

# Random Models and Guarded Logic

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## Abstract

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Building on ideas of Gurevich and Shelah for the Gödel Class, we present a new probabilistic proof of the finite model property for the Guarded Fragment of First-Order Logic. Our proof is conceptually simple and yields the optimal doubly-exponential upper bound on the size of minimal models. We precisely analyse the obtained bound, up to constant factors in the exponents, and construct sentences that enforce models of tightly matching size. The probabilistic approach adapts naturally to the Triguarded Fragment, an extension of the Guarded Fragment that also subsumes the Two-Variable Fragment. Finally, we derandomise the probabilistic proof by providing an explicit model construction which replaces randomness with deterministic hash functions.

**2012 ACM Subject Classification** Theory of computation → Finite Model Theory

**Keywords and phrases** guarded fragment, finite model property, probabilistic method

**Digital Object Identifier** 10.4230/LIPIcs.STACS.2026.37

**Related Version** *Full Version*: <https://arxiv.org/abs/2601.05247>

**Funding** Polish National Science Center, grant No. 2021/41/B/ST6/00996.

**Acknowledgements** I am deeply grateful to Prof. Emanuel Kieroński for his guidance and supervision throughout this work. I also thank the third reviewer of ICALP 2025 for providing detailed and valuable feedback on an earlier version of this paper.

## 1 Introduction

In this work we consider First-Order Logic (FO) without function symbols of positive arity. The Guarded Fragment (GF) is a fragment of FO in which quantifiers are relativised by atomic formulas. Syntactically, it is obtained by restricting quantification to the forms:

$$\forall \bar{x} (\gamma(\bar{x}, \bar{y}) \rightarrow \psi(\bar{x}, \bar{y})) \quad \text{and} \quad \exists \bar{x} (\gamma(\bar{x}, \bar{y}) \wedge \psi(\bar{x}, \bar{y})),$$

where  $\gamma(\bar{x}, \bar{y})$  is an atomic formula, called a *guard*, mentioning all variables in  $\bar{x}$  and  $\bar{y}$ .

For example, the following sentence, describing a professor–student scenario, is in GF:

$$\forall p, s (\text{supervises}(p, s) \rightarrow (\neg \text{graduate}(s) \wedge \exists t (\text{prepares}(s, t) \wedge \text{thesis}(t)))), \quad (1)$$

where  $\text{supervises}(p, s)$  and  $\text{prepares}(s, t)$  serve as guards. In contrast, the following sentence is not in GF, since the quantifier  $\forall p, s$  is not guarded by a single atomic formula:

$$\forall p, s ((\text{professor}(p) \wedge \text{student}(s)) \rightarrow \exists t (\text{better-than-in}(p, s, t) \wedge \text{topic}(t))). \quad (2)$$

Andréka, van Benthem, and Némethi [2] introduced the Guarded Fragment as a generalisation of modal logic, aiming to transfer its key properties into the richer framework of First-Order Logic. They established the decidability of satisfiability, and Grädel [18] later proved the complexity to be 2-EXPTIME-complete; under bounded number of variables or bounded arity of relation symbols, the complexity drops to EXPTIME-complete.

The decidability of GF is impressively robust: it is preserved under numerous extensions, including fixed points [19], transitive or equivalence guards [25, 24], and (negated) conjunctive queries [5]. Further decidable fragments have been obtained by relaxing the notion of a guard. They include the Loosely, Packed, and Clique-Guarded Fragments [30, 26, 17], the Guarded Negation Fragment [6], and the Triguarded Fragment [29].

**Motivations.** In addition to decidability, a central question in the study of logical fragments is the *finite model property*: namely, whether every satisfiable sentence admits a finite model. For the Guarded Fragment, the finite model property is known to hold, with a doubly exponential upper bound (in the length of the sentence) on the size of minimal models.

The first proof of the finite model property for GF was given by Grädel [18], relying on a deep combinatorial theorem by Herwig [21]. While Grädel’s approach was elegant in its logical formulation – using Herwig’s result as a black box – the underlying construction is technically involved. Moreover, this proof yields only a triply-exponential upper bound on the size of minimal models, which is far from being optimal. A significant improvement came from Bárány, Gottlob, and Otto [4], who established an optimal doubly-exponential bound. Their approach involves analysing finite guarded bisimilar covers of hypergraphs and relational structures, substantially generalising Rosati’s finite chase [28]. In fact, their result extends beyond GF, covering the richer setting where (negations of) conjunctive queries are also allowed. A simplified version tailored specifically to GF appears in Pratt-Hartmann’s book [27], though we believe that even this version remains challenging to follow.

As the existing proofs of the finite model property for GF are unexpectedly difficult, it is natural to ask: *Can we find simpler proofs of this fundamental result?*

A perspective that connects abstract model theory with applied computer science arises in knowledge representation and reasoning, a subfield of artificial intelligence. In this context, the Guarded Fragment can serve as a logical foundation for decidable reasoning frameworks, subsuming and extending the basic description logics (DLs) of the  $\mathcal{ALC}$  family (for an introduction to DLs, see, e.g., [3]). Here, objects from applications such as databases, knowledge bases, or computer programs are represented as logical structures, while formulas act as a declarative specification language describing their properties. Algorithms for satisfiability then become reasoning engines: given a formula, they decide whether an object satisfying the imposed logical constraints exists.

Over the years, a variety of algorithms solving satisfiability for the Guarded Fragment have been developed, ranging from purely theoretical decision procedures (e.g., [18]) to practically implementable methods based on resolution, saturation, or tableau (e.g., [9, 23, 22]). A key aspect is that many of these algorithms not only decide satisfiability but also produce a finite combinatorial object: a certificate of formula consistency.

Our interest lies in the subsequent step: *How to turn such certificates into explicit finite models?* Since smaller models are typically more useful – both for computational efficiency and for practical interpretability – the central challenge is to generate finite models as small as possible: not only close to the theoretical doubly exponential bounds, typically considered up to polynomial slack in the second-level exponent, but actually sharpened to within constant factors and detailed structural parameters of sentences.

**Main Contribution.** In this work, we give a new proof of the finite model property for GF, yielding the optimal doubly exponential bound on the size of minimal models. To the best of our knowledge, no earlier proof exhibits comparable simplicity and self-containment.

We employ a probabilistic approach inspired by Gurevich and Shelah’s proof of the finite model property for the Gödel Class [20]. The central idea is the following: given a formula  $\varphi$  and a (possibly infinite) model  $\mathfrak{A}$  with  $\mathfrak{A} \models \varphi$ , we define a random procedure that generates a finite structure  $\mathfrak{B}_n$  with domain of size  $n \in \mathbb{N}$ . We then prove that, once  $n$  exceeds a certain threshold depending only on  $\varphi$ , the probability that  $\mathfrak{B}_n \models \varphi$  becomes strictly positive. Consequently, some finite  $\mathfrak{B}_n$  must be a model of  $\varphi$ , yielding the finite model property with an upper bound on the minimal model size that matches this threshold.

Applying this probabilistic method to the Guarded Fragment, we establish our main result: every satisfiable GF-sentence  $\varphi$  has a finite model whose domain size is  $2^{2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}}$ .

In this work, we assume that  $|\varphi|$  measures the *uniform length* of  $\varphi$ : formulas are viewed as words over an infinite alphabet consisting of parentheses, logical connectives, quantifiers, variables, relation symbols, and constants, with each symbol contributing 1 to the length.<sup>1</sup>

To witness the tightness of our upper bound, we construct a family of sentences  $(\varphi_n)_{n \in \mathbb{N}}$  whose minimal models have domains of size at least  $2^{2^{\Omega(|\varphi_n| \cdot \log |\varphi_n|)}}$ . It is an improvement upon earlier approaches for enforcing large models in GF, as they only achieved domain sizes of  $2^{2^{\Omega(\sqrt{|\varphi_n|})}}$  (cf. [18]). Consequently, our probabilistic model construction yields an essentially optimal upper bound, up to constant factors in the second-level exponent.

► **Theorem 1.** *There exist universal constants  $0 < C_{\text{lb}} < C_{\text{ub}}$  such that the following holds.*

1. *Every satisfiable GF-sentence  $\varphi$  has a finite model whose domain size is at most*

$$2^{2^{C_{\text{ub}} \cdot |\varphi| \cdot \log |\varphi|}}.$$

2. *For every  $n \in \mathbb{N}$ , there exists a satisfiable GF-sentence  $\varphi_n$  with  $|\varphi_n| \geq n$  such that any model of  $\varphi_n$  must have domain size at least*

$$2^{2^{C_{\text{lb}} \cdot |\varphi_n| \cdot \log |\varphi_n|}}.$$

The upper bound of Theorem 1 is clean and elegant, but it is stated solely in terms of the length of  $\varphi$  and an unspecified constant in the second-level exponent. We complement this result by deriving a more precise bound on the size of minimal models. For a concise formulation, we express it in terms of the number of (*atomic*) *k*-types – that is, maximal consistent configurations of literals over *k* variables (see Section 2 for a formal definition).

Given a sentence  $\varphi$ , its *induced signature*  $\sigma$  is the set of all relation and constant symbols occurring in  $\varphi$ . The *width* of  $\sigma$ , denoted  $\text{wd}(\sigma)$ , is the maximum arity of any relation symbol in  $\sigma$ . We define the *expanded normal-form signature*  $\sigma_{\text{nf}}$  from  $\sigma$  as follows: for every subformula  $\chi$  of  $\varphi$  that begins with a maximal block of quantifiers and which is not a sentence (i.e.,  $\chi$  has free variables), we introduce a fresh relation symbol  $R_\chi$  whose arity equals the number of free variables of  $\chi$  (which is at most  $\text{wd}(\sigma)$ , since  $\chi$  is required to be guarded).

Note that the construction of  $\sigma_{\text{nf}}$  preserves width, i.e.,  $\text{wd}(\sigma_{\text{nf}}) = \text{wd}(\sigma)$ , does not introduce new constants, and keeps the overall size of  $\sigma_{\text{nf}}$  linear in  $|\varphi|$ .

► **Theorem 2.** *There exist sequences  $(C_t)_{t \in \mathbb{N}}$  and  $(\varepsilon_t)_{t \in \mathbb{N}}$  with  $C_t > 0$ ,  $\varepsilon_t \in (0, 1/e)$ , and  $\varepsilon_t \rightarrow 0$  such that the following holds for every  $t \in \mathbb{N}$ .*

*Let  $\varphi$  be any satisfiable GF-sentence, and let  $\sigma_{\text{nf}}$  be its expanded normal-form signature. Denote  $k = \text{wd}(\sigma_{\text{nf}})$ . If  $k \geq t \geq 2$ , then  $\varphi$  has a finite model whose domain size is at most*

$$C_t \cdot |\tau_k^{\sigma_{\text{nf}}}|^{1-1/e+\varepsilon_t},$$

where  $\tau_k^{\sigma_{\text{nf}}}$  is the set of all *k*-types over  $\sigma_{\text{nf}}$ .

The exponent in Theorem 2 converges to  $1 - 1/e \approx 0.63$  as  $t \rightarrow \infty$ . Informally, Theorem 2 can thus be read as stating that every satisfiable GF-sentence admits a finite model of domain size proportional to  $|\tau_k^{\sigma_{\text{nf}}}|^{0.63}$ , provided that the width  $k = \text{wd}(\sigma_{\text{nf}})$  is sufficiently large.

<sup>1</sup> The uniform length differs from the bit-length of formulas (finite alphabet), in which  $n$  distinct variables or symbols requires  $\Theta(n \cdot \log n)$  bits.

At first sight, working with the expanded normal-form signature rather than the induced one may seem unintuitive. However, this choice is in fact a consequence of the semantics of guarded sentences: a quantified subformula  $\chi(\bar{y})$  with free variables  $\bar{y}$  naturally defines a relation in a structure  $\mathfrak{A}$ , namely  $R_\chi^\mathfrak{A} = \{\bar{a} \mid \mathfrak{A} \models \chi(\bar{a})\}$ . To simplify the quantifier structure of a sentence, a standard technique is to transform it into a suitable *normal form* (see Section 3 for a definition). The normal-form reduction makes such implicitly defined relations explicit by introducing corresponding relation symbols. As a result, it distinguishes elements that are locally – i.e., at the level of atomic types – indistinguishable in the induced signature, but differ in their satisfaction of quantified subformulas.

► **Example 3.** To illustrate, consider the following (slightly abstract) GF-sentence:

$$\exists x (U(x) \wedge \exists y, z (P(x, y, z) \wedge \neg U(y))) \wedge \exists u (U(u) \wedge \forall v, w (P(u, v, w) \rightarrow U(v))).$$

This sentence is satisfied in a model  $\mathfrak{A}$  with domain  $\{1, 2, 3\}$ , where  $U^\mathfrak{A} = \{1, 3\}$  and  $P^\mathfrak{A} = \{(1, 2, 3)\}$ . The variables  $x, y, z$  are witnessed by the elements 1, 2, 3; and the variable  $u$  by element 3. Notice, however, that in the induced signature  $\sigma = \{P, U\}$ , elements 1 and 3 are indistinguishable at the level of 1-types: both satisfy only the unary predicate  $U$ .

Let  $\mu$  and  $\nu$  denote the subformulas that begin with quantifiers  $\exists y, z$  and  $\forall v, w$ , respectively. The expanded normal-form signature introduces two additional predicates  $R_\mu$  and  $R_\nu$ , which record the different roles of 1 and 3 in the model:  $R_\mu^\mathfrak{A} = \{1\}$  and  $R_\nu^\mathfrak{A} = \{2, 3\}$ .

Finally, note that the subformulas starting with  $\exists x$  or  $\exists u$  are subsentences; and those with  $\exists z$  or  $\forall w$  do not begin with a maximal block of quantifiers. Hence we do not introduce fresh relation symbols for them. Thus the resulting signature is  $\sigma_{\text{nf}} = \{P, U, R_\mu, R_\nu\}$ .

**Triguarded Fragment.** The Guarded Fragment (GF) and the Two-Variable Fragment ( $\text{FO}^2$ ) are among the most prominent fragments of First-Order Logic. Both capture a wide range of modal and description logics and are decidable, yet they differ substantially and are incomparable in expressive power. In particular, GF cannot express certain basic properties, expressible in  $\text{FO}^2$ , such as  $\forall s, d ((\text{student}(s) \wedge \text{dean}(d)) \rightarrow \text{knows}(s, d))$ .

To unify these two perspectives, Rudolph and Šimkus [29] introduced the *Triguarded Fragment* (TGF), building on related ideas developed earlier in [23, 8].

The key idea of TGF is to relax the quantification restrictions of GF: formulas with at most two free variables may be quantified freely, while the guardedness requirement is retained for formulas with three or more free variables (hence the name “tri-guarded”). This way, TGF subsumes both  $\text{FO}^2$  and GF, while also capturing properties beyond their reach. For instance, in formula (2), the quantifier  $\forall p, s$  is admissible in TGF because the quantified subformula has only two free variables,  $p$  and  $s$ , whereas the quantifier  $\exists t$  is required to be guarded, and indeed is by the atom `better-than-in`( $p, s, t$ ). Hence (2) belongs to TGF.

An important distinction concerns the role of equality. While both  $\text{FO}^2$  and GF remain decidable in the presence of equality, the satisfiability problem for TGF with equality is undecidable. The reason is that TGF is expressive enough to encode the Gödel Class (i.e., the prefix class  $\forall\forall\exists^*$ ), which was shown to be undecidable in the presence of equality by Goldfarb [13]. Excluding equality suffices to restore decidability of satisfiability, with complexity 2-EXPTIME without constants and 2-NEXPTIME when constants are allowed [29].

Returning to the finite model property, Kieroński and Rudolph [24] proved that the equality-free fragment of TGF has the finite model property, with an optimal doubly exponential bound on minimal model size. Their proof, however, is technically intricate: it uses the finite model property for GF as a black box and adds a further combinatorial construction that carefully glues several structures into a single model.

In contrast, our probabilistic approach to GF extends seamlessly to TGF, yielding a considerably simpler proof of the finite model property for this broader fragment. Moreover, it also achieves the optimal doubly-exponential upper bound on minimal model size.

► **Theorem 4.** *There exists a universal constant  $C > 0$  such that the following holds. Let  $\varphi$  be a satisfiable equality-free sentence of TGF. Then  $\varphi$  has a finite model whose domain size is at most*

$$2^{2^{C \cdot |\varphi| \cdot \log |\varphi|}}.$$

**Derandomisation.** As mentioned earlier, the satisfiability problem for the Guarded Fragment is 2-EXPTIME-complete [18]. Yet, although the probabilistic construction shows that every satisfiable sentence has a model of doubly exponential size, this alone does not imply that such a model can be *constructed* in time matching the decision complexity. Indeed, the number of structures of doubly exponential size is triply exponential.

From a practical point of view, this gap is minor: our randomised construction succeeds with probability at least 1/2 while sampling structures over a domain of doubly exponential size. Nevertheless, from a theoretical standpoint it leaves a conceptual separation between *reasoning* and *model building*: deterministic vs. randomised time complexity.

We close this gap by providing a constructive, fully deterministic version of Theorem 1. The central idea is to replace the random choices by carefully selected deterministic ones, using families of hash functions that mimic the statistical properties of true randomness. We then prove that these deterministic choices always produce a valid model.

In our formulation, we rely on the notion of a *witness of satisfiability*: a finite combinatorial object – namely, a set of  $k$ -types satisfying specific closure and consistency properties – that certifies the satisfiability of a sentence. Its definition is deferred to Section 3; for the present discussion it suffices to know that such a witness can be computed from a satisfiable sentence in 2-EXPTIME [18, 27]. On this basis we obtain the following constructive guarantee:

► **Proposition 5.** *There exists a deterministic algorithm with the following property: given a witness of satisfiability  $\mathcal{W}$  for a GF-sentence  $\varphi$ , the algorithm constructs a structure  $\mathfrak{B}$  with domain of size  $n \in \mathbb{N}$  such that  $\mathfrak{B} \models \varphi$ , where the parameter  $n$  satisfies  $n = 2^{2^{O(|\varphi| \cdot \log |\varphi|)}}$ . The algorithm runs in doubly exponential time in the length of  $\varphi$ .*

**Related Work.** Beyond Gurevich and Shelah’s approach for the Gödel class [20], the probabilistic method has been employed in several other proofs of the finite model property.

One of the earliest and most influential applications of probabilistic techniques in logic is Fagin’s proof of the 0–1 law for First-Order Logic [10]. A notable consequence of Fagin’s argument is that any finite subset of the theory of the Rado graph admits a finite model.

Goldfarb, Gurevich, and Shelah subsequently extended the approach of the latter two authors to the so-called subminimal Gödel class with identity [16]. Goldfarb later developed probabilistic proofs for additional decidable fragments, namely a solvable Skolem class [15] and the Maslov class [14]. For recent applications of the probabilistic method, see [11, 12].

**Technical Overview.** We begin in Section 2 by introducing the basic notation and conventions used throughout the paper. Section 3 provides the necessary background on GF, including its syntax, the normal form, and a satisfiability criterion. Building on this, Section 4 develops probabilistic constructions of finite models for GF. In Section 5, we construct sentences enforcing models that tightly match the upper bound established in Section 4.

Section 6 demonstrates that our methods extend naturally to TGF. In Section 7, we give a deterministic procedure for constructing finite models. Finally, Section 8 concludes the paper and discusses directions for future research.

The missing material is provided in the full version of the paper. Beyond the technical details omitted here, the full version also contains additional results, remarks, and examples.

## 2 Preliminaries and Notation

We denote the set of natural numbers including 0 by  $\mathbb{N}$ . For  $k \in \mathbb{N}$ , the notation  $[k]$  stands for the set  $\{1, \dots, k\}$ , with the convention that  $[0] = \emptyset$ . More generally, we use interval notation  $[a, b] \subseteq \mathbb{N}$  to denote the set  $\{a, a+1, \dots, b\}$  whenever  $a \leq b$ , and the empty set  $\emptyset$  whenever  $a > b$ . For a set  $S$ , we denote by  $2^S$  the powerset of  $S$ , and by  $\binom{S}{k}$  the set of all  $k$ -subsets of  $S$ . If  $S \subseteq \mathbb{N}$ , we also use  $\binom{S}{k}$  for the set  $\{(a_1, \dots, a_k) \in S^k \mid a_1 < \dots < a_k\}$ .

**First-Order Logic.** We assume general familiarity with First-Order Logic (FO). The logical symbols are  $=, \perp, \top, \vee, \wedge, \neg, \rightarrow, \leftrightarrow$ , and the quantifiers  $\forall, \exists$ . Formulas may also use non-logical symbols: relation symbols of arbitrary arity (from a countably infinite set), constant symbols (also from a countably infinite set), and variables (again countably many). We do not allow function symbols. The *length* of a formula  $\varphi$ , denoted  $|\varphi|$ , is defined as the total number of symbols it contains, where each occurrence of a symbol – be it a variable, relation symbol, or constant – contributes 1. We use  $\text{fv}(\varphi)$  to denote the set of *free* variables of  $\varphi$ .

A *signature*  $\sigma$  is a finite set of symbols, partitioned as  $\sigma = \text{Rels} \uplus \text{Cons}$ , where  $\text{Rels}$  is the set of relation symbols and  $\text{Cons}$  is the set of constant symbols. We require  $\text{Rels} \neq \emptyset$ . Every relation symbol  $R \in \text{Rels}$  comes with associated arity, denoted  $\text{ar}(R)$ . We require  $\text{ar}(R) \geq 1$ . The *width* of  $\sigma$ , denoted  $\text{wd}(\sigma)$ , is the maximum arity of any symbol in  $\text{Rels}$ . The *signature of a formula* is the finite set of relation and constant symbols that appear in the formula.

We use Fraktur letters such as  $\mathfrak{A}, \mathfrak{B}, \dots$  to denote structures, and the corresponding Roman letters  $A, B, \dots$  for their domains. A  $\sigma$ -structure  $\mathfrak{A}$  is a structure that interprets the symbols in  $\sigma$ : a relation symbol  $R$  as a relation  $R^{\mathfrak{A}} \subseteq A^k$  with  $k$  denoting the arity of  $R$ ; and a constant symbol  $c$  as an element  $c^{\mathfrak{A}} \in A$ . If  $B \subseteq A$ , we write  $\mathfrak{A} \upharpoonright B$  for the *restriction* of  $\mathfrak{A}$  to the subdomain  $B$ . Note that  $B$  must include all interpretations of constant symbols to remain a  $\sigma$ -structure. The structure  $\mathfrak{A}$  is *empty* if it contains no facts, i.e.,  $R^{\mathfrak{A}} = \emptyset$  for all relation symbols  $R$ ; however, we do not insist (and even forbid) that the domain is the empty set  $\emptyset$ . The *size* of a structure is the cardinality of its domain.

With respect to satisfiability, equality between constants can be eliminated by a straightforward reduction: if a formula has a model  $\mathfrak{A}$  satisfying  $c_1^{\mathfrak{A}} = c_2^{\mathfrak{A}}$ , then  $c_2$  can be replaced by  $c_1$  throughout the formula, without affecting satisfiability. Thus we work under the *standard name assumption* requiring that constants are interpreted in structures by themselves. Given a  $\sigma$ -structure  $\mathfrak{A}$ , we partition its domain  $A$  as  $A_0 \uplus \text{Cons}$ , where  $A_0$  and  $\text{Cons}$  are disjoint sets of *unnamed* and, respectively, *named* elements.

**Types.** Fix a signature  $\sigma = \text{Rels} \uplus \text{Cons}$ . For  $k \in \mathbb{N}$ , let  $\text{Lit}_k(\sigma)$  denote the set of all non-equality literals over relation and constant symbols from  $\sigma$  and variables  $x_1, \dots, x_k$ .

For  $k \in \mathbb{N}$ , a *k-type* over  $\sigma$  is a maximal consistent subset of  $\text{Lit}_k(\sigma)$ . A *type* is simply a *k-type* for some  $k \in \mathbb{N}$ . We write  $\tau_k^\sigma$  for the set of all *k-types* over  $\sigma$ , and  $\tau^\sigma := \bigcup_{k \in \mathbb{N}} \tau_k^\sigma$ .

▷ **Claim 6.** For every  $k \in \mathbb{N}$ , we have

$$|\tau_k^\sigma| = \prod_{R \in \text{Rels}} 2^{(k+|\text{Cons}|)^{\text{ar}(R)}} \quad \text{and thus} \quad 2^{k^{\text{wd}(\sigma)}} \leq |\tau_k^\sigma| \leq 2^{|\text{Rels}| \cdot (k+|\text{Cons}|)^{\text{wd}(\sigma)}}.$$

We denote by  $\tau_{\text{all-neg}}$  the type consisting exclusively of negative literals (the value of  $k$  will be clear from the context).

If  $k \leq \ell$ , we write  $\tau_2 \models \tau_1$  to indicate that a  $k$ -type  $\tau_1$  is contained in an  $\ell$ -type  $\tau_2$ .

For  $\tau \in \tau_k^\sigma$ , we define:  $\partial\tau := \{\gamma \in \tau \mid \text{fv}(\gamma) = \{x_1, \dots, x_k\}\}$ , and  $\text{int}(\tau) := \tau \setminus \partial\tau$ , that is, the *boundary* and *interior* of  $\tau$ . The set of all *boundary  $k$ -types* is  $\partial\tau_k^\sigma := \{\partial\tau \mid \tau \in \tau_k^\sigma\}$ .

A  $k$ -type  $\tau$  is called *guarded* if either  $k \leq 1$  or  $\tau$  contains a positive literal  $\gamma$  with  $\text{fv}(\gamma) = \{x_1, \dots, x_k\}$ , i.e.,  $\gamma \in \partial\tau$ . Note that necessarily  $k \leq \text{wd}(\sigma)$ .

Given a  $k$ -tuple of distinct unnamed elements  $\langle a_1, \dots, a_k \rangle$  of a  $\sigma$ -structure  $\mathfrak{A}$ , we write  $\text{tp}^{\mathfrak{A}}[a_1, \dots, a_k]$  for the unique  $k$ -type *realised* by  $\langle a_1, \dots, a_k \rangle$  in  $\mathfrak{A}$ . Formally,  $\text{tp}^{\mathfrak{A}}[a_1, \dots, a_k]$  is the set of literals  $\gamma \in \text{Lit}_k(\sigma)$  such that  $\mathfrak{A}, f \models \gamma$ , where  $f$  is the assignment that sends  $x_i \mapsto a_i$  for every  $i \in [k]$ . The collection of all  $k$ -types realised in  $\mathfrak{A}$  is denoted

$$\tau_k^{\mathfrak{A}} := \{ \text{tp}^{\mathfrak{A}}[a_1, \dots, a_k] \mid a_1, \dots, a_k \in A \setminus \text{Cons are pairwise distinct} \}.$$

It is convenient to view  $k$ -types themselves as  $\sigma$ -structures over the canonical domain  $\{x_1, \dots, x_k\} \cup \text{Cons}$ . This allows us to use structure-like notation; for instance, if  $\tau \in \tau_3^\sigma$ , we may write  $\text{tp}^\tau[x_1, x_3]$  for the 2-type induced by  $\tau$  on the variables  $x_1$  and  $x_3$ .

### 3 Technical Background on Guarded Fragment

In this section we formally introduce the syntax of the Guarded Fragment, together with the normal form and a satisfiability criterion. The material presented here is mostly a direct adaptation of Grädel's work [18] and is included primarily for the reader's convenience. In particular, no claims of novelty are made here. For a comprehensive introduction to the Guarded Fragment, see also Chapter 4 in Pratt-Hartmann's monograph [27].

**Syntax of GF.** We formalise the syntax of the Guarded Fragment in Definition 7.

► **Definition 7.** *The Guarded Fragment (GF) is the set of formulas in First-Order Logic generated by the following rules:*

- (i) *Every atomic formula belongs to GF.*
- (ii) *GF is closed under Boolean connectives.*
- (iii) *Let  $\bar{x}$  be a tuple of variables, let  $\psi$  be a formula in GF, and let  $\gamma$  be an atomic formula. If  $\text{fv}(\psi) \subseteq \text{fv}(\gamma)$  and  $\bar{x} \subseteq \text{fv}(\gamma)$ , then both  $\exists \bar{x} (\gamma \wedge \psi)$  and  $\forall \bar{x} (\gamma \rightarrow \psi)$  belong to GF.*

*The atom  $\gamma$  in rule (iii) is called a guard.*

Note that equality may serve as a guard: the formula  $\forall x (x = x \rightarrow \psi)$  is in GF whenever  $\psi \in \text{GF}$  and  $\text{fv}(\psi) \subseteq \{x\}$ . This allows for free quantification over individual elements.

**Normal Form.** Let  $\varphi$  be a sentence in GF. Then  $\varphi$  is in *normal form* if it is a conjunction of a finite number of guarded *existential*, guarded *universal*, and guarded *Skolem* sentences:

$$\bigwedge_t \exists \bar{x} (\alpha_t(\bar{x}) \wedge \psi_t(\bar{x})) \wedge \bigwedge_t \forall \bar{x} (\alpha_t(\bar{x}) \rightarrow \psi_t(\bar{x})) \wedge \bigwedge_t \forall \bar{x} (\alpha_t(\bar{x}) \rightarrow \exists \bar{y} (\beta_t(\bar{x}, \bar{y}) \wedge \psi_t(\bar{x}, \bar{y}))),$$

where  $\bar{x}$  and  $\bar{y}$  are disjoint tuples of distinct variables,  $\alpha_t(\bar{x})$  and  $\beta_t(\bar{x}, \bar{y})$  are guards, and  $\psi_t$  are quantifier-free formulas. The notation  $\psi(\bar{x})$  only highlights that the set of free variables of  $\psi$  is contained in the tuple  $\bar{x}$ . However, it does not imply that all of them are actually used by  $\psi$ . In particular, when  $\psi$  is an atom, the variables can occur in any order and with repetitions. Also the actual sets of free variables of  $\alpha_t$ ,  $\beta_t$ , and  $\psi_t$  can vary with  $t$ . Nevertheless, we require that each conjunct is properly guarded. For every relevant index  $t$ , the following

must hold: if  $t$  corresponds to an existential or universal conjunct, then  $\text{fv}(\psi_t) \subseteq \text{fv}(\alpha_t) = \bar{x}$ ; if  $t$  corresponds to a Skolem conjunct, then  $\text{fv}(\alpha_t) = \bar{x}$  and  $\text{fv}(\psi_t) \cup \bar{y} \subseteq \text{fv}(\beta_t) \subseteq \bar{x} \cup \bar{y}$ , where in both cases the tuples  $\bar{x}$  and  $\bar{y}$  may depend on  $t$ .

Lemma 8 reduces the finite model property for GF to the normal-form case. Moreover, any upper bound on minimal model size that depends only on the length and signature, and is proved for normal-form sentences, naturally carries over to arbitrary GF-sentences.

► **Lemma 8.** *Let  $\varphi$  be a GF-sentence over a signature  $\sigma$ . Then there exists a normal-form GF-sentence  $\varphi_{\text{nf}}$  over an expanded signature  $\sigma_{\text{nf}} \supseteq \sigma$  such that the following conditions hold.*

1. *The sentences  $\varphi$  and  $\varphi_{\text{nf}}$  are equisatisfiable.*
2. *If  $\mathfrak{B}$  is a  $\sigma_{\text{nf}}$ -structure with  $\mathfrak{B} \models \varphi_{\text{nf}}$ , then the  $\sigma$ -reduct of  $\mathfrak{B}$  is a model of  $\varphi$ .*
3.  *$|\varphi_{\text{nf}}| \leq C \cdot |\varphi|$  for some fixed constant  $C > 0$ .*
4. *The expanded signature  $\sigma_{\text{nf}}$  is obtained from  $\sigma$  as follows: for every subformula  $\chi$  of  $\varphi$  that begins with a maximal block of quantifiers and is not a sentence (i.e., has free variables), introduce a fresh relation symbol  $R_\chi$  whose arity equals the number of free variables of  $\chi$  (which is at most  $\text{wd}(\sigma)$ , since  $\chi$  is required to be guarded).*

**Satisfiability Criterion.** We now formulate a sufficient criterion for the satisfiability of sentences in GF. This allows us to modularise the argument and clearly separate the general properties of guarded logic from the new components of our proof.

Let  $\sigma = \text{Rels} \uplus \text{Cons}$  be a signature. A set of types  $\tau \subseteq \tau^\sigma$  is said to be *closed (under reductions)* if the following holds: for every  $\tau \in \tau$  which is a  $k$ -type for some  $k$ , and for every  $\ell \in [0, k]$  and choice of distinct indices  $i_1, \dots, i_\ell \in [k]$ , the type  $\text{tp}^\tau[x_{i_1}, \dots, x_{i_\ell}]$  also belongs to  $\tau$ . Further, the set  $\tau$  is said to be *consistent* if it contains a unique 0-type.

► **Definition 9.** *Let  $\sigma = \text{Rels} \uplus \text{Cons}$  be a signature, and let  $\tau = \bigcup_{k=0}^{\text{wd}(\sigma)} \tau_k$  be a family of sets of  $k$ -types over  $\sigma$ , where  $\tau_k \subseteq \tau_k^\sigma$  for each  $k \in [0, \text{wd}(\sigma)]$ . Let  $\mathfrak{A}$  be a  $\sigma$ -structure with domain  $A$ , and let  $A_k = \{\langle a_1, \dots, a_k \rangle \in (A \setminus \text{Cons})^k \mid a_1, \dots, a_k \text{ are pairwise distinct}\}$ .*

- (i) *We say that  $\mathfrak{A}$  is  $\tau$ -guarded if, for every  $k \in [0, \text{wd}(\sigma)]$  and every  $\bar{a} \in A_k$ , whenever the  $k$ -type  $\text{tp}^{\mathfrak{A}}[\bar{a}]$  is guarded, it holds that  $\text{tp}^{\mathfrak{A}}[\bar{a}] \in \tau_k$ .*
- (ii) *We say that  $\mathfrak{A}$  has the  $\tau$ -extension property if, for every  $k \in [0, \text{wd}(\sigma)-1]$  and every pair  $(\tau_1, \tau_2) \in \tau_k \times \tau_{k+1}$  such that  $\tau_2 \models \tau_1$ , the following holds: whenever  $\bar{a} \in A_k$  satisfies  $\text{tp}^{\mathfrak{A}}[\bar{a}] = \tau_1$ , there exists an element  $b \in A \setminus (\bar{a} \cup \text{Cons})$  such that  $\text{tp}^{\mathfrak{A}}[\bar{a}, b] = \tau_2$ .*

► **Lemma 10.** *Let  $\varphi$  be a normal-form GF-sentence over a signature  $\sigma = \text{Rels} \uplus \text{Cons}$ . Suppose  $\varphi$  is satisfiable. Then there exists a closed and consistent family of sets of  $k$ -types  $\tau^* = \bigcup_{k=0}^{\text{wd}(\sigma)} \tau_k^*$ , where  $\tau_k^* \subseteq \tau_k^\sigma$  for each  $k \in [0, \text{wd}(\sigma)]$ , such that the following holds: whenever a  $\sigma$ -structure  $\mathfrak{B}$  is  $\tau^*$ -guarded and satisfies the  $\tau^*$ -extension property, then  $\mathfrak{B} \models \varphi$ .*

Let  $\varphi$  be a GF-sentence in normal form. If a family of sets of  $k$ -types  $\tau^* = \bigcup_{k=0}^{\text{wd}(\sigma)} \tau_k^*$  satisfies the conditions of Lemma 10, then we call it a *satisfiability witness* for  $\varphi$ . Assuming that  $\varphi$  is satisfiable, one may obtain a satisfiability witness for  $\varphi$  by taking the collection of all  $k$ -types, with  $k \leq \text{wd}(\sigma)$ , that are realised in some (possibly infinite) model of  $\varphi$ . This argument is inherently non-constructive; see [27] for a procedure that decides satisfiability and, if so, computes a corresponding satisfiability witness in 2-EXPTIME.

W.l.o.g. we will always assume  $\tau_k^* \neq \emptyset$  for all  $k \in [0, \text{wd}(\sigma)]$ . By Claim 6, we have that

$$|\tau^*| = \sum_{k=0}^{\text{wd}(\sigma)} |\tau_k^*| \leq \sum_{k=0}^{\text{wd}(\sigma)} 2^{|\text{Rels}| \cdot (k + |\text{Cons}|)^{\text{wd}(\sigma)}} = 2^{2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}}, \quad (3)$$

provided that  $\sigma = \text{Rels} \uplus \text{Cons}$  is the induced signature of  $\varphi$ .

## 4 Probabilistic Model Constructions

In this section we establish Theorems 1 and 2. We begin with an informal overview of the probabilistic approach (Subsection 4.1), and then develop the technical details in two stages. First, in Subsection 4.2, we present the baseline method of *Independent Sampling*, which already yields a doubly-exponential upper bound, sufficient for proving Theorem 1. Next, in Subsection 4.3, we refine the construction via *Markovian Sampling*, achieving a sharper bound that is necessary for proving Theorem 2.

### 4.1 Informal Overview

To gain intuition, let us start with Gurevich and Shelah’s probabilistic proof of the finite model property for the Gödel Class [20]. Consider a sentence  $\varphi$  in the shape of  $\forall x_1, x_2 \exists y \psi$ , where  $\psi$  is quantifier-free and uses only predicates of arity 1 and 2. Assuming that  $\varphi$  has a model  $\mathfrak{A}$ , we generate a finite random structure  $\mathfrak{B}$  as follows. First prepare a domain  $B$  of size  $n \in \mathbb{N}$  and assign to its elements the 1-types realised in  $\mathfrak{A}$ , so that each 1-type is assigned roughly to the same number of elements. Then we set the 2-types: for each pair of distinct elements  $b_1, b_2$  choose a 2-type randomly from those 2-types realised in  $\mathfrak{A}$  whose endpoints agree with the already defined 1-types of  $b_1$  and  $b_2$ . When  $n$  is large enough, the structure  $\mathfrak{B}$  becomes a finite model of  $\varphi$ . To see this, consider any two elements  $b_1, b_2$  of  $\mathfrak{B}$ . Let  $\langle a_1, a_2 \rangle$  be a pair in  $\mathfrak{A}$  realising the same 2-type as  $\langle b_1, b_2 \rangle$ . With probability at least  $1 - 2^{-\delta n}$ , for a certain fixed independent of  $n$  number  $\delta > 0$ , some element  $b_3$  of  $\mathfrak{B}$  extends the 2-type of  $\langle b_1, b_2 \rangle$  to a 3-type in the same way as a correct witness  $a_3$  in  $\mathfrak{A}$  extends the 2-type of  $\langle a_1, a_2 \rangle$ . By the union bound over all such pairs  $\langle b_1, b_2 \rangle$ , we get  $\mathbb{P}[\mathfrak{B} \not\models \varphi] \leq n^2 \cdot 2^{-\delta n} \rightarrow 0$ .

Imagine now a hypothetical generalisation of this method to sentences in the shape of  $\varphi = \forall x_1, x_2, x_3 \exists y \psi$ , where  $\psi$  uses symbols of arbitrary arity. We first assign 1-types, then randomly choose 2-types, . . . and when trying to assign 3-types, we get stuck. Indeed, the independent random choices could generate a configuration of 2-types on pairs  $\langle b_1, b_2 \rangle$ ,  $\langle b_2, b_3 \rangle$ , and  $\langle b_1, b_3 \rangle$  which is not induced by any 3-type realised in  $\mathfrak{A}$ .

Suppose now that  $\varphi$  is a guarded sentence in normal form. If we decide not to put any facts speaking about  $\{b_1, b_2, b_3\}$  or any of its supersets, then a guarded sentence cannot notice that this unintended pattern has occurred. Therefore, the following strategy appears natural. We first assign the 1-types, then proceed by randomly choosing the 2-types, and for the 3-types we act as follows. For each triple  $\langle b_1, b_2, b_3 \rangle$ , we randomly select a 3-type  $\tau$ . However, before assigning  $\tau$  to  $\langle b_1, b_2, b_3 \rangle$ , we first check whether  $\tau$  is *compatible* with the pattern induced by the previously defined 2-types on  $b_1, b_2, b_3$ . (We make the notion of compatibility precise in a moment.) If it is indeed compatible, then we set the 3-type of  $\langle b_1, b_2, b_3 \rangle$  to  $\tau$ ; otherwise, we simply omit this triple and continue with the remaining ones. In a similar manner, we can then specify the  $k$ -types also for  $k > 3$ .

Assuming an appropriate choice of the parameter  $n$  and relying on careful probability estimates, we show that, although many tuples are omitted due to conflicts between  $k$ -types, there is still a significant chance that all required witnesses exist for all tuples. Consequently, we conclude that there is a sequence of random choices that produces a finite model  $\mathfrak{B}$  for  $\varphi$ .

### 4.2 Baseline: Independent Sampling

In this subsection we prove Proposition 11. Combined with Lemmas 8 and 10, it yields the upper bound of Theorem 1: every satisfiable GF-sentence  $\varphi$  admits a finite model whose domain has cardinality  $2^{2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}}$ . (The lower bound of Theorem 1 is proven in Section 5.)

■ **Algorithm 1** Model generation from a satisfiability witness via *Independent Sampling*.

---

**Input:** signature  $\sigma = \text{Rels} \uplus \text{Cons}$ , satisfiability witness  $\{\tau_k^*\}_{k=0}^{\text{wd}(\sigma)}$ , parameter  $n \in \mathbb{N}$   
**Output:** random  $\sigma$ -structure  $\mathfrak{B}$  with domain  $B := \{1, 2, \dots, n\} \uplus \text{Cons}$

- 1 initialise  $\mathfrak{B}$  as the empty  $\sigma$ -structure with domain  $B$ ;
- 2 assign the unique 0-type from  $\tau_0^*$  to  $\mathfrak{B}$ ;
- 3 **for**  $k = 1, 2, \dots, \text{wd}(\sigma)$  **do**
- 4     **foreach**  $b_1, \dots, b_k \in [n]$  such that  $b_1 < \dots < b_k$  **do**
- 5         choose a  $k$ -type  $\tau \in \tau_k^*$  uniformly at random;
- 6         **if**  $\tau$  is compatible with  $\text{tp}^{\mathfrak{B}}[b_1, \dots, b_k]$  **then**
- 7             assign  $\text{tp}^{\mathfrak{B}}[b_1, \dots, b_k] := \tau$ ;

---

► **Proposition 11.** *There exists a constant  $C > 0$  such that the following holds. Let  $\varphi$  be a normal-form GF-sentence with signature  $\sigma = \text{Rels} \uplus \text{Cons}$ . If  $\tau^* \subseteq \tau^\sigma$  is a satisfiability witness for  $\varphi$ , then  $\varphi$  has a finite model with unnamed domain of size at most*

$$C \cdot |\tau^*|^{2^{\text{wd}(\sigma)}}.$$

To prove Proposition 11, we describe a procedure that generates a  $\sigma$ -structure  $\mathfrak{B}$  from a witness of satisfiability  $\tau^* = \bigcup_{k=0}^{\text{wd}(\sigma)} \tau_k^*$ . The procedure is given in Algorithm 1. In addition to  $\tau^*$ , the algorithm takes as input a parameter  $n \in \mathbb{N}$ , specifying the size of the unnamed part of the domain of  $\mathfrak{B}$ . The analysis of the algorithm is carried out in Lemma 12.

In Algorithm 1, we rely on the following notion to determine a set of  $k$ -types to which a given configuration of atoms can be extended. Let  $k \in \mathbb{N}$  and let  $\tau_1, \tau_2$  be  $k$ -types. We say that  $\tau_1$  and  $\tau_2$  are *compatible* if these  $k$ -types agree on every literal that mentions a strict subset of the variables  $\{x_1, \dots, x_k\}$ , i.e., the interiors of  $\tau_1$  and  $\tau_2$  are the same:  $\text{int}(\tau_1) = \text{int}(\tau_2)$ . For instance, 2-types are compatible if they agree on the 1-types corresponding to their endpoints  $x_1$  and  $x_2$  but possibly differ on some literals that use both  $x_1$  and  $x_2$ ; similarly 3-types are compatible if they agree on the 2-types induced on  $\langle x_1, x_2 \rangle$ ,  $\langle x_2, x_3 \rangle$ , and  $\langle x_1, x_3 \rangle$  but possibly differ on some literals that use simultaneously  $x_1$ ,  $x_2$ , and  $x_3$ ; and so on for larger  $k$ . In particular, 1-types are compatible if they agree on constants.

► **Lemma 12.** *There exists a fixed constant  $C > 0$  such that the following holds: if  $\sigma$  is the signature of  $\varphi$ ,  $\tau^* = \bigcup_{k=0}^{\text{wd}(\sigma)} \tau_k^*$  is a satisfiability witness for  $\varphi$ , and the parameter  $n$  is chosen so that*

$$n \geq C \cdot |\tau^*|^{2^{\text{wd}(\sigma)}},$$

*then the structure  $\mathfrak{B}$  generated by Algorithm 1 satisfies  $\mathfrak{B} \models \varphi$  with probability at least  $1/2$ .*

**Proof of Lemma 12.** W.l.o.g. assume  $|\tau^*| \geq 3$ .<sup>2</sup> Set  $\delta = |\tau^*|^{-2^{\text{wd}(\sigma)-1}}$ .

For the analysis, we introduce a family of random variables  $\mathcal{X} = \{\mathcal{X}(\bar{b}) \mid \bar{b} \in \bigcup_{k=1}^{\text{wd}(\sigma)} \binom{[n]}{k}\}$ . For each  $k \in [\text{wd}(\sigma)]$  and each  $k$ -tuple  $\bar{b} \in \binom{[n]}{k}$ , the variable  $\mathcal{X}(\bar{b})$  records the  $k$ -type  $\tau$  chosen by Algorithm 1 at Line 5 when processing  $\bar{b}$ . Thus,  $\mathcal{X}(\bar{b})$  is uniformly distributed over  $\tau_k^*$ , and the variables in  $\mathcal{X}$  are mutually independent.

<sup>2</sup> If  $\tau^* = \{\tau_0\}$  with  $\tau_0$  the unique 0-type, then  $\varphi$  is satisfiable already in the domain  $\text{Cons}$ . If  $\tau^* = \{\tau_0, \tau_1\}$  where  $\tau_0$  is a 0-type and  $\tau_1$  a 1-type, then  $\varphi$  is satisfiable in the domain  $\text{Cons} \cup \{1\}$ .

Note that each  $k$ -tuple  $\bar{b} = \langle b_1, \dots, b_k \rangle \in \binom{[n]}{k}$  is assumed to be ordered increasingly. For a permutation  $\rho: [k] \rightarrow [k]$ , we will naturally write  $\mathcal{X}(b_{\rho(1)}, \dots, b_{\rho(k)}) = \text{tp}^{\mathcal{X}(\bar{b})}[x_{\rho(1)}, \dots, x_{\rho(k)}]$ , i.e.,  $\mathcal{X}(\rho(\bar{b}))$  is the  $k$ -type  $\mathcal{X}(\bar{b})$  reindexed according to  $\rho$ . This convention is adopted mainly for convenience in later proofs. As the witness of satisfiability  $\tau^*$  is closed under reductions (and hence under permutations of  $k$ -types), this convention will not be problematic.

▷ **Claim 13.** For every  $k \in [\text{wd}(\sigma)]$  and every  $k$ -tuple  $\langle b_1, \dots, b_k \rangle \in \binom{[n]}{k}$ , the following holds:

1. For any  $k$ -type  $\tau \in \tau_k^*$ , we have  $\text{tp}^{\mathfrak{B}}[b_1, \dots, b_k] = \tau$  whenever

$$\text{tp}^\tau[x_{i_1}, \dots, x_{i_t}] = \mathcal{X}(b_{i_1}, \dots, b_{i_t}) \text{ for every } t \in [k] \text{ and every } 1 \leq i_1 < \dots < i_t \leq k.$$

2. If  $\partial\text{-tp}^{\mathfrak{B}}[b_1, \dots, b_k] \neq \partial\tau_{\text{all-neg}}$ , then  $\text{tp}^{\mathfrak{B}}[b_1, \dots, b_k] = \mathcal{X}(b_1, \dots, b_k)$ .

*Proof.* Observe first that Algorithm 1 initialises  $\mathfrak{B}$  with facts only on the constants, and subsequently proceeds by monotonically adding new facts in each iteration. More precisely, when processing a tuple  $\bar{b}$ , there are two possibilities: either  $\mathfrak{B}$  remains unchanged, or the  $k$ -type on  $\bar{b}$  is set to  $\text{tp}^{\mathfrak{B}}[\bar{b}] = \mathcal{X}(\bar{b})$ . Crucially, the latter occurs only if  $\mathcal{X}(\bar{b})$  is *compatible* with the atoms already defined in  $\mathfrak{B}$ , i.e.,  $\text{int}(\text{tp}^{\mathfrak{B}}[\bar{b}]) = \text{int}(\mathcal{X}(\bar{b}))$ . Thus, only the boundary of  $\text{tp}^{\mathfrak{B}}[\bar{b}]$  – that is, the facts  $R(\bar{a})$  whose scope satisfies  $\bar{b} \subseteq \bar{a} \subseteq \bar{b} \cup \text{Cons}$  – may be modified:  $\partial\text{-tp}^{\mathfrak{B}}[\bar{b}]$  is set to  $\partial\mathcal{X}(\bar{b})$ . Moreover, boundaries corresponding to distinct tuples  $\bar{b}$  are disjoint, and therefore never interfere with each other. Consequently, for every  $\bar{b}$ , we have either

$$\partial\text{-tp}^{\mathfrak{B}}[\bar{b}] = \partial\mathcal{X}(\bar{b}) \quad \text{and} \quad \text{int}(\text{tp}^{\mathfrak{B}}[\bar{b}]) = \text{int}(\mathcal{X}(\bar{b})),$$

or else  $\partial\text{-tp}^{\mathfrak{B}}[\bar{b}]$  remains equal to  $\partial\tau_{\text{all-neg}}$  throughout.

The claim follows easily from the above explanation. Item 1 can be proven by induction over the subtuples of  $\bar{b} = \langle b_1, \dots, b_k \rangle$ . Assume that every proper subtuple  $\langle b_{i_1}, \dots, b_{i_t} \rangle$  has already been assigned the  $t$ -type  $\text{tp}^\tau[x_{i_1}, \dots, x_{i_t}]$ . Since  $\mathcal{X}(\bar{b}) = \tau$ , the compatibility test at Line 6 succeeds, and the algorithm assigns the type  $\mathcal{X}(\bar{b})$  to  $\bar{b}$ . Item 2 is immediate. ◁

The proof of Lemma 12 rests on three auxiliary claims. Claim 14 establishes that  $\mathfrak{B}$  is  $\tau^*$ -guarded with certainty. Claims 15 and 16 establish that  $\mathfrak{B}$  satisfies the  $\tau^*$ -extension property with probability at least 1/2. By Lemma 10, these two properties imply  $\mathfrak{B} \models \varphi$ .

▷ **Claim 14.** The structure  $\mathfrak{B}$  is  $\tau^*$ -guarded in every realisation, i.e., deterministically.

*Proof.* It is readily verified that the 0-type of  $\mathfrak{B}$  is the unique element of  $\tau_0^*$ . Likewise, every 1-type realised in  $\mathfrak{B}$  comes from  $\tau_1^*$ : for every  $b \in [n]$ , its 1-type  $\text{tp}^{\mathfrak{B}}[b]$  is given by  $\mathcal{X}(b) \in \tau_1^*$ .

For  $k \in [2, \text{wd}(\sigma)]$ , we fix a  $k$ -tuple  $\bar{b} \in \binom{[n]}{k}$  such that its  $k$ -type  $\text{tp}^{\mathfrak{B}}[\bar{b}]$  is guarded, i.e., it contains a positive literal  $\gamma$  with  $\text{fv}(\gamma) = \{x_1, \dots, x_k\}$ . As  $\gamma$  belongs to the boundary of  $\text{tp}^{\mathfrak{B}}[\bar{b}]$ , we have that  $\partial\text{-tp}^{\mathfrak{B}}[\bar{b}] \neq \partial\tau_{\text{all-neg}}$ . From Claim 13, it follows that  $\text{tp}^{\mathfrak{B}}[\bar{b}] = \mathcal{X}(\bar{b}) \in \tau_k^*$ . ◁

▷ **Claim 15.** Let  $k \in [0, \text{wd}(\sigma) - 1]$  and let  $\bar{b} \in \binom{[n]}{k}$ . Fix a  $k$ -type  $\tau_1 \in \tau_k^*$  and a  $(k+1)$ -type  $\tau_2 \in \tau_{k+1}^*$  satisfying that  $\tau_2 \models \tau_1$ . Define  $\mathcal{F}\text{ail}(\tau_2 \models \tau_1; \bar{b})$  to be the probabilistic event that

$$\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1 \quad \text{and} \quad \text{tp}^{\mathfrak{B}}[\bar{b}, b'] \neq \tau_2 \text{ for every } b' \in [n] \setminus \bar{b}.$$

Then  $\mathbb{P}[\mathcal{F}\text{ail}(\tau_2 \models \tau_1; \bar{b})] \leq e^{-\delta \cdot (n - \text{wd}(\sigma))}$ .

*Proof.* Assume that  $\mathbb{P}[\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1] > 0$ , as otherwise  $\mathbb{P}[\mathcal{F}\text{ail}(\tau_2 \models \tau_1; \bar{b})] = 0$  and the claim holds trivially. We condition on the probabilistic event  $\{\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1\}$ .

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Enumerate the elements of  $\bar{b}$  as  $b_1, \dots, b_k$ . Let us fix  $b_{k+1} \in [n] \setminus \bar{b}$ . We argue that, with probability at least  $\delta$ , it happens that  $\{\text{tp}^{\mathfrak{B}}[\bar{b}, b_{k+1}] = \tau_2\}$ . From Claim 13, it follows that  $\text{tp}^{\mathfrak{B}}[b_1, \dots, b_{k+1}] = \tau_2$  whenever, for every  $t \in [k+1]$  and every  $1 \leq i_1 < \dots < i_t \leq k+1$ , we have that  $\mathcal{X}(b_{i_1}, \dots, b_{i_t}) = \text{tp}^{\tau_2}[x_{i_1}, \dots, x_{i_t}]$ . Since  $\tau_2 \models \tau_1$  and  $\{\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1\}$  is fixed, we can restrict our attention to subsequences having  $i_t = k+1$ . Consequently, the random event  $\{\text{tp}^{\mathfrak{B}}[\bar{b}, b_{k+1}] = \tau_2\}$ , under the condition  $\{\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1\}$ , can be expanded as

$$\prod_{\substack{0 \leq t \leq k \\ 1 \leq i_1 < \dots < i_t \leq k}} \{ \mathcal{X}(b_{i_1}, \dots, b_{i_t}, b_{k+1}) = \text{tp}^{\tau_2}[x_{i_1}, \dots, x_{i_t}, x_{k+1}] \}. \quad (4)$$

Using the independence of variables in  $\mathcal{X}$ , the probability  $\mathbb{P}[\text{tp}^{\mathfrak{B}}[\bar{b}, b_{k+1}] = \tau_2 \mid \text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1]$  can be lower bounded by  $\delta$  as follows:

$$\begin{aligned} & \prod_{\substack{0 \leq t \leq k \\ 1 \leq i_1 < \dots < i_t \leq k}} \mathbb{P}[\mathcal{X}(b_{i_1}, \dots, b_{i_t}, b_{k+1}) = \text{tp}^{\tau_2}[x_{i_1}, \dots, x_{i_t}, x_{k+1}] \mid \text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1] \\ &= \prod_{t=0}^k |\tau_{t+1}^*|^{-\binom{k}{t}} \geq \prod_{t=0}^k |\tau^*|^{-\binom{k}{t}} = |\tau^*|^{-2^k} \geq |\tau^*|^{-2^{\text{wd}(\sigma)-1}} = \delta. \end{aligned} \quad (5)$$

We keep the  $k$ -tuple  $\bar{b}$ , the  $k$ -type  $\tau_1$ , and the  $(k+1)$ -type  $\tau_2$  fixed as before, as well as the condition  $\{\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1\}$ . We now bound the probability that no candidate  $b'$  for  $b_{k+1}$  makes  $\text{tp}^{\mathfrak{B}}[\bar{b}, b_{k+1}] = \tau_2$ . Since  $\{\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1\}$  is fixed, the events  $\{\text{tp}^{\mathfrak{B}}[\bar{b}, b'] = \tau_2\}$ , where  $b'$  ranges the set  $[n] \setminus \bar{b}$ , are mutually independent. Indeed, by (4), they are generated by disjoint subsets of  $\mathcal{X}$ . Hence the event  $\mathcal{F}ail(\tau_2 \models \tau_1; \bar{b})$  happens with probability at most

$$\prod_{b' \in [n] \setminus \{b_1, \dots, b_k\}} \mathbb{P}[\text{tp}^{\mathfrak{B}}[\bar{b}, b'] \neq \tau_2 \mid \text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1] \leq (1 - \delta)^{n-k} \leq e^{-\delta \cdot (n - \text{wd}(\sigma))}. \quad (6)$$

In the last inequality, we use that  $1 - \delta \leq e^{-\delta}$ , for any  $\delta \in \mathbb{R}$ , to move  $\delta$  to the exponent.

We conclude that  $\mathbb{P}[\mathcal{F}ail(\tau_2 \models \tau_1; \bar{b})] \leq e^{-\delta \cdot (n - \text{wd}(\sigma))}$ , as claimed.  $\triangleleft$

$\triangleright$  **Claim 16.** Let  $K = 8 \cdot \delta^{-1} \cdot (\text{wd}(\sigma) + \ln |\tau^*|)$ . If  $n \geq K \cdot \ln K$ , then the structure  $\mathfrak{B}$  satisfies the  $\tau^*$ -extension property with probability at least  $1/2$ .

*Proof.* The structure  $\mathfrak{B}$  satisfies the  $\tau^*$ -extension property precisely when none of the events  $\mathcal{F}ail(\tau_2 \models \tau_1; \bar{b})$  occur. Applying the union bound, we estimate the probability of the complementary event – that at least one of the events  $\mathcal{F}ail(\tau_2 \models \tau_1; \bar{b})$  occurs – as follows:

$$\sum_{\langle \bar{b}, \tau_1, \tau_2 \rangle \in \Omega} \mathbb{P}[\mathcal{F}ail(\tau_2 \models \tau_1; \bar{b})] \leq n^{\text{wd}(\sigma)} \cdot |\tau^*| \cdot e^{-\delta \cdot (n - \text{wd}(\sigma))}, \quad (7)$$

where the summation ranges over the set

$$\Omega = \bigcup_{k=0}^{\text{wd}(\sigma)-1} \{ \langle \bar{b}, \tau_1, \tau_2 \rangle \in \binom{[n]}{k} \times \tau_k^* \times \tau_{k+1}^* \mid \tau_2 \models \tau_1 \}. \quad (8)$$

The notation  $\bar{b} \in \binom{[n]}{k}$  implicitly assumes that  $\bar{b}$  is sorted in increasing order. Since  $\tau^*$  is closed, it suffices to consider the events  $\mathcal{F}ail(\tau_2 \models \tau_1; \bar{b})$  only for such tuples.

By Claim 15, we have the estimate on probability:  $\mathbb{P}[\mathcal{F}ail(\tau_2 \models \tau_1; \bar{b})] \leq e^{-\delta \cdot (n - \text{wd}(\sigma))}$ . Since  $\tau_2$  determines  $\tau_1$  and  $\sum_{k=0}^{\text{wd}(\sigma)-1} \binom{n}{k} \leq n^{\text{wd}(\sigma)}$  for all  $n \geq 2$ , we have the estimate on size:  $|\Omega| \leq n^{\text{wd}(\sigma)} \cdot |\tau^*|$ . Combining these estimates yields (7).

Let  $p$  denote the right side of (7). We require  $p \leq 1/e < 1/2$ . Taking natural logarithms on both sides of  $p \leq 1/e$  yields

$$\text{wd}(\sigma) \cdot \ln n + \ln |\tau^*| - \delta \cdot (n - \text{wd}(\sigma)) \leq -1. \quad (9)$$

Rearranging (9) to isolate  $n$  gives

$$n \geq \delta^{-1} \cdot (\text{wd}(\sigma) \cdot \ln n + \ln |\tau^*| + 1) + \text{wd}(\sigma). \quad (10)$$

By collecting the terms in (10) into two groups  $\nu$  and  $\mu$ , we can rewrite the inequality as

$$n \geq \nu \cdot \ln n + \mu, \quad \text{where } \nu = \delta^{-1} \cdot \text{wd}(\sigma) \quad \text{and} \quad \mu = \delta^{-1} \cdot (\ln |\tau^*| + 1) + \text{wd}(\sigma).$$

The inequality  $n \geq \nu \cdot \ln n + \mu$  holds whenever  $n \geq 4 \cdot (\nu + \mu) \cdot \ln(\nu + \mu)$ . Since

$$\nu + \mu = \delta^{-1} \cdot (\text{wd}(\sigma) + \ln |\tau^*| + 1) + \text{wd}(\sigma) \leq 2 \cdot \delta^{-1} \cdot (\text{wd}(\sigma) + \ln |\tau^*|) \leq K/4,$$

it follows that choosing  $n \geq K \cdot \ln K$  ensures  $p \leq 1/e$ , thereby establishing the claim.  $\triangleleft$

We conclude now the proof of Lemma 12. Since  $|\tau^*| \geq 3$  and  $\delta^{-1} = |\tau^*|^{2^{\text{wd}(\sigma)-1}}$ , it holds  $\text{wd}(\sigma) \leq \ln(\delta^{-1})$  and  $\ln |\tau^*| \leq \ln(\delta^{-1})$ . In consequence, the number  $K$  from Claim 16 satisfies  $K \leq 16 \cdot \delta^{-1} \cdot \ln \delta^{-1}$ . Hence there exists a constant  $C > 0$  such that

$$K \cdot \ln K \leq 16 \cdot \delta^{-1} \cdot \underbrace{\ln \delta^{-1} \cdot \ln (16 \cdot \delta^{-1} \cdot \ln \delta^{-1})}_{\text{asymptotically dominated by } \delta^{-1}} \leq C \cdot \delta^{-2}. \quad (11)$$

We derive the threshold of Lemma 12 directly from Claim 16 and inequality (11):

$$n \geq C \cdot \delta^{-2} = C \cdot |\tau^*|^{2 \cdot 2^{\text{wd}(\sigma)-1}} = C \cdot |\tau^*|^{2^{\text{wd}(\sigma)}}. \quad (12)$$

$\blacktriangleleft$

### 4.3 Alternative: Markovian Sampling

It is important to note that Proposition 11, proved in Section 4.2, is not strong enough to yield Theorem 2. To achieve the sharper bound stated in Theorem 2, we refine the method of Section 4.2 and establish Proposition 17. Theorem 2 then follows from the observation that  $|\tau_{\text{wd}(\sigma)-1}^\sigma| \geq |\tau_{\text{wd}(\sigma)}^\sigma|^{1/e-\varepsilon}$ , where  $\varepsilon \rightarrow 0$  as the width  $\text{wd}(\sigma) \rightarrow \infty$ .

► **Proposition 17.** *There exists a constant  $C > 0$  such that the following holds. Let  $\varphi$  be a normal-form GF-sentence with signature  $\sigma = \text{Rels} \uplus \text{Cons}$ . If  $\varphi$  is satisfiable, then it has a finite model with unnamed domain of size at most*

$$C \cdot \frac{|\tau_{\text{wd}(\sigma)}^\sigma|}{|\tau_{\text{wd}(\sigma)-1}^\sigma|} \cdot (\ln |\tau_{\text{wd}(\sigma)}^\sigma|)^2.$$

Let us first see where the bottleneck of Algorithm 1 occurs. When processing a tuple  $\bar{b}$ , Algorithm 1 samples a  $k$ -type  $\tau$  uniformly at random, independently of all previous choices. Only a posteriori it verifies whether the chosen  $\tau$  is compatible with the already defined structure on  $\text{tp}^{\text{B}}[\bar{b}]$ . This leads to a loss of probability, as many samples are rejected.

To improve this, we incorporate information from previous decisions directly into the sampling process. The key idea is to replace independent sampling with a Markovian scheme: when considering a tuple  $\bar{b}$ , we first determine whether the pattern induced by the already

■ **Algorithm 2** Model generation from a satisfiability witness via *Markovian Sampling*.

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**Input:** signature  $\sigma = \text{Rels} \uplus \text{Cons}$ , satisfiability witness  $\{\tau_k^*\}_{k=0}^{\text{wd}(\sigma)}$ , parameter  $n \in \mathbb{N}$   
**Output:** random  $\sigma$ -structure  $\mathfrak{B}$  with domain  $B := \{1, 2, \dots, n\} \uplus \text{Cons}$

- 1 initialise  $\mathfrak{B}$  as the empty  $\sigma$ -structure with domain  $B$ ;
- 2 assign the unique 0-type from  $\tau_0^*$  to  $\mathfrak{B}$ ;
- 3 **for**  $k = 1, 2, \dots, \text{wd}(\sigma)$  **do**
- 4     **foreach**  $b_1, \dots, b_k \in [n]$  such that  $b_1 < \dots < b_k$  **do**
- 5         let  $\tau$  be the set  $\{\tau \in \tau_k^* \mid \tau \text{ is compatible with } \text{tp}^{\mathfrak{B}}[b_1, \dots, b_k]\}$ ;
- 6         **if**  $\tau \neq \emptyset$  **then**
- 7             choose a  $k$ -type  $\tau \in \tau$  uniformly at random;
- 8             assign  $\text{tp}^{\mathfrak{B}}[b_1, \dots, b_k] := \tau$ ;

---

fixed structure on  $\text{tp}^{\mathfrak{B}}[\bar{b}]$  can be extended to a  $k$ -type in  $\tau_k^*$ . If so, we choose uniformly at random among the  $k$ -types that are compatible with this pattern; if not, we simply omit the tuple  $\bar{b}$ , forcing all atoms involving precisely those unnamed elements to be false. In this way, every random choice has a higher chance of being accepted, boosting the overall success probability of the procedure.

We formalise this idea in Algorithm 2. Its analysis follows the same general scheme as that of Algorithm 1 (Lemma 12). In particular, Claims 13 and 14 remain valid without modification. The analogue of Claim 15 requires exploiting the Markov property in place of full independence. The details are deferred to the full version.

## 5 Tightness Analysis: Enforcing Large Models

Setting  $\text{wd}(\sigma) = \Theta(|\varphi|)$ , Proposition 11 provides an upper bound of  $2^{2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}}$  on the size of minimal finite models for sentences in GF. In this section we show that this bound is essentially tight: we explicitly construct a family of GF-sentences whose minimal models match this doubly exponential upper bound, up to constant factors in the second-level exponent. Hence we establish the lower bound of Theorem 1: for specific sentences  $\varphi$ , models of size  $2^{2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}}$  are necessary.

► **Proposition 18.** *There exists a family of GF-sentences  $(\varphi_n)_{n \in \mathbb{N}}$  such that each  $\varphi_n$  contains neither constants nor equality, is over a signature  $\sigma_n$  with  $\text{wd}(\sigma_n) = n+4$ , satisfies  $|\varphi_n| \leq C \cdot n$  for some fixed constant  $C > 0$ , and is satisfiable, but only in domains of size at least  $2^{n!}$ .*

The general strategy for enforcing large models is to encode within them a combinatorial object of the desired size. A well-known technique for enforcing models of size  $2^{2^n}$  is to encode  $2^n$ -bit numbers from the range  $[0, 2^{2^n} - 1]$ . However, let us highlight that the standard constructions appearing in the literature (e.g., [18, 27]) yield formulas of size  $\Theta(n^2)$ . Consequently, they enforce models of size only  $2^{2^{\mathcal{O}(\sqrt{n})}}$ , which is asymptotically much smaller than the upper bound  $2^{2^{\mathcal{O}(|\varphi| \cdot \log |\varphi|)}}$  from Proposition 11; even when compared at the doubly logarithmic scale, since  $\sqrt{n}/(n \cdot \log n) \rightarrow 0$ . That said, it should be noted that these lower bounds were developed primarily to establish complexity lower bounds, rather than to enforce large domain sizes.

The bottleneck arises from the way these encodings represent  $2^n$ -bit numbers: fix two distinct elements  $a, b$ . A pair  $(a, b)$  is used to encode a  $2^n$ -bit number  $v$ . The  $k$ -th bit of  $v$  is set to 1 precisely when the atom  $\text{Bit}(c_1, \dots, c_n)$  holds, where the tuple  $\langle c_1, \dots, c_n \rangle \in \{a, b\}^n$  corresponds to the binary representation of  $k$ . Otherwise the  $k$ -th bit of  $v$  is set to 0.

Yet, specifying that some two sequences  $\langle c_1, \dots, c_n \rangle$  and  $\langle c'_1, \dots, c'_n \rangle$  correspond to respectively the  $k$ -th and  $(k+1)$ -st bits of  $v$  requires examining  $n$  different cases to account for potential carries, which forces formulas of size  $\Omega(n^2)$ .

To reach the optimal lower bound of  $2^{2^{\Omega(|\varphi| \cdot \log |\varphi|)}}$  a different construction is required. Our approach is to encode the powerset of the set of all permutations of  $n$  elements, which has cardinality  $2^{n!} = 2^{2^{\Theta(n \cdot \log n)}}$ . We use a certain decomposition of permutations from [12] to achieve this with formulas of length  $\mathcal{O}(n)$ , thereby ensuring the desired growth rate.

**Proof of Proposition 18.** Fix an integer  $n \geq 3$ . We shall construct a GF-sentence  $\varphi_n$  designed to encode the powerset of  $S_n$  (the set of permutations of  $\{1, \dots, n\}$ ). The signature of  $\varphi_n$  contains the following relation symbols:

$$\sigma_n = \{Wit, Mem, Adj, Gen, Inc, Dec, Succ, P_1, \dots, P_n, Q_1, \dots, Q_{n-1}\}, \quad (13)$$

where the arity of  $Wit$  and  $Mem$  is  $n+1$ ; of  $Adj$  is  $n+2$ ; of  $Gen$ ,  $Inc$ , and  $Dec$  is  $n+3$ ; of  $Succ$  is  $n+4$ ; and the symbols  $P_i$  and  $Q_i$  are all unary. We write  $\bar{x} = \langle x_1, \dots, x_n \rangle$  and  $\bar{y} = \langle y_1, y_2 \rangle$  for the short-hand tuples that occur repeatedly below. Let  $\pi_{\text{cyc}}$  denote the cyclic permutation  $1 \mapsto 2 \mapsto \dots \mapsto n \mapsto 1$ , and let  $\pi_{\text{swp}}$  denote the transposition swapping 1 and 2.

The construction relies on encoding subsets of the symmetric group  $S_n$ . We begin with a fixed “ground set” of size  $n$ , represented by some  $n$ -tuple of elements, denoted  $\bar{a}$ .

For every subset  $F \subseteq S_n$  we postulate the existence of a *witness element*  $w_F$ . The witness  $w_F$  is linked uniformly to all permutations of the ground set: for each  $\pi \in S_n$  we require  $Wit(\pi(\bar{a}), w_F)$ . The actual membership of a permutation  $\pi$  in  $F$  is then expressed via the predicate  $Mem$ :  $Mem(\pi(\bar{a}), w_F)$  if and only if  $\pi \in F$ . This forces distinct subsets  $F, F'$  to be represented by distinct witnesses  $w_F \neq w_{F'}$ . Thus, if we succeed in enforcing the existence of witnesses for all subsets of  $S_n$ , the model will contain at least  $2^{n!}$  distinct elements.

To navigate between different subsets, observe that one can transform any subset  $F \subseteq S_n$  into any other  $F' \subseteq S_n$  by toggling permutations one by one. We capture this stepwise modification using the relation  $Adj$ : the atom  $Adj(\pi(\bar{a}), w_F, w_{F'})$  is intended to hold precisely when the symmetric difference of  $F$  and  $F'$  is the singleton  $\{\pi\}$ .

To axiomatise the relation  $Adj$ , we employ a certain decomposition of permutations, which will be implemented using the remaining auxiliary predicates. Details of this decomposition and its encoding will be presented below.

We first postulate the existence of the ground set and an initial witness:

$$\exists \bar{x}, y \left( Wit(\bar{x}, y) \wedge \bigwedge_{i=1}^n P_i(x_i) \right). \quad (14)$$

In what follows, we denote the tuple witnessing for  $\bar{x}$  by  $\bar{a} = \langle a_1, \dots, a_n \rangle$ . We require the components of  $\bar{a}$  to be pairwise distinct. For this we axiomatise that  $\neg P_i(x) \vee \neg P_j(x)$  holds for distinct  $i \neq j$ . However, the obvious sentence would be of size  $\Theta(n^2)$ . To keep the formula of size linear in  $n$ , we use the auxiliary predicates  $Q_1, \dots, Q_{n-1}$ :

$$\bigwedge_{i=2}^n \forall x (P_i(x) \rightarrow Q_{i-1}(x)) \wedge \bigwedge_{i=2}^{n-1} \forall x (Q_i(x) \rightarrow Q_{i-1}(x)) \wedge \bigwedge_{i=1}^{n-1} \forall x (Q_i(x) \rightarrow \neg P_i(x)). \quad (15)$$

Next, we ensure that witnesses respect the intended invariant: for every witness  $y$  and every permutation  $\pi \in S_n$  we must have  $Wit(\pi(\bar{a}), y)$ . Since  $S_n$  is generated by  $\pi_{\text{cyc}}$  and  $\pi_{\text{swp}}$ , it suffices to enforce closure under these two transformations:

$$\forall \bar{x}, y \left( Wit(\bar{x}, y) \rightarrow \left( Wit(\pi_{\text{swp}}(\bar{x}), y) \wedge Wit(\pi_{\text{cyc}}(\bar{x}), y) \right) \right). \quad (16)$$

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We now enforce that every set  $F$  can be extended to a set  $F'$  differing precisely on a single permutation  $\pi$ . Formally, we require that for each permutation tuple  $\pi(\bar{a})$  and each witness  $y_1$ , there exists another witness  $y_2$  that differs exactly on the membership of  $\pi$ :

$$\forall \bar{x}, y_1 \left( \text{Wit}(\bar{x}, y_1) \rightarrow \exists y_2 \left( \text{Adj}(\bar{x}, y_1, y_2) \wedge \text{Wit}(\bar{x}, y_2) \wedge \left( \text{Mem}(\bar{x}, y_1) \oplus \text{Mem}(\bar{x}, y_2) \right) \right) \right). \quad (17)$$

To axiomatise that the witnesses  $y_1$  and  $y_2$  agree on every other permutation  $\pi' \neq \pi$ , i.e.,  $\text{Mem}(\pi'(a), y_1) \leftrightarrow \text{Mem}(\pi'(a), y_2)$ , we employ the following combinatorial lemma.

► **Lemma 19** (Lemma 7.2 in [12]). *Let  $\pi, \pi' \in S_n$  be permutations. Then  $\pi' \neq \pi$  if and only if  $\pi' = \pi_{\text{cyc}}^{-j} \circ \rho \circ \pi_{\text{cyc}}^k \circ \pi$  for some  $0 \leq j < k < n$  and some  $\rho \in S_n$  fixing  $n$  (i.e.,  $\rho(n) = n$ ).*

We implement the decomposition of Lemma 19 using the auxiliary predicates  $\text{Gen}$ ,  $\text{Inc}$ ,  $\text{Dec}$ , and  $\text{Succ}$ . Intuitively:  $\text{Inc}$  increases the counter along powers of  $\pi_{\text{cyc}}$ ;  $\text{Gen}$  introduces the action of  $\rho$  restricted to  $[n-1]$ , generated by  $\pi_{\text{swp}}$  and  $\pi'_{\text{cyc}}: 1 \mapsto 2 \mapsto \dots \mapsto n-1 \mapsto 1$ ;  $\text{Dec}$  decreases the counter via powers of  $\pi_{\text{cyc}}^{-1}$ ; and  $\text{Succ}$  provides the successor relation on the counter elements. The counter itself is represented by the last argument, where an element marked by  $P_i$  with  $1 \leq i \leq n-1$  indicates the difference  $i := k - j$  ( $j, k$  as in Lemma 19). This ensures that we never reach 0 or  $n$ , thereby avoiding the trivial case  $\pi' = \pi$ .

We axiomatise  $\text{Succ}$  as:

$$\forall \bar{x}, \bar{y}, z, z' \left( \text{Succ}(\bar{x}, \bar{y}, z, z') \rightarrow \bigvee_{i=1}^{n-2} \left( P_i(z) \wedge P_{i+1}(z') \right) \right). \quad (18)$$

Next, we generate permutations of the form  $\pi_{\text{cyc}}^k \circ \pi$  for  $0 < k < n$ :

$$\forall \bar{x}, \bar{y} \left( \text{Adj}(\bar{x}, \bar{y}) \rightarrow \exists z \left( \text{Inc}(\pi_{\text{cyc}}(\bar{x}), \bar{y}, z) \wedge P_1(z) \right) \right), \quad (19)$$

$$\forall \bar{x}, \bar{y}, z \left( \text{Inc}(\bar{x}, \bar{y}, z) \rightarrow \left( P_{n-1}(z) \vee \exists z' \left( \text{Succ}(\bar{x}, \bar{y}, z, z') \wedge \text{Inc}(\pi_{\text{cyc}}(\bar{x}), \bar{y}, z') \right) \right) \right). \quad (20)$$

Then, we generate permutations of the form  $\rho \circ \pi_{\text{cyc}}^k \circ \pi$ , where  $\rho$  fixes  $n$ . The subgroup  $\{\rho \in S_n \mid \rho(n) = n\}$  is induced by  $\pi_{\text{swp}}$  and  $\pi'_{\text{cyc}}: 1 \mapsto 2 \mapsto \dots \mapsto n-1 \mapsto 1$ . We enforce:

$$\forall \bar{x}, \bar{y}, z \left( \text{Inc}(\bar{x}, \bar{y}, z) \rightarrow \text{Gen}(\bar{x}, \bar{y}, z) \right), \quad (21)$$

$$\forall \bar{x}, \bar{y}, z \left( \text{Gen}(\bar{x}, \bar{y}, z) \rightarrow \left( \text{Gen}(\pi'_{\text{cyc}}(\bar{x}), \bar{y}, z) \wedge \text{Gen}(\pi_{\text{swp}}(\bar{x}), \bar{y}, z) \right) \right). \quad (22)$$

Finally, we generate permutations of the form  $\pi_{\text{cyc}}^{-j} \circ \rho \circ \pi_{\text{cyc}}^k \circ \pi$  with  $0 \leq j < k < n$ :

$$\forall \bar{x}, \bar{y}, z \left( \text{Gen}(\bar{x}, \bar{y}, z) \rightarrow \text{Dec}(\bar{x}, \bar{y}, z) \right), \quad (23)$$

$$\forall \bar{x}, \bar{y}, z \left( \text{Dec}(\bar{x}, \bar{y}, z) \rightarrow \left( P_1(z) \vee \exists z' \left( \text{Succ}(\bar{x}, \bar{y}, z', z) \wedge \text{Dec}(\pi_{\text{cyc}}^{-1}(\bar{x}), \bar{y}, z') \right) \right) \right). \quad (24)$$

By Lemma 19, every  $\pi' \neq \pi$  admits a representation such that  $\text{Dec}(\pi'(\bar{a}), y_1, y_2, z)$  holds. We then enforce membership agreement between  $y_1$  and  $y_2$  on all such permutations:

$$\forall \bar{x}, y_1, y_2, z \left( \text{Dec}(\bar{x}, y_1, y_2, z) \rightarrow \left( \text{Mem}(\bar{x}, y_1) \leftrightarrow \text{Mem}(\bar{x}, y_2) \right) \right). \quad (25)$$

We define  $\varphi_n$  to be the conjunction of the sentences (14)–(25). Since each conjunct has length  $\mathcal{O}(n)$ , the resulting sentence  $\varphi_n$  satisfies  $|\varphi_n| = \mathcal{O}(n)$ . Moreover, the signature  $\sigma_n$  of  $\varphi_n$  has width  $n+4$ , as required. Finally, by the discussion above,  $\varphi_n$  guarantees the existence of a distinguished tuple  $\bar{a}$  and, for every subset  $F \subseteq S_n$ , a corresponding witness  $w_F$ . Since the elements  $w_F$  and  $w_{F'}$  are distinct whenever  $F \neq F'$ , every model of  $\varphi_n$  contains at least  $2^{n!}$  distinct elements. This completes the proof of Proposition 18. ◀

## 6 Extension to a Stronger Fragment: Triguarded Fragment

The *Triguarded Fragment* (TGF) extends the Guarded Fragment by admitting one additional rule in the syntax of GF (Definition 7):

- (iv) Let  $x, y$  be variables and let  $\psi$  be a formula in TGF. If  $\text{fv}(\psi) \subseteq \{x, y\}$ , then both  $\exists x \psi$  and  $\forall x \psi$  belong to TGF.

Rule (iv) relaxes the guarding requirement of GF in the case of formulas with at most two free variables. Thus, TGF permits unguarded quantification over pairs of elements. To compare, recall that GF allows for free quantification only over individual elements.

In this section we establish Theorem 4, showing that the equality-free subfragment of TGF enjoys the finite model property with a doubly exponential upper bound on model size.

**Proof of Theorem 4.** To unify the syntax of GF and TGF, it is convenient (following [29]) to introduce an auxiliary fragment, denoted GFU. This is simply GF over signatures extended by a *distinguished* binary symbol  $U$ , interpreted as the full binary relation on the domain. In the terminology of description logics,  $U$  is referred to as a *universal role*. With this convention, rule (iv) can be viewed as a special case of the guarded quantifier rule (iii), with  $U$  acting as a dummy guard for formulas with at most two free variables. In particular, Theorem 4 is obtained as a corollary of Theorem 1, in combination with the following lemma.

► **Lemma 20.** *Let  $\tau^* = \bigcup_{k=0}^{\text{wd}(\sigma)} \tau_k^*$  be a satisfiability witness, and let  $n \in \mathbb{N}$  be arbitrary. Assume that  $\tau^*$  satisfies the following condition:*

*for every pair of 1-types  $\tau_1, \tau_2 \in \tau_1^*$ , there exists a 2-type  $\tau_{1,2} \in \tau_2^*$  such that its endpoints are  $\tau_1$  and  $\tau_2$ , that is,  $\text{tp}^{\tau_{1,2}}[x_1] = \tau_1$  and  $\text{tp}^{\tau_{1,2}}[x_2] = \tau_2$ .*

*Then Algorithm 2 given  $\tau^*$  and  $n$  generates the structure  $\mathfrak{B}$  with the property that between every pair of elements the 2-type is defined, i.e.,  $\text{tp}^{\mathfrak{B}}[a, b] \in \tau_2^*$  for all distinct  $a, b \in [n]$ .*

The constraint on the satisfiability witness  $\tau^*$  in Lemma 20 guarantees that the symbol  $U$  is interpreted as a universal relation: it ensures that Algorithm 2 always has at least one compatible 2-type to assign to every pair of elements.<sup>3</sup>

The model construction for GFU is clearly *sound*: given a witness  $\tau^*$  satisfying the constraint of Lemma 20, the algorithm produces a valid model  $\mathfrak{B}$ . A natural question is whether this condition is also *complete* – i.e., whether for every satisfiable GFU-sentence we can always find a model  $\mathfrak{A}$  of  $\varphi$  and extract a witness  $\tau^*$  from  $\mathfrak{A}$  that satisfies this constraint.

The sufficient condition for completeness is straightforward: the only problematic case arises when both 1-types are identical. Hence the requirement reduces to ensuring that, for every 1-type realised in a model, there exist at least two distinct unnamed elements realising it.<sup>4</sup> A standard argument shows that such models always exist for equality-free sentences: given a structure  $\mathfrak{A}$ , we define  $2 \times \mathfrak{A}$  to be a structure with domain  $\{0, 1\} \times A$ , where each constant  $c$  is identified with  $(c, 0)$ . For every relation symbol  $R$  and every sequence  $i_1, \dots, i_k \in \{0, 1\}$ , we put  $R^{2 \times \mathfrak{A}}((a_1, i_1), \dots, (a_k, i_k))$  whenever  $R^{\mathfrak{A}}(a_1, \dots, a_k)$ . Then  $\mathfrak{A}$  and  $2 \times \mathfrak{A}$  satisfy the same equality-free sentences (see Lemma 6.2.26 in [7]).

Consequently, by the reduction to the equality-free GFU, this establishes Theorem 4. ◀

<sup>3</sup> Note that this property does not hold for Algorithm 1, since it selects 2-types from the entire space rather than restricting to those compatible with the already assigned 1-types.

<sup>4</sup> Recall that, by our (non-standard) convention, 1-types are defined only for unnamed elements.

■ **Algorithm 3** Deterministic model generation from a satisfiability witness.

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**Input:** signature  $\sigma = \text{Rels} \uplus \text{Cons}$ ; satisfiability witness  $\{\tau_k^*\}_{k=0}^{\text{wd}(\sigma)}$ ;  
hash parameters  $p_1, \dots, p_{2^{\text{wd}(\sigma)}-1} \in \mathbb{N}$ ; number  $M := \prod_{i=1}^{2^{\text{wd}(\sigma)}-1} p_i$

**Output:**  $\sigma$ -structure  $\mathfrak{B}$  with domain  $B := [0, \text{wd}(\sigma) - 1] \times [0, M - 1] \uplus \text{Cons}$

- 1 initialise  $\mathfrak{B}$  as the empty  $\sigma$ -structure with domain  $B$ ;
- 2 assign the unique 0-type from  $\tau_0^*$  to  $\mathfrak{B}$ ;
- 3 **for**  $k = 1, 2, \dots, \text{wd}(\sigma)$  **do**
- 4     let  $\tau_k^*(0), \dots, \tau_k^*(|\tau_k^*| - 1)$  enumerate  $\tau_k^*$ ;
- 5     **foreach**  $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k) \in [0, \text{wd}(\sigma) - 1] \times [0, M - 1]$  with  $\alpha_1 < \dots < \alpha_k$  **do**
- 6         compute index  $i := 2^{\alpha_1} + \dots + 2^{\alpha_k}$ ;
- 7         compute hash  $h := ((\beta_1 + \dots + \beta_k) \bmod p_i) \bmod |\tau_k^*|$ ;
- 8         let  $\tau := \tau_k^*(h)$ ;
- 9         **if**  $\tau$  is compatible with  $\text{tp}^{\mathfrak{B}}[(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)]$  **then**
- 10             assign  $\text{tp}^{\mathfrak{B}}[(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)] := \tau$ ;

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## 7 Derandomisation: Explicit Model Construction

In Section 4 using probabilistic arguments we established Theorem 1: every satisfiable sentence  $\varphi$  of GF admits a finite model of doubly exponential size. However, the probabilistic proof is inherently non-constructive: it does not produce a concrete model of  $\varphi$ , but instead shows only that models of  $\varphi$  constitute a non-zero fraction of all structures of the given size.

We now turn to a constructive variant of Theorem 1. Our strategy is to derandomise the probabilistic proof by giving explicit values for the random choices in Algorithm 1. The resulting deterministic procedure (Algorithm 3) relies on algebraic hash functions to simulate randomness, and we prove that with appropriate parameters, it always produces a valid model (Lemma 21).

► **Lemma 21.** *Let  $\sigma$  be the signature of  $\varphi$ , and let  $\tau^* = \bigcup_{k=0}^{\text{wd}(\sigma)} \tau_k^*$  be a satisfiability witness for  $\varphi$ . Suppose the hash parameters  $p_1, \dots, p_{2^{\text{wd}(\sigma)}-1}$  are chosen so that*

1.  $p_1, \dots, p_{2^{\text{wd}(\sigma)}-1}$  are all prime, and
2.  $|\tau_{\text{wd}(\sigma)}^*| \leq p_1 < \dots < p_{2^{\text{wd}(\sigma)}-1}$ .

*Then the structure  $\mathfrak{B}$  generated by Algorithm 3 satisfies  $\mathfrak{B} \models \varphi$ .*

**Proof of Lemma 21.** Let us assume that every type in  $\tau^*$  is guarded.

Set  $M = \prod_{i=1}^{2^{\text{wd}(\sigma)}-1} p_i$ . For each  $k \in [\text{wd}(\sigma)]$ , let  $\Omega_k$  denote the set of  $k$ -tuples processed in the inner loop at Line 5, during the  $k$ th iteration of the outer loop at Line 3:

$$\Omega_k = \{ \langle (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k) \rangle \mid 0 \leq \alpha_1 < \dots < \alpha_k < \text{wd}(\sigma), \beta_1, \dots, \beta_k \in [0, M - 1] \}.$$

For technical reasons, we also declare  $\Omega_0$  as the set with the unique zero-length tuple.

For each  $k \in [\text{wd}(\sigma)]$  and each  $k$ -tuple  $\bar{b} \in \Omega_k$ , let  $\mathcal{X}(\bar{b})$  denote the  $k$ -type  $\tau$  chosen by Algorithm 3 at Line 8 when processing  $\bar{b}$ . For a permutation  $\rho: [k] \rightarrow [k]$ , we naturally write  $\mathcal{X}(\rho(\bar{b})) = \text{tp}^{\mathcal{X}(\bar{b})}[x_{\rho(1)}, \dots, x_{\rho(k)}]$ , that is, the  $k$ -type  $\mathcal{X}(\bar{b})$  reindexed according to  $\rho$ .

The values  $\mathcal{X}(\bar{b})$  can be viewed as a particular instantiation of the random variables in Algorithm 1. Hence by the same reasoning as in Claim 14 the structure  $\mathfrak{B}$  is  $\tau^*$ -guarded. If we show that  $\mathfrak{B}$  also satisfies the  $\tau^*$ -extension property, then Lemma 10 will imply that  $\mathfrak{B} \models \varphi$ . The main technical step is Claim 22, being a deterministic replacement for Claim 15.

▷ **Claim 22.** Let  $k \in [0, \text{wd}(\sigma) - 1]$ , and let  $\tau \in \tau_{k+1}^*$  be a  $(k+1)$ -type. Consider a tuple  $\langle (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k) \rangle \in \Omega_k$  together with an index  $\alpha_{k+1} \in [0, \text{wd}(\sigma) - 1] \setminus \{\alpha_1, \dots, \alpha_k\}$ . Then there exists some  $\beta_{k+1} \in [0, M - 1]$  such that the following holds: for every  $t \in [0, k]$  and every sequence  $1 \leq j_1 < \dots < j_t \leq k$ ,

$$\mathcal{X}((\alpha_{j_1}, \beta_{j_1}), \dots, (\alpha_{j_t}, \beta_{j_t}), (\alpha_{k+1}, \beta_{k+1})) = \text{tp}^\tau[x_{j_1}, \dots, x_{j_t}, x_{k+1}].$$

*Proof.* To prove the claim, we need to find  $\beta_{k+1} \in [0, M - 1]$  such that for every subset  $S \subseteq [0, k + 1]$  with  $k + 1 \in S$  the following holds.

Let  $m = |S|$ , and let  $r_1, \dots, r_m$  enumerate  $S$  so that  $\alpha_{r_1} < \dots < \alpha_{r_m}$ . The hash value computed at Line 7 when processing  $\langle (\alpha_{r_1}, \beta_{r_1}), \dots, (\alpha_{r_m}, \beta_{r_m}) \rangle \in \Omega_m$  shall satisfy:

$$((\beta_{r_1} + \dots + \beta_{r_m}) \bmod p_S) \bmod |\tau_m^*| = h_S, \quad (26)$$

where  $p_S := p_i$  for the index  $i$  as computed at Line 6, i.e.,  $i = 2^{\alpha_{r_1}} + \dots + 2^{\alpha_{r_m}}$ ; and  $h_S$  is the position of the  $m$ -type  $\text{tp}^\tau[x_{r_1}, \dots, x_{r_m}]$  in the enumeration of  $\tau_m^*$  as specified at Line 4, i.e.,  $\tau_m^*(h_S) = \text{tp}^\tau[x_{r_1}, \dots, x_{r_m}]$ . Note that  $\beta_{k+1}$  occurs in (26), as  $k + 1 \in S$ .

Since  $h_S < |\tau_m^*| \leq |\tau_{\text{wd}(\sigma)}^\sigma| \leq p_S$ , the solvability of (26) reduces to

$$\beta_{r_1} + \dots + \beta_{r_m} \equiv h_S \pmod{p_S}. \quad (27)$$

Now, observe a natural correspondence: the subset  $\{\alpha_{r_1}, \dots, \alpha_{r_m}\} \subseteq [0, \text{wd}(\sigma) - 1]$  is encoded in the binary digits of the index  $i = 2^{\alpha_{r_1}} + \dots + 2^{\alpha_{r_m}}$ .

In consequence, the moduli of (27) are distinct primes for distinct subsets  $S$ . By the Chinese remainder theorem the set of congruences specified by (27) admits a solution  $\beta_{k+1} \in [0, M - 1]$ , thereby proving the claim. ◁

Note that Claim 22 applies only to tuples from the sets  $\Omega_k$ . Since all types in  $\tau^*$  are guarded, the  $\tau^*$ -extension property therefore needs to be verified only for such tuples.

Let  $k \in [0, \text{wd}(\sigma) - 1]$ . Consider a tuple  $\bar{b} = \langle (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k) \rangle \in \Omega_k$ , a  $k$ -type  $\tau_1 \in \tau_k^*$ , and a  $(k+1)$ -type  $\tau_2 \in \tau_{k+1}^*$  with  $\tau_2 \models \tau_1$ . Assume that  $\text{tp}^{\mathfrak{B}}[\bar{b}] = \tau_1$ . Choose arbitrarily  $\alpha_{k+1} \in [0, \text{wd}(\sigma) - 1] \setminus \{\alpha_1, \dots, \alpha_k\}$ . Let  $\beta_{k+1} \in [0, M - 1]$  be the element obtained from Claim 22. One then verifies that  $\text{tp}^{\mathfrak{B}}[\bar{b}, b_{k+1}] = \tau_2$  for  $b_{k+1} = (\alpha_{k+1}, \beta_{k+1})$ . ◀

**Hash Parameters.** To constructively derive Theorem 1 from Lemma 21, it remains to show that suitable hash parameters  $p_1, \dots, p_{2^{\text{wd}(\sigma)} - 1}$  can indeed be chosen. A simple approach is to invoke the classical Bertrand's postulate (see: Chapter 2 in [1]): for every integer  $m \geq 2$ , there exists a prime  $p$  with  $m < p < 2 \cdot m$ .

We start from  $|\tau_{\text{wd}(\sigma)}^\sigma|$  and apply Bertrand's postulate iteratively  $2^{\text{wd}(\sigma)} - 1$  times. This yields distinct primes  $p_1, \dots, p_{2^{\text{wd}(\sigma)} - 1}$  such that  $2^{i-1} \cdot |\tau_{\text{wd}(\sigma)}^\sigma| < p_i < 2^i \cdot |\tau_{\text{wd}(\sigma)}^\sigma|$  for all  $i \in [2^{\text{wd}(\sigma)} - 1]$ . This is sufficient to obtain the upper bound stated in Theorem 1:

$$\prod_{i=1}^{2^{\text{wd}(\sigma)} - 1} p_i \leq 2^{1 + \dots + 2^{\text{wd}(\sigma) - 1}} \cdot |\tau_{\text{wd}(\sigma)}^\sigma|^{2^{\text{wd}(\sigma)} - 1} \leq 2^{4^{\text{wd}(\sigma)}/2} \cdot |\tau_{\text{wd}(\sigma)}^\sigma|^{2^{\text{wd}(\sigma)} - 1}. \quad (28)$$

## 8 Conclusions

In this work, we presented a new probabilistic proof of the finite model property for the Guarded Fragment. Our methods yield tight bounds on the size of minimal models and extend naturally to the Triguarded Fragment. To the best of our knowledge, no previous work on the Guarded Fragment has employed a similar probabilistic technique.

Several natural directions remain open. A particularly appealing challenge is to extend the applicability of probabilistic methods to stronger fragments. One candidate is the Clique Guarded Fragment (CGF), which is known to enjoy the finite model property [4]. Here, however, a direct probabilistic approach seems out of reach. For instance, in a graph-theoretic setting with a single relation  $E$ , CGF can express the absence of triangles via the sentence

$$\forall x, y, z \left( (E(x, y) \wedge E(y, z) \wedge E(z, x)) \rightarrow \perp \right).$$

Yet if edges were placed obliviously at random, a triangle would appear almost with certainty. This suggests that some additional underlying structure is needed, and that future work may profit from combining probabilistic reasoning with classical model constructions.

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