

# Lax Modal Lambda Calculi

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## Abstract

Intuitionistic modal logics (IMLs) extend intuitionistic propositional logic with modalities such as the box and diamond connectives. Advances in the study of IMLs have inspired several applications in programming languages via the development of corresponding type theories with modalities. Until recently, IMLs with diamonds have been misunderstood as somewhat peculiar and unstable, causing the development of type theories with diamonds to lag behind type theories with boxes. In this article, we develop a family of typed-lambda calculi corresponding to sublogics of a peculiar IML with diamonds known as Lax logic. These calculi provide a modal logical foundation for various strong functors in typed-functional programming. We present possible-world and categorical semantics for these calculi and constructively prove normalization, equational completeness and proof-theoretic inadmissibility results. Our main results have been formalized using the proof assistant Agda.

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## 1 Introduction

In modal logic, a modality is a unary logical connective that exhibits some logical properties. Two such modalities are the connectives  $\Box$  (“box”) and  $\Diamond$  (“diamond”). Intuitively, a formula  $\Box A$  can be understood as “necessarily  $A$ ” and a formula  $\Diamond A$  as “possibly  $A$ ”. In classical modal logic, the most basic logic K extends classical propositional logic (CPL) with the box modality, the *necessitation* rule (if  $A$  is a theorem then so is  $\Box A$ ) and the K axiom ( $\Box(A \Rightarrow B) \Rightarrow \Box A \Rightarrow \Box B$ ). The diamond modality can be encoded in this logic as a dual of the box modality:  $\Diamond A \equiv \neg \Box \neg A$ . That is,  $\Diamond A$  is true if and only if  $\neg \Box \neg A$  is true.

In intuitionistic modal logic (IML), there is no consensus on one logic as the most basic logic. We instead find a variety of different IMLs based on different motivations. The  $\Box$  and  $\Diamond$  modalities are independent connectives in IML [35, Requirement 5], just as  $\wedge$  and  $\vee$  are independent connectives that are not inter-definable in intuitionistic propositional logic (IPL). In contrast to  $\Box$ , however, the logical properties of  $\Diamond$  vary widely in IML literature. This has misconstrued  $\Diamond$  as a controversial and unstable modality. It had been incorrectly assumed until recently that several IMLs with both  $\Box$  and  $\Diamond$  coincided (i.e. were conservative extensions of their sublogics) only in the  $\Diamond$ -free fragment, suggesting some sort of stability of  $\Box$ -only logics. Fortunately, misconceptions concerning intuitionistic diamonds have been broken in recent results [16, 18] and we are approaching a better understanding of it.

Advances in IML have led to a plethora of useful applications in programming languages through the development of corresponding type theories with modalities. Modal lambda calculi [32, 13] with box modalities have found applications in staged meta-program-



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ming [17, 31, 23], reactive programming [5], safe usage of temporal resources [2] and checking productivity of recursive definitions [10]. Two particular box axioms that have received plenty of attention in these developments are the axioms  $T : \Box A \Rightarrow A$  and  $4 : \Box A \Rightarrow \Box \Box A$ . Dual-context modal calculi [32, 24] which admit one or both of these axioms are well-understood. These calculi enjoy a rich meta-theory, including confluent reduction, normalization and a comprehensive analysis of provability. Fitch-style modal lambda calculi [13] admitting axioms  $T$  and  $4$  further enjoy an elegant categorical interpretation, possible-world semantics, and results showing how categorical models of these calculi can be constructed using possible-world semantics of their corresponding logics [38].

Lambda calculi with diamond modalities in comparison have received much less attention from the type-theoretic perspective. The controversy surrounding the diamond modality in IML appears to have restricted the development of type theories with diamonds. For example, Kavvos [25] cites Simpson’s survey [35] of IMLs and restricts the development of dual-context modal calculi “to the better-behaved, and seemingly more applicable box modality” arguing that the “computational interpretation [of  $\Diamond$ ] is not very crisp”. Recent breakthroughs in intuitionistic modal logic have made it clear that diamonds are no more problematic than boxes. In this article, we further the type-theoretic account of a special class of diamond modalities with compelling applications in programming languages.

*Propositional lax logic* (PLL) is an intuitionistic modal logic introduced independently by Fairtlough and Mendler [20] and Benton, Bierman and de Paiva [7]. PLL extends IPL with a diamond modality  $\Diamond$ , known as the lax modality, which exhibits a peculiar modal axiom  $S$  (for “strength”), in addition to axioms  $R$  (for “return”) and  $J$  (for “join”) that are well-known in classical modal logic as duals to the box axioms  $T$  and  $4$  respectively.

$$S : A \times \Diamond B \Rightarrow \Diamond(A \times B) \qquad R : A \Rightarrow \Diamond A \qquad J : \Diamond \Diamond A \Rightarrow \Diamond A$$

It is known that PLL corresponds to a typed-lambda calculus (we call  $\lambda_{ML}$ ) known as Moggi’s *monadic metalanguage* [30], which models side effects in functional programming using *strong monads* from category theory. Benton, Bierman and de Paiva [7], and later Pfenning and Davies [32], show that a judgment is provable in a natural deduction proof system for PLL if and only if there exists a typing derivation for its corresponding judgment in  $\lambda_{ML}$ . However, in contrast to the comprehensive treatment of box modalities mentioned above, there remain several gaps in our understanding of the lax modality:

1. It has remained unclear as to whether type theories can exist for sublogics of PLL or whether the axioms of PLL in combination happen to coincidentally enjoy a status of “well-behavedness”. What happens if we drop one or more of the modal axioms  $R$  and  $J$ ? Does a corresponding type theory still exist?
2. A satisfactory account of the correspondence between the possible-world semantics of PLL and the categorical semantics of  $\lambda_{ML}$  is still missing. In particular, how can we leverage the possible-world semantics of PLL to construct models of  $\lambda_{ML}$ ?

The first objective of this article is to develop corresponding type theories for sublogics of PLL that drop one or both of axioms  $R$  and  $J$ . From the type-theoretic perspective, this corresponds to type theories for non-monadic strong functors, which are prevalent in functional programming. For example, in Haskell, the array data type (in `Data.Array`) is a strong functor that neither exhibits return (axiom  $R$ ) nor join (axiom  $J$ ). Several other Haskell data types exhibit return<sup>1</sup> or join<sup>2</sup>, but not both<sup>3</sup>. We are interested in developing a uniform modal logical foundation for the axioms of non-monadic strong functors.

<sup>1</sup> <https://hackage.haskell.org/package/pointed-5.0.5/docs/Data-Pointed.html>

<sup>2</sup> <https://hackage.haskell.org/package/semigroupoids-6.0.1/docs/Data-Functor-Bind.html#g:4>

<sup>3</sup> [https://wiki.haskell.org/Why\\_not\\_Pointed%3F](https://wiki.haskell.org/Why_not_Pointed%3F)

The second objective of this article is to study the connection between possible-world semantics of PLL and its sublogics and categorical models of their corresponding type theories. Possible-world semantics for logics are concerned with provability of formulas and not about proofs themselves. Categorical models of lambda calculi, on the other hand, distinguish different proofs (terms) of the same proposition (type). Mitchell and Moggi [29] show the connection between these two different semantics using a categorical refinement of possible-world semantics for the simply-typed lambda calculus (STLC). They note that their refined semantics, which we shall call *proof-relevant possible-world semantics*, makes it “easy to devise Kripke counter-models” since they “seem to support a set-like intuition about lambda terms better than arbitrary cartesian closed categories”. We wish to achieve this technical convenience in model construction for all the modal lambda calculi in this article.

Towards our first objective, we formulate three new modal lambda calculi as subsystems of  $\lambda_{\text{ML}}$ :  $\lambda_{\text{SL}}$ ,  $\lambda_{\text{SRL}}$ ,  $\lambda_{\text{SJL}}$ . The calculus  $\lambda_{\text{SL}}$  models *strong* functors and corresponds to a logic SL (for “S-lax Logic”) that admits axiom S, but neither R nor J. The calculus  $\lambda_{\text{SRL}}$  models strong *pointed* functors and corresponds to a logic SRL (for “SR-lax Logic”) that admits axioms S and R, but not J. The calculus  $\lambda_{\text{SJL}}$  models strong *semimonads* and corresponds to a logic SJL (for “SJ-lax Logic”) that admits axioms S and J, but not R. We refer to all four calculi collectively as *lax modal lambda calculi*. Towards our second objective, we extend Mitchell and Moggi’s proof-relevant possible-world semantics to lax modal lambda calculi and show that it is complete for their equational theories. We further show that all four calculi are normalizing by constructing *Normalization by Evaluation* models as instances of possible-world semantics and prove completeness and inadmissibility results as corollaries. All the theorems in this article have been verified correct [37] using the proof assistant Agda [1].

## 2 Overview of PLL and its corresponding lambda calculus $\lambda_{\text{ML}}$

In this section, we define the syntax and semantics of PLL and its sublogics as extensions of the *negative*, i.e. disjunction and absurdity-free, fragment of IPL. This section is a summary of the background presumed in this article and is based on previously published work [20, 30].

### 2.1 Syntax and semantics of PLL

**Syntax.** The language of (the negative fragment of) PLL consists of formulas defined inductively by propositional atoms ( $p, q, r$ , etc.), a constant  $\mathbf{1}$  and logical connectives  $\times$ ,  $\Rightarrow$  and  $\Diamond$ . The connective  $\Diamond$  has the highest operator precedence, and is followed by  $\times$  and  $\Rightarrow$ . Following the usual convention, we suppose that  $\times$  and  $\Rightarrow$  associate to the right.

$$\text{Prop} \quad A, B ::= p, q, r, \dots \mid \mathbf{1} \mid A \times B \mid A \Rightarrow B \mid \Diamond A \quad \text{Ctx} \quad \Gamma, \Delta ::= \cdot \mid \Gamma, A$$

The constant  $\mathbf{1}$  denotes universal truth, the binary connectives  $\times$  and  $\Rightarrow$  respectively denote conjunction and implication, and the unary connective  $\Diamond$  denotes the lax modality. Intuitively, a formula  $\Diamond A$  may be understood as qualifying the truth of formula  $A$  under *some* constraint. A context  $\Gamma$  is a multiset of formulas  $A_1, A_2, \dots, A_n$ , where  $\cdot$  denotes the empty context.

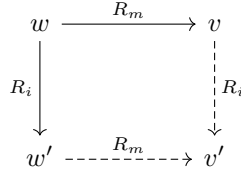
A Hilbert-style axiomatization of PLL can be given by extending the usual axioms and rules of deduction for IPL with the modal axioms S, R, and J in Section 1.

**Semantics.** The possible-world semantics of PLL defines the truth of PLL-formulas in a model using gadgets known as *frames*. A PLL-frame  $F = (W, R_i, R_m)$  is a triple that consists of a set  $W$  of *worlds* and two reflexive-transitive relations  $R_i$  (for “intuitionistic”) and  $R_m$  (for “modal”) on worlds satisfying two compatibility conditions:

- Forward confluence:  $R_i^{-1}; R_m \subseteq R_m; R_i^{-1}$
- Inclusion:  $R_m \subseteq R_i$

The relation  $R_i^{-1}$  is the converse of relation  $R_i$ , and is defined as  $R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\}$ . The operator  $;$  denotes composition of relations and is defined for two relations  $R_1$  and  $R_2$  on worlds as  $R_1; R_2 = \{(x, z) \mid \text{there exists } y \in W \text{ such that } (x, y) \in R_1 \text{ and } (y, z) \in R_2\}$ .

We may intuitively understand worlds as nodes in a graph denoting the “state of assumptions”, relation  $R_i$  as paths denoting increase in assumptions, and relation  $R_m$  as paths denoting constraining of assumptions. That is,  $w R_i w'$  denotes the increase in assumptions from world  $w$  to  $w'$ , and  $w R_m v$  denotes a constraining of  $w$  by  $v$  such that  $v$  is reachable from  $w$  when the constraint can be satisfied. Under this reading, the inclusion condition  $R_m \subseteq R_i$  states that imposing a constraint increases assumptions.



The forward confluence condition  $R_i^{-1}; R_m \subseteq R_m; R_i^{-1}$  states that constraints can be “transported” over an increase in assumptions. It can be visualized as depicted above, where the dotted lines represent “there exists”. This condition does not appear in Fairtlough and Mendler’s original work [20], but can be found in earlier work on intuitionistic diamonds by Božić and Došen [12, §8] and Plotkin and Stirling [33]. It simplifies the interpretation of  $\diamond$  and is satisfied by all the models we will construct in this article to prove completeness. We return to the discussion on forward confluence in Section 6.

A model  $\mathcal{M} = (F, V)$  couples a frame  $F$  with a *valuation* function  $V$  that assigns to each propositional atom  $p$  a set  $V(p)$  of worlds hereditary in  $R_i$ , i.e. if  $w R_i w'$  and  $w \in V(p)$  then  $w' \in V(p)$ . The truth of a formula in a model  $\mathcal{M}$  is defined by the *satisfaction* relation  $\Vdash$  for a given world  $w \in W$  by induction on a formula as:

- $\mathcal{M}, w \Vdash p$       iff  $w \in V(p)$
- $\mathcal{M}, w \Vdash 1$       iff true
- $\mathcal{M}, w \Vdash A \times B$  iff  $\mathcal{M}, w \Vdash A$  and  $\mathcal{M}, w \Vdash B$
- $\mathcal{M}, w \Vdash A \Rightarrow B$  iff for all  $w' \in W$  such that  $w R_i w'$ ,  $\mathcal{M}, w' \Vdash A$  implies  $\mathcal{M}, w' \Vdash B$
- $\mathcal{M}, w \Vdash \diamond A$     iff there exists  $v \in W$  with  $w R_m v$  and  $\mathcal{M}, v \Vdash A$

We write  $\mathcal{M}, w \Vdash \Gamma$  to denote  $\mathcal{M}, w \Vdash A_i$  for all formulas  $A_i$  with  $1 \leq i \leq n$  in context  $\Gamma = A_1, \dots, A_n$ , and write  $\Gamma \models A$  to denote  $\mathcal{M}, w \Vdash \Gamma$  implies  $\mathcal{M}, w \Vdash A$  for all worlds  $w$  in all models  $\mathcal{M}$ . Furthermore, we write  $\mathcal{M} \models A$  to denote  $\mathcal{M}, w \Vdash A$  for all worlds  $w$  in  $\mathcal{M}$ .

The soundness of PLL for its semantics can be shown using the following key properties:

- **Proposition 1.** *For an arbitrary model  $\mathcal{M}$  of PLL*
- *if  $w R_i w'$  and  $\mathcal{M}, w \Vdash A$  then  $\mathcal{M}, w' \Vdash A$ , for all worlds  $w, w'$  and formulas  $A$*
- $\mathcal{M} \models A \times \diamond B \Rightarrow \diamond(A \times B)$ , for all formulas  $A, B$
- $\mathcal{M} \models A \Rightarrow \diamond A$ , for all formulas  $A$
- $\mathcal{M} \models \diamond \diamond A \Rightarrow \diamond A$ , for all formulas  $A$

**Proof.** The first property, sometimes called “monotonicity”, states that the truth of a formula  $A$  persists as knowledge increases. This property can be proved by induction on the formula  $A$ , using the forward confluence condition for the case of  $\diamond A$ . The remaining properties can be proved using the definition of the relations  $\models$  and  $\Vdash$ , by respectively using the inclusion condition  $R_m \subseteq R_i$ , reflexivity of  $R_m$ , and transitivity of  $R_m$ . ◀

## 2.2 Syntax and semantics of $\lambda_{\text{ML}}$

**Syntax.** The calculus  $\lambda_{\text{ML}}$  is a typed  $\lambda$ -calculus that was developed by Moggi [30] before PLL. The language of  $\lambda_{\text{ML}}$  consists of types, contexts and terms, and can be understood as an extension of STLC with a unary type constructor  $\diamond$  that exhibits the PLL axioms S, R and J. Types and contexts in  $\lambda_{\text{ML}}$  are defined inductively by the following grammars:

$$\text{Ty } A, B ::= \iota \mid 1 \mid A \times B \mid A \Rightarrow B \mid \diamond A \quad \text{Ctx } \Gamma, \Delta ::= \cdot \mid \Gamma, x : A \quad (x \text{ not in } \Gamma)$$

The type  $\iota$  denotes an uninterpreted base (or “ground”) type,  $1$  denotes the unit type,  $A \times B$  denotes product types,  $A \Rightarrow B$  denotes function types, and  $\diamond A$  denotes *modal* types. A modal type  $\diamond A$  can be understood as the type of a computation that performs some side-effects to return a value of type  $A$ . A context (or “typing environment”)  $\Gamma$  is a list  $x_1 : A_1, \dots, x_n : A_n$  of unique type assignments to variables, where  $\cdot$  denotes the empty context.

The term language of STLC consists of variables ( $x, y, \dots$ ) and term constructs for the unit type (**unit**), product types (**pair**, **fst**, **snd**) and function types (**lam**, **app**). The term language of  $\lambda_{\text{ML}}$  extends that of STLC with the constructs **return<sub>ML</sub>** and **let<sub>ML</sub>** for the modal types. The *typing judgments*  $\Gamma \vdash t : A$  defined in Figure 1 identify *well-typed* terms in  $\lambda_{\text{ML}}$ . We say that a term  $t$  is *well-typed for type  $A$  under context  $\Gamma$*  when there exists a derivation, called the *typing derivation*, of the typing judgment  $\Gamma \vdash t : A$ . In this article, we are only concerned with well-typed terms and assume that every term  $t$  is well-typed for some type  $A$  under some context  $\Gamma$ . The *equality judgments*  $\Gamma \vdash t \sim t' : A$  defined in Figure 1 identify the equivalence between well-typed terms  $t$  and  $t'$  and specify the *equational theory* for  $\lambda_{\text{ML}}$ .

In Figure 1, the notation  $t[u/x]$  denotes the *substitution* of term  $u$  for the variable  $x$  in term  $t$ , and the notation  $\uparrow t$  denotes the *weakening* of a term  $\Gamma \vdash t : A$  by embedding it into a larger context  $\Gamma \subseteq \Gamma'$  as  $\Gamma' \vdash \uparrow t : A$ . We write  $\Gamma \subseteq \Gamma'$  when the context  $\Gamma$  is a sub-list of context  $\Gamma'$ , meaning  $\Gamma'$  contains at least the variable-type assignments in  $\Gamma$ . Substitution and weakening are both well-typed operations on terms that are admissible in  $\lambda_{\text{ML}}$ :

- Substitution: If  $\Gamma, x : A \vdash t : B$  and  $\Gamma \vdash u : A$ , then  $\Gamma \vdash t[u/x] : B$
- Weakening: If  $\Gamma \vdash t : A$  and  $\Gamma \subseteq \Gamma'$ , then  $\Gamma' \vdash \uparrow t : A$

We say that a type  $X$  is *derivable* (or *can be derived*) in  $\lambda_{\text{ML}}$  when there exists a typing derivation of the judgment  $\cdot \vdash t : X$  for some term  $t$ . The types corresponding to the PLL axioms S, R and J are each derivable in  $\lambda_{\text{ML}}$ , as witnessed by the terms below:

- $\cdot \vdash \text{lam } x. \text{let}_{\text{ML}} y = (\text{snd } x) \text{ in } (\text{return}_{\text{ML}} (\text{pair } (\text{fst } x) y)) : A \times \diamond B \Rightarrow \diamond(A \times B)$
- $\cdot \vdash \text{lam } x. \text{return}_{\text{ML}} x : A \Rightarrow \diamond A$
- $\cdot \vdash \text{lam } x. \text{let}_{\text{ML}} y = x \text{ in } y : \diamond \diamond A \Rightarrow \diamond A$

**Semantics.** The semantics of  $\lambda_{\text{ML}}$  is given using categories. A categorical model of  $\lambda_{\text{ML}}$  is a cartesian-closed category equipped with a strong monad  $\diamond$  (defined in Appendix A). Given a categorical model  $\mathcal{C}$  of  $\lambda_{\text{ML}}$ , we interpret types and contexts in  $\lambda_{\text{ML}}$  as  $\mathcal{C}$ -objects and terms  $\Gamma \vdash t : A$  in  $\lambda_{\text{ML}}$  as  $\mathcal{C}$ -morphisms  $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$ , by induction on types, contexts and terms respectively. The interpretation of the term constructs **return<sub>ML</sub>** and **let<sub>ML</sub>** (and in turn the modal axioms S, R and J) is given by the structure of the strong monad  $\diamond$ .

► **Proposition 2** (Categorical semantics for  $\lambda_{\text{ML}}$ ). *Given two terms  $t, u$  in  $\lambda_{\text{ML}}$ ,  $\Gamma \vdash t \sim u : A$  if and only if for all categorical models  $\mathcal{C}$  of  $\lambda_{\text{ML}}$   $\llbracket t \rrbracket = \llbracket u \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$  in  $\mathcal{C}$ .*

**Proof.** Follows by induction on the judgment  $\Gamma \vdash t \sim u : A$  in one direction, and by a term model construction (see for e.g., [13, Section 3.2]) in the converse. ◀

$$\begin{array}{c}
\text{VAR-ZERO} \\
\frac{}{\Gamma, x : A \vdash_{\text{VAR}} x : A} \\
\\
\text{VAR-SUCC} \\
\frac{\Gamma \vdash_{\text{VAR}} x : A \quad (y \text{ not in } \Gamma)}{\Gamma, y : B \vdash_{\text{VAR}} x : A} \\
\\
\text{VAR} \\
\frac{\Gamma \vdash_{\text{VAR}} x : A}{\Gamma \vdash x : A} \\
\\
\text{1-INTRO} \\
\frac{}{\Gamma \vdash \text{unit} : 1} \\
\\
\text{\(\times\)-INTRO} \\
\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \text{pair } t u : A \times B} \\
\\
\text{\(\times\)-ELIM-1} \\
\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \text{fst } t : A} \\
\\
\text{\(\times\)-ELIM-2} \\
\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash \text{snd } t : B} \\
\\
\text{\(\Rightarrow\)-INTRO} \\
\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \text{lam } x. t : A \Rightarrow B} \\
\\
\text{\(\Rightarrow\)-ELIM} \\
\frac{\Gamma \vdash t : A \Rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash \text{app } t u : B} \\
\\
\text{ML/\(\diamond\)-RETURN} \\
\frac{\Gamma \vdash t : A}{\Gamma \vdash \text{return}_{\text{ML}} t : \diamond A} \\
\\
\text{ML/\(\diamond\)-LET} \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : \diamond B}{\Gamma \vdash \text{let}_{\text{ML}} x = t \text{ in } u : \diamond B} \\
\\
\text{1-\(\eta\)} \\
\frac{}{\Gamma \vdash t \sim \text{unit} : 1} \\
\\
\text{\(\times\)-\(\eta\)} \\
\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t \sim \text{pair } (\text{fst } t) (\text{snd } t) : A \times B} \\
\\
\text{\(\times\)-\(\beta_1\)} \\
\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \text{fst } (\text{pair } t u) \sim t : A} \\
\\
\text{\(\times\)-\(\beta_2\)} \\
\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B}{\Gamma \vdash \text{snd } (\text{pair } t u) \sim u : B} \\
\\
\text{\(\Rightarrow\)-\(\eta\)} \\
\frac{\Gamma \vdash t : A \Rightarrow B}{\Gamma \vdash t \sim \text{lam } x. (\text{app } (\uparrow t) x) : A \Rightarrow B} \\
\\
\text{\(\Rightarrow\)-\(\beta\)} \\
\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A}{\Gamma \vdash \text{app } (\text{lam } x. t) u \sim t[u/x] : B} \\
\\
\text{ML/\(\diamond\)-\(\beta\)} \\
\frac{\Gamma \vdash t : A \quad \Gamma, x : A \vdash u : \diamond B}{\Gamma \vdash \text{let}_{\text{ML}} x = (\text{return}_{\text{ML}} t) \text{ in } u \sim u[t/x] : \diamond B} \\
\\
\text{ML/\(\diamond\)-\(\eta\)} \\
\frac{\Gamma \vdash t : \diamond A}{\Gamma \vdash t \sim \text{let}_{\text{ML}} x = t \text{ in } (\text{return}_{\text{ML}} x) : \diamond A} \\
\\
\text{ML/\(\diamond\)-ASS} \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : \diamond B \quad \Gamma, y : B \vdash u' : \diamond C}{\Gamma \vdash \text{let}_{\text{ML}} y = (\text{let}_{\text{ML}} x = t \text{ in } u) \text{ in } u' \sim \text{let}_{\text{ML}} x = t \text{ in } (\text{let}_{\text{ML}} y = u \text{ in } (\uparrow u')) : \diamond C}
\end{array}$$

■ **Figure 1** Well-typed terms and equational theory for  $\lambda_{\text{ML}}$ .

### 2.3 Sublogics of PLL and corresponding lambda calculi

The minimal sublogic of PLL, which we call SL, can be axiomatized by extending the usual axioms and rules of IPL with (only) the modal axiom S. Furthermore, we axiomatize:

- the logic SRL by extending SL with axiom R
- the logic SJJ by extending SL with axiom J
- the logic PLL by extending SL with axioms R and J (as defined previously)

The semantics for SL, SRL and SJJ is given as before for PLL by restricting the definitions of frames. An SL-frame  $F = (W, R_i, R_m)$  is a triple that consists of a set  $W$  of worlds, a reflexive-transitive relation  $R_i$ , and a relation  $R_m$  (that need not be reflexive or transitive), satisfying the forward confluence and inclusion conditions. Furthermore, an SL-frame is

- an SRL-frame when  $R_m$  is reflexive
- an SJJ-frame when  $R_m$  is transitive
- a PLL-frame when  $R_m$  is reflexive and transitive (as defined previously)

$$\begin{array}{c}
\text{SL}/\diamond\text{-LETMAP} \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : B}{\Gamma \vdash \text{letmap}_{\text{SL}} x = t \text{ in } u : \diamond B} \\
\\
\text{SL}/\diamond\text{-}\eta \\
\frac{\Gamma \vdash t : \diamond A}{\Gamma \vdash t \sim \text{letmap}_{\text{SL}} x = t \text{ in } x : \diamond A} \\
\\
\text{SL}/\diamond\text{-}\beta \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : B \quad \Gamma, y : B \vdash u' : C}{\Gamma \vdash \text{letmap}_{\text{SL}} y = (\text{letmap}_{\text{SL}} x = t \text{ in } u) \text{ in } u' \sim \text{letmap}_{\text{SL}} x = t \text{ in } (\uparrow u')[u/y] : \diamond C}
\end{array}$$

■ **Figure 2** Well-typed terms and equational theory for  $\lambda_{\text{SL}}$  (omitting those of STLC).

In the upcoming section, we will define a corresponding modal lambda calculus for each of PLL's sublogics (Section 3). We develop proof-relevant possible-world semantics for these calculi and show the connection to categorical semantics by studying the properties of presheaf categories determined by proof-relevant frames (Section 4). We leverage this connection to then construct Normalization by Evaluation models for the calculi, and show as corollaries completeness and inadmissibility theorems (Section 5).

### 3 The calculi $\lambda_{\text{SL}}$ , $\lambda_{\text{SRL}}$ and $\lambda_{\text{S JL}}$

We define the calculi  $\lambda_{\text{SL}}$ ,  $\lambda_{\text{SRL}}$  and  $\lambda_{\text{S JL}}$ —akin to the calculus  $\lambda_{\text{ML}}$  in Section 2.2—as extensions of STLC with a unary type constructor  $\diamond$  that exhibits the characteristic axioms of their corresponding logics. The types and contexts of all four calculi are identical to  $\lambda_{\text{ML}}$ , while the term constructs and equational theory for the respective modal fragments vary.

**The calculus  $\lambda_{\text{SL}}$ .** The well-typed terms and equational theory of the modal fragment of  $\lambda_{\text{SL}}$  are defined in Figure 2. The calculus  $\lambda_{\text{SL}}$  extends STLC with a construct  $\text{letmap}_{\text{SL}}$  and two equations  $\text{SL}/\diamond\text{-}\eta$  and  $\text{SL}/\diamond\text{-}\beta$ . Observe that the typing rule for  $\text{letmap}_{\text{SL}}$  in  $\lambda_{\text{SL}}$  differs from  $\text{let}_{\text{ML}}$  in  $\lambda_{\text{ML}}$ : a term  $\text{letmap}_{\text{SL}} x = t \text{ in } u$  “maps” a term  $\Gamma, x : A \vdash u : B$  over a term  $\Gamma \vdash t : \diamond A$  to yield a term well-typed for type  $\diamond B$  under context  $\Gamma$ . This difference disallows a typing derivation for axiom J, while allowing axiom S to be derived as below:

$$\cdot \vdash \text{lam } x. \text{letmap}_{\text{SL}} y = (\text{snd } x) \text{ in } (\text{pair } (\text{fst } x) y) : A \times \diamond B \Rightarrow \diamond(A \times B)$$

Note how this derivation differs from the one in Section 2.2 for  $\lambda_{\text{ML}}$ : it uses  $\text{letmap}_{\text{SL}}$  in place of  $\text{let}_{\text{ML}}$  without a need for  $\text{return}_{\text{ML}}$ . Axiom R, however, cannot be derived in  $\lambda_{\text{SL}}$ .

A categorical model of  $\lambda_{\text{SL}}$  is a cartesian-closed category equipped with a strong functor  $\diamond$  (that need not be a monad). Given a categorical model  $\mathcal{C}$  of  $\lambda_{\text{SL}}$ , we interpret types and contexts in  $\lambda_{\text{SL}}$  as  $\mathcal{C}$ -objects and terms  $\Gamma \vdash t : A$  as  $\mathcal{C}$ -morphisms  $\llbracket t \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$  as before with  $\lambda_{\text{ML}}$  by induction on types and terms respectively. The interpretation of the term construct  $\text{letmap}_{\text{SL}}$  (and in turn the modal axiom S) is given by the tensorial strength of functor  $\diamond$ , which gives us a morphism  $X \times \diamond Y \rightarrow \diamond(X \times Y)$  for all objects  $X, Y$  in  $\mathcal{C}$ .

► **Proposition 3** (Categorical semantics for  $\lambda_{\text{SL}}$ ). *Given two terms  $t, u$  in  $\lambda_{\text{ML}}$ ,  $\Gamma \vdash t \sim u : A$  if and only if for all categorical models  $\mathcal{C}$  of  $\lambda_{\text{SL}}$   $\llbracket t \rrbracket = \llbracket u \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$  in  $\mathcal{C}$ .*

**The calculus  $\lambda_{\text{SRL}}$ .** The well-typed terms and equational theory for the modal fragment of  $\lambda_{\text{SRL}}$  are defined in Figure 3. The calculus  $\lambda_{\text{SRL}}$  extends STLC with two constructs  $\text{return}_{\text{SRL}}$  and  $\text{letmap}_{\text{SRL}}$ , and three equations  $\text{SRL}/\diamond\text{-}\eta$ ,  $\text{SRL}/\diamond\text{-}\beta_1$  and  $\text{SRL}/\diamond\text{-}\beta_2$ .

$$\begin{array}{c}
\text{SRL}/\diamond\text{-RETURN} \\
\frac{\Gamma \vdash t : A}{\Gamma \vdash \text{return}_{\text{SRL}} t : \diamond A} \\
\\
\text{SRL}/\diamond\text{-LETMAP} \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : B}{\Gamma \vdash \text{letmap}_{\text{SRL}} x = t \text{ in } u : \diamond B} \\
\\
\text{SRL}/\diamond\text{-}\eta \\
\frac{\Gamma \vdash t : \diamond A}{\Gamma \vdash t \sim \text{letmap}_{\text{SRL}} x = t \text{ in } x : \diamond A} \\
\\
\text{SRL}/\diamond\text{-}\beta_1 \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : B \quad \Gamma, y : B \vdash u' : C}{\Gamma \vdash \text{letmap}_{\text{SRL}} y = (\text{letmap}_{\text{SRL}} x = t \text{ in } u) \text{ in } u' \sim \text{letmap}_{\text{SRL}} x = t \text{ in } (\uparrow u')[u/y] : \diamond C} \\
\\
\text{SRL}/\diamond\text{-}\beta_2 \\
\frac{\Gamma \vdash t : A \quad \Gamma, x : A \vdash u : B}{\Gamma \vdash \text{letmap}_{\text{SRL}} x = (\text{return}_{\text{SRL}} t) \text{ in } u \sim \text{return}_{\text{SRL}} (u[t/x]) : \diamond B}
\end{array}$$

■ **Figure 3** Well-typed terms and equational theory for  $\lambda_{\text{SRL}}$  (omitting those of STLC).

Observe that the typing rule of the construct  $\text{letmap}_{\text{SRL}}$  in  $\lambda_{\text{SRL}}$  is identical to  $\text{letmap}_{\text{SL}}$  in  $\lambda_{\text{SL}}$ . As a result, axiom S can be derived in  $\lambda_{\text{SRL}}$  similar to  $\lambda_{\text{SL}}$ :

$$\cdot \vdash \text{lam } x. \text{letmap}_{\text{SRL}} y = (\text{snd } x) \text{ in } (\text{pair } (\text{fst } x) y) : A \times \diamond B \Rightarrow \diamond(A \times B)$$

Similarly, observe that the typing rule of the construct  $\text{return}_{\text{SRL}}$  in  $\lambda_{\text{SRL}}$  is identical to  $\text{return}_{\text{ML}}$  in  $\lambda_{\text{ML}}$ . As a result axiom R can as well be derived in  $\lambda_{\text{SRL}}$  similar to  $\lambda_{\text{ML}}$ :

$$\cdot \vdash \text{lam } x. \text{return}_{\text{SRL}} x : A \Rightarrow \diamond A$$

Axiom J cannot be derived in  $\lambda_{\text{SRL}}$  since there is no counterpart for  $\text{let}_{\text{ML}}$  in  $\lambda_{\text{SRL}}$ .

A categorical model of  $\lambda_{\text{SRL}}$  is a cartesian-closed category equipped with a strong pointed functor  $\diamond$ . The term construct  $\text{letmap}_{\text{SRL}}$  (and in turn axiom S) is interpreted in a model  $\mathcal{C}$  of  $\lambda_{\text{SRL}}$  using the tensorial strength of functor  $\diamond$ , as before with  $\lambda_{\text{SL}}$ . The interpretation of the term construct  $\text{return}_{\text{SRL}}$  (and in turn axiom R) is given by the pointed structure of the functor  $\diamond$ , which gives us a morphism  $X \rightarrow \diamond X$  for all objects  $X$  in  $\mathcal{C}$ .

► **Proposition 4** (Categorical semantics for  $\lambda_{\text{SRL}}$ ). *Given two terms  $t, u$  in  $\lambda_{\text{SRL}}$ ,  $\Gamma \vdash t \sim u : A$  if and only if for all categorical models  $\mathcal{C}$  of  $\lambda_{\text{SRL}}$   $\llbracket t \rrbracket = \llbracket u \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$  in  $\mathcal{C}$ .*

**The calculus  $\lambda_{\text{SJL}}$ .** The well-typed terms and equational theory for the modal fragment of  $\lambda_{\text{SJL}}$  are defined in Figure 4. The calculus  $\lambda_{\text{SJL}}$  extends STLC with two constructs  $\text{letmap}_{\text{SJL}}$  and  $\text{let}_{\text{SJL}}$ , and five equations  $\text{SJL}/\diamond\text{-}\eta$ ,  $\text{SJL}/\diamond\text{-}\beta_1$ ,  $\text{SJL}/\diamond\text{-}\beta_2$ ,  $\text{SJL}/\diamond\text{-com}$  and  $\text{SJL}/\diamond\text{-ass}$ .

Observe that the typing rule of the construct  $\text{letmap}_{\text{SJL}}$  in  $\lambda_{\text{SJL}}$  is once again identical to  $\text{letmap}_{\text{SL}}$  in  $\lambda_{\text{SL}}$ . As a result, axiom S can once again be derived in  $\lambda_{\text{SJL}}$  similar to  $\lambda_{\text{SL}}$ :

$$\cdot \vdash \text{lam } x. \text{letmap}_{\text{SJL}} y = (\text{snd } x) \text{ in } (\text{pair } (\text{fst } x) y) : A \times \diamond B \Rightarrow \diamond(A \times B)$$

Similarly, observe that the typing rule of the construct  $\text{let}_{\text{SJL}}$  in  $\lambda_{\text{SJL}}$  is identical to construct  $\text{let}_{\text{ML}}$  in  $\lambda_{\text{ML}}$ . As a result, axiom J can be derived in  $\lambda_{\text{SJL}}$  similar to  $\lambda_{\text{ML}}$ :

$$\cdot \vdash \text{lam } x. \text{let}_{\text{SJL}} y = x \text{ in } y : \diamond \diamond A \Rightarrow \diamond A$$

Axiom R cannot be derived in  $\lambda_{\text{SJL}}$  as there is no counterpart for  $\text{return}_{\text{ML}}$  in  $\lambda_{\text{SJL}}$ .

$$\begin{array}{c}
\text{SJL}/\diamond\text{-LETMAP} \qquad \text{SJL}/\diamond\text{-LET} \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : B}{\Gamma \vdash \text{letmap}_{\text{SJL}} x = t \text{ in } u : \diamond B} \qquad \frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : \diamond B}{\Gamma \vdash \text{lets}_{\text{SJL}} x = t \text{ in } u : \diamond B} \\
\\
\text{SJL}/\diamond\text{-}\eta \\
\frac{\Gamma \vdash t : \diamond A}{\Gamma \vdash t \sim \text{letmap}_{\text{SJL}} x = t \text{ in } x : \diamond A} \\
\\
\text{SJL}/\diamond\text{-}\beta_1 \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : B \quad \Gamma, y : B \vdash u' : C}{\Gamma \vdash \text{letmap}_{\text{SJL}} y = (\text{letmap}_{\text{SJL}} x = t \text{ in } u) \text{ in } u' \sim \text{letmap}_{\text{SJL}} x = t \text{ in } (\uparrow u')[u/y] : \diamond C} \\
\\
\text{SJL}/\diamond\text{-}\beta_2 \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : B \quad \Gamma, y : B \vdash u' : \diamond C}{\Gamma \vdash \text{lets}_{\text{SJL}} y = (\text{letmap}_{\text{SJL}} x = t \text{ in } u) \text{ in } u' \sim \text{lets}_{\text{SJL}} x = t \text{ in } (\uparrow u')[u/y] : \diamond C} \\
\\
\text{SJL}/\diamond\text{-COM} \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : \diamond B \quad \Gamma, y : B \vdash u' : C}{\Gamma \vdash \text{letmap}_{\text{SJL}} y = (\text{lets}_{\text{SJL}} x = t \text{ in } u) \text{ in } u' \sim \text{lets}_{\text{SJL}} x = t \text{ in } (\text{letmap}_{\text{SJL}} y = u \text{ in } (\uparrow u')) : \diamond C} \\
\\
\text{SJL}/\diamond\text{-ASS} \\
\frac{\Gamma \vdash t : \diamond A \quad \Gamma, x : A \vdash u : \diamond B \quad \Gamma, y : B \vdash u' : \diamond C}{\Gamma \vdash \text{lets}_{\text{SJL}} y = (\text{lets}_{\text{SJL}} x = t \text{ in } u) \text{ in } u' \sim \text{lets}_{\text{SJL}} x = t \text{ in } (\text{lets}_{\text{SJL}} y = u \text{ in } (\uparrow u')) : \diamond C}
\end{array}$$

■ **Figure 4** Well-typed terms and equational theory for  $\lambda_{\text{SJL}}$  (omitting those of STLC).

A categorical model of  $\lambda_{\text{SJL}}$  is a cartesian-closed category equipped with a strong semi-monad  $\diamond$ . We interpret the term construct  $\text{letmap}_{\text{SJL}}$  (and in turn axiom S) in a categorical model  $\mathcal{C}$  of  $\lambda_{\text{SJL}}$ , using the tensorial strength of functor  $\diamond$  as before with  $\lambda_{\text{SL}}$  and  $\lambda_{\text{SRL}}$ . The interpretation of the term construct  $\text{lets}_{\text{SJL}}$  (and in turn axiom J) is given by the semimonad structure of functor  $\diamond$ , which gives us a morphism  $\diamond\diamond X \rightarrow \diamond X$  for all objects  $X$  in  $\mathcal{C}$ .

► **Proposition 5** (Categorical semantics for  $\lambda_{\text{SJL}}$ ). *Given two terms  $t, u$  in  $\lambda_{\text{SJL}}$ ,  $\Gamma \vdash t \sim u : A$  if and only if for all categorical models  $\mathcal{C}$  of  $\lambda_{\text{SJL}}$   $\llbracket t \rrbracket = \llbracket u \rrbracket : \llbracket \Gamma \rrbracket \rightarrow \llbracket A \rrbracket$  in  $\mathcal{C}$ .*

## 4 Proof-relevant possible-world semantics

In Section 2, possible-world semantics was given for the logic PLL and its sublogics in a classical meta-language using sets and relations. In this section, we will give proof-relevant possible-world semantics for lax modal lambda calculi, for which we will instead work in a constructive dependent type-theory loosely based on the proof assistant Agda.

We will use a type  $X : \mathbf{Type}$  in place of a set  $X$  and values  $x : X$  in place of elements  $x \in X$ . The arrow  $\rightarrow$  denotes functions, and quantifications  $\forall x$  and  $\Sigma_x$  denote universal and existential quantification respectively, where  $x : X$  is a value of some type  $X : \mathbf{Type}$  that is left implicit. A value of type  $\forall x. P(x)$  for some predicate  $P : X \rightarrow \mathbf{Type}$  is a function  $\lambda x. p$  with  $p : P(x)$ . When the expression  $p$  does not mention the variable  $x$  we will leave the abstraction implicit and simply write  $p$  as a value of  $\forall x. P(x)$ . A value of type  $\Sigma_x. P(x)$  is a tuple  $(x, p)$ , but we will similarly leave the witness  $x$  implicit at times and write  $(\_, p)$  or simply  $p$  for brevity.

**Semantics for  $\lambda_{\text{SL}}$ .** A proof-relevant  $\lambda_{\text{SL}}$ -frame  $F = (W, R_i, R_m)$  is a triple that consists of a *type*  $W : \text{Type}$  of worlds and two proof-relevant relations  $R_i, R_m : W \rightarrow W \rightarrow \text{Type}$  with

- functions  $\text{refl}_i : \forall w. w R_i w$  and  $\text{trans}_i : \forall w, w', w''. w R_i w' \rightarrow w' R_i w'' \rightarrow w R_i w''$  respectively proving the reflexivity and transitivity of  $R_i$  such that
  - $\text{trans}_i \text{refl}_i i = i$  and  $\text{trans}_i i \text{refl}_i = i$
  - $\text{trans}_i (\text{trans}_i i i') i'' = \text{trans}_i i (\text{trans}_i i' i'')$
- function  $\text{factor} : \forall w, w', v. w R_i w' \rightarrow w R_m v \rightarrow \Sigma_{v'} . (w' R_m v' \times v R_i v')$  such that
  - $\text{factor} \text{refl}_i m = (m, \text{refl}_i)$
  - $\text{factor} (\text{trans}_i i_1 i_2) m = (m'_2, (\text{trans}_i i'_1 i'_2))$   
where  $(m'_1, i'_1) = \text{factor} i_1 m$  and  $(m'_2, i'_2) = \text{factor} i_2 m'_1$ .
- function  $\text{incl} : \forall w, v. w R_m v \rightarrow w R_i v$  such that
  - $\text{trans}_i i (\text{incl} m') = \text{trans}_i (\text{incl} m) i'$ , where  $(i', m') = \text{factor} i m$

The function  $\text{refl}_i$  and  $\text{trans}_i$  are the proof-relevant encoding of reflexivity and transitivity of  $R_i$  respectively. These functions are subject to the accompanying coherence laws, which state that the proof computed by  $\text{refl}_i$  must be the unit of  $\text{trans}_i$ , i.e.  $R_i$  must form a category  $\mathcal{W}_i$ . The coherence laws facilitate a sound interpretation of  $\lambda_{\text{SL}}$ 's equational theory.

The functions  $\text{factor}$  and  $\text{incl}$  are proof-relevant encodings of the forward confluence ( $R_i^{-1}; R_m \subseteq R_m; R_i^{-1}$ ) and inclusion ( $R_m \subseteq R_i$ ) conditions respectively. Given a proof of  $w R_i w'$  (i.e.  $w' R_i^{-1} w$ ) and  $w R_m v$ ,  $\text{factor}$  returns a pair of proofs for some world  $v'$ :  $w' R_m v'$  and  $v R_i v'$  (i.e.  $v' R_i^{-1} v$ ). Similarly, given a proof of  $w R_m v$ ,  $\text{incl}$  returns a proof of  $w R_i v$ . These functions are also accompanied by the stated coherence laws.

The proof-relevant relation  $R_i$  in a  $\lambda_{\text{SL}}$ -frame determines a category  $\mathcal{W}_i$  whose objects are given by worlds and morphisms by proofs of  $R_i$ , with  $\text{refl}_i$  witnessing the identity morphisms and  $\text{trans}_i$  witnessing the composition of morphisms. This determines a category  $\widehat{\mathcal{W}}_i$  of covariant presheaves indexed by  $\mathcal{W}_i$ . The objects in the category  $\widehat{\mathcal{W}}_i$  are presheaves and the morphisms are natural transformations. A presheaf  $P$  is given by a family of meta-language types  $P_w : \text{Type}$  indexed by worlds  $w : W$ , accompanied by “transportation” functions  $\text{mon}_P : \forall w, w'. w R_i w' \rightarrow P_w \rightarrow P_{w'}$  subject to the “functoriality” conditions that  $\text{mon}_P \text{refl}_i p = p$  and  $\text{mon}_P (\text{trans}_i i i') p = \text{mon}_P i' (\text{mon}_P i p)$ , for arbitrary values  $i : w R_i w'$ ,  $i' : w' R_i w''$  and  $p : P_w$ . A natural transformation  $f : P \rightarrow Q$  is a family of functions  $\forall w. P_w \rightarrow Q_w$  subject to a “naturality” condition that  $f (\text{mon}_P i p) = \text{mon}_Q i (f p)$ .

► **Proposition 6** ( $\diamond$  Strong Functor). *The presheaf category  $\widehat{\mathcal{W}}_i$  determined by a  $\lambda_{\text{SL}}$ -frame exhibits a strong endofunctor  $(\diamond P)_w = \Sigma_{v'} . w R_m v' \times P_{v'}$  for some world  $w$  and presheaf  $P$ .*

**Proof.** We show that the family  $\diamond P$  is a presheaf by defining the function  $\text{mon}_{\diamond P}$  as:

$$\text{mon}_{\diamond P} i (v, m, p) = (v', m', \text{mon}_P i' p) \quad \text{where } (v', m' : w' R_m v', i' : v R_i v') = \text{factor} i m$$

The functoriality conditions follow from the coherence conditions on  $\text{factor}$ . To show the operator  $\diamond$  is a functor on presheaves, we must show that for every natural transformation  $f : P \rightarrow Q$  there exists a natural transformation  $\diamond f : \diamond P \rightarrow \diamond Q$ . This natural transformation is defined as  $\diamond f (v, m, p) = (v, m, (f_v p))$  by applying  $f$  at the world  $v$  witnessing the  $\Sigma$  quantification. The laws of the functor  $\diamond$  follow immediately. The strength of the functor  $\diamond$  can be defined using the function  $\text{incl}$  and its accompanying coherence conditions. ◀

Propositions 3 and 6 give us that  $\widehat{\mathcal{W}}_i$  is a categorical model of  $\lambda_{\text{SL}}$ . For clarity, we elaborate on this consequence by giving a direct interpretation of  $\lambda_{\text{SL}}$  in  $\widehat{\mathcal{W}}_i$ .

A proof-relevant possible-world *model*  $M = (F, V)$  couples a proof-relevant frame  $F$  with a valuation function  $V$  that assigns to a base type  $\iota$  a presheaf  $V_\iota : W \rightarrow \mathit{Type}$ . Given such a model, the types in  $\lambda_{\text{SL}}$  are interpreted as presheaves, i.e. we interpret a type  $A$  as a family  $\llbracket A \rrbracket_w : \mathit{Type}$  indexed by an arbitrary world  $w : W$  – as shown on the left below.

$$\begin{array}{l} \llbracket \iota \rrbracket_w = V_{\iota, w} \\ \llbracket \mathbf{1} \rrbracket_w = \top \\ \llbracket A \times B \rrbracket_w = \llbracket A \rrbracket_w \times \llbracket B \rrbracket_w \\ \llbracket A \Rightarrow B \rrbracket_w = \forall w'. w R_i w' \rightarrow \llbracket A \rrbracket_{w'} \rightarrow \llbracket B \rrbracket_{w'} \\ \llbracket \diamond A \rrbracket_w = \sum_v . w R_m v \times \llbracket A \rrbracket_v \end{array} \quad \begin{array}{l} \llbracket \cdot \rrbracket_w = \top \\ \llbracket \Gamma, x : A \rrbracket_w = \llbracket \Gamma \rrbracket_w \times \llbracket A \rrbracket_w \end{array}$$

The interpretation of the base type  $\iota$  is given by the valuation function  $V$ , and the unit, product and function types are interpreted as usual using their semantic counterparts. We interpret the  $\diamond$  modality using the proof-relevant quantifier  $\sum$ : the interpretation of a type  $\diamond A$  at a world  $w$  is given by the interpretation of  $A$  at some modal future world  $v$  along with a proof of  $w R_m v$  witnessing the connection from  $w$  to  $v$  via  $R_m$ . The typing contexts are interpreted as usual by taking the cartesian product of presheaves.

The terms in  $\lambda_{\text{SL}}$  are interpreted as natural transformations by induction on the typing judgment. Interpretation of STLC terms follows the usual routine: we interpret variables by projecting the environment  $\gamma : \llbracket \Gamma \rrbracket_w$  using a function *lookup*, the unit and pair constructs (unit, pair, fst, snd) with their semantic counterparts  $((), (-, -), \pi_1, \pi_2)$ , and the function constructs (lam, app) with appropriate semantic function abstraction and application.

$$\begin{array}{l} \llbracket - \rrbracket : \Gamma \vdash A \rightarrow (\forall w. \llbracket \Gamma \rrbracket_w \rightarrow \llbracket A \rrbracket_w) \\ \llbracket x \rrbracket \gamma = \mathit{lookup} \ x \ \gamma \\ \llbracket \mathbf{unit} \rrbracket \gamma = () \\ \llbracket \mathbf{pair} \ t \ u \rrbracket \gamma = (\llbracket t \rrbracket \gamma, \llbracket u \rrbracket \gamma) \\ \llbracket \mathbf{fst} \ t \rrbracket \gamma = \pi_1(\llbracket t \rrbracket \gamma) \\ \llbracket \mathbf{snd} \ t \rrbracket \gamma = \pi_2(\llbracket t \rrbracket \gamma) \\ \llbracket \mathbf{lam} \ x. t \rrbracket \gamma = \lambda i. \lambda a. \llbracket t \rrbracket (\mathit{mon}_{[\Gamma]} \ i \ \gamma, a) \\ \llbracket \mathbf{app} \ t \ u \rrbracket \gamma = (\llbracket t \rrbracket \gamma) \ \mathit{refl}_i \ (\llbracket u \rrbracket \gamma) \\ \llbracket \mathbf{letmap}_{\text{SL}} \ x = t \ \mathbf{in} \ u \rrbracket \gamma = (m, \llbracket u \rrbracket (\mathit{mon}_{[\Gamma]} (\mathit{incl} \ m) \ \gamma, a)) \\ \text{where } (m : w R_m v, a : \llbracket A \rrbracket_v) = \llbracket t \rrbracket \gamma \end{array}$$

The interesting case is that of  $\mathbf{letmap}_{\text{SL}}$ : given terms  $\Gamma \vdash t : \diamond A$  and  $\Gamma, x : A \vdash u : B$ , and an environment  $\gamma : \llbracket \Gamma \rrbracket_w$ , we must produce an element of type  $\llbracket \diamond B \rrbracket_w = \sum_v . w R_m v \times \llbracket B \rrbracket_v$ . Recursively interpreting  $t$  gives us a pair  $(m : w R_m v, a : \llbracket A \rrbracket_v)$ , using the former of which we transport  $\gamma$  along  $R_m$  to the world  $v$ , as  $\mathit{mon}_{[\Gamma]} (\mathit{incl} \ m) \ \gamma : \llbracket \Gamma \rrbracket_v$ , which is in turn used to recursively interpret  $u$ , thus obtaining the desired element of type  $\llbracket B \rrbracket_v$ .

**Semantics for  $\lambda_{\text{SRL}}$ .** A proof-relevant  $\lambda_{\text{SRL}}$ -frame  $(W, R_i, R_m)$  is a  $\lambda_{\text{SL}}$ -frame that exhibits:

- function  $\mathit{refl}_m : \forall w. w R_m w$ , such that
  - factor  $i \ \mathit{refl}_m = (\mathit{refl}_m, i)$
  - $\mathit{incl} \ \mathit{refl}_m = \mathit{refl}_i$

► **Proposition 7** ( $\diamond$  Strong Pointed). *The strong functor  $\diamond$  on the category of presheaves  $\widehat{\mathcal{W}}_i$  determined by a  $\lambda_{\text{SRL}}$ -frame is strong pointed.*

**Proof.** To show that  $\diamond$  is pointed, we define  $\mathit{point} : P \dot{\rightarrow} \diamond P$  using function  $\mathit{refl}_m$ , and then use the coherence law  $\mathit{incl} \ \mathit{refl}_m = \mathit{refl}_i$  to show that  $\mathit{point}$  is a strong natural transformation. ◀

Propositions 4 and 7 give us that  $\widehat{\mathcal{W}}_i$  is a categorical model of  $\lambda_{\text{SRL}}$  for  $\lambda_{\text{SRL}}$ -frames. The interpretation of the modal fragment of  $\lambda_{\text{SRL}}$  can be given explicitly in  $\widehat{\mathcal{W}}_i$  as:

$$\begin{aligned} \llbracket \text{return}_{\text{SRL}} t \quad \rrbracket \gamma &= (\text{refl}_m, \llbracket t \rrbracket \gamma) \\ \llbracket \text{letmap}_{\text{SRL}} x = t \text{ in } u \rrbracket \gamma &= (m, \llbracket u \rrbracket (\text{mon}_{\llbracket \Gamma \rrbracket} (\text{incl } m) \gamma, a)) \\ &\text{where } (m : w R_m v, a : \llbracket A \rrbracket_v) = \llbracket t \rrbracket \gamma \end{aligned}$$

**Semantics for  $\lambda_{\text{SJL}}$ .** A proof-relevant  $\lambda_{\text{SJL}}$ -frame  $(W, R_i, R_m)$  is a  $\lambda_{\text{SL}}$ -frame that exhibits:

- function  $\text{trans}_m : \forall u, v, w. u R_m v \rightarrow v R_m w \rightarrow u R_m w$ , such that
  - $\text{factor } i (\text{trans}_m m_1 m_2) = (\text{trans}_m m'_1 m'_2, i'_2)$   
where  $(m'_1, i'_1) = \text{factor } i m_1$  and  $(m'_2, i'_2) = \text{factor } i'_1 m_2$ .
  - $\text{trans}_m (\text{trans}_m m_1 m_2) m_3 = \text{trans}_m m_1 (\text{trans}_m m_2 m_3)$
  - $\text{incl} (\text{trans}_m m_1 m_2) = \text{trans}_i (\text{incl } m_1) (\text{incl } m_2)$

► **Proposition 8** ( $\diamond$  Strong Semimonad). *The strong functor  $\diamond$  on the category of presheaves  $\widehat{\mathcal{W}}_i$  determined by a  $\lambda_{\text{SJL}}$ -frame is a strong semimonad.*

**Proof.** We define  $\mu : \diamond \diamond P \rightarrow \diamond P$  using the function  $\text{trans}_m$  to show  $\diamond$  is a semimonad, and then use the coherence law  $\text{incl} (\text{trans}_m m_1 m_2) = \text{trans}_i (\text{incl } m_1) (\text{incl } m_2)$  to show that  $\mu$  is a strong natural transformation – giving us that  $\mu$  is a strong semimonad. ◀

Propositions 5 and 8 give us that  $\widehat{\mathcal{W}}_i$  is a categorical model of  $\lambda_{\text{SJL}}$  for  $\lambda_{\text{SJL}}$ -frames. The interpretation of the modal fragment of  $\lambda_{\text{SJL}}$  can be given explicitly in  $\widehat{\mathcal{W}}_i$  as:

$$\begin{aligned} \llbracket \text{letmap}_{\text{SJL}} x = t \text{ in } u \rrbracket \gamma &= (m, \llbracket u \rrbracket (\text{mon}_{\llbracket \Gamma \rrbracket} (\text{incl } m) \gamma, a)) \\ &\text{where } (m : w R_m v, a : \llbracket A \rrbracket_v) = \llbracket t \rrbracket \gamma \\ \llbracket \text{let}_{\text{SJL}} x = t \text{ in } u \quad \rrbracket \gamma &= (\text{trans}_m m m', b) \\ &\text{where } (m : w R_m v, a : \llbracket A \rrbracket_v) = \llbracket t \rrbracket \gamma \\ &\quad (m' : v R_m v', b : \llbracket B \rrbracket_{v'}) = \llbracket u \rrbracket (\text{mon}_{\llbracket \Gamma \rrbracket} (\text{incl } m) \gamma, a) \end{aligned}$$

**Semantics for  $\lambda_{\text{ML}}$ .** A proof-relevant  $\lambda_{\text{ML}}$ -frame  $F = (W, R_i, R_m)$  is both a  $\lambda_{\text{SRL}}$ -frame and  $\lambda_{\text{SJL}}$ -frame that further exhibits the unit laws  $\text{trans}_m \text{refl}_m m = m$  and  $\text{trans}_m m \text{refl}_m = m$ . That is, proofs of  $R_m$  now form a category  $\mathcal{W}_m$  with a functor  $\mathcal{W}_m \rightarrow \mathcal{W}_i$  given by function  $\text{incl}$ .

► **Proposition 9** ( $\diamond$  Strong Monad). *The strong functor  $\diamond$  on the category of presheaves  $\widehat{\mathcal{W}}_i$  determined by a  $\lambda_{\text{ML}}$ -frame is a strong monad.*

**Proof.** We apply Propositions 6–8 to show that the functor  $\diamond$  is a strong pointed semimonad. We then use the unit laws of the category  $\mathcal{W}_m$  to prove the unit laws of the monad  $\diamond$ . ◀

Propositions 2 and 9 give us that  $\widehat{\mathcal{W}}_i$  is a categorical model of  $\lambda_{\text{ML}}$  for  $\lambda_{\text{ML}}$ -frames. The interpretation of the modal fragment of  $\lambda_{\text{ML}}$  can be given explicitly in  $\widehat{\mathcal{W}}_i$  as:

$$\begin{aligned} \llbracket \text{return}_{\text{ML}} t \quad \rrbracket \gamma &= (\text{refl}_m, \llbracket t \rrbracket \gamma) \\ \llbracket \text{let}_{\text{ML}} x = t \text{ in } u \rrbracket \gamma &= (\text{trans}_m m m', b) \\ &\text{where } (m : w R_m v, a : \llbracket A \rrbracket_v) = \llbracket t \rrbracket \gamma \\ &\quad (m' : v R_m v', b : \llbracket B \rrbracket_{v'}) = \llbracket u \rrbracket (\text{mon}_{\llbracket \Gamma \rrbracket} (\text{incl } m) \gamma, a) \end{aligned}$$

► **Theorem 10** (Soundness of proof-relevant possible-world semantics). *For any two terms  $\Gamma \vdash t, u : A$  in  $\lambda_{\text{SL}}/\lambda_{\text{SRL}}/\lambda_{\text{SJL}}/\lambda_{\text{ML}}$ , if  $\Gamma \vdash t \sim u : A$  then  $\llbracket t \rrbracket = \llbracket u \rrbracket$  for an arbitrary proof-relevant possible-world model determined by the respective  $\lambda_{\text{SL}}/\lambda_{\text{SRL}}/\lambda_{\text{SJL}}/\lambda_{\text{ML}}$ -frames.*

**Proof.** Applying Propositions 6–9 to Propositions 2–5 accordingly gives us that the category  $\widehat{\mathcal{W}}_i$  determined by a  $\lambda_{\text{SL}}/\lambda_{\text{SRL}}/\lambda_{\text{SJL}}/\lambda_{\text{ML}}$ -frame is a categorical model of the respective calculus. As a result, we get the soundness of the equational theory for possible-world models via soundness of the equational theory for categorical models. ◀

$$\begin{array}{c}
\text{NE/VAR} \\
\frac{\Gamma \vdash_{\text{VAR}} x : A}{\Gamma \vdash_{\text{NE}} x : A} \\
\\
\text{NF/UP} \\
\frac{\Gamma \vdash_{\text{NE}} n : \iota}{\Gamma \vdash_{\text{NF}} \text{up } n : \iota} \\
\\
\text{NF/1-INTRO} \\
\Gamma \vdash_{\text{NF}} \text{unit} : 1 \\
\\
\text{NE}/\times\text{-ELIM-1} \\
\frac{\Gamma \vdash_{\text{NE}} n : A \times B}{\Gamma \vdash_{\text{NE}} \text{fst } n : A} \\
\\
\text{NE}/\times\text{-ELIM-2} \\
\frac{\Gamma \vdash_{\text{NE}} n : A \times B}{\Gamma \vdash_{\text{NE}} \text{snd } n : B} \\
\\
\text{NF}/\times\text{-INTRO} \\
\frac{\Gamma \vdash_{\text{NF}} n : A \quad \Gamma \vdash_{\text{NF}} m : B}{\Gamma \vdash_{\text{NF}} \text{pair } n m : A \times B} \\
\\
\text{NF}/\Rightarrow\text{-INTRO} \\
\frac{\Gamma, x : A \vdash_{\text{NF}} n : B}{\Gamma \vdash_{\text{NF}} \text{lam } x. n : A \Rightarrow B} \\
\\
\text{NE}/\Rightarrow\text{-ELIM} \\
\frac{\Gamma \vdash_{\text{NE}} n : A \Rightarrow B \quad \Gamma \vdash_{\text{NF}} m : A}{\Gamma \vdash_{\text{NE}} \text{app } n m : B} \\
\\
\text{NF/SL}/\diamond\text{-LETMAP} \\
\frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad \Gamma, x : A \vdash_{\text{NF}} m : B}{\Gamma \vdash_{\text{NF}} \text{letmap}_{\text{SL}} x = n \text{ in } m : \diamond B}
\end{array}$$

■ **Figure 5** Neutral terms and Normal forms for  $\lambda_{\text{SL}}$ .

## 5 Normalization, completeness and inadmissibility results

Catarina Coquand [14, 15] proved normalization for STLC in the proof assistant Alf [28] by constructing an instance of Mitchell and Moggi’s proof-relevant possible-world semantics. This model-based approach to normalization, known as Normalization by Evaluation (NbE) [9, 8], dispenses with tedious syntactic reasoning that typically complicate normalization proofs. In this section, we extend Coquand’s result to lax modal lambda calculi and observe corollaries including completeness and inadmissibility of irrelevant modal axioms.

The objective of NbE is to define a function *norm* :  $\Gamma \vdash A \rightarrow \Gamma \vdash_{\text{NF}} A$ , assigning a *normal form* to every term in the calculus. We write  $\Gamma \vdash A$  to denote all terms  $\Gamma \vdash t : A$  and  $\Gamma \vdash_{\text{NF}} A$  to denote all normal forms  $\Gamma \vdash_{\text{NF}} n : A$ . The normal form judgments  $\Gamma \vdash_{\text{NF}} n : A$  are defined in Figure 5 alongside *neutral term* judgments  $\Gamma \vdash_{\text{NE}} n : A$ , which can be roughly understood as “straight-forward” inferences that do not involve introduction rules.

To define *norm* for  $\lambda_{\text{SL}}$ , we construct a possible-world model  $(N, V)$ , known as the NbE model, with a  $\lambda_{\text{SL}}$ -frame  $N = (\text{Ctx}, \subseteq, \triangleleft_{\text{SL}})$  consisting of contexts for worlds, the context inclusion relation  $\subseteq$  for  $R_i$ , and the accessibility relation  $\triangleleft_{\text{SL}}$  for  $R_m$ . The valuation is given by neutral terms as  $V_{\iota, \Gamma} = \Gamma \vdash_{\text{NE}} \iota$ . The relation  $\triangleleft_{\text{SL}}$  can be defined inductively as below. This definition states that  $\Gamma \triangleleft_{\text{SL}} \Delta$  if and only if  $\Delta = \Gamma, x : A$  for some variable  $x$  (not in  $\Gamma$ ) and type  $A$  such that there exists a neutral term  $\Gamma \vdash n : \diamond A$ .

$$\frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad (x \text{ not in } \Gamma)}{\text{single } n : \Gamma \triangleleft_{\text{SL}} (\Gamma, x : A)}$$

The proof-relevant relation  $\triangleleft_{\text{SL}}$  is neither reflexive nor transitive, but is included in the relation  $\subseteq$  since we can define a function *incl* :  $\forall \Gamma, \Delta. \Gamma \triangleleft_{\text{SL}} \Delta \rightarrow \Gamma \subseteq \Delta$ . We can also show that the  $\lambda_{\text{SL}}$ -frame  $N$  satisfies the forward confluence condition by defining a function *factor* :  $\forall \Gamma, \Gamma', \Delta. \Gamma \subseteq \Gamma' \rightarrow \Gamma \triangleleft_{\text{SL}} \Delta \rightarrow \exists \Delta'. (\Gamma' \triangleleft_{\text{SL}} \Delta' \times \Delta \subseteq \Delta')$ .

By construction, we obtain an interpretation of terms  $\llbracket - \rrbracket : \Gamma \vdash A \rightarrow (\forall \Delta. \llbracket \Gamma \rrbracket_{\Delta} \rightarrow \llbracket A \rrbracket_{\Delta})$  in the NbE model as an instance of the generic interpreter for an arbitrary possible-world model (Section 4). This model exhibits two type-indexed functions characteristic of NbE models known as *reify* and *reflect*, which are defined for the modal fragment as follows:

$$\begin{array}{c}
\text{NF/SRL}/\diamond\text{-RETURN} \\
\frac{\Gamma \vdash_{\text{NF}} n : A}{\Gamma \vdash_{\text{NF}} \text{return}_{\text{SRL}} n : \diamond A}
\end{array}
\qquad
\begin{array}{c}
\text{NF/SRL}/\diamond\text{-LETMAP} \\
\frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad \Gamma, x : A \vdash_{\text{NF}} m : B}{\Gamma \vdash_{\text{NF}} \text{letmap}_{\text{SRL}} x = n \text{ in } m : \diamond B}
\end{array}$$

$$\begin{array}{c}
\text{NF/SJL}/\diamond\text{-LETMAP} \\
\frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad \Gamma, x : A \vdash_{\text{NF}} m : B}{\Gamma \vdash_{\text{NF}} \text{letmap}_{\text{SJL}} x = n \text{ in } m : \diamond B}
\end{array}
\qquad
\begin{array}{c}
\text{NF/SJL}/\diamond\text{-LET} \\
\frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad \Gamma, x : A \vdash_{\text{NF}} m : \diamond B}{\Gamma \vdash_{\text{NF}} \text{let}_{\text{SJL}} x = n \text{ in } m : \diamond B}
\end{array}$$

$$\begin{array}{c}
\text{NF/ML}/\diamond\text{-RETURN} \\
\frac{\Gamma \vdash_{\text{NF}} n : A}{\Gamma \vdash_{\text{NF}} \text{return}_{\text{ML}} n : \diamond A}
\end{array}
\qquad
\begin{array}{c}
\text{NF/ML}/\diamond\text{-LET} \\
\frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad \Gamma, x : A \vdash_{\text{NF}} m : \diamond B}{\Gamma \vdash_{\text{NF}} \text{let}_{\text{ML}} x = n \text{ in } m : \diamond B}
\end{array}$$

■ **Figure 6** Normal forms for modal fragments of  $\lambda_{\text{SRL}}$ ,  $\lambda_{\text{SJL}}$  and  $\lambda_{\text{ML}}$ .

$$\begin{array}{l}
\text{reify}_A : \forall \Gamma. \llbracket A \rrbracket_{\Gamma} \rightarrow \Gamma \vdash_{\text{NF}} A \\
\dots \\
\text{reify}_{\diamond A; \Gamma} (\text{single } n : \Gamma \triangleleft_{\text{SL}} (\Gamma, x : B), a : \llbracket A \rrbracket_{\Gamma, x : B}) = \text{letmap}_{\text{SL}} x = n \text{ in } (\text{reify}_{A; (\Gamma, x : B)} a) \\
\text{reflect}_A : \forall \Gamma. \Gamma \vdash_{\text{NE}} A \rightarrow \llbracket A \rrbracket_{\Gamma} \\
\dots \\
\text{reflect}_{\diamond A; \Gamma} n = (\text{single } n, \text{reflect}_{A; (\Gamma, x : A)} x)
\end{array}$$

The function *reify* is a type-indexed natural transformation, which for the case of type  $\diamond A$  in some context  $\Gamma$ , is given as argument an element of type  $\llbracket \diamond A \rrbracket_{\Gamma}$ , which is  $\Sigma_{\Delta}. \Gamma \triangleleft_{\text{SL}} \Delta \times \llbracket A \rrbracket_{\Delta}$ . The first component gives us a neutral  $\Gamma \vdash_{\text{NE}} n : \diamond B$ , and recursively reifying the second component gives us a normal form of  $\Gamma, x : B \vdash_{\text{NF}} \text{reify}_{A; (\Gamma, x : B)} a : A$ . We use these to construct the normal form  $\Gamma \vdash_{\text{NF}} \text{letmap}_{\text{SL}} x = n \text{ in } (\text{reify}_{A; (\Gamma, x : B)} a) : \diamond A$ , which is the desired result. The function *reflect*, on the other hand, constructs a value pair of type  $\llbracket \diamond A \rrbracket_{\Gamma} = \Sigma_{\Delta}. \Gamma \triangleleft_{\text{SL}} \Delta \times \llbracket A \rrbracket_{\Delta}$  using the given neutral  $\Gamma \vdash_{\text{NE}} n : \diamond A$  and picking  $\Gamma, x : A$  (with a fresh variable  $x$  not in  $\Gamma$ ) as the witness of  $\Delta$  to obtain a value of type  $\llbracket A \rrbracket_{\Gamma, x : A}$  by reflecting the variable  $x$  as a neutral term  $\Gamma, x : A \vdash_{\text{NE}} x : A$ . These functions are key to defining *quote* :  $(\forall \Delta. \llbracket \Gamma \rrbracket_{\Delta} \rightarrow \llbracket A \rrbracket_{\Delta}) \rightarrow \Gamma \vdash_{\text{NF}} A$ , which in turn gives us the function *norm*:

$$\text{norm } t = \text{quote } \llbracket t \rrbracket$$

NbE models can be constructed likewise for the calculi  $\lambda_{\text{SRL}}$ ,  $\lambda_{\text{SJL}}$  and  $\lambda_{\text{ML}}$ . The normal forms of these calculi are defined in Figure 6. To construct the model, we uniformly pick contexts for worlds, the relation  $\subseteq$  for  $R_i$ , and the respective modal accessibility relation defined in Figure 7 for  $R_m$ . As before, we also pick neutrals terms for valuation.

Observe that relation  $\triangleleft_{\text{PLL}}$  satisfies the inclusion condition (we can define function *incl*) and is reflexive (we can define *refl<sub>m</sub>*) and transitive (we can define *trans<sub>m</sub>*). On the other hand, the relations  $\triangleleft_{\text{SRL}}$  and  $\triangleleft_{\text{SJL}}$  both satisfy the inclusion condition and are respectively reflexive and transitive, but not the other way around. The main idea behind the definitions of these relations is that they imitate the binding structure of the normal forms in Figure 6.

► **Theorem 11** (Correctness of normalization). *For all terms  $\Gamma \vdash t : A$  in  $\lambda_{\text{SL}}/\lambda_{\text{SRL}}/\lambda_{\text{SJL}}/\lambda_{\text{ML}}$ , there exists a normal form  $\Gamma \vdash_{\text{NF}} n : A$  such that  $t \sim n$ .*

**Proof.** By virtue of the function *norm*, we get that every term  $t$  has a normal form *norm*  $t$ . Using a standard logical relation based argument we can further show that  $t \sim \text{norm } t$ . ◀

$$\begin{array}{c}
\text{nil} : \Gamma \triangleleft_{\text{SRL}} \Gamma \qquad \frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad (x \text{ not in } \Gamma)}{\text{single } n : \Gamma \triangleleft_{\text{SRL}} \Gamma, x : A} \\
\\
\frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad (x \text{ not in } \Gamma)}{\text{single } n : \Gamma \triangleleft_{\text{SJL}} \Gamma, x : A} \qquad \frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad m : \Gamma, x : A \triangleleft_{\text{SJL}} \Delta}{\text{cons } nm : \Gamma \triangleleft_{\text{SJL}} \Delta} \\
\\
\text{nil} : \Gamma \triangleleft_{\text{PLL}} \Gamma \qquad \frac{\Gamma \vdash_{\text{NE}} n : \diamond A \quad m : \Gamma, x : A \triangleleft_{\text{PLL}} \Delta}{\text{cons } nm : \Gamma \triangleleft_{\text{PLL}} \Delta}
\end{array}$$

■ **Figure 7** Modal accessibility relations for  $\lambda_{\text{SRL}}$ ,  $\lambda_{\text{SJL}}$  and  $\lambda_{\text{ML}}$ .

► **Corollary 12** (Completeness of proof-relevant possible-world semantics). *For any two terms  $\Gamma \vdash t, u : A$  in  $\lambda_{\text{SL}}/\lambda_{\text{SRL}}/\lambda_{\text{SJL}}/\lambda_{\text{ML}}$ , if  $\llbracket t \rrbracket = \llbracket u \rrbracket$  in all proof-relevant possible-world models determined by the respective  $\lambda_{\text{SL}}/\lambda_{\text{SRL}}/\lambda_{\text{SJL}}/\lambda_{\text{ML}}$ -frames, then  $\Gamma \vdash t \sim u : A$ .*

**Proof.** In the respective NbE model, we know  $\llbracket t \rrbracket = \llbracket u \rrbracket$  implies  $\text{norm } t = \text{norm } u$  by definition of  $\text{norm}$ . By Theorem 11, we also know  $t \sim \text{norm } t$  and  $u \sim \text{norm } u$ , thus  $t \sim u$ . ◀

► **Corollary 13** (Inadmissibility of irrelevant modal axioms). *The axiom R is not derivable in  $\lambda_{\text{SL}}$  or  $\lambda_{\text{SJL}}$ , and similarly the axiom J is not derivable in  $\lambda_{\text{SL}}$  or  $\lambda_{\text{SRL}}$ .*

**Proof.** We first observe that for any neutral term  $\Gamma \vdash_{\text{NE}} n : A$ , the type  $A$  is a subformula of some type in context  $\Gamma$ . We then show by case analysis that there cannot exist a derivation of the judgment  $\cdot \vdash_{\text{NE}} A \Rightarrow \diamond A$  in  $\lambda_{\text{SL}}$  or  $\lambda_{\text{SJL}}$ , and thus there cannot exist a derivation of axiom R in either calculus – because every term must have a normal form, as shown by the normalization function. A similar argument can be given for axiom J in  $\lambda_{\text{SL}}$  and  $\lambda_{\text{SRL}}$ . ◀

## 6 Related and further work

Simpson [35, Chapter 3] gives a comprehensive summary of several IMLs alongside a detailed discussion of their characteristic axioms and possible-world semantics. Notable early work on IMLs can be traced back to Fischer-Servi [22, 34], Božić and Došen [12], Sotirov [36], Plotkin and Stirling [33], Wijesekera [39], and many others since.

**Global vs local interpretation.** Fairtlough and Mendler [20] give a different presentation of PLL. The truth of their lax modality  $\circ$  is defined “globally” as follows:

$$\mathcal{M}, w \Vdash \circ A \text{ iff for all } w' \text{ s.t. } w R_i w', \text{ there exists } v \text{ with } w' R_m v \text{ and } \mathcal{M}, v \Vdash A$$

Their semantics does not require the forward confluence condition  $R_i^{-1}; R_m \subseteq R_m; R_i^{-1}$  since monotonicity follows immediately the definition of the satisfaction relation. In the presence of forward confluence, this definition is equivalent to the “local” one we have chosen in Section 2 for the  $\diamond$  modality [35, 18], which means  $\circ A$  is true if and only if  $\diamond A$  is true. This observation can also be extended to the respectively determined presheaf functors:

► **Proposition 14.** *The presheaf functors  $\diamond$  and  $\circ$  are naturally isomorphic.*

In modal logic, the forward confluence condition forces the axiom  $\diamond(A \vee B) \implies \diamond A \vee \diamond B$  to be true [6], which does not hold generally for strong functors. This observation, however, presupposes that the satisfaction clause for the disjunction connective is defined as follows:

$$\mathcal{M}, w \Vdash A \vee B \text{ iff } \mathcal{M}, w \Vdash A \text{ or } \mathcal{M}, w \Vdash B$$

This “Kripke-style” interpretation of disjunction is not suitable for our purposes given that our objective is to constructively prove completeness for lambda calculi using possible-world semantics. Completeness in the presence of sum types in lambda calculi is a notorious matter [4, 21] that requires further investigation in the presence of the lax modality.

**Box modality in lax logic.** Fairtlough and Mendler [20] note that “there is no point” in defining a  $\Box$  modality for PLL since it “yields nothing new”. With the following standard extension of the satisfaction clause for the  $\Box$  modality:

$$\mathcal{M}, w \Vdash \Box A \text{ iff for all } w', v \text{ s.t. } w R_i w' \text{ and } w' R_m v, \mathcal{M}, v \Vdash A$$

it follows that  $\mathcal{M}, w \Vdash A$  if and only if  $\mathcal{M}, w \Vdash \Box A$  for an arbitrary model  $\mathcal{M}$  of PLL, making the connective  $\Box$  a logically meaningless addition to PLL.

**Proof-relevant semantics.** Alechina et al. [3] study a connection between categorical and possible-world models of lax logic. They show that a PLL-*modal algebra* determines a possible-world model of PLL [3, Theorem 4] via the Stone representation, and observe that a modal algebra is a “thin” categorical model, whose morphisms are given by the partial-order relation of the algebra. This connection, while illuminating, does not satisfy an important requirement motivating Mitchell and Moggi’s [29] work: to construct models of lambda calculi by leveraging the possible-world semantics of the corresponding logic. Our proof-relevant possible-world semantics satisfies this requirement and is the key to constructing NbE models.

Kavvos [27, 26] develops proof-relevant possible-world semantics (calling it “Two-dimensional Kripke semantics”) for the modal logic  $\text{IK}_\diamond$  of Galois connections due to Dzik et al [19], which corresponds to the minimal Fitch-style calculus [11, 13]. Kavvos adopts a categorical perspective and shows that profunctors determine an adjunction  $\diamond \dashv \Box$  on presheaves, which can be used to model  $\text{IK}_\diamond$ . Kavvos’ profunctor condition is the proof-relevant refinement of Sotirov’s [36] bimodule frame condition which states that  $R_i; R_m; R_i \subseteq R_m$

Proof-relevant possible-world semantics and its connection to NbE for modal lambda calculi is a novel consideration in our work. Valliappan et al [38] prove normalization for Fitch-style modal lambda calculi [11, 13], consisting of the necessity modality  $\Box$  and its left adjoint  $\diamond$  using possible-world semantics with a proof-*irrelevant* relation  $R_m$ .

**Frame correspondence.** The study of necessary and sufficient frame conditions for modal axioms, known as *frame correspondence*, appears to be tricky in the proof-relevant setting. Plotkin and Stirling [33] prove a remarkably general correspondence theorem (Theorem 2.1) that tells us that the reflexivity of  $R_m; R_i^{-1}$  corresponds to axiom R and  $R_m^2 \subseteq R_m; R_i^{-1}$  corresponds to axiom J. We have not studied frame correspondence in this article, but leave it as a matter for future work. The categorical methods of Kavvos [27] might help here.

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## A Definitions of strong functors

A *strong* functor  $F : \mathcal{C} \rightarrow \mathcal{C}$  for a cartesian category  $\mathcal{C}$  is an endofunctor on  $\mathcal{C}$  with a natural transformation  $\theta_{P,Q} : P \times FQ \rightarrow F(P \times Q)$  natural in  $\mathcal{C}$ -objects  $P$  and  $Q$  such that the following diagrams stating coherence conditions commute:

$$\begin{array}{ccc}
 1 \times FP & \xrightarrow{\theta_{1,P}} & F(1 \times P) \\
 & \searrow \pi_2 & \swarrow F\pi_2 \\
 & & FP \\
 (P \times Q) \times FR & \xrightarrow{\theta_{P \times Q, R}} & F((P \times Q) \times R) \\
 \downarrow \alpha_{P,Q,FR} & & \downarrow F\alpha_{P,Q,R} \\
 P \times (Q \times FR) & \xrightarrow{id_P \times \theta_{Q,R}} P \times F(Q \times R) & \xrightarrow{\theta_{P, Q \times R}} F(P \times (Q \times R))
 \end{array}$$

Observe that the terminal object  $1$ , the projection morphism  $\pi_2 : P \times Q \rightarrow Q$  and the associator morphism  $\alpha_{P,Q,R} : (P \times Q) \times R \rightarrow P \times (Q \times R)$  (for all  $\mathcal{C}$ -objects  $P, Q, R$ ) live in the cartesian category  $\mathcal{C}$ .

A *pointed* functor  $F : \mathcal{C} \rightarrow \mathcal{C}$  on a category  $\mathcal{C}$  is an endofunctor on  $\mathcal{C}$  equipped with a natural transformation  $point : Id \rightarrow F$  from the identity functor  $Id$  on  $\mathcal{C}$ .

A strong and pointed functor  $F$  is said to be *strong pointed*, when it satisfies an additional coherence condition that  $point$  is a strong natural transformation, meaning that the following diagram stating a coherence condition commutes:

$$\begin{array}{ccc}
 & P \times Q & \\
 id_P \times point_Q \swarrow & & \searrow point_{P \times Q} \\
 P \times FQ & \xrightarrow{\theta_{P,Q}} & F(P \times Q)
 \end{array}$$

A *semimonad*  $F : \mathcal{C} \rightarrow \mathcal{C}$ , or *joinable* functor, on a category  $\mathcal{C}$  is an endofunctor on  $\mathcal{C}$  that forms a semigroup in the sense that it is equipped with a “multiplication” natural transformation  $\mu : F^2 \dot{\rightarrow} F$  that is “associative” as  $\mu_P \circ \mu_{FP} = \mu_P \circ F(\mu_P) : F^3 P \rightarrow FP$ .

A strong functor  $F$  that is also a semimonad is a *strong semimonad* when  $\mu$  is a strong natural transformation, meaning that the following coherence condition diagram commutes:

$$\begin{array}{ccc}
 P \times FFQ & \xrightarrow{\theta_{P,FQ}} & F(P \times FQ) \xrightarrow{F\theta_{P,Q}} FF(P \times Q) \\
 \downarrow id_P \times \mu_Q & & \downarrow \mu_{P \times Q} \\
 P \times FQ & \xrightarrow{\theta_{P,Q}} & F(P \times Q)
 \end{array}$$

A *monad*  $F : \mathcal{C} \rightarrow \mathcal{C}$  on a category  $\mathcal{C}$  is a semimonad that is pointed, such that the natural transformation *point* :  $Id \dot{\rightarrow} F$  is the left and right unit of multiplication  $\mu : F^2 \dot{\rightarrow} F$  in the sense that  $\mu_P \circ F\text{point}_P = id_{FP}$  and  $\mu_P \circ \text{point}_{FP} = id_{FP}$  for some  $\mathcal{C}$ -object  $P$ .

A strong functor  $F$  that is also a monad is a *strong monad* when the natural transformations *point* and  $\mu$  of the monad are both strong natural transformations, making  $F$  both a strong pointed functor and a strong semimonad.

## B Auxiliary definitions

► **Definition 15** (Context Inclusion). *The relation  $\subseteq$  is defined inductively on contexts:*

$$\frac{}{base : \cdot \subseteq \cdot} \quad \frac{i : \Gamma \subseteq \Gamma' \quad (x \text{ not in } \Gamma')}{drop_A i : \Gamma \subseteq \Gamma', x : A} \quad \frac{i : \Gamma \subseteq \Gamma' \quad (x \text{ not in } \Gamma')}{keep_A i : \Gamma, x : A \subseteq \Gamma', x : A}$$

■ The relation  $\subseteq$  is reflexive and transitive, as witnessed by functions:

$$\begin{array}{lll}
 refl_{\subseteq} : \forall \Gamma. \Gamma \subseteq \Gamma & trans_{\subseteq} : \forall \Gamma, \Gamma', \Gamma''. \Gamma \subseteq \Gamma' \rightarrow \Gamma' \subseteq \Gamma'' \rightarrow \Gamma \subseteq \Gamma'' & \\
 refl_{\subseteq} = base & trans_{\subseteq} i = base = i & \\
 refl_{\subseteq \Gamma, x:A} = keep_A refl_{\subseteq \Gamma} & trans_{\subseteq} (drop i) = drop (trans_{\subseteq} i i') & \\
 & trans_{\subseteq} (keep i) = keep (trans_{\subseteq} i i') & \\
 & trans_{\subseteq} (keep i) = keep (trans_{\subseteq} i i') & 
 \end{array}$$

■ The function *factor* for the NbE model of  $\lambda_{SL}$  is defined as:

$$\begin{array}{l}
 factor : \forall \Gamma, \Gamma', \Delta. \Gamma \subseteq \Gamma' \rightarrow \Gamma \triangleleft_{SL} \Delta \rightarrow \exists \Delta'. (\Gamma' \triangleleft_{SL} \Delta' \times \Delta \subseteq \Delta') \\
 factor \ i \ (\text{single } n) = (\text{single } (\uparrow n), \text{keep } i)
 \end{array}$$

■ The function *incl* for the NbE model of  $\lambda_{SL}$  is defined as:

$$\begin{array}{l}
 incl : \forall \Gamma, \Delta. \Gamma \triangleleft_{SL} \Delta \rightarrow \Gamma \subseteq \Delta \\
 incl \ (\text{single } (n : \Gamma \vdash_{NE} \diamond A)) = drop_A refl_{\subseteq \Gamma}
 \end{array}$$