A Polynomial Bound on the Pathwidth of Graphs Edge-Coverable by k Shortest Paths

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Abstract

Dumas, Foucaud, Perez and Todinca [SIAM J. Disc. Math., 2024] recently proved that every graph whose edge set can be covered by k shortest paths has pathwidth at most 3^k. In this paper, we improve this upper bound on the pathwidth to a polynomial bound; namely, we show that every graph whose edge set can be covered by k shortest paths has pathwidth $O(k^4)$, answering a question from the same paper. Moreover, we also prove that when $k \leq 3$, every such graph has pathwidth at most k (and this bound is tight). Eventually, we show that even though there exist graphs with arbitrary large treewidth whose vertex set can be covered by 2 isometric trees, every graph whose set of edges can be covered by 2 isometric trees has treewidth at most 2.

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1 Introduction

Graph covering is an old and recurrent topic in graph theory. For a fixed class $\mathcal{H}$ of graphs, the covering literature aims to answer two types of problems. First are optimization problems: given a graph G, what is the minimum number of graphs from $\mathcal{H}$ needed to "cover"[1] G?

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[1] Various notions of coverings can usually be considered here.

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We refer to [17] for a more general overview on the topic. The second type are structural problems: for the fixed class $\mathcal{H}$ of graphs and some $k \geq 1$, what is the structure of the graphs that can be "covered" by (at most) k graphs from $\mathcal{H}$?

We say that a graph G is *edge-coverable* (resp. *vertex-coverable*) by some subgraphs $H_1, \ldots, H_k$ if every *edge* (resp. *vertex*) of G belongs to at least one of the subgraphs H_i. Motivated by some algorithmic applications to the ISOMETRIC PATH COVER problem and its variants, Dumas, Foucaud, Perez and Todinca [10] recently worked on a particular metric version of such covering problems. They proved that every graph edge-coverable (resp. vertex-coverable) by few shortest paths shares structural similarities with a simple path. More precisely, if a graph is edge-coverable or vertex-coverable by k shortest paths, its *pathwidth* is smaller than an exponential in k (see Section 2 for a definition of pathwidth).

▶ **Theorem 1** ([10]). *Let $k \geq 1$ and G be a graph vertex-coverable by k shortest paths. Then* $\mathrm{pw}(G) = O(k \cdot 3^k)$. *Moreover, if G is edge-coverable by k shortest paths, then* $\mathrm{pw}(G) = O(3^k)$.

The proof of Theorem 1 uses a branching approach to show that in every graph vertex-coverable (resp. edge-coverable) by at most k shortest paths, for every vertex u and every $i \geq 1$, the number of vertices at distance exactly i from u in G is $O(k \cdot 3^k)$ (resp. $O(3^k)$). We also mention that a "coarse" variant of Theorem 1 has recently been proved in [14]. The authors of [10] asked if the bounds from Theorem 1 can be made polynomial in k. Our main result is a positive answer in the edge-coverable case.

▶ **Theorem 2.** *Let $k \geq 1$, and G be a graph edge-coverable by k shortest paths. Then* $\mathrm{pw}(G) = O(k^4)$.

Our proof uses a different approach from [10]. Roughly speaking, it consists in finding some separator X of polynomial size, such that for every component C of $G - X$, there exists one path P from the covering set of paths, such that C has a "nice" layering with respect to P.

In terms of lower bounds, one can show for every $k \geq 1$ the existence of a graph edge-coverable by k shortest paths and with pathwidth equal to k.[2] To our knowledge, no better lower bound is known. Our second main result is that this lower bound is tight for $k \leq 3$.

▶ **Theorem 3.** *Let $k \leq 3$, and G be a graph edge-coverable by k shortest paths. Then* $\mathrm{pw}(G) \leq k$.

Given a graph G, a subgraph H of G is *isometric* if for every two vertices x, y of H, the distances between x and y are the same in G and H. In other words, if x and y are connected in G, then there must exist a shortest path between them in G which is included in H. For instance, shortest paths in G are isometric subgraphs of G.

In view of the previous results, the following question naturally arises: given a family $\mathcal{H}$ of graphs, what is the structure of a graph G edge/vertex-coverable by at most k *isometric* subgraphs all isomorphic to graphs in $\mathcal{H}$? For instance, if $\mathcal{H}$ is the family of all trees, does G somehow look like a tree?

Extending a result of Aigner and Fromme [1] for paths, Ball, Bell, Guzman, Hanson-Colvin and Schonsheck [2] proved that every graph vertex-coverable by k isometric subtrees has *cop number* at most k. In particular, it is well-known that the cop number of a graph is always

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[2] For completeness, we give such a construction in the updated arXiv version of our paper (`https://arxiv.org/abs/2510.02901`). We thank one of the anonymous reviewers for pointing out that no such construction was explicitely given in [10], as we wrongly asserted in a previous version of this work.

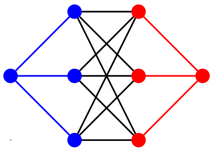

Figure 1 An example of a graph vertex-coverable by 2 isometric trees and containing a $K_{3,3}$.

upper-bounded by its *treewidth* (see Section 5 for a definition of treewidth). In analogy to Theorem 1, a natural question is then the following: is it true that every graph which is vertex/edge-coverable by a small number of isometric subtrees has small treewidth? This question turns out to have a negative answer when considering vertex-coverability, as shown by the following example, pointed to us by Marcin Briański (personal communication): for every $t \geq 1$, let G_t be the graph on $2t+2$ vertices obtained from the biclique $K_{t,t}$ after adding two vertices a, b of degree t respectively adjacent to all vertices from each of the sides of $K_{t,t}$ (see Figure 1 for the case $t = 3$). Then, G_t is vertex-coverable by the two induced stars S_a, S_b respectively centered in a and b (which are in particular isometric subgraphs of G_t), but has treewidth at least t, as it contains $K_{t,t}$ as a minor. Our third result implies that such a construction does not exist anymore when considering graphs which are edge-coverable by two isometric trees.

▶ **Theorem 4.** *Let G be a graph which is edge-coverable by 2 isometric trees. Then* $\mathrm{tw}(G) \leq 2$.

Algorithmic motivations. *Routing problems* play a central role not only in algorithmic graph theory and complexity, but also in structural graph theory. One of the best examples is the Disjoint Paths problem, appearing in the seminal Graph Minor series of papers of Robertson and Seymour [19], in which one is given as an input a graph G, an integer k and k pairs of vertices, and one must decide whether there exists k pairwise disjoint paths connecting all pairs of vertices. This problem has been shown to be solvable in FPT time, when parameterized by the number k of shortest paths (see [19, 15]), and this result has been a key ingredient to design efficient algorithms deciding the existence of *minors* in graphs. The metric variant Disjoint Shortest Path of this problem introduced in [11] (in which one further wants to decide whether there exist k pairwise disjoint shortest paths connecting all input pairs of vertices) has attired a lot of attention, and was recently proved to be solvable in XP time, when parameterized by k [5, 18, 4].[3]

A number of metric variants of routing problems of similar flavor has been studied over the last decades. Among them, Isometric Path Cover studied in [1, 12] takes as input a graph G and an integer k, and asks whether G is vertex-coverable by k-shortest paths. Its routing version Isometric Path Cover With Terminals takes as input a graph G, some integer k, and k pairs of vertices, and asks whether there exist k shortest paths that vertex-cover G and connect all the input pairs of vertices. Both problems are NP-Complete [7, 10]. Combining Theorem 1 with Courcelle's theorem [9], the authors from [10] proved respectively the existence of some algorithms running in XP and FPT time when parameterized by k, to solve respectively Isometric Path Cover and Isometric Path Cover With Terminals. In particular, it is still open whether Isometric Path Cover problem admits an FPT algorithm parameterized by k [10]. We also refer to [6] for more results on Isometric Path Cover, and to [8] for some recent work on similar metric versions of routing problems.

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[3] It is worth mentioning that all the aforementioned problems are NP-hard [13, 18].

Organization of the paper

Section 2 contains all preliminary definitions and some basic results related to pathwidth. Section 3 contains a proof of Theorem 2, Section 4 contains a proof of Theorem 3, and Section 5 contains a proof of Theorem 4. Sections 3–5 are independent of each other, and can be read in any order. Section 6 contains some discussion and additional questions left open by this work. This paper being an extended abstract of our work, throughout the paper, we will adopt the convention that every statement colored in dark green is proven in the extended version.

2 Preliminaries

2.1 Notations and Basic Definitions

Graphs. In this work, we consider undirected finite simple graphs, without loops. For a graph G, we denote with $V(G)$ its set of *vertices*, and $E(G)$ its set of *edges*. For any set $X \subseteq V(G)$ of vertices, $G[X]$ denotes the subgraph of G induced by X, that is, the vertex set of $G[X]$ is X, and $G[X]$ contains all edges of G with both end-points in X. $G - X$ denotes the subgraph of G induced by $V(G) \setminus X$. For simplicity, for every vertex x, we let $G - x$ denote the graph $G - \{x\}$. For all subsets $X, Y \subseteq V(G)$ of vertices, we denote $E_G(X, Y) := \{xy \in E(G) : x \in X \setminus Y, y \in Y \setminus X\}$. Let G' be a graph, whose vertex set potentially intersects the one of G. The union of G and G' is the graph $G \cup G' := (V(G) \cup V(G'), E(G) \cup E(G'))$.

For every $v \in V(G)$, we let $N_G(v)$ (or simply $N(v)$ when the context is clear) denote the set of neighbors of v. Similarly, for every subset $X \subseteq V(G)$ of vertices, we let $N_G(X)$ (or simply $N(X)$) denote the set of vertices in $V(G) \setminus X$ having at least one neighbor in X. If $U \subseteq V(G)$, we similarly let $N_U(X)$ denote the set of vertices in $U \setminus X$ having at least one neighbor in X.

For every three subsets $X, A, B \subseteq V(G)$, we say that X *separates* A from B in G if no component of $G - X$ contains both a vertex from A and a vertex from B.

Paths. A path P is a sequence of pairwise distinct vertices $v_1, \ldots, v_k$ such that for each $i < k$, $v_i v_{i+1} \in E(G)$. Note that by definition, a path is a sequence of vertices; however, by abuse of notations, we will often identify a path $P = v_1, \ldots, v_k$ of G and its corresponding subgraph in G (that is, the graph with vertices $v_1, \ldots, v_k$ and edges $v_i v_{i+1}$ for each i such that $1 \leq i < k$). A path connecting two vertices $v_1 = x$ and $v_k = y$ will also be called an *xy-path*. We call the vertices v_1 and v_k the *endvertices* of P, while all its other vertices are its *internal vertices*. The *length* of P corresponds to its number of edges. For each $i \leq j$, we let $P[v_i, v_j]$ denote the subpath $v_i, v_{i+1}, \ldots, v_j$ of P, and we let P^{-1} denote the path $v_k, \ldots, v_1$. For every two paths $P = v_1, \ldots, v_p$ and $Q = w_1, \ldots, w_q$ such that $v_p = w_1$, and such that $P - v_p$ is disjoint from $Q - w_1$, we define the path $P \cdot Q := v_1, \ldots, v_p = w_1, \ldots, w_q$. If $P = v_1, \ldots, v_k$ is a path, for every $1 \leq i < j \leq k$, we say that v_i is *before* v_j on P, and that v_j is *after* v_i on P. A path P is *vertical* with respect to a vertex u, if for every $i \in \mathbb{N}$, P has at most one vertex at distance i from u in G. Note that every path vertical with respect to some vertex u is a shortest path, and that every shortest path starting from u is vertical with respect to u.

Covers. Let G be a graph and $H_1, \ldots, H_k$ be subgraphs of G. We say that the graphs $H_1, \ldots, H_k$ *edge-cover* G if $E(G) = \bigcup_{i=1}^{k} E(H_i)$. Similarly, we say that $H_1, \ldots, H_k$ *vertex-cover* G if $V(G) = \bigcup_{i=1}^{k} V(H_i)$.

Metrics. We let $d_G(\cdot,\cdot)$ (or simply $d(\cdot,\cdot)$ when the context is clear) denote the shortest-path metric of a graph G. Recall that a graph H is an *isometric subgraph* of G if H is a subgraph of G such that every shortest path in H is also a shortest path in G. For every vertex v of a graph G, we let $\operatorname{ecc}(v) := \max\{d_G(u,v) : u \in V(G)\} \in \mathbb{N} \cup \{\infty\}$ denote the *eccentricity* of v.

Orders. Let $(X, \le)$ be a partially ordered set. For every two disjoint subsets $A, B \subseteq X$, we write $A < B$ if $a < b$ for every $(a,b) \in A \times B$. Moreover, for every $x \in X$ and every $A \subseteq X$, we let $A|_{\le x} := \{a \in A : a \le x\}$ and $A|_{\ge x} := \{a \in A : a \ge x\}$. We define similarly $A|_{<x}$ and $A|_{>x}$. A *chain* is a sequence $(x_1, \dots, x_k)$ of distinct elements of X such that $x_1 \le x_2 \le \dots \le x_k$.

2.2 Path-decompositions

We collect in this section a number of basic properties of pathwidth.

Path-decomposition. A path-decomposition of a graph G is a sequence $\mathcal{P} = (X_1, X_2, \dots, X_q)$ of vertex subsets of G, called bags, such that $V(G) = \bigcup_{i=1}^{q} X_i$, for every edge $\{x,y\} \in E$ there is at least one bag containing both endpoints, and for every vertex $x \in V$, the bags containing x form a continuous subsequence of $\mathcal{P}$. The width of $\mathcal{P}$ is $\max\{|X_i| - 1 : 1 \le i \le q\}$, and the *pathwidth* $\operatorname{pw}(G)$ of G is the smallest width of a decomposition, among all path-decomposition of G. The bags X_1, X_q are called the *extremities* of $\mathcal{P}$.

▶ Remark 5. For a graph G and $X \subseteq V(G)$, $\operatorname{pw}(G) \le |X| + \operatorname{pw}(G - X)$.

Vertex-separation number. Let G be a graph and $<$ denote a total ordering on $V(G)$. The vertex-separation number of $(G,<)$ is the value $\min_{w \in V(G)} |\{u \in V(G) : u < w \text{ and } \exists v \ge w, uv \in E(G)\}|$. The vertex separation number of G is the minimum over the vertex separation numbers of $(G,<)$, for all possible linear orderings $<$ of G.

▶ **Theorem 6** ([16]). *The vertex separation number of a graph equals its pathwidth.*

Layerings. A *layering* of a graph G is a partition of its vertices into a sequence $(V_1, \dots, V_t)$ of subsets of $V(G)$ such that each graph $G[V_i]$ is an independent set, and for every $i, j \in \{1, \dots, t\}$ such that $|j - i| \ge 2$, we have $|E_G(V_i, V_j)| = 0$.[4] A *k-layering* of a graph G is a partition of $V(G)$ into a sequence $(V_1, \dots, V_t)$ of sets such that each graph $G[V_i]$ is an independent set, and for every $0 < i < t$, we have $|E_G(V_i, V_{i+1})| \le k$. Note that in particular, it implies that $|N_{V_{i+1}}(V_i)| \le k$ and $|N_{V_i}(V_{i+1})| \le k$. The bags V_1 and V_t are the *extremities* of the layering.

▶ **Lemma 7.** Let G be a graph admitting a k-layering. Then G has pathwidth at most k. Moreover, if G is connected, then G has a path-decomposition of width at most k, whose extremities are exactly the extremities of the k-layering.

3 A General Polynomial Bound

In this section, we prove Theorem 2.

▶ **Theorem 2.** *Let $k \ge 1$, and G be a graph edge-coverable by k shortest paths. Then* $\operatorname{pw}(G) = O(k^4)$.

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[4] Note that our definition of layering differs from the one which is commonly used in the literature, as one usually allows in a layering edges between vertices from a same layer.

Throughout this section, we let G be a graph whose edge set can be covered by a set $\mathcal{P} := \{P_1, \ldots, P_k\}$ of k isometric paths. Our main result is the following, which will immediately imply a proof of Theorem 2.

▶ **Theorem 8.** *For every $i \in \{1, \ldots, k\}$, there exists a set of vertices $X_i \subseteq V$ of size at most $720k^3 + 4k$ such that every connected component of $G - X_i$ that intersects P_i has pathwidth at most $6k$.*

Proof that Theorem 8 implies Theorem 2. For every $1 \leq i \leq k$, we use Theorem 8 to obtain the set X_i. We consider their union $X := \bigcup_{i=1}^{k} X_i$. Observe that $|X| \leq \sum_{i\in[k]}|X_i| \leq k \cdot (720k^3 + 4k) = 720k^4 + 4k^2$. We now let C be a connected component of $G - X$. As $P_1, \ldots, P_k$ edge-cover (and thus also vertex-cover) G, there is some $i \in \{1, \ldots, k\}$ such that the path P_i intersects C. In particular, as C is included in a connected component of $G - X_i$, it implies that C has pathwidth smaller than $6k$, and thus that $\mathrm{pw}(G - X) \leq 6k$. Using Remark 5 we conclude that $\mathrm{pw}(G) \leq |X| + \mathrm{pw}(G - X) \leq 720k^4 + 4k^2 + 6k$. ◀

The remainder of the section will consist in a proof of Theorem 8. In Subsection 3.1 we prove some preliminary results, and show that our proof reduces to the problem of finding some special kind of separations in a subgraph G_0 of G which is $6k$-layered (see Lemma 15). In Subsection 3.2, we give a proof of Lemma 15, which is the most technical part of our proof.

Without loss of generality, we will assume from now on that $i = 1$, i.e., we will construct a set X of size at most $720k^3 + 4k$ such that every component of $G - X$ that intersects P_1 has pathwidth at most $6k$.

3.1 Preliminary Results

Parallel paths. Let P be a shortest path between two vertices a, b of G. A path Q of G is *parallel* to P in G if Q is a subpath of a shortest ab-path of G.

Note that this does not define a symmetric relation in general: the fact that a path Q is parallel to another path P does not necessarily imply that P is parallel to Q. However, in the remainder of the proof, we will always consider the property of being parallel to the path P_1, which is fixed and connects vertices a_1 and b_1. In particular, the following simple observation implies that every graph which is edge-coverable by a few number of paths that are all parallel to P_1 has a very restrained structure.

▶ **Lemma 9.** Let $\{Q_1, \ldots, Q_\ell\}$ be a collection of paths in G that are all parallel to P_1. Then the graph $\bigcup_{j=1}^{\ell} Q_j$ is ℓ-layered.

We consider the partial order $\leq_1$ on $V(G)$ defined by setting for every two vertices $u, v \in V(G)$, $u \leq_1 v$ if and only if $d_G(a_1, u) \leq d_G(a_1, v)$. Note in particular that the vertex set of every path P which is parallel to P_1, forms a chain with respect to $\leq_1$.

▶ **Lemma 10.** *Let $a \leq_1 b \leq_1 c$ be three vertices such that there exist two paths P, Q both parallel to P_1, such that P connects a to b and Q connects b to c. Then $P \cdot Q$ is parallel to P_1.*

Proof. Note that $P \cdot Q$ must be a shortest ac path, as for each $i \geq 0$, it contains at most one vertex at distance exactly i from a_1. In particular, it is not hard to conclude that $P \cdot Q$ is parallel to P_1. ◀

Reducing $\mathcal{P}$. The first step in the proof of Theorem 8 consists in modifying the covering family $\mathcal{P}$ in order to make it reduced (see definition below). The intuition behind this step is that our proof of Theorem 8 works particularly well when all paths in $\mathcal{P}$ intersect P_1. Informally, the reduction operation consists in making $\mathcal{P}$ contain as much paths that intersect P_1 as possible. This step will turn out to be useful at the very end of our proof (see the proof of Lemma 17).

A shortest path Q of G is called a *reducing path* if

- Q is disjoint from P_1, and admits a subpath P parallel to P_1, that connects two vertices u, v, such that $u <_1 v$,
- there exists a shortest path R in G from u to v that intersects P_1.

Figure 2 depicts a reducing path. If a family $\mathcal{Q}$ of shortest paths has no reducing path, then we say that $\mathcal{Q}$ is *reduced*.

▶ **Proposition 11.** *Let $\mathcal{Q}$ be a family of shortest paths. Then there exists a family $\mathcal{Q}'$ of shortest paths which is reduced, such that $|\mathcal{Q}'| \leq 2|\mathcal{Q}|$ and that the paths of $\mathcal{Q}'$ cover all the edges covered by the paths of $\mathcal{Q}$, i.e., such that*

$$\bigcup_{P \in \mathcal{Q}} E(P) \subseteq \bigcup_{P \in \mathcal{Q}'} E(P).$$

Proof. Assume that $\mathcal{Q}$ is not reduced, and let Q be a reducing path in $\mathcal{Q}$. We also let P be a subpath of Q with endvertices u, v such that $u <_1 v$, and R be a shortest path in G from u to v intersecting P_1. We let R_1, R_2 denote two shortest paths respectively from a_1 to u and from v to b_1. We also write $Q = Q_1 \cdot Q_2 \cdot Q_3$, so that Q_2 is the uv-subpath of Q parallel to P_1 (see Figure 2). Observe now that both paths $Q' := Q_1 \cdot R \cdot Q_3$ and $Q'' := R_1 \cdot Q_2 \cdot R_2$ are shortest paths in G that both intersect P_1 and cover all the edges of Q. In particular, it implies that the family of paths $\mathcal{Q}' := (\mathcal{Q} \setminus \{Q\}) \cup \{Q', Q''\}$ covers all the edges covered by the paths of $\mathcal{Q}$, and has strictly fewer paths than $\mathcal{Q}$ that do not intersect P_1. We apply iteratively the same operation as long as the obtained family $\mathcal{Q}'$ is not reduced. In particular, as at each step, the number of paths of $\mathcal{Q}$ which are disjoint with P_1 strictly decreases, then after at most $|\mathcal{Q}|$ iterations, we obtain a reduced family with the desired properties. ◀

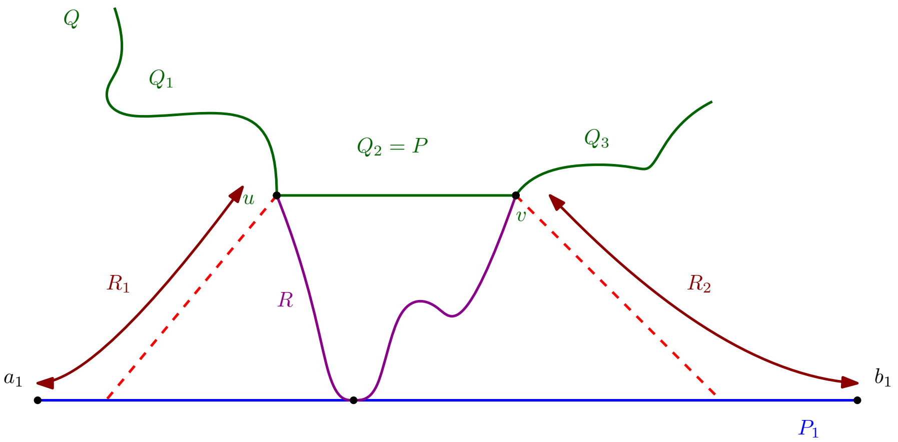

Figure 2 The path Q represented in green is reducing. P_1 is represented in blue.

Up to applying Proposition 11 to $\mathcal{P}$, we will assume from now on that $\mathcal{P}$ is a reduced family of at most $2k$ shortest paths of G that edge-cover G.

Good and bad vertices. We now partition the set $\mathcal{P} := \{P_i : 1 \le i \le 2k\}$ into the subset $\mathcal{P}_1$ of the paths in $\mathcal{P}$ that intersect P_1, and $\mathcal{P}_2 := \mathcal{P} \setminus \mathcal{P}_1$.

For every $P \in \mathcal{P}_1$, we let C_P denote the maximal subpath of P with both endvertices a_P, b_P in $V(P_1)$, so that $a_P \le_1 b_P$ (up to considering P^{-1} instead of P, we also assume that a_P is before b_P on P). Note that $C_{P_1} = P_1$, and that by definition of $\mathcal{P}_1$, each C_P has at least one vertex, and is parallel to P_1. We moreover consider for every $P \in \mathcal{P}_1$ the two vertices u_P, v_P that respectively maximize $d_G(u_P, a_P)$ and $d_G(b_P, v_P)$, such that u_P, a_P, b_P, v_P appear in this order on P, and such that the paths $P[u_P, a_P]$ and $P[b_P, v_P]$ are parallel to P_1. We now let $\widetilde{C}_P := P[u_P, v_P]$ and A_P, B_P denote the two subpaths of P such that $P = A_P \cdot \widetilde{C}_P \cdot B_P$ (see Figure 3). We stress out that in general, the path $\widetilde{C}_P$ is not necessarily parallel to P_1.

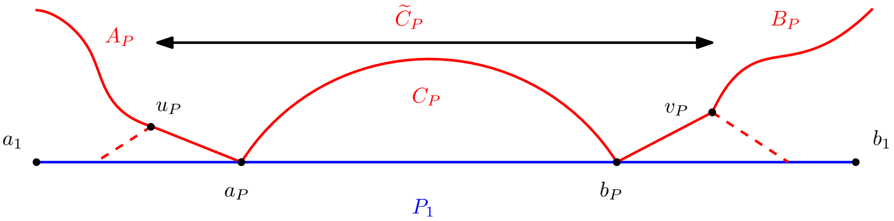

Figure 3 The red path represents some path $P \in \mathcal{P}_1$. The blue path is P_1.

We let $\mathcal{A} := \{A_P^{-1} : P \in \mathcal{P}_1\} \cup \{B_P : P \in \mathcal{P}_1\}$ and $\mathcal{B} := \{\widetilde{C}_P : P \in \mathcal{P}_1\}$. In particular, $P_1 \in \mathcal{B}$. Note that by definition, every path in $\mathcal{B}$ is the concatenation of at most 3 paths that are parallel to P_1, and recall that $|\mathcal{P}| \le 2k$, hence we have $|\mathcal{B}| \le 2k$.

We say that the edges of the paths in $\mathcal{A} \cup \mathcal{P}_2$ are *bad*, and we call a vertex $v \in V(G)$ *bad* if it is the endvertex of a bad edge. We call every vertex which is not bad *good*, and we let V_b and V_g denote respectively the set of bad and good vertices of G. We moreover set $X_0 := \{a_P : P \in \mathcal{P}_1\} \cup \{b_P : P \in \mathcal{P}_1\}$ and $X_1 := \{u_P : P \in \mathcal{P}_1\} \cup \{v_P : P \in \mathcal{P}_1\}$. Note that in particular, $|X_0| \le 4k$ and that the only bad vertices that belong to P_1 are the vertices a_P, b_P for $P \in \mathcal{P}_1$, hence $V(P_1) \cap V_b \subseteq X_0$.

We now consider the subgraph $G_0 := \bigcup_{P \in \mathcal{B}} P$ of G. The following remark immediately follows from the definition of $\mathcal{B}$, the fact that $|\mathcal{P}_1| \le |\mathcal{P}| \le 2k$, and that every path in $\mathcal{B}$ is edge-covered by at most 3 paths that are parallel to P_1.

▶ Remark 12. G_0 is edge-covered by at most $6k$ paths that are parallel to P_1.

▶ **Lemma 13.** *We have $V_g \subseteq V(G_0)$. Moreover, $G[V_g]$ is a subgraph of G_0.*

Proof. Let u be a good vertex. As G is covered by $\mathcal{P}$, there exists some path $P \in \mathcal{P}$ such that $u \in V(P)$. In particular, as u is a good vertex, we must have $P \in \mathcal{P}_1$, and moreover u must be a vertex of $\widetilde{C}_P$. It implies that $u \in V(G_0)$, and thus that $V_g \subseteq V(G_0)$. The "Moreover" part follows from the fact that by definition, no bad edge can be incident to a good vertex. In particular, note that every edge from $E(G) \setminus E(G_0)$ is bad, hence $G[V_g]$ is indeed a subgraph of G_0. ◀

Our goal from now on will be to prove the following lemma.

▶ **Lemma 14** (Main). *There exists a set $X \subseteq V(G)$ of size at most $720k^3 + 4k$ that separates every vertex on P_1 from V_b in G.*

Proof of Theorem 8 using Lemma 14. We let X be given by Lemma 14. In particular, by Lemma 13, every component of $G - X$ that contains a vertex of P_1 induces a subgraph of G_0. By Lemma 7, Lemma 9 and Remark 12, G_0 is $6k$-layered, and thus has pathwidth at most $6k$, allowing us to conclude the proof of Theorem 8. ◀

Reduction to a separation problem in the graph G_0. We will now show that, in order to prove Lemma 14, it is enough to separate $V(P_1)$ from V_b in the subgraph G_0 of G. We show that Lemma 14 amounts to prove the following key lemma, that we will prove in Section 3.2.

▶ **Lemma 15** (Key lemma). *Let P be a path of G_0 which is parallel to P_1, and $A \in \mathcal{A} \cup \mathcal{P}_2$ which intersects P. Then there exists some set $X_{A,P} \subseteq V(G_0)$ of vertices of size $30k$ such that $X_{A,P}$ separates every vertex of $V(A) \cap V(P)$ from the vertices of P_1 in G_0.*

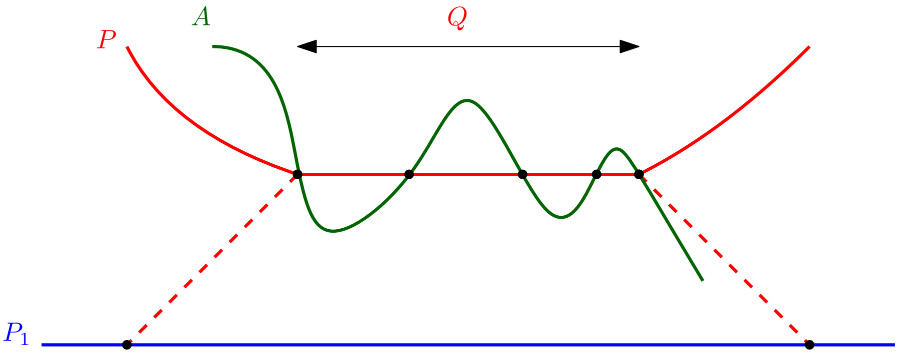

Figure 4 Configuration of Lemma 15.

Proof of Lemma 14 using Lemma 15. Recall that every path of $\mathcal{B}$ is the concatenation of at most 3 paths parallel to P_1. We thus consider a set $\mathcal{B}'$ of size at most $3|\mathcal{B}|$ of paths parallel to P_1 that cover the edges of the paths from $\mathcal{B}$. We let I denote the set of all pairs (A, P) such that $A \in \mathcal{A} \cup \mathcal{P}_2$, $P \in \mathcal{B}'$, and $V(A) \cap V(P) \neq \emptyset$.

We set

$$X := X_0 \cup \left(\bigcup_{(A,P)\in I} X_{A,P} \right),$$

where the sets $X_{A,P}$ are given by Lemma 15. Note that $|\mathcal{A} \cup \mathcal{P}_2| \leq 2|\mathcal{P}_1| + |\mathcal{P}_2| \leq |\mathcal{P}_1| + |\mathcal{P}| \leq 4k$ and $|\mathcal{B}'| \leq 3|\mathcal{B}| = 3|\mathcal{P}_1| \leq 6k$, thus $|I| \leq 24k^2$ and $|X| \leq 720k^3 + 4k$.

We now show that X separates $V(P_1)$ from V_b in G. Suppose, for contradiction, that there exists a path R in $G - X$ connecting a vertex v from V_b to a vertex u on P_1. Among those paths, choose a path R with minimum length. Since $V(P_1) \cap V_b \subseteq X_0 \subseteq X$, we have $v \notin V(P_1)$ and thus R has length at least 1. Also, since R is a path of minimum length connecting V_b to P_1, every vertex in $R - v$ must be good. Since every vertex in $R - v$ is good, there is no bad edge in R. Hence, as $\mathcal{P}$ edge-covers G, all the edges of R must be covered by the paths from $\mathcal{B}$, implying that R is also a path of G_0.

This implies in particular that $v \in V(G_0) \cap V_b$. Thus by definition of G_0 and V_b, there exist some paths $P \in \mathcal{B}$ and $A \in \mathcal{A} \cup \mathcal{P}_2$ such that $v \in V(P) \cap V(A)$. Lemma 15 then implies that R intersects $X_{A,P} \subseteq X$, giving a contradiction. ◀

3.2 Proof of Lemma 15

We fix some path P in G_0 which is parallel to P_1, and some path $A \in \mathcal{A} \cup \mathcal{P}_2$ which intersects P. We will in fact prove that one can find a set $X_{A,P}$ of vertices with the desired size that separates in G_0 all the vertices of Q from P_1. We let Q denote the minimal subpath of P containing all vertices in $V(A) \cap V(P)$, and let b, c denote the endvertices of Q such that $b \leq_1 c$. By Remark 12, G_0 is coverable by at most $6k$ paths which are all parallel to P_1. In particular, for every $i \geq 0$, the set $V_i := \{v \in V(G_0) : d_G(v, a_1) = i\}$ contains at most $6k$ vertices, and separates in G_0 the sets $\bigcup_{j \leq i} V_j$ and $\bigcup_{j \geq i} V_j$. We also may assume without loss of generality that Q contains at least 2 vertices, hence $b <_1 c$ (otherwise, we conclude by choosing $X_{A,P} := V(Q) = \{b\}$).

For every $v \in V(G_0)$, we let $\iota(v)$ denote the unique index $i \geq 0$ such that $v \in V_i$. For every path L in G_0 which is parallel to P_1, if u, v denote the endvertices of L with $u \leq_1 v$, we define the *projection* of L on P_1 as the set $U_L := \{x \in V(P_1) : \iota(u) < \iota(x) < \iota(v)\}$. We say that L is *$\nearrow$-free* if there does not exist in G_0 a path R parallel to P_1 that connects a vertex $x \in U_L$ to a vertex $y \in V(L)$ with $x \leq_1 y$ (see Figure 5). Symmetrically, we say that L is *$\searrow$-free* if there does not exist in G_0 a path R parallel to P_1 that connects a vertex $x \in U_L$ to a vertex $y \in V(L)$ with $x \geq_1 y$. Our main result in this subsection will be Lemma 18, which states that if Q is $\nearrow$-free or $\searrow$-free, then we can separate Q from P_1 in G_0 after removing $30k$ vertices. Before proving this, we will first show in the next two lemmas that Q can be covered by at most two subpaths which are each $\nearrow$-free or $\searrow$-free.

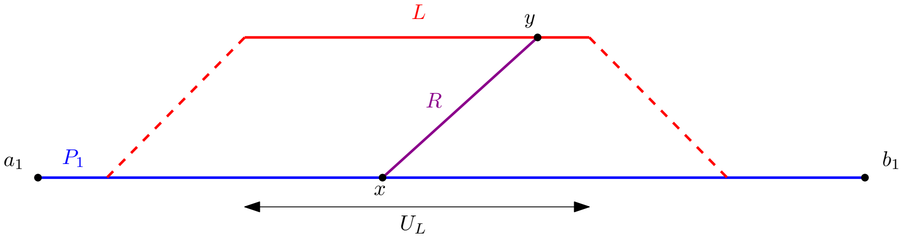

Figure 5 The path L (in red) is not $\nearrow$-free.

▶ **Lemma 16.** *If $A \in \mathcal{A}$, then Q is $\nearrow$-free or $\searrow$-free.*

Proof. We assume first that $A = A_S^{-1}$ for some $S \in \mathcal{P}_1$, and claim that the case where $A = B_S$ is symmetric. We moreover assume that $A[u_S, b]$ is a subpath of $A[u_S, c]$, as depicted in Figure 6, and claim that the case where $A[u_S, c]$ is a subpath of $A[u_S, b]$ is symmetric. We show that in this case, Q must be $\nearrow$-free. As P, and thus Q are parallel to P_1, note that every vertex $v \in V(Q)$ satisfies $\iota(v) \in [\iota(b), \iota(c)]$. We let A' denote the subpath of S^{-1} which starts at a_S and contains A as a subpath.

We assume for sake of contradiction that there exists some path R in G_0 which is parallel to P_1 in G, and which connects a vertex $x \in U_Q$ to a vertex $y \in V(Q)$, with $x \leq_1 y$. We consider two cases according to whether $a_S \leq_1 x$ or not, described in Figure 6.

If $a_S \leq_1 x$, then by Lemma 10, the path $P_1[a_S, x] \cdot R \cdot Q[y, c]$ is parallel to P_1. On the other hand, recall that by definition of u_S, and as $c \neq u_S$ (this is because $b \neq c$ and $A[u_S, c]$ contains b), $A'[a_S, c]$ cannot be parallel to P_1. In particular, $A'[a_S, c]$ must be strictly longer than $P_1[a_S, x] \cdot R \cdot Q[y, c]$, implying a contradiction as A' is a shortest path in G.

If $a_S >_1 x$, then as $x \in U_Q$, $x >_1 b$. Then, the path $R \cdot Q[y, c]$ is vertical with respect to a_1, and thus strictly shorter than Q. Note in this case, the path $A'[a_S, b]$ is also strictly longer than $P_1[a_S, x]$. It thus implies that $P_1[a_S, x] \cdot R \cdot Q[y, c]$ is strictly shorter than the path $A'[a_S, c]$, contradicting again that A' is a shortest path in G. ◀

▶ **Lemma 17.** *If $A \in \mathcal{P}_2$, then there exists subpaths Q_1, Q_2 such that $Q = Q_1 \cdot Q_2$, Q_1 is $\searrow$-free and Q_2 is $\nearrow$-free.*

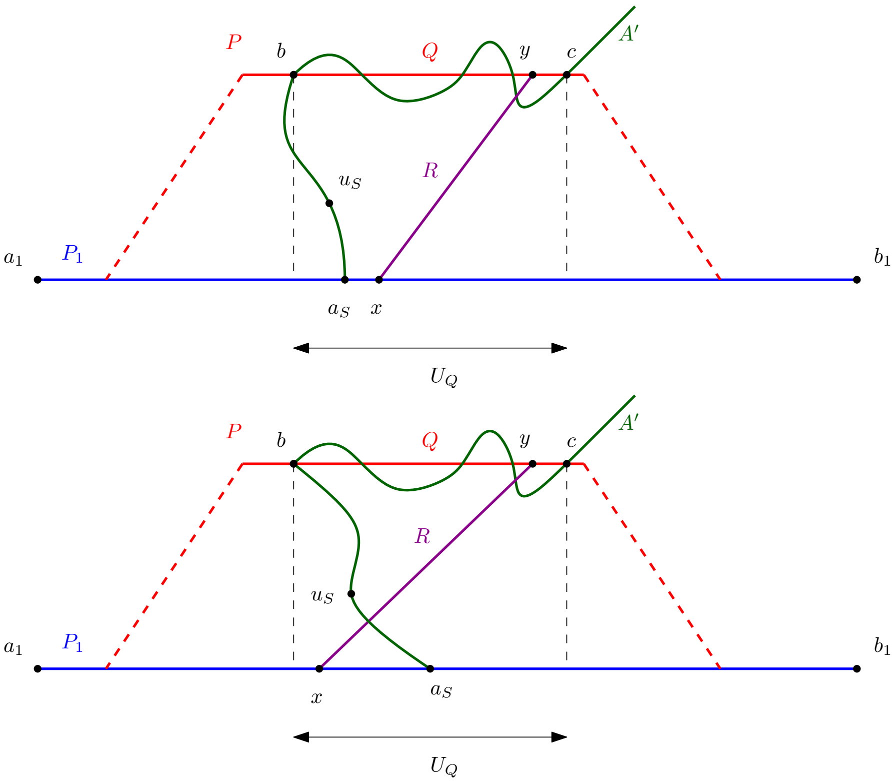

Figure 6 Top: configuration in the proof of Lemma 16 when $a_S \leq_1 x$. Bottom: configuration in the proof of Lemma 16 when $a_S >_1 x$.

Proof. If Q is $\nearrow$-free, then we immediately conclude by choosing Q_1 to be the path having b as a single vertex, thus we may assume that there exists some path R parallel to P_1 connecting some vertex $x \in U_Q$ to a vertex $y \in V(Q)$ such that $x \leq_1 y$. We moreover choose such a path so that x is maximal with respect to $\leq_1$. We let z be the unique vertex of Q such that $\iota(z) = \iota(x)$, and set $Q_1 := Q[b, z], Q_2 := Q[z, c]$. By maximality of x, note that Q_2 must be $\nearrow$-free. We moreover claim that Q_1 is $\searrow$-free. Assume for a contradiction that it is not the case, and that there exists a shortest path R' parallel to P_1 connecting a vertices $y' \leq_1 x'$, with $y' \in V(Q_1)$ and $x' \in U_{Q_1}$ (see Figure 7). Then the path $T := Q_1[b, y'] \cdot R' \cdot P_1[x', x] \cdot R \cdot Q_2[y, c]$ is a path in G from b to c that intersects P_1 and which is parallel to P_1. In particular, as $b, c \in V(A)$ the existence of T implies that A is a reducing path belonging to $\mathcal{P}$, contradicting our assumption that $\mathcal{P}$ is reduced. ◀

The next lemma is the crucial part from the proof of Lemma 15. As its proof is slightly technical, we included it in the long version.

▶ **Lemma 18.** Let L be a path in G_0 that is parallel to P_1. If L is $\nearrow$-free or $\searrow$-free, then there exists $X \subseteq V(G_0)$ of size at most $18k$ that separates in G_0 the vertices of L from the ones of P_1.

Proof of Lemma 15. By Lemmas 16 and 17, there exist two subpaths Q_1, Q_2 of Q such that $Q = Q_1 \cdot Q_2$, and such that for each $i \in \{1, 2\}$, Q_i is either $\nearrow$-free or $\searrow$-free. In particular, by Lemma 18, there exist two sets X_1, X_2 of size at most $18k$ such that for each $i \in \{1, 2\}$, X_i separates in G_0 the set $V(Q_i)$ from $V(P_1)$. In particular, the set $X := X_1 \cup X_2$ then separates

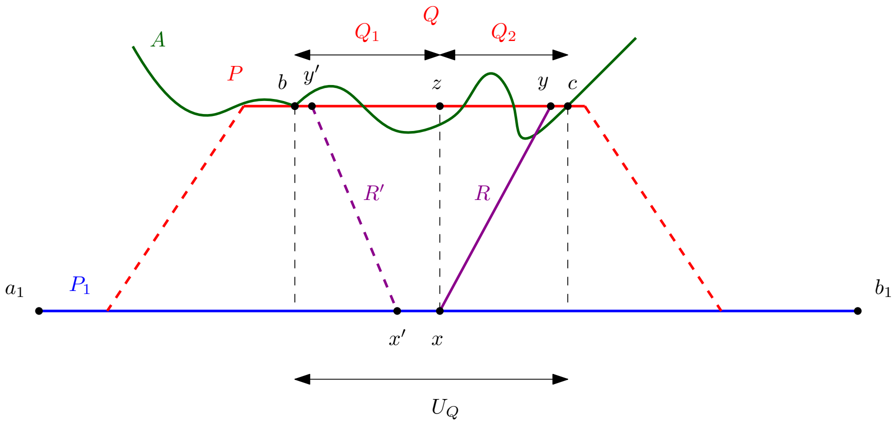

Figure 7 The configuration at the end of the proof of Lemma 17. The purple dashed path R' represents a shortest path which is parallel to P_1, and connects vertices $y' \leq_1 x'$, with $y' \in V(Q_1)$ and $x' \in U_{Q_1}$.

$V(Q)$ from $V(P_1)$. Moreover, we claim that when going through the proof of Lemma 18, one gets that X_1 and X_2 both contain a common layer V_i (for $i := \iota(c_{Q_1}) = \iota(b_{Q_2})$), allowing us to slightly reduce our upper bound on $|X_1 \cup X_2|$ to obtain $|X_1 \cup X_2| \leq 30k$ (instead of $36k$). ◀

4 Exact Bounds for $k \leq 3$

In this section, we prove some optimal upper-bounds on the pathwidth of graphs that are edge-coverable by k shortest paths, when $k \in \{2, 3\}$. As our proof of the case $k = 3$ is fairly technical, with multiple different cases to consider separately, we refer to the long version for proofs of all the next lemmatas.

▶ **Theorem 3.** *Let $k \leq 3$, and G be a graph edge-coverable by k shortest paths. Then* $\mathrm{pw}(G) \leq k$*.*

4.1 Two Paths

As a warm-up, we start describing the structure of graphs edge-coverable by 2 shortest paths, implying an immediate proof of Theorem 3 when $k = 2$.

▶ **Definition 19.** *A* skewer *is a graph G for which there exists a sequence of pairwise distinct vertices $x_1, \ldots, x_\ell$ ($\ell \geq 1$) such that:*

- *for each $i \in \{1, \ldots, \ell - 1\}$, the vertex x_i is connected to the vertex x_{i+1} by two internally disjoint paths Q_i, R_i having the same length;*
- *for each $i \in \{1, \ldots, \ell - 1\}$, the paths Q_i, R_i are separated from $G - (Q_i \cup R_i)$ by the set $\{x_i, x_{i+1}\}$;*
- *there exist four internally disjoint paths Q_0, R_0, Q_ℓ, R_ℓ (which do not necessarily have the same length), such that Q_0, R_0 (resp. Q_ℓ, R_ℓ) are separated from $G - (Q_0 \cup R_0)$ (resp. $G - (Q_\ell \cup R_\ell)$) by the set $\{x_1\}$ (resp. $\{x_\ell\}$).*

See Figure 8 for an illustration of a skewer. For each $i \in \{0, \ldots, \ell\}$, the graph $C_i := Q_i \cup R_i$ is called the *piece* of x_i. Note that we allow the paths Q_i and R_i to have length 1, in which case the graph C_i is just the edge $x_i x_{i+1}$ (recall that the graphs we consider are simple), while if

Q_i and R_i have length at least 2, then C_i is an even cycle. Note that every skewer G has a 2-layering, and thus pathwidth at most 2, and that moreover, for every $i \in \{0, \dots, \ell\}$, and every pair of vertices a, b with $a \in V(Q_i)$ and $b \in V(R_i)$, there exists a path-decomposition of G of width 2 in which there exists a bag equals to $\{a, b\}$.

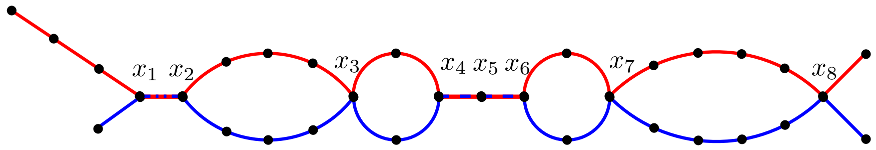

Figure 8 A skewer. The blue paths represent the paths R_i and the red paths represent the paths Q_i.

▶ **Lemma 20.** *Let P, Q be two shortest paths in a graph G. Then there cannot exist three vertices $a, b, c \in V(P) \cap V(Q)$ such that b is an internal vertex of $P[a, c]$ and such that c is an internal vertex of $Q[a, b]$.*

Proof. Assume for sake of contradiction that there exist three such vertices $a, b, c \in V(P) \cap V(Q)$. Since b is an internal vertex of $P[a, c]$, $d(a, b) < d(a, c)$. But since c is an internal vertex of $Q[a, b]$, $d(a, c) < d(a, b)$, a contradiction. ◀

▶ **Proposition 21.** *Every connected graph G which is edge-coverable by two isometric paths is a skewer. In particular,* $\mathrm{pw}(G) \leq 2$.

Proof. We let P_1, P_2 be two isometric paths that edge-cover G, and let $x_1, \dots, x_\ell$ denote the vertices from $V(P_1) \cap V(P_2)$, such that for each $i < j$, x_i is before x_j on P_1. By Lemma 20, for each $i < j$, x_i is also before x_j on P_2. It thus implies that each x_i separates in G all vertices appearing before it in P_1 and P_2 from all vertices appearing after it in P_1 and P_2. In particular, it easily follows that G is a skewer. ◀

4.2 Three Paths

This subsection consists in a proof of Theorem 3 when $k = 3$, which we restate here for convenience.

▶ **Theorem 22.** *Let G be a graph which is edge-coverable by 3 isometric paths. Then* $\mathrm{pw}\,(G) \leq 3$.

In the remainder of the subsection, we let G denote a graph which is edge-coverable by three shortest paths P_1, P_2, P_3. We also assume that for each $i \in \{1, 2, 3\}$, P_i is a $a_i b_i$-path for some vertices $a_i, b_i \in V(G)$. Moreover, note that if the P_i's do not all belong to the same connected component of G, then every connected component of G is coverable by at most 2 shortest paths, hence by Proposition 21, $\mathrm{pw}(G) \leq 2$. We thus moreover assume that G is connected.

For two sequences $\mathcal{P}_1 = (X_1, \dots, X_k)$ and $\mathcal{P}_2 = (Y_1, \dots, Y_\ell)$ of vertex subsets such that $X_k = Z = Y_\ell$, we say that $\mathcal{P} = (X_1, \dots, X_k, Z, Y_1, \dots Y_\ell)$ is *the glue of* $\mathcal{P}_1$ and $\mathcal{P}_2$ *by the bag* Z. The following three lemmas deal with some first easy cases.

▶ **Lemma 23.** If the intersection of two of the paths P_1, P_2, P_3 is included in the third one, then $\mathrm{pw}(G) \leq 3$.

▶ **Lemma 24.** If one of the three paths P_1, P_2, P_3 intersects the other two at most twice in total, then $\mathrm{pw}(G) \leq 3$.

▶ **Lemma 25.** If there are at least two vertices that simultaneously belong to the three paths P_1, P_2, P_3, then $\mathrm{pw}(G) \leq 3$

We say that a path P *bounces* between two paths Q_1, Q_2 if there exists $i \in \{1, 2\}$ such that P has a subpath P' with both endvertices on Q_i, and such that one of the internal vertices of P' intersects Q_{3-i}.

By Lemma 24, we can assume from now on that P_1 intersects the other paths at least twice, and by Lemma 25, that at most one of the vertices in these intersections lies simultaneously on the three paths P_1, P_2, P_3. In particular, up to exchanging a_1 and b_1, and P_2 and P_3, we may assume that if u_0 denotes the vertex on P_1 which is the closest to a_1, and belongs to one of the two paths P_2, P_3, then $u_0 \in V(P_1) \cap V(P_2) \setminus V(P_3)$. We set $A_i := P_i[a_i, u_0]$ and $B_i := P_i[u_0, b_i]$ for $i = 1, 2$.

▶ Remark 26. Observe that our choice of u_0 implies that $V(A_1) \cap (V(P_2) \cup V(P_3)) = \{u_0\}$.

By Lemma 23 and Lemma 24, we can assume that P_3 intersects $B_1 - u_0$, and at least one of the paths between $A_2 - u_0$ and $B_2 - u_0$, say $B_2 - u_0$ by symmetry. In particular, at least one of the following situations occurs (up to symmetrically exchange the roles of A_2 and B_2)

(1) P_3 bounces between two of the paths A_2, B_2 and B_1,
(2) P_3 does not intersect A_2,
(3) P_3 intersects the three paths A_2, B_2, B_1 in this order,
(4) P_3 intersects the three paths A_2, B_1, B_2 in this order.

To conclude the proof of Theorem 3, we will prove that $\mathrm{pw}(G) \leq 3$ in each of these cases in the following separate lemmas. This is the most technical part of the proof, especially for cases (1) and (2). The next four lemmas respectively deal with each of the four cases.

▶ **Lemma 27.** If P_3 bounces between two of the paths in $\{A_2, B_1, B_2\}$, then $\mathrm{pw}(G) \leq 3$.

▶ **Lemma 28.** If P_3 and A_2 are disjoint, then $\mathrm{pw}(G) \leq 3$.

▶ **Lemma 29.** If P_3 intersects the subpaths A_2, B_1, B_2 in this order, then $\mathrm{pw}(G) \leq 3$.

▶ **Lemma 30.** If P_3 intersects the subpaths A_2, B_2, B_1 in this order, then $\mathrm{pw}(G) \leq 3$.

Proof of Theorem 3. As explained above, observe that, up to replacing P_2 by P_2^{-1}, one of the four cases between (1), (2), (3) and (4) should occur. We thus conclude that $\mathrm{pw}(G) \leq 3$ after applying according accordingly Lemma 27, Lemma 28, Lemma 29 and Lemma 30. ◀

5 Coverings by Isometric Subtrees

Treewidth. A *tree-decomposition* of a graph G is a pair $(T, \mathcal{V})$ where T is a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ is a family of subsets V_t of $V(G)$ such that:

- $V(G) = \bigcup_{t \in V(T)} V_t$;
- for every nodes t, t', t'' such that t' is on the unique path of T from t to t'', $V_t \cap V_{t''} \subseteq V_{t'}$;
- every edge $e \in E(G)$ is contained in an induced subgraph $G[V_t]$ for some $t \in V(T)$.

The subsets V_t are called the *bags* of the tree-decomposition $(T, \mathcal{V})$, and the tree T is called the *decomposition tree*. The *width* of a tree-decomposition is the maximum size of a bag minus 1. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$ is defined as the minimum possible width of a tree-decomposition of G. Note that path-decompositions of G correspond exactly to the tree-decompositions of G whose decomposition tree is a path. In particular, we have in general $\mathrm{tw}(G) \leq \mathrm{pw}(G)$.

Recall that a graph G is 2-*connected* if it is connected of order at least 3, and has no *cutvertex*, that is, no vertex v such that $G - v$ is not connected anymore. The *blocks* of G are the inclusion-wise maximal sets of vertices $X \subseteq V(G)$ such that $G[X]$ is connected and has no cutvertex[5]. The following is a well-known and easy fact.

▶ **Proposition 31** (folklore). *The treewidth of a graph is equal to the maximum treewidth of its blocks (maximal* 2*-connected components). Put differently, for every* $v \in V(G)$*, if* $X_1, \ldots, X_k$ *denote the connected components of* $G - v$*, then* $\operatorname{tw}(G) = \max_{1 \leq i \leq k} \operatorname{tw}(G[X_i \cup \{v\}])$.

The following remark is immediate.

▶ Remark 32. Let G be a graph edge-coverable by k isometric trees, and X be a block of G. Then $G[X]$ is also edge-coverable by k isometric trees.

We now prove our main result in this section, which we restate.

▶ **Theorem 4.** *Let* G *be a graph which is edge-coverable by* 2 *isometric trees. Then* $\operatorname{tw}(G) \leq 2$.

Proof. Let T_1, T_2 denote two isometric subtrees of G that cover all its edges. By Proposition 31 and Remark 32, we may assume without loss of generality that G is 2-connected. In particular, G has no vertex of degree 1. Moreover, up to removing some of the edges incident to the leaves of T_1 and T_2, we may further assume that every edge of T_1 which is incident to a leaf of T_1 is not an edge of T_2. Let u be a leaf of T_1. As G has no vertex of degree 1, u must also be a vertex of T_2.

▷ Claim 33 (proof: vertical). Every edge of G is vertical with respect to u.

We now consider the subtree T_1' of T_1 formed by taking the union of all paths P between u and a vertex of T_1 such that all internal vertices of P are in $V(T_1) \setminus V(T_2)$. As G is 2-connected, all leaves of T_1' are in T_2. Let G_v be the subgraph of G induced by the descendants of v, i.e., by vertices x for which there exists some ux-path P which contains v and which is vertical with respect to u. Note that in particular, $v \in V(G_v)$.

▷ Claim 34. Each leaf v of T_1' which is distinct from u separates G_v from the rest of G.

As G is 2-connected, Claim 34 implies that for each leaf v of T_1' distinct from u, we have $V(G_v) = \{v\}$. In particular, as we initially chose u to be a leaf of T_1, it then implies that $T_1 = T_1'$, so the only vertices of T_1 that also belong to T_2 are its leaves. Observe that up to reproducing symmetrically the same reasoning, where the roles of T_1 and T_2 are exchanged, we may moreover assume that the only vertices of T_2 belonging to T_1 are its leaves. In other words, we are left with the case where T_1 and T_2 have no common internal nodes, and intersect exactly at their leaves.

In the remainder of the proof, we will say that a graph H has a *mirror-decomposition* if there exist trees T, T' such that

- $H = T \cup T'$;
- the trees T and T' have no common internal nodes;
- there exists a graph isomorphism ι from T to T' such that for every $v \in V(T_1) \cap V(T_2)$, we have $\iota(v) = v$.

The proof of Theorem 4 will be an immediate consequence of the following two Claims.

▷ Claim 35. G has a mirror-decomposition, with respect to the trees T_1 and T_2.

▷ Claim 36. Every graph admitting a mirror-decomposition has treewidth at most 2. ◀

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[5] Note that here, blocks are not necessarily 2-connected, as we allow them to have order less than 3.

6 Conclusion

As mentioned in the introduction, the best lower bound we know on the possible pathwidth of graphs edge-coverable by k shortest paths is k. Together with Theorem 3, it suggests that the upper bound from Theorem 2 could be potentially improved to a linear bound.

▶ **Question 37.** *Let G be a graph edge-coverable by k shortest paths. Is it true that $\mathrm{pw}(G) = O(k)$? And that $\mathrm{pw}(G) \leq k$?*

Another observation going in the direction of Question 37 is that graphs edge/vertex-coverable by k shortest paths have *cop number* at most k [1]. In particular, it is well-known (and not hard to prove) that the cop number of a graph is upper bounded by its treewidth, and thus also by its pathwidth.

The second question which is obviously left open by our work (and asked in [10]) concerns the existence of a polynomial upper-bound on the pathwidth of graphs which are vertex-coverable by k shortest paths. At the moment, it does not seem for us that the ideas from Section 3 generalise to this case.

Eventually, we still do not know if Theorem 4 generalizes in some way to larger values of k:

▶ **Question 38.** *Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k \geq 3$ and every graph G edge-coverable by k isometric subtrees, we have $\mathrm{tw}(G) \leq f(k)$? If yes, can f be polynomial?*

However, Bastide, Duron, Hodor, Liu and Nie [3] found constructions of graphs edge-coverable by 4 isometric subtrees that contain arbitrarily large subdivided walls as a subgraph, implying a negative answer to Question 38. A question left open by their work concerns the existence of graphs edge-coverable by 3 isometric subtrees and with arbitrarily large treewidth. It might also be interesting to find some restricted hypothesis under which Question 38 could hold.

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