerate path categories where dependent homotopy exponentials simplify to dependent exponentials. We do this by proving an isomorphism of 2-categories between path categories and an existing notion of weak semantics: display map categories. Then we apply the results of Lumsdaine and Warren [33] and Bocquet [12] to strictify those path categories that satisfy the additional logical framework (LF) condition, giving a biequivalence with models satisfying this condition. We also introduce a new and more fine-grained notion – that of a *display map path category*, which distinguishes types and telescopes – and prove similar correspondences:

| | |
|---|--------------------------------------|
| Syntax | Semantics |
| axiomatic =, (1), (Σ), (Π) | suitable display map path categories |

This result is more modular without introducing significant additional complexity: display map path categories only have axiomatic =-types by default and can be extended with any combination of 1, Σ , and Π in a relatively simple way.

Our contribution. Our main contribution is twofold: (a) providing a categorical formulation of the weak models of axiomatic =-types, and exploiting it to obtain a coherence result for the class of path categories, and (b) a characterisation of the strict models of axiomatic type theory in terms of path categories. In detail, we show that the 2-category of path categories is isomorphic to a 2-category of suitable display map categories. We consider display map categories to phrase the semantics of type theory because they form a good middle ground: they allow for a formulation of type formers that closely follows the syntax, while being expressive enough to consider both weak and strict models. Additionally, they are relatively simple and close to the homotopical notions.

The idea of formulating the semantics of axiomatic =-types using path categories originates from the work of Van den Berg and Moerdijk [9, 6]. Our contribution lies in clarifying precisely in what sense path categories serve as models for these axiomatic =-types: every path category is *rooted* – that is, every context is essentially build by extending the empty context with finitely many types – and models axiomatic =-types, and extensional 1- and Σ -types. We show that, when they are *structured* (come equipped with choices of substitutions), and satisfy the LF condition, they can be strictified into actual models.

Additionally, we introduce a more fine-grained notion, that of a display map path category, which distinguishes between types and telescopes. Because of this distinction, they do not model extensional 1- and Σ -types, but still model axiomatic =-types. This makes them a suitable semantics for a minimal ATT – only axiomatic =-types – which can be extended with axiomatic 1-types, Σ -types, and Π -types.

Related work. This paper is positioned at the intersection of two goals: (a) finding a well-structured categorical description of models for ATT, in such a way that there is a modular correspondence [34, 24] between certain categories, where syntax is not explicitly articulated, and specific type theories, and (b) proving coherence theorems. From this perspective, our work is in line with that of Maietti [34] and that of Clairambault and Dybjer [16]. Maietti’s work is based on characterising models of certain theories via universal properties. This is largely due to the fact that her work focuses on extensional theories. To extend this to axiomatic theories, we need to use a weaker notion of universal properties in the form of *homotopy (co)limits*. Such an approach seems necessary for inductive types such as the natural numbers type and W-types. For =-types, 1-types, and Σ -types, we show that it suffices to ask the constructor to be an equivalence: a homotopy universal property

already follows from this requirement. Another approach consists in describing axiomatic type formers by means of higher dimensional universal properties in higher dimensional structures, as done in [44]. We also reference Van der Weide’s work [48], which generalizes both Maietti’s and Clairambault and Dybjer’s results to univalent categories.

The correspondence that we establish is similar to the one by Clairambault and Dybjer [16], since finitely complete categories are precisely the path categories in which the only equivalences are the isomorphisms and every morphism is a fibration. However, a formulation of our result as a generalization of the former is hard to find. On the one hand, finitely complete categories induce *pseudo-stable* type formers in the corresponding display map category, which is enough to obtain a split display map category with stable type formers, hence a genuine model, by *right-adjoint* splitting. On the other hand, a general path category only provides the corresponding display map category with *weakly stable* type formers, which forces us to base our splitting strategy on the *left-adjoint* splitting functor, as described by Lumsdaine and Warren [33] and Bocquet [12]. However, the latter requires the input display map category to be structured, which in general does not follow from our hypotheses. Hence, we cannot hope to link general path categories to models of ATT, but only to weak models with weakly stable structure, whereas Clairambault and Dybjer’s biequivalence automatically involves pseudo-stable structure – and hence falls within the domain of the right-adjoint splitting. Our solution consists in restricting our biequivalence between path categories and weak models to the *structured*, or *cloven*, case in both sides, so that the latter fall within the domain of the left-adjoint splitting: in this way we deduce a biequivalence between cloven path categories and strict models of our type theory.

Structure of the paper. In Section 2 we describe *axiomatic =-types* syntactically, and explain why the based version is preferred in a setting without Π -types. In Section 3, we review the notion of a display map category as well as the semantic notion of axiomatic =-types, and explain *strict* and *weak* stability in both the general and the structured case. Analogously, Section 4 recalls the notion of a path category together with its basic properties.

In Section 5 we describe how path categories extend the notion of a *clan* with based axiomatic =-types. We use this fact to show that path categories are rooted display map categories with weakly stable based axiomatic =-types and extensional $\mathbb{1}$ - and Σ -types. Secondly, we prove that these conditions on a display map category are themselves enough to recover the structure of a path category, and that this correspondence extends to an isomorphism between the corresponding 2-categories. In Section 6, we introduce the notion of a display map path category adding a distinction between display maps and fibrations, and prove a similar 2-isomorphism. Consequence of our correspondence are explored in Section 7, where we strictify to obtain genuine models.

2 Syntax

We will assume some familiarity with the syntax of Martin-Löf type theory; for an overview, see [37, 47]. Our main focus will be a minimal type theory with only propositional equality (the =-types) and no definitional equality (the \equiv -judgement). For the structural rules, the unit type ($\mathbb{1}$), dependent product types (Σ) and dependent function types (Π), see Appendix A. Type formers in intensional Martin-Löf type theory are given by formation, introduction, elimination, and computation (β) rules. For the =-types, these are:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a : A \quad \Gamma \vdash a' : A}{\Gamma \vdash a =_A a' \text{ type}} =F, \quad \frac{\Gamma \vdash a : A}{\Gamma \vdash \text{refl}(a) : a =_A a} =I,$$

$$\frac{\begin{array}{l} \Gamma, x : A, x' : A, \chi : x =_A x' \vdash C \text{ type} \\ \Gamma, x : A \vdash d : C[x/x', \text{refl}(x)/\chi] \\ \Gamma \vdash \alpha : a =_A a' \end{array}}{\Gamma \vdash \text{ind}_d^{\overline{\alpha}} : C[a/x, a'/x', \alpha/\chi]} = \text{E}, \quad \frac{\begin{array}{l} \Gamma, x : A, x' : A, \chi : x =_A x' \vdash C \text{ type} \\ \Gamma, x : A \vdash d : C[x/x', \text{refl}(x)/\chi] \\ \Gamma \vdash a : A \end{array}}{\Gamma \vdash \text{ind}_d^{\overline{\text{refl}(a)}} \equiv d[a/x]} = \beta.$$

In extensional type theory, we also add a uniqueness (η) rule:

$$\frac{\begin{array}{l} \Gamma, x : A, x' : A, \chi : x =_A x' \vdash c : C \\ \Gamma \vdash \alpha : a =_A a' \end{array}}{\Gamma \vdash \text{ind}_{c[x/x', \text{refl}(x)/\chi]}^{\overline{\alpha}} \equiv c[a/x, a'/x', \alpha/\chi]} = \eta.$$

The η -rule implies that propositional equality and definitional equality coincide, making type checking undecidable. We are mainly interested in the other direction – a minimal type theory without definitional equality – so we modify intensional $=$ -types in two ways: we weaken the β -rule (definitional equality) to a β -axiom (propositional equality), and fix the variable x in place as a constant term a in the eliminator:

$$\frac{\begin{array}{l} \Gamma, x' : A, \chi : a =_A x' \vdash C \text{ type} \\ \Gamma \vdash d : C[a/x', \text{refl}(a)/\chi] \\ \Gamma \vdash \alpha : a =_A a' \end{array}}{\Gamma \vdash \text{ind}_d^{\overline{\alpha}} : C[a'/x', \alpha/\chi]} = \text{E}_{\text{based}}, \quad \frac{\begin{array}{l} \Gamma, x' : A, \chi : a =_A x' \vdash C \text{ type} \\ \Gamma \vdash d : C[a/x', \text{refl}(a)/\chi] \end{array}}{\Gamma \vdash \beta_d^{\overline{\alpha}} : \text{ind}_d^{\overline{\alpha}}(\text{refl}(a)) = d} = \beta_{\text{based}}^{\text{ax}}.$$

These two modifications are orthogonal. The main focus of this paper is on the first one: weakening the β -rule to a β -axiom, that is, only requiring the $=$ -type to be inhabited instead of requiring a judgemental equality. This weaker notion of $=$ -type is called an *axiomatic $=$ -type*, and is the version that appears in cubical type theory [18] as well as in axiomatic type theory (type theory without reductions) [8]. The other modification, fixing one of the two endpoints, is a technical variation known as *based* path induction or the Paulin-Mohring eliminator [41]. With Π -types, based and unbased path induction are equivalent. However, without Π -types, the based version is stronger and better behaved. For example, the transport $\alpha_* b : B[a'/x]$ of a term $b : B[a/x]$ along an equality $\alpha : a =_A a'$ can be defined as $\text{ind}_b^{\overline{\alpha}}(\alpha)$ using based path induction, while it is not definable using unbased path induction.

Alternatively, to give the unbased version the same strength as the based version, we can extend it with a telescope Ω . This is called the *parametrized* or Frobenius version of path induction. For example, axiomatic parametrized unbased path induction is given by:

$$\frac{\begin{array}{l} \Gamma, x : A, x' : A, \chi : x =_A x', \Omega \vdash C \text{ type} \\ \Gamma, x : A, \Omega[x/x', \text{refl}(x)/\chi] \vdash d : C[x/x', \text{refl}(x)/\chi] \\ \Gamma \vdash \alpha : a =_A a' \end{array}}{\Gamma, \Omega[a/x, a'/x', \alpha/\chi] \vdash \text{ind}_d^{\overline{\alpha}} : C[a/x, a'/x', \alpha/\chi]} = \text{E}_{\text{param}}, = \beta_{\text{param}}^{\text{ax}}.$$

$$\Gamma, \Omega[a/x, a'/x', \alpha/\chi] \vdash \beta_d^{\overline{\alpha}} : \text{ind}_d^{\overline{\alpha}}(\text{refl}(a)) = d$$

We have equivalences between axiomatic/intensional/extensional versions of path induction:

$$\text{unparametrized based} \leftrightarrow \text{parametrized based} \leftrightarrow \text{parametrized unbased},$$

while the original unparametrized unbased version is weaker without Π -types. We prove a semantic version of this in Proposition 14.

In general we use the following terminology: a type former is *axiomatic* if we only have a β -axiom, *intensional* if it has a β -rule, and *extensional* if it also has an η -rule. Note that the β -axiom generally implies the existence of an η -axiom [3, 4]. The type theory is *axiomatic/intensional/extensional* if every type former is. Note that having extensional $=$ -types implies that every other type former is extensional.

3 Display map categories

There is a big zoo of semantic frameworks for the structural part of dependent type theory; see [1] for an overview of the equivalences. These range from frameworks that closely follow the syntax (such as categories with families, which have primitive notions for contexts, types, and terms) to frameworks that simplify the structure (such as display map categories, where a type A in context Γ is identified with the display map or context projection $\Gamma.A \rightarrow \Gamma$, and a term a of A is identified with the induced context map $\Gamma \rightarrow \Gamma.A$). We will be working in the later setting for two reasons: it allows us to talk about both strict and weak models, and can easily be extended using ideas of homotopy theory as we will see in Section 4.

► **Definition 1** (Display map category [28, 46]). *A display map category is a category \mathcal{C} with a subclass of maps $\mathcal{C}^{\text{disp}} \subseteq \mathcal{C}^{\rightarrow}$ called the display maps (\rightarrow) that is replete (closed under isomorphism in the arrow category $\mathcal{C}^{\rightarrow}$), and closed under pullback along any map.*

The objects of a display map category are called *contexts*. A *type* A in context Γ consists of a context suggestively written $\Gamma.A$ together with a display map $p_A : \Gamma.A \rightarrow \Gamma$. A *term* a of A is a section of $p_A : \Gamma.A \rightarrow \Gamma$, that is, a map $a : \Gamma \rightarrow \Gamma.A$ such that $p_A a = 1_\Gamma$. More generally, a *term* a of A for a context map $\sigma : \Delta \rightarrow \Gamma$ is a map $a : \Delta \rightarrow \Gamma.A$ such that $pa = \sigma$. Such a term of A for σ is equivalent to having a term of the pullback $A[\sigma]$ of A along σ :

$$\begin{array}{ccc} \Delta.A[\sigma] & \xrightarrow{\sigma^\nabla} & \Gamma.A \\ p_{A[\sigma]} \downarrow & \lrcorner & \downarrow p_A \\ \Delta & \xrightarrow{\sigma} & \Gamma. \end{array}$$

We call $A[\sigma]$ the *re-indexing* or *substitution* of A along σ , and σ^∇ the *weakening* of σ for A . We extend both of these notions further. If a is a term of type A , then, for any choice of $A[\sigma]$, we write $a[\sigma]$ for the unique term of type $A[\sigma]$ such that $(\sigma^\nabla)(a[\sigma]) = a\sigma$:

$$\begin{array}{ccc} \Delta.A[\sigma] & \xrightarrow{\sigma^\nabla} & \Gamma.A \\ a[\sigma] \lrcorner \downarrow & \lrcorner & \downarrow \lrcorner a \\ \Delta & \xrightarrow{\sigma} & \Gamma. \end{array}$$

For a type B in context Γ we write $B^\nabla := B[p_A]$ for the induced type in an extended context $\Gamma.A$; and for a term b of B we write $b^\nabla := b[p_A]$ for the induced term of B^∇ . Finally, for any choice of B^∇ , we use the categorical notation $(a, b) : \Gamma \rightarrow \Gamma.A.B^\nabla$ for the term of the pullback induced by terms $a : \Gamma \rightarrow \Gamma.A$ and $b : \Gamma \rightarrow \Gamma.B$. We extend this to the dependent case by writing $(a, b) : \Gamma \rightarrow \Gamma.A.B$ for the term $a^\nabla b$ induced by terms $a : \Gamma \rightarrow \Gamma.A$ and $b : \Gamma \rightarrow \Gamma.B[a]$. For a type A we have always the *variable* term or *diagonal* $\delta_a := (1_{\Gamma.A}, 1_{\Gamma.A}) : \Gamma.A \rightarrow \Gamma.A.A^\nabla$.

► **Definition 2** (Root). *We call a sequence of types $A_0 \dots A_{n-1}$, such that A_i is a type in context $\Gamma.A_0 \dots A_{i-1}$, a telescope of Γ , and a composition of display maps $\Gamma.A_0 \dots A_{n-1} \rightarrow \dots \rightarrow \Gamma$ a fibration (\rightarrow). A display map category is called rooted if it has a terminal object 1 (the empty context or root), and any map $\Gamma \rightarrow 1$ is a fibration¹.*

Type formers follow the syntactic specification. For example, an $=$ -type is given by:

¹ Note that having a root is the correct way to state democracy in the absence of extensional 1- and Σ -types. Democracy means that every context has a representation, and is usually stated as every context being isomorphic to $1.A$ for a type A in the empty context [16, Definition 2.6].

► **Definition 3** (=type). We say that a display map category has axiomatic (based) =-types if, for every type A in context Γ , there exists the data of a =-type for A :

F. a type ld_A in context $\Gamma.A.A^\nabla$ (for some choice of re-indexing),

I. a term refl_A of type $\text{ld}_A[\delta_A]$ in context $\Gamma.A$, where $\delta_A := (1_{\Gamma.A}, 1_{\Gamma.A}) : \Gamma.A \rightarrow \Gamma.A.A^\nabla$:

$$\begin{array}{ccc} \Gamma.A.\text{ld}_A[\delta_A] & \xrightarrow{\delta_A^\nabla} & \Gamma.A.A^\nabla.\text{ld}_A \\ \text{refl}_A \curvearrowright \downarrow & \lrcorner & \downarrow \\ \Gamma.A & \xrightarrow{\delta_A} & \Gamma.A.A^\nabla, \end{array}$$

E. for every term a of type A , every type C in context $\Gamma.A.\text{ld}_A[a^\nabla]$, and every term d of type $C[a, \text{refl}[a]]^2$, a term ind_d^- of C :

$$\begin{array}{ccc} \Gamma.C[a, \text{refl}[a]] & \xrightarrow{(a, \text{refl}[a])^\nabla} & \Gamma.A.\text{ld}_A[a^\nabla].C \\ d \curvearrowright \downarrow & \lrcorner & \downarrow \text{ind}_d^- \\ \Gamma & \xrightarrow{(a, \text{refl}[a])} & \Gamma.A.\text{ld}_A[a^\nabla], \end{array}$$

β . for every a, C, d as before, $\text{ind}_d^-[a, \text{refl}[a]]$ is propositionally equal to d , that is, we have a term β_d^- of type $\text{ld}_{C[a, \text{refl}[a]]}[\text{ind}_d^-[a, \text{refl}[a]], d]$.

We are interested in the axiomatic notion, but for clarity we also explain stronger notions. We call an =-type for a type A *intensional* if it satisfies the β -rule: for every C, d as before we have that $\beta_d^- = \text{refl}[d]$, and therefore $\text{ind}_d^-[a, \text{refl}[a]] = d$. We call the =-type *extensional* if it also satisfies the η -rule: for every C and term c of C we have that $\text{ind}_{c^-}^- = c$.

We say that a display map category with =-types has *weakly stable* =-types if for every type A in context Γ there exist an =-type ld_A and for every $\sigma : \Delta \rightarrow \Gamma$ we can equip $\text{ld}_A[\sigma^{\nabla\nabla}]$ and

$$\Delta.A[\sigma] \xrightarrow{\text{refl}_A[\sigma^{\nabla\nabla}]} \Delta.A[\sigma].\text{ld}_A[\delta_A][\sigma^{\nabla\nabla}] \cong \Delta.A[\sigma].\text{ld}_A[\sigma^{\nabla\nabla}][\delta_A[\sigma]],$$

with the additional data required to form an =-type. To explain the intuition for this notion, we first need to explain the idea of an equivalence. A (*homotopy*) *equivalence* between types A and B fibered over a context Γ is a term $b : \Gamma.A \rightarrow \Gamma.B$ of B for p_A such that there exists a term $a : \Gamma.B \rightarrow \Gamma.A$ of A for p_B so that the two compositions are pointwise propositionally equal to the identity, that is, we have terms of $\text{ld}_A[ab, 1_{\Gamma.A}]$ and $\text{ld}_B[ba, 1_{\Gamma.B}]$. One can observe that between any two =-types $\text{ld}_A, \text{ld}'_A$ for A there exists an equivalence that respects the data of an =-type, and that any type that is equivalent to an =-type for A can be given the additional data to form an =-type. Hence, weak stability is equivalent to the statement that =-types are preserved by re-indexing up to equivalence. This certainly holds in the syntax because the substitution of an =-type is still an =-type.

As explained in [1], there exists a 2-category DispCat of display map categories, functors preserving display maps and pullbacks of display maps, and natural transformations. We can specialize this to 0-cells that are rooted and 1-cells that preserve the terminal object. Another specialization is to 0-cells with additional weakly stable type formers and 1-cells that

² Note that $\text{refl}[a]$ is a morphism of the form $\Gamma \rightarrow \Gamma.\text{ld}[\delta_A][a]$ and because $\delta_A a = (1, 1)a = (a, a) = a^\nabla a$ we can also write the codomain as $\Gamma.\text{ld}[a^\nabla][a]$ so that $(a, \text{refl}[a]) : \Gamma \rightarrow \Gamma.A.\text{ld}_A[a^\nabla]$. However, a structured display map category (see below) does not have to choose the same object for $\Gamma.\text{ld}[\delta_A][a]$ and $\Gamma.\text{ld}[a^\nabla][a]$. So, there we should understand the statement as leaving a canonical isomorphism between the chosen objects implicit. A strict display map category is guaranteed to choose the same re-indexing.

weakly preserve them; see Appendix B. In Section 5, we show that the 2-category of path categories is isomorphic to the 2-category $\text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{root}}$ of rooted display map categories with weakly stable axiomatic $=$ -types, extensional $\mathbb{1}$ -types, and extensional Σ -types.

Strict display map categories

In the syntax, substitution is defined recursively and therefore satisfies definitional equalities. These equalities are not even expressible in a display map category because substitution is only specified up to isomorphism, and $=$ -types are only specified up to equivalence. So, to model type theory strictly, we need specified choices: a *structured* or *cloven* display map category is equipped with a subclass of the display maps that generates the display maps by repletion (the *strict* display maps), and choices for re-indexing strict display maps to strict display maps. We call it a *strict* or *split* display map category if these choices respect identity and composition: $A[1_\Gamma] = A$ and $A[\tau\sigma] = A[\tau][\sigma]$. A strict rooted display map category also has choices to write every $\Gamma \rightarrow 1$ as a composition of strict display maps and isomorphisms. Given specified choices for a type former, we can consider a range of conditions [33]:

- *weakly stable*: preserved by re-indexing up to equivalence over the context.
- *pseudo-stable*: preserved by re-indexing up to isomorphism over the context.
- *strictly stable*: preserved by re-indexing.

For example, specified choices for $=$ -types consist of: for every strict type A a strict type Id_A and a term refl_A , together with for every strict type C and term d , terms ind_d^- and β_d^- . These choices are *strictly stable* if for any σ we have the following equalities:

$$\text{Id}_A[\sigma^{\nabla\nabla}] = \text{Id}_{A[\sigma]}, \quad \text{ind}_d^-[\sigma^{\nabla\nabla}] = \text{ind}_{d[\sigma]}^-, \quad \text{refl}_A[\sigma^{\nabla}] = \text{refl}_{A[\sigma]}, \quad \beta_d^-[\sigma^{\nabla}] = \beta_{d[\sigma]}^-.$$

A *weak model* of a type theory is a display map category with weakly stable type formers, and a *strict model* is a strict display map category with strictly stable type formers.

4 Path categories

We start with the following notion from abstract homotopy theory:

► **Definition 4** (Clan [30]). *A clan is a category \mathcal{C} with a subclass of maps $\mathcal{C}^{\text{fib}} \subseteq \mathcal{C}^{\rightarrow}$ called the fibrations ($- \twoheadrightarrow$) that is replete, closed under identity, composition, and pullback along any map. In addition, there exists a terminal object 1 and every map $A \rightarrow 1$ is a fibration.*

We have a forgetful functor from rooted display map categories to clans (only remembering the fibrations, that is, the compositions of display maps), and conversely every clan gives a rooted display map category in a cofree way: the entire class of fibrations can be taken as the display maps. To emphasize the topological view, we start writing the pullback of $\Omega \twoheadrightarrow \Gamma$ and $\sigma : \Delta \rightarrow \Gamma$ as $\Delta \times_\Gamma \Omega$ instead of $\Delta.\Omega[\sigma]$. We extend clans with equivalences:

► **Definition 5** (Path category [9]). *A path category is a clan with an additional subclass $\mathcal{C}^{\text{eq}} \subseteq \mathcal{C}^{\rightarrow}$ of (weak) equivalences (\simeq). If a map is both a fibration and an equivalence, we call it a trivial fibration ($\simeq \twoheadrightarrow$). In addition, we require:*

1. *isomorphisms are weak equivalences,*
2. *equivalences satisfy 2-out-of-6: if for $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ we have that both gf and hg are equivalences, then so are f , g , h , and hgf ,*
3. *the pullback of a trivial fibration along any map is also a trivial fibration,*
4. *every trivial fibration has a section,*
5. *every object A has a path object: an object PA together with a fibration $(s, t) : PA \twoheadrightarrow A \times A$ and an equivalence $r : A \simeq PA$ with $(s, t)r = (1, 1)$.*

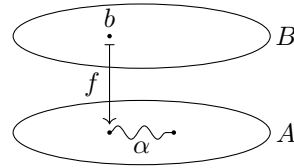
This is a strengthening of Brown’s notion of a category of fibrant objects [14] in two ways: (a) we have 2-out-of-6 instead of the weaker 2-out-of-3 (if for $A \rightarrow B \rightarrow C$ two of the three maps are equivalences, then so is the third), which is equivalent to saturation [42], and (b) every trivial fibration has a section, which is equivalent to every object being cofibrant.

A trivial example of a path category is a category with finite limits where every morphism is a fibration, and only isomorphisms are equivalences. More interesting examples arise by taking the fibrant-cofibrant objects of a model category where every object is cofibrant. In addition, the effective topos can be seen as the homotopy category of a path category [7].

Four main results for path categories are: factorization, slicing, lifting, and saturation, which we will show here; for proofs, see [9]. First, notice that finite products exist and that the projections are pullbacks of fibrations and therefore fibrations. In addition, we see that the source and target maps $s, t : PA \rightarrow A$ are always trivial fibrations: equivalences by 2-out-of-3 on $sr = 1$ and $tr = 1$, and fibrations as compositions of fibrations because $s = \pi_0(s, t)$ and $t = \pi_1(s, t)$. Now we can state:

► **Proposition 6 (Factorization).** *We can factor any map $f : B \rightarrow A$ as an equivalence $w_f : B \xrightarrow{\simeq} Lf$ (a section of a trivial fibration) followed by a fibration $p_f : Lf \twoheadrightarrow A$.*

Using generalized elements, we think of the *mapping path space* $Lf := B \times_s PA$ as the space of pairs (b, α) with b in B and α a path in A with source fb :



Then $w_f := (1_B, rf)$ maps b in B to the pair consisting of b and the trivial path on fb , and $p_f := t\pi_1$ maps (b, α) to the target of α .

Proof. The projection $\pi_0 : B \times_s PA \rightarrow B$ is the pullback of s and therefore a trivial fibration. So, with 2-out-of-3 on $\pi_0(1_B, rf) = 1_B$ we see that $w_f := (1_B, rf)$ is an equivalence. In addition, we see that the following square is pullback:

$$\begin{array}{ccc}
 B \times_s PA & \xrightarrow{\pi_1} & PA \\
 (\pi_0, t\pi_1) \downarrow & \lrcorner & \downarrow \downarrow (s,t) \\
 B \times A & \xrightarrow{(f\pi_0, \pi_1)} & A \times A
 \end{array}$$

So, $(\pi_0, t\pi_1)$ is a fibration, and therefore $p_f := t\pi_1 = \pi_1(\pi_0, t\pi_1)$ is as well. ◀

► **Definition 7 (Slice category of fibrations).** *If \mathcal{C} is a path category then for any object Γ in \mathcal{C} we define $\mathcal{C}/^{\text{fib}}\Gamma$ as the full subcategory of \mathcal{C}/Γ whose objects are fibrations to Γ . This is a path category when notions of fibration and equivalence are copied from \mathcal{C} . The path object $A \simeq P_\Gamma A \twoheadrightarrow A \times_\Gamma A$ of a fibration $A \twoheadrightarrow \Gamma$ is a factorisation of $\delta_A : A \rightarrow A \times_\Gamma A$ in \mathcal{C} .*

► **Definition 8 (Homotopy).** *We write $f \simeq g$ for $f, g : A \rightarrow B$ if there exists an homotopy: an $h : A \rightarrow PB$ such that $(s, t)h = (f, g)$. A homotopy equivalence is a map $f : A \rightarrow B$ such that there exists a $g : B \rightarrow A$ with $gf \simeq 1$ and $fg \simeq 1$. More generally, if $B \twoheadrightarrow \Gamma$ then we write $f \simeq_\Gamma g$ if there exists a fibrewise homotopy: an $h : A \rightarrow P_\Gamma B$ such that $(s, t)h = (f, g)$.*

► **Theorem 9 (Saturation).** *The homotopy equivalences are precisely the weak equivalences.*

► **Theorem 10** (Lifting). *Suppose that we have the commutative outer square:*

$$\begin{array}{ccc}
 B & \xrightarrow{f} & A \\
 w \downarrow & \simeq_{\Gamma} \dashrightarrow l & \downarrow p \\
 \Delta & \xrightarrow{\sigma} & \Gamma.
 \end{array}$$

Then there exists a map $l : \Delta \rightarrow A$ such that the lower triangle commutes and the upper triangle commutes up to fibrewise homotopy: we have $pl = \sigma$ and $lw \simeq_{\Gamma} f$. Moreover, such a lifting l is unique up to fibrewise homotopy.

► **Corollary 11.** *Let $f : B \rightarrow A$ be any morphism in a path category and let PB and PA be path objects. Then there is a morphism $Pf : PB \rightarrow PA$ commuting with the morphisms (s_B, t_B) and (s_A, t_A) and such that $(Pf)r_B \simeq_{A \times A} r_A f$. In particular, homotopy is independent of the choice of path objects (consider $f = 1_A$).*

Although they are 1-categories, path categories already contain higher structure. In particular, they are enriched in ∞ -groupoids [40]: a map between morphisms $f, f' : A \rightarrow B$ is an homotopy, that is, a morphism $h : A \rightarrow PB$, and so on.

We have a 2-category PathCat of path categories, functors preserving the structure (fibrations, trivial fibrations, pullbacks, and the terminal object), and natural transformations. Such functors also preserve equivalences and therefore path objects by Brown's lemma [14].

We call path categories *structured* if there is a choice of fibrations (the *strict* fibrations) that generates the fibrations by repletion, and choices of path fibrations, pullbacks of strict fibrations to strict fibrations, and terminal object. We call it *strict* if these choices strictly respect identity, composition, and path objects.

5 From path categories to weak models and back

We anticipated that path categories would provide a natural semantics for axiomatic $=$ -types. In this section, we establish the rigorous correspondence by rephrasing a path category as a display map category with weakly-stable axiomatic $=$ -types and three other properties: a root, (pseudo-stable) extensional $\mathbf{1}$ -types, and (pseudo-stable) extensional Σ -types. In Section 6, we introduce a version of path categories that does not necessarily model extensional $\mathbf{1}$ -types and Σ -types, and hence provides semantics for a minimal axiomatic type theory with only axiomatic $=$ -types. We will also see how we can extend this with other axiomatic type formers. Let us start from these last properties:

► **Proposition 12.** *A display map category has:*

- *extensional $\mathbf{1}$ -types iff all identity maps are display maps;*
- *extensional Σ -types iff display maps are closed under composition.*

So, a display map category is equal to a clan (as categories with a specified subclass of maps) iff it is rooted and has extensional $\mathbf{1}$ - and Σ -types. In addition, extensional $\mathbf{1}$ - and Σ -types are always pseudo-stable.

Proof. If we have extensional $\mathbf{1}$ -types, then for any Γ , the display map $\Gamma.\mathbf{1} \rightarrow \Gamma$ is an isomorphism by the introduction and β, η -rules. As any isomorphism of \mathcal{C} is isomorphic to the identity in $\mathcal{C}^{\rightarrow}$, the latter is a display map as well. Similarly, if we have extensional Σ -types then for any composable display maps $\Gamma.A.B \rightarrow \Gamma.A \rightarrow \Gamma$, we have a display map $\Gamma.\Sigma_A B \rightarrow \Gamma$ that is isomorphic in $\mathcal{C}^{\rightarrow}$ to the composition, hence the latter is a display map as well. Vice versa, define $\mathbf{1}_{\Gamma}$ as $\mathbf{1}_{\Gamma} : \Gamma \rightarrow \Gamma$ and $\Sigma_A B$ as $\Gamma.A.B \rightarrow \Gamma.A \rightarrow \Gamma$.

Pseudo-stability follow from the fact that isomorphisms are preserved by re-indexing. ◀

To show that path categories also have axiomatic $=$ -types, we adapt [6, Proposition 4.5]:

► **Proposition 13.** *A path category has weakly stable based axiomatic $=$ -types. Therefore – by considering every fibration to be a display map – a path category is a rooted display map category with weakly stable axiomatic $=$ -types, and extensional $\mathbb{1}$ - and Σ -types.*

Proof Sketch. Formation and introduction for a type A in context Γ are provided by a path object for the fibration $A \rightarrow \Gamma$. Elimination and the β -axiom follow by the lifting theorem (Theorem 10). Weak stability follows from the fact that the re-indexing of a path object is a path object itself. See [39, Appendix C] for a complete proof. ◀

For the converse, we first translate different notions of axiomatic $=$ -types:

► **Proposition 14.** *For a display map category, the existence of the following weakly stable axiomatic identity types is equivalent:*

$$\text{unparametrized based} \leftrightarrow \text{parametrized based} \leftrightarrow \text{parametrized unbased},$$

whereas the existence of weakly stable unparametrized unbased axiomatic $=$ -types is weaker: transport is not definable with unbased path induction (and no Π -types).

Proof. The proof of the first equivalence is obtained by adapting Bocquet’s argument [11, Section 3] to non-strict models. Regarding the second equivalence, a parametrized unbased axiomatic $=$ -type for a type A in a given context Γ can be defined as a specific re-indexing of a parametrized based axiomatic $=$ -type for *some* A^∇ over $\Gamma.A$ – namely, along the corresponding morphism δ_A^∇ .

The strategy to prove that transport is not admissible in a theory with only the Martin-Löf elimination and computation rules and no Π -types – and hence that weakly stable unparametrized unbased axiomatic $=$ -types are weaker than the other notions – adapts an argument by Bocquet [11, page 15]. We also refer the reader to [21].

We refer the reader to [39, Appendix C] for a complete proof. ◀

► **Proposition 15.** *A rooted display map with weakly stable based axiomatic $=$ -types and extensional $\mathbb{1}$ - and Σ -types can be given the structure of a path category where the weak equivalences are defined as the homotopy equivalences.*

Proof Sketch. By Proposition 14 and Propositional 12 – by looking at its display map class as a class of fibrations – we infer that \mathcal{C} is a clan and is endowed with weakly stable parametrized unbased axiomatic $=$ -types. Now, one can observe that weak stability implies the existence of arrows:

$$\text{Id}_\Omega[\sigma^{\nabla\nabla}] \rightarrow \text{Id}_{\Omega[\sigma]} \quad \text{and} \quad \text{Id}_{\Omega[\sigma]} \rightarrow \text{Id}_\Omega[\sigma^{\nabla\nabla}] \quad (\text{logical equivalence})$$

over the canonical isomorphism $\Omega[\sigma] \times_\Delta \Omega[\sigma] \cong (\Omega \times_\Gamma \Omega)[\sigma^\nabla]$, for every fibration $\Omega \rightarrow \Gamma$ and every substitution $\Delta \xrightarrow{\sigma} \Gamma$. In other words, the equivalence relations represented by $\text{Id}_\Omega[\sigma^{\nabla\nabla}]$ and $\text{Id}_{\Omega[\sigma]}$ coincide. We provide full verification of this fact in [39, Appendix C]. Van den Berg [6, Theorem 5.16] proves that these data on \mathcal{C} are in fact equivalent to the existence and the choice of a class of weak equivalences on \mathcal{C} – in fact provided by its homotopy equivalences – making the clan \mathcal{C} into a path category, concluding our argument. ◀

The correspondence delineated by Proposition 13 and Proposition 15 extends to an equality of the 2-categories; for which we provide details in [39, Appendix C]:

► **Theorem 16.** *There exists an isomorphism of 2-categories: $\text{PathCat} \cong \text{DispCat}_{=_{\text{axi}}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{root}}$.*

6 Display map path categories

To define a more fine-grained semantics that does not necessarily model extensional 1-types and extensional Σ -types, we will add some additional structure to path categories: we exchange the clan for a display map category to distinguish display maps and fibrations.

► **Definition 17** (Display map path category). *A display map path category is a display map category \mathcal{C} together with a class $\mathcal{C}^{\text{eq}} \subseteq \mathcal{C}^{\rightarrow}$ of (weak) equivalences (\simeq) such that:*

1. *isomorphisms are equivalences,*
2. *equivalences satisfy 2-out-of-6: if for $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ we have that both gf and hg are equivalences, then so are f , g , h , and hgf ,*
3. *the pullback of a trivial fibration along any map is also a trivial fibration,*
4. *every trivial fibration has a section,*
5. *every display map $A \rightarrow \Gamma$ has a path display map:
a display map $(s, t) : P_{\Gamma}A \rightarrow A \times_{\Gamma} A$ and equivalence $r : A \simeq P_{\Gamma}A$ with $(s, t)r = (1, 1)$.*
- PF. *every fibration $\Delta \rightarrow \Gamma$ has a path fibration:
a fibration $(s, t) : P_{\Gamma}\Delta \rightarrow \Delta \times_{\Gamma} \Delta$ and equivalence $r : \Delta \simeq P_{\Gamma}\Delta$ with $(s, t)r = (1, 1)$.*

We call a display map path category *rooted* if the underlying display map category is. Every rooted display map path category is in particular a path category, and every path category gives a cofree rooted display map path category by considering every fibrations to be a display map. Now a natural question is: are path display maps $P_{\Gamma}A \rightarrow A \times_{\Gamma} A$ already enough to define path fibrations $P_{\Gamma}\Delta \rightarrow \Delta \times_{\Gamma} \Delta$? This is the case in the syntax: with =-types (and transport) we can define =-telescopes. For a telescope $\vec{x} : \vec{A}$ of Γ we define the =-telescope $\vec{\chi} : \vec{\chi}_* \vec{x} =_{\vec{A}} \vec{x}'$ of Γ , $\vec{x} : \vec{A}$, $\vec{x}' : \vec{A}$ where:

$$\chi_i : (\chi_{i-1})_* \cdots (\chi_0)_* x_i =_{A_i[x'_0, \dots, x'_{i-1}/x_0, \dots, x_{i-1}]} x'_i.$$

The same is true in general. However, we do need to assume a notion of transport:

- **Theorem 18.** *In the presence of axioms 1–5 and a root, the following are equivalent:*
- PO. (path objects) *every object Γ has a path object $P\Gamma$.*
 - PF. (path fibrations) *every fibration $\Delta \rightarrow \Gamma$ has a path fibration $P_{\Gamma}\Delta$.*
 - F. (factorisation) *every map $f : B \rightarrow A$ has a factorisation $B \simeq Lf \rightarrow A$.*
 - T. (transport) *for any $p : A \rightarrow \Gamma$ and path object $P\Gamma$ of Γ there exists a map $\tau : Lp \rightarrow A$ for $Lp := A_p \times_s P\Gamma$ such that $p\tau = t\pi_1$ and $\tau(1, rp) \simeq_{\Gamma} 1$.*
 - L. (lifting) *if we have a commutative outer square*

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ w \downarrow \wr & \simeq_{\Gamma} \nearrow l & \downarrow p \\ \Delta & \xrightarrow{\sigma} & \Gamma, \end{array}$$

then there exists a map $l : \Delta \rightarrow A$ such that the lower triangle commutes and the upper triangle commutes up to fibrewise homotopy, that is, we have $pl = \sigma$ and $lw \simeq_{\Gamma} f$. Moreover, such a lifting is unique up to fibrewise homotopy.

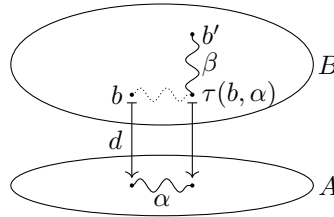
Proof. We have the following implications:

- (PO \rightarrow PF \rightarrow F \rightarrow PO) follows by Proposition 6 and the fact that the path object $P\Gamma$ is a factorisation of the diagonal $\delta_{\Gamma} : \Gamma \rightarrow \Gamma \times \Gamma$.
- (PO \rightarrow L) by the lifting theorem for path categories (Theorem 10).

- $(\mathbf{L} \rightarrow \mathbf{T})$ because a transport map is a lift for the following square:

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ w_p \downarrow \wr & \simeq_{\Gamma} \nearrow \tau & \downarrow p \\ Lp & \xrightarrow{p_p} & \Gamma, \end{array}$$

- $(\mathbf{T} \rightarrow \mathbf{PO})$ follows with induction on the number of display maps in the fibration $\Gamma \rightarrow 1$. In the base case we take the path object to be 1. In the successor case we will combine a path object $PA \rightarrow^n A \times A$ for $A \rightarrow^n 1$ with a path object $P_A B \rightarrow B \times_A B$ for $p : B \rightarrow A$ into a path object $PB \rightarrow^{n+1} B \times B$ for $B \rightarrow^{n+1} 1$. Take $Lp := B \times_p \times_s PA$ and let $\tau : Lp \rightarrow A$ be a transport map. The intuition for PB will be that it consists of quadruples (b, b', α, β) as in the following picture:



More precisely, we define PB using the two pullbacks:

$$\begin{array}{ccccc} P_A B & \xleftarrow{\pi_1} & PB & & \\ (s,t) \downarrow & & \downarrow \pi_0 & & \\ B \times_A B & \xleftarrow{(\tau(\pi_0 \pi_0, \pi_1), \pi_1 \pi_0)} & (B \times B) \times_{A \times A} P_A & \xrightarrow{\pi_1} & P_A \\ & & \pi_0 \downarrow_n & \lrcorner & \downarrow_n (s,t) \\ & & B \times B & \xrightarrow{p \times p} & A \times A \end{array}$$

and we take $r_B := (((1, 1), rp), r)$ and $(s_B, t_B) := \pi_0 \pi_0$. To see that r_B is an equivalence, we note that there is a more concise isomorphic definition of PB as the mapping path space $L_A \tau$ of $\tau : Ld \rightarrow B$ in \mathcal{C}_A :

$$L_A \tau := Ld \times_{\tau \times_s} P_A B = (B \times_p \times_s P_A) \times_{\tau \times_s} P_A B.$$

We see that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{r_B} & PB \\ w_d \downarrow \wr & & \downarrow | (\pi_0 \times 1_{P_A B}) \times 1_{P_A} \\ Ld & \xrightarrow[\sim]{\pi_0} & L_A \tau, \end{array}$$

so, with 2-out-of-3, we see that r_B is an equivalence. ◀

So, the main results for path categories turn out to be equivalent to the existence of path fibrations. Another observation is that the slice \mathcal{C}/Γ of a (rooted) display path category is generally only a display map path category, whereas $\mathcal{C}/^{\text{fib}}\Gamma$ is a rooted display map path category. So, both the rooted and the unrooted version are important regardless of the starting category, which we exploit in the following:

► **Theorem 19.** *A display map path category has weakly stable =-types, and every display map category with weakly stable =-types can be given a class of weak equivalences that turns it into a display map path category: the fibrewise homotopy equivalences. This extends to an isomorphism of the 2-categories: $\text{DispPathCat}^{\text{root}} \cong \text{DispCat}_{=\text{axi}}^{\text{root}}$.*

Proof. If we have a display map path category, then for every $A \rightarrow \Gamma$ we have a path object $P_\Gamma A \rightarrow A \times_\Gamma A$ in the rooted display path category $\mathcal{C}/^{\text{fib}}\Gamma$. This is in particular a path category, so by Proposition 13 it is a $=$ -type in $\mathcal{C}/^{\text{fib}}\Gamma$, which implies that it is a $=$ -type in \mathcal{C} . The $=$ -types are weakly stable because we see that display path objects are preserved by pullback using (the proof of) Brown’s lemma [14, page 428].

Conversely, if we have a display map category with weakly stable axiomatic $=$ -types, then we say that a map $f : A \rightarrow B$ is an equivalence if there exists a Γ such that $A \rightarrow \Gamma$ and $B \rightarrow \Gamma$ and f is a homotopy equivalence over Γ . We see that this satisfies axioms **1–4** so the main difficulty is showing that every trivial fibration has a section and that every fibration has a path fibration. For a (trivial) fibration $f : \Delta \simeq \Omega$ there exists a Γ such that f is an equivalence over Γ . So, we consider $\mathcal{C}/^{\text{fib}}\Gamma$, which has a root, and Van den Berg [6, Section 6] shows that parametrized unbased axiomatic $=$ -types make it a path category whose equivalences are again the homotopy equivalences. So, the remaining properties follow.

If there is a root, then we know that saturation holds (Theorem 9), and the fibrewise homotopy equivalences become the homotopy equivalences, so this is the only option for the weak equivalences. In a similar way as for Theorem 16 we see that the 1-cells agree. ◀

► **Proposition 20.** *A (display map) path category:*

- *has weakly stable axiomatic $\mathbb{1}$ -types iff the identity maps are homotopic to display maps;*
- *has weakly stable axiomatic Σ -types iff the display maps are closed under composition up to homotopy;*
- *has weakly stable axiomatic Π -types iff for every $B \rightarrow A \rightarrow \Gamma$ there exist $\Pi_A B \rightarrow \Gamma$ and $\text{app}_{A,B} : (\Pi_A B) \times_\Gamma A \rightarrow B$ over Γ such that for every $\sigma : \Delta \rightarrow \Gamma$ the map*

$$(\mathcal{C}/\Gamma)(\Delta, \Pi_A B) \xrightarrow{-\times_\Gamma A} (\mathcal{C}/A)(\Delta \times_\Gamma A, (\Pi_A B) \times_\Gamma A) \xrightarrow{\text{app}^\circ} (\mathcal{C}/A)(\Delta \times_\Gamma A, B),$$

is essentially surjective. Here we view the domain and codomain as groupoids, where the maps are given by pointwise homotopies up to higher homotopy. $\Pi_A B$ satisfies the function extensionality axiom iff the map is full, which implies that it is an equivalence.

Proof. For the $\mathbb{1}$ - and Σ -types, the introduction and β -axiom give equivalences $\Gamma.\mathbb{1} \simeq \Gamma$ and $\Gamma.\Sigma_A.B \simeq \Gamma.A.B$ over Γ . So, the identity and composition are homotopic to display maps. Conversely, we can define $\mathbb{1}_\Gamma$ as the display map homotopic to $1_\Gamma : \Gamma \rightarrow \Gamma$ and $\Sigma_A B$ as the display map homotopic to $B \rightarrow A \rightarrow \Gamma$. They are always weakly stable because display maps and equivalences are preserved by re-indexing.

For the Π -types, these correspond to the weak/strong homotopy exponentials of Den Besten [10]. Essentially surjective gives a form of the introduction and β -axiom with weak stability build in: if we have a substitution $\sigma : \Delta \rightarrow \Gamma$ and a term $b : \Delta.A[\sigma] \rightarrow \Gamma.B$ over $\sigma p_{A[\sigma]}$ then we get a term $\lambda b : \Delta \rightarrow \Gamma.\Pi_A B$ over σ such that $\text{app}(\lambda b, 1_{\Gamma.A})$ is homotopic to b . That the map being full implies that it is an equivalence, and that this is equivalent the axiom of function extensionality is shown by Den Besten [10, Proposition 5.4]. ◀

7 Strictification and coherence

In this last section, we use our isomorphisms to prove a coherence result: a biequivalence between a certain class of path categories and the class of (strict) models of a specific theory. First, we recall the LF condition that Lumsdaine and Warren [33] use to strictify:

► **Definition 21** (Logical framework (LF)). *A display map category satisfies LF if it has finite products and if, whenever $A \rightarrow \Gamma$ is a display map and $B \rightarrow A$ is either a display map or a product projection, the categorical dependent exponent $\Pi_A B$ exists: there exist $\Pi_A B \rightarrow \Gamma$ and $\text{app}_{A,B} : (\Pi_A B) \times_\Gamma A \rightarrow B$ over Γ such that for every $\sigma : \Delta \rightarrow \Gamma$ the map:*

$$(\mathcal{C}/\Gamma)(\Delta, \Pi_A B) \xrightarrow{-\times_\Gamma A} (\mathcal{C}/A)(\Delta \times_\Gamma A, (\Pi_A B) \times_\Gamma A) \xrightarrow{\text{app}^-} (\mathcal{C}/A)(\Delta \times_\Gamma A, B) \quad (\text{LF})$$

is an isomorphism.

This condition is very close to the existence of Π -types for (display map) path categories but distinct in two important ways: we do not ask for $\Pi_A B \rightarrow \Gamma$ to be a display map (so $\Pi_A B$ is not a type), and we ask for an isomorphism instead of a full map or equivalence. As a consequence, it captures a situation that is better viewed as a logical framework [25] or a 2-level type theory [35, 2] where the ambient theory has strong Π -types regardless of the internal theory that we consider.

► **Theorem 22** (Path categories provide models). *Let \mathcal{C} be a structured path category satisfying the LF condition. Then, seen as a structured display map category, it is equivalent in $\text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{structured}, \text{LF}, \text{root}}$ to an object $\mathcal{C}_!$ of $\text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{strict}, \text{LF}, \text{root}}$ (a strict model).³*

This follows by combining Theorem 16 with the work of Lumsdaine and Warren [33], and Bocquet [12]. As they work in different frameworks, we provide the details in the full version [39, Appendix D].

► **Corollary 23** (Biequivalence of path categories and axiomatic Martin-Löf type theories). *There is a biequivalence $\text{PathCat}^{\text{structured}, \text{LF}} \simeq \text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{strict}, \text{LF}, \text{root}}$.*

Proof. There is a choice of an equivalence $\eta_{\mathcal{C}} : \mathcal{C} \simeq \mathcal{C}_!$ in $\text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{structured}, \text{LF}, \text{root}}$, natural in \mathcal{C} : the unit of the left-adjoint splitting. Now, if \mathcal{D} is an object of $\text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{strict}, \text{LF}, \text{root}}$ then $\eta_{\mathcal{D}}$ – where \mathcal{D} is included into $\text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{structured}, \text{LF}, \text{root}}$ – witnesses an equivalence $\mathcal{D}_! \simeq \mathcal{D}$ living in $\text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{strict}, \text{LF}, \text{root}}$. Therefore there is a biequivalence which, pre-composed by the restriction to the structured case of the biequivalence at Theorem 16, yields the statement. ◀

► **Corollary 24** (Coherence for path categories). *Every structured LF path category is equivalent to a strict LF path category.*

See [39, Appendix D] for the proof. These results work for general type formers, so we also get statements for display map path categories. The diagram summarises this:

$$\begin{array}{ccccc} \text{PathCat}_{(\Pi_{\text{axi}})}^{\text{structured}, \text{LF}} & \cong & \text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}, (\Pi_{\text{axi}})}^{\text{structured}, \text{LF}, \text{root}} & \simeq & \text{DispCat}_{=\text{axi}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}, (\Pi_{\text{axi}})}^{\text{strict}, \text{LF}, \text{root}} \\ \text{forget} \left(\uparrow \downarrow \right) \text{cofree} & & \text{free} \left(\uparrow \downarrow \right) \text{forget} & & \\ \text{DispPathCat}_{(\mathbb{1}_{\text{axi}}, (\Sigma_{\text{axi}}), (\Pi_{\text{axi}}))}^{\text{structured}, \text{LF}, \text{root}} & \cong & \text{DispCat}_{=\text{axi}, (\mathbb{1}_{\text{axi}}), (\Sigma_{\text{axi}}), (\Pi_{\text{axi}})}^{\text{structured}, \text{LF}, \text{root}} & \simeq & \text{DispCat}_{=\text{axi}, (\mathbb{1}_{\text{axi}}), (\Sigma_{\text{axi}}), (\Pi_{\text{axi}})}^{\text{strict}, \text{LF}, \text{root}} \end{array}$$

Note that the adjunctions, although labelled differently, are the same: depending on the point of view, the left adjoint is either forgetful (it forgets display maps while remembering the fibrations), or free (it takes the identity and composition closure of the display maps), while the right adjoint is either cofree (it takes every fibration to be a display map) or forgetful (it forgets the $\mathbb{1}$ and Σ structure).

³ Note that if we are willing to assume the axiom of choice in the metatheory, we do not need to assume that the display map category is structured.

8 Conclusion

We have established a biequivalence between path categories and models of axiomatic \equiv -types extended with extensional $\mathbb{1}$ - and Σ -types. This result provides a clean categorical formulation of the semantics of axiomatic \equiv -types, allowing us to construct strict models directly from weak homotopy-theoretic structures. Furthermore, we introduced the novel notion of a display map path category. This more fine-grained structure distinguishes between types and telescopes, and offers a semantics for minimal axiomatic type theories that do not include extensional type formers. As we saw, this leaves room to add other axiomatic type formers, such as axiomatic $\mathbb{1}$ -, Σ -, and Π -types, providing a modular semantics for axiomatic type theory. Our framework lays the groundwork for future investigations into the semantics of higher inductive types and computational interpretations of univalence.

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A Full syntax and semantics of axiomatic type theory

The structural rules of axiomatic type theory are:

$$\frac{}{\epsilon \text{ ctx}} \text{ ctx-empty}, \quad x \text{ fresh} \frac{\Gamma \vdash A \text{ type}}{\Gamma, x : A \text{ ctx}} \text{ ctx-extend}, \quad \frac{\Gamma, x : A, \Omega \text{ ctx}}{\Gamma, x : A, \Omega \vdash x : A} \text{ var}.$$

These are modelled by any display map category with a designated empty context. In a rooted display map category this is the terminal object.

A.1 =-types

The rules for axiomatic based =-types are:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash a : A \quad \Gamma \vdash a' : A}{\Gamma \vdash a =_A a' \text{ type}} =\text{F}, \quad \frac{\Gamma \vdash a : A}{\Gamma \vdash \text{refl}(a) : a =_A a} =\text{I},$$

$$\frac{\Gamma, x' : A, \chi : a =_A x' \vdash C \text{ type} \quad \Gamma \vdash d : C[a/x', \text{refl}(a)/\chi] \quad \Gamma \vdash \alpha : a =_A a'}{\Gamma \vdash \text{ind}_d^{\bar{}}(\alpha) : C[a/x, a'/x', \alpha/\chi]} =\text{E}_{\text{based}}, \quad \frac{\Gamma, x' : A, \chi : a =_A x' \vdash C \text{ type} \quad \Gamma \vdash d : C[a/x', \text{refl}(a)/\chi]}{\Gamma \vdash \beta_d^{\bar{}} : \text{ind}_d^{\bar{}}(\text{refl}(a)) = d} =\beta_{\text{based}}^{\text{ax}}.$$

This induces a notion in display map categories:

► **Definition 25** (=types). We say that a display map category has axiomatic (based) =-types if, for every type A in context Γ , there exists:

F. a type Id_A in context $\Gamma.A.A^\nabla$,

I. a term refl_A of type $\text{Id}_A[\delta_A]$ in context $\Gamma.A$, where δ_A is the diagonal map $(1_{\Gamma.A}, 1_{\Gamma.A}) : \Gamma.A \rightarrow \Gamma.A.A^\nabla$:

$$\begin{array}{ccc} \Gamma.A.\text{Id}_A[\delta_A] & \xrightarrow{\delta_A^\nabla} & \Gamma.A.A^\nabla.\text{Id}_A \\ \text{refl}_A \uparrow \downarrow & \lrcorner & \downarrow \\ \Gamma.A & \xrightarrow{\delta_A} & \Gamma.A.A^\nabla, \end{array}$$

E. for every term a of type A , every type C in context $\Gamma.A.\text{Id}_A[a^\nabla]$, and every term d of type $C[a, \text{refl}[a]]$, a term ind_d^- of C :

$$\begin{array}{ccc} \Gamma.C[a, \text{refl}[a]] & \xrightarrow{(a, \text{refl}[a])^\nabla} & \Gamma.A.\text{Id}_A[a^\nabla].C \\ d \uparrow \downarrow & \lrcorner & \downarrow \uparrow \text{ind}_d^- \\ \Gamma & \xrightarrow{(a, \text{refl}[a])} & \Gamma.A.\text{Id}_A[a^\nabla], \end{array}$$

β . for every a, C, d as before, $\text{ind}_d^-[a, \text{refl}[a]]$ is propositionally equal to d , that is, we have a term β_d^- of type $\text{Id}_{C[a, \text{refl}[a]]}[\text{ind}_d^-[a, \text{refl}[a]], d]$.

Any such choice of data for a given type A is also called an axiomatic =-type for A . Additionally, we call an =-type Id_A intensional if it satisfies the β -rule: for every a, C, d as before we have that $\beta_d^- = \text{refl}[d]$, and therefore that $\text{ind}_d^-[a, \text{refl}[a]] = d$. We call the =-type extensional if it also satisfies the η -rule: for every C and term c of C we have that $\text{ind}_{c[a, \text{refl}[a]]}^- = c$.

In a strict display map category, we call a choice of =-types (strictly) stable if for every type A in context Γ and every morphism $\sigma : \Delta \rightarrow \Gamma$, we have:

$$\begin{array}{ll} \text{Id}_A[\sigma^{\nabla\nabla}] = \text{Id}_{A[\sigma]}, & \text{ind}_d^-[\sigma^{\nabla\nabla}] = \text{ind}_{d[\sigma^\nabla]}^-, \\ \text{refl}_A[\sigma^\nabla] = \text{refl}_{A[\sigma]}, & \beta_d^-[\sigma^\nabla] = \beta_{d[\sigma^\nabla]}^-. \end{array}$$

We say that the =-types are weakly stable if for every type A in context Γ and every morphism $\sigma : \Delta \rightarrow \Gamma$ we can equip $\text{Id}_A[\sigma^{\nabla\nabla}]$ and

$$\Delta.A[\sigma] \xrightarrow{\text{refl}_A[\sigma^\nabla]} \Delta.A[\sigma].\text{Id}_A[\delta_A][\sigma^\nabla] \cong \Delta.A[\sigma].\text{Id}_A[\sigma^{\nabla\nabla}][\delta_{A[\sigma]}],$$

with the additional data required to form a =-type.

A.2 1-types

The rules for axiomatic 1-types are:

$$\begin{array}{ll} \frac{\Gamma \text{ ctx}}{\Gamma \vdash \mathbb{1} \text{ type}} \mathbb{1}\text{F}, & \frac{}{\Gamma \vdash 0_{\mathbb{1}} : \mathbb{1}} \mathbb{1}\text{I}, \\ \frac{\Gamma, w : \mathbb{1} \vdash C \text{ type}}{\Gamma \vdash d : C[0_{\mathbb{1}}/w]} & \frac{\Gamma, w : \mathbb{1} \vdash C \text{ type}}{\Gamma \vdash d : C[0_{\mathbb{1}}/w]} \\ \frac{\Gamma \vdash i : \mathbb{1}}{\Gamma \vdash \text{ind}_d^1(i) : C[i/w]} \mathbb{1}\text{E}, & \frac{\Gamma \vdash d : C[0_{\mathbb{1}}/w]}{\Gamma \vdash \beta_d^1 : \text{ind}_d^1(0_{\mathbb{1}}) = d} \mathbb{1}\beta^{\text{axi}}. \end{array}$$

This induces a notion in display map categories:

► **Definition 26** ($\mathbb{1}$ -types). *We say that a display map category has axiomatic $\mathbb{1}$ -types if, for every context Γ , there exists:*

- F. a type $\mathbb{1}_\Gamma$ in context Γ ,
- I. a term $0_\mathbb{1}$ of type $\mathbb{1}$:

$$\begin{array}{c} \Gamma.\mathbb{1} \\ \downarrow \kappa \\ \Gamma, \end{array} \left. \vphantom{\begin{array}{c} \Gamma.\mathbb{1} \\ \downarrow \kappa \\ \Gamma, \end{array}} \right\} 0_\mathbb{1}$$

E. for every type C in context $\Gamma.\mathbb{1}$ and every term d of type $C[\mathbb{1}]$ a term $\text{ind}_d^\mathbb{1}$ of C :

$$\begin{array}{ccc} \Gamma.C[0_\mathbb{1}] & \xrightarrow{\text{pair}^\nabla} & \Gamma.\mathbb{1}.C \\ d \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & \lrcorner & \downarrow \kappa \\ \Gamma & \xrightarrow{0_\mathbb{1}} & \Gamma.\mathbb{1}, \end{array} \left. \vphantom{\begin{array}{ccc} \Gamma.C[0_\mathbb{1}] & \xrightarrow{\text{pair}^\nabla} & \Gamma.\mathbb{1}.C \\ d \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) & \lrcorner & \downarrow \kappa \\ \Gamma & \xrightarrow{0_\mathbb{1}} & \Gamma.\mathbb{1}, \end{array}} \right\} \text{ind}_d^\mathbb{1}$$

β . for every C and d as before, $\text{ind}_d^\mathbb{1}[0_\mathbb{1}]$ is propositionally equal to d , that is, we have a term $\beta_d^\mathbb{1}$ of type $\text{Id}_{C[0_\mathbb{1}]}[\text{ind}_d^\mathbb{1}[0_\mathbb{1}], d]$.

We call a $\mathbb{1}$ -type $\mathbb{1}_\Gamma$ *intensional* if it satisfies the β -rule: for every C and d we have that $\beta_d^\mathbb{1} = \text{refl}[d]$, and therefore that $\text{ind}_d^\mathbb{1}[0_\mathbb{1}] = d$. We call the $\mathbb{1}$ -type *extensional* if it also satisfies the η -rule: for every C and c of type C we have $\text{ind}_c^\mathbb{1}[0_\mathbb{1}] = c$.

In a strict display map category, we call a choice of $\mathbb{1}$ -types (*strictly*) *stable* if for every context Γ and every morphism $\sigma : \Delta \rightarrow \Gamma$, we have:

$$\begin{array}{ll} (\mathbb{1}_\Gamma)[\sigma] = \mathbb{1}_\Delta, & \text{ind}_d^\mathbb{1}[\sigma^\nabla] = \text{ind}_{d[\sigma^\nabla]}^\mathbb{1}, \\ 0_\mathbb{1}[\sigma] = 0_\Delta, & \beta_d^\mathbb{1}[\sigma^\nabla] = \beta_{d[\sigma^\nabla]}^\mathbb{1}. \end{array}$$

We say that the $\mathbb{1}$ -types are *weakly stable* if for every context Γ and every morphism $\sigma : \Delta \rightarrow \Gamma$, we can equip $\mathbb{1}_\Gamma[\sigma]$ and $0_\mathbb{1}[\sigma]$ with the additional data required to form a $\mathbb{1}$ -type.

A.3 Σ -types

The rules for axiomatic Σ -types are:

$$\begin{array}{ll} \frac{\Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash \Sigma_{x:A} B \text{ type}} \Sigma\text{F}, & \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B[a/x]}{\Gamma \vdash \langle a, b \rangle : \Sigma_{x:A} B} \Sigma\text{I}, \\ \\ \frac{\begin{array}{l} \Gamma, w : \Sigma_{x:A} B \vdash C \text{ type} \\ \Gamma, x : A, y : B \vdash d : C[\langle x, y \rangle/w] \\ \Gamma \vdash p : \Sigma_{x:A} B \end{array}}{\Gamma \vdash \text{ind}_d^\Sigma(p) : C[p/w]} \Sigma\text{E}, & \frac{\begin{array}{l} \Gamma, w : \Sigma_{x:A} B \vdash C \text{ type} \\ \Gamma, x : A, y : B \vdash d : C[\langle x, y \rangle/w] \\ \Gamma \vdash a : A \quad \Gamma \vdash b : B[a/x] \end{array}}{\Gamma \vdash \beta_d^\Sigma(a, b) : \text{ind}_d^\Sigma(\langle a, b \rangle) = d[a/x, b/y]} \Sigma\beta^{\text{axi}}. \end{array}$$

This induces a notion in display map categories:

► **Definition 27** (Σ -types). *We say that a display map category has axiomatic Σ -types if, for every type A in context Γ and type B in context $\Gamma.A$, there exists:*

- F. a type $\Sigma_A B$ in context Γ ,

I. a (generalised) term $\text{pair}_{A,B}$ of type $\Sigma_A B$ for $p_{APB} : \Gamma.A.B \twoheadrightarrow \Gamma$:

$$\begin{array}{ccc} \Gamma.A.B & \xrightarrow{\text{pair}_{A,B}} & \Gamma.\Sigma_A B \\ \downarrow & & \downarrow \\ \Gamma.A & \longrightarrow & \Gamma, \end{array}$$

E. for every type C in context $\Gamma.\Sigma_A B$ and every term d of type $C[\text{pair}]$ a term ind_d^Σ of C :

$$\begin{array}{ccc} \Gamma.A.B.C[\text{pair}] & \xrightarrow{\text{pair}^\nabla} & \Gamma.\Sigma_A B.C \\ d \left(\downarrow \right) & \lrcorner & \left(\downarrow \right) \text{ind}_d^\Sigma \\ \Gamma.A.B & \xrightarrow{\text{pair}} & \Gamma.\Sigma_A B, \end{array}$$

β . for every C and d as before, $\text{ind}_d^\Sigma[\text{pair}]$ is propositionally equal to d , that is, we have a term β_d^Σ of type $\text{Id}_{C[\text{pair}]}[\text{ind}_d^\Sigma[\text{pair}], d]$.

We call a Σ -type $\Sigma_A B$ *intensional* if it satisfies the β -rule: for every C and d we have that $\beta_d^\Sigma = \text{refl}[d]$, and therefore that $\text{ind}_d^\Sigma[\text{pair}] = d$. We call the Σ -type *extensional* if it also satisfies the η -rule: for every C and c of type C we have $\text{ind}_c^\Sigma[\text{pair}] = c$.

In a strict display map category, we call a choice of Σ -types (strictly) *stable* if for every type A in context Γ , every type B in context $\Gamma.A$ and every morphism $\sigma : \Delta \rightarrow \Gamma$, we have:

$$\begin{array}{ll} (\Sigma_A B)[\sigma] = \Sigma_{A[\sigma]}(B[\sigma^\nabla]), & \text{ind}_d^\Sigma[\sigma^\nabla] = \text{ind}_{d[\sigma^\nabla]}^\Sigma, \\ \text{pair}_{A,B}[\sigma] = \text{pair}_{A[\sigma],B[\sigma^\nabla]}, & \beta_d^\Sigma[\sigma^\nabla] = \beta_{d[\sigma^\nabla]}^\Sigma. \end{array}$$

We say that the Σ -types are *weakly stable* if for every type A in context Γ , every type B in context $\Gamma.A$, and every morphism $\sigma : \Delta \rightarrow \Gamma$ we can equip $\Sigma_A[\sigma]$ and $\text{pair}[\sigma]$ with the additional data required to form a Σ -type.

A.4 Π -types

The rules for axiomatic Π -types are:

$$\begin{array}{ll} \frac{\Gamma, x : A \vdash B \text{ type}}{\Gamma \vdash \Pi_{x:A} B \text{ type}} \text{PIIF}, & \frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda_{x:A} b : \Pi_{x:A} B} \text{PIII}, \\ \frac{\Gamma \vdash f : \Pi_{x:A} B}{\Gamma \vdash f a : B} \text{PIIE}, & \frac{\Gamma, x : A \vdash b : B}{\Gamma \vdash \beta_a^\Pi(b) : (\lambda_{x:A} b) a = b[a/x]} \text{PII}^{\beta\text{axi}}. \end{array}$$

This induces a notion in display map categories:

► **Definition 28** (Π -types). We say that a display map category has axiomatic Π -types if, for every type A in context Γ and type B in context $\Gamma.A$, there exists:

F. a type $\Pi_A B$ in context Γ .

I. for every term b of type B in context $\Gamma.A$ a term λb of type $\Pi_A B$ in context Γ :

$$\begin{array}{ccc} \Gamma.A.B & & \Gamma.\Pi_A B \\ b \left(\downarrow \right) & & \left(\downarrow \right) \lambda b \\ \Gamma.A & \longrightarrow & \Gamma, \end{array}$$

E. a term $\mathbf{app}_{A,B}$ of type B for $p_{\Pi_A B}^\nabla : \Gamma.\Pi_A B.A^\nabla \rightarrow \Gamma.A$:

$$\begin{array}{ccc} \Gamma.\Pi_A B.A^\nabla & \xrightarrow{\mathbf{app}_{A,B}} & \Gamma.A.B \\ & \searrow p_{\Pi_A B}^\nabla & \swarrow \\ & \Gamma.A, & \end{array}$$

β . for any term b of type B , $\mathbf{app} \circ (\lambda b)^\nabla$ is propositionally equal to b , that is, we have a term β_b^Π of type $\mathbf{Id}_B[\mathbf{app} \circ (\lambda b)^\nabla, b]$.

We call a Π -type *intensional* if it satisfies the β -rule: for every b we have that $\beta_b^\Pi = \mathbf{refl}[b]$, and therefore that $\mathbf{app} \circ (\lambda b)^\nabla = b$. We call the Π -type *extensional* if it also satisfies the η -rule: for every f of $\Pi_A B$ in context Γ we have $\lambda(\mathbf{app} \circ f^\nabla) = f$.

In a strict display map category, we call a choice of Π -types *strictly stable* under our choices for substitutions if for every B over $\Gamma.A$ and $\sigma : \Delta \rightarrow \Gamma$:

$$\begin{aligned} (\Pi_A B)[\sigma] &= \Pi_{A[\sigma]}(B[\sigma^\nabla]), & \mathbf{app}_{A,B}[\sigma^\nabla] &= \mathbf{app}_{A[\sigma],B[\sigma^\nabla]}, \\ (\lambda b)[\sigma] &= \lambda(b[\sigma^\nabla]), & \beta_d^\Pi[\sigma^\nabla] &= \beta_{b[\sigma^\nabla]}^\Pi. \end{aligned}$$

We say that the Π -types are *weakly stable* if for any B over $\Gamma.A$ and $\sigma : \Delta \rightarrow \Gamma$ we can equip $(\Pi_A B)[\sigma]$ and $\mathbf{app}_{A,B}[\sigma^\nabla]$ with the additional data required to form a Π -type.

In addition, we consider the (*dependent*) *function extensionality axiom* stating for every $f, f' : \Pi_{x:A} B$ that the canonical function $(f = f') \rightarrow \Pi_{x:A}(f x = f' x)$ is an equivalence [23, 32]. This is where our naming convention of axiomatic/intensional/extensional type formers can be a bit confusing: the axiom of function extensionality is independent from Π -types being extensional (satisfying β - and η -rules). However, if the type theory is extensional – every type former is extensional or equivalently the $=$ -types are extensional – then the axiom of function extensionality holds.

B Additional details on the 2-isomorphism

In this appendix we present the ingredients involved in the statement of the 2-isomorphism of between the 2-categories $\mathbf{PathCat}$ and $\mathbf{DispCat}_{=_{\text{axi}}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{root}}$. In Section 3 and Appendix A we recalled the notion of a display map category and described how it can be equipped with the right structure to interpret a theory with specified type formers. Similarly, in Section 4 we outlined the notion of a path category. Here we expand on this by giving a detailed account of the 2-categories involved.

Display map categories. The 2-category $\mathbf{DispCat}$ consists of:

0-cells. The display map categories.

1-cells. If \mathcal{C} and \mathcal{D} are display map categories, the 1-cells from \mathcal{C} to \mathcal{C}' are functors $G : \mathcal{C} \rightarrow \mathcal{C}'$ that map display maps to display maps and pullback squares of display maps into pullback squares of display maps.

2-cells. If G and G' are parallel 1-cells from \mathcal{C} to \mathcal{D} , the 2-cells from G to G' are natural transformations $G \Rightarrow G'$.

This can be specialised to several suc-2-categories where the 0-cells are endowed with additional weakly stable type formers, and the 1-cells weakly preserve them. For example, we write $\mathbf{DispCat}_{=_{\text{axi}}}/\mathbf{DispCat}_{=_{\text{int}}}/\mathbf{DispCat}_{=_{\text{ext}}}$ for the 2-categories with:

0-cells. The display map categories equipped with weakly stable based axiomatic/intensional/extensional $=$ -types.

1-cells. If \mathcal{C} and \mathcal{C}' are 0-cells then the 1-cells from \mathcal{C} to \mathcal{C}' are those 1-cells of DispCat that weakly preserve the based axiomatic/intensional/extensional $=$ -types – see Section 3.

2-cells. The ones of DispCat .

We follow the same pattern for other type formers ($\mathbb{1}$, Σ , Π). We write $\text{DispCat}^{\text{root}}$ for the sub-2-category where the 0-cells are rooted, and the 1-cells preserve the terminal object. Similarly, $\text{DispCat}^{\text{LF}}$ is the sub-2-category where the 0-cells satisfy the logical framework (LF) condition, and the 1-cells preserve the products and dependent exponents present. All of these restrictions can be combined in various ways; for example $\text{DispCat}_{=_{\text{axi}}, \mathbb{1}_{\text{ext}}, \Sigma_{\text{ext}}}^{\text{root}}$.

Lastly, we write $\text{DispCat}^{\text{structured}}$ and $\text{DispCat}^{\text{strict}}$ for the full sub-2-categories of DispCat spanned by structured and strict display map categories, respectively. If this is combined with type formers, then the 0-cells of $\text{DispCat}^{\text{structured}}$ also have choices for the type formers, but they are still only weakly stable under re-indexing, and only weakly preserved by 1-cells. The 0-cells of $\text{DispCat}^{\text{strict}}$ have choices for the type formers, and they are strictly stable under re-indexing, but the 1-cells only weakly preserve them. Note that these are the same choices that Clairambault and Dybjer [16] make for their 2-categories.

Path categories. The 2-category PathCat consists of:

0-cells. Path categories.

1-cells. Functors preserving fibrations, trivial fibrations, pullbacks of fibrations, and the terminal object (and therefore also equivalences and path objects [14]).

2-cells. Natural transformations.

Again, in $\text{PathCat}^{\text{LF}}$ the 1-cells preserve the products and dependent exponents present, while $\text{PathCat}^{\text{structured}}$ and $\text{PathCat}^{\text{strict}}$ are full sub-2-categories.