


Planting and MCMC Sampling from the Potts Model

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Abstract

We consider the problem of sampling from the ferromagnetic q -state Potts model on the random d -regular graph with parameter $\beta > 0$. A key difficulty that arises in sampling from the model is the existence of a “metastability” window $\beta \in (\beta_u, \beta'_u)$, where roughly the distribution has two competing modes, the so-called disordered and ordered phases. This causes classical Markov-chain algorithms to be slow mixing from worst-case initialisations. Nevertheless, Helmuth, Jenssen and Perkins (SODA '19) designed a sampling algorithm that works for all β , when $d \geq 5$ and $q = d^{\Omega(d)}$, using polymers and cluster expansion methods; more recently, their analysis technique has been adapted to show that a Markov chain (random-cluster dynamics) mixes fast when initialised appropriately, in the same regime of q, d, β .

Despite these positive algorithmic results, a well-known bottleneck behind cluster-expansion arguments is that they inherently only work for large q , whereas it is widely conjectured that sampling on the random d -regular graph is possible for all $q, d \geq 3$. The only result so far that applies to general $q, d \geq 3$ is by Blanca and Gheissari who showed that the random-cluster dynamics mixes fast in the “uniqueness” regime $\beta < \beta_u$ where roughly only the disordered mode exists. For $\beta \geq \beta_u$ however, a second subdominant mode emerges creating bottlenecks and giving rise to correlations which have been hard to handle, especially for small values of q and d .

Our main contribution is to perform a delicate analysis of the Potts distribution and the random-cluster dynamics that goes beyond the threshold β_u . We use planting as the main tool, a technique used in the analysis of random CSPs to capture how the space of solutions is correlated with the structure of the random instance. While planting arguments provide only weak sampling guarantees generically, here we instead combine planting with the analysis of random-cluster dynamics to obtain significantly stronger guarantees. We are thus able to show that the random-cluster dynamics initialised from all-out mixes fast for all integers $q, d \geq 3$ beyond the uniqueness threshold β_u , all the way to the optimal threshold $\beta_c \in (\beta_u, \beta'_u)$ where the dominant mode switches from disordered to ordered. A more involved analysis also applies to the ordered regime $\beta > \beta_c$ where we obtain an algorithm for all $d \geq 3$ and $q \geq (5d)^5$, improving significantly upon the previous range of q, d by Carlson, Davies, Fraiman, Kolla, Potukuchi, and Yap (FOCS'22).

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1 Introduction

The Potts model is a weighted model assigning probabilities to (non-proper) q -colourings. The model originated in statistical physics but has since been a central object of study in various contexts; here, we focus on the computational problem of sampling from the model.

For an integer $q \geq 3$ and a graph $G = (V, E)$, the q -state Potts model on G with parameter β is a probability distribution $\mu = \mu_{G,q,\beta}$ on $[q]^V$ where $[q] = \{1, \dots, q\}$. Each configuration $\sigma \in [q]^V$ has weight $\mu(\sigma) = e^{\beta m(\sigma)} / Z_G$ where $m(\sigma)$ is the number of monochromatic edges under σ ; the normalising factor Z_G is the partition function. Throughout, we consider the “ferromagnetic” case $\beta > 0$, where configurations with many monochromatic edges are favoured in the distribution.

From a computational complexity perspective, sampling from the ferromagnetic Potts model has various twists relative to other similar models and the complexity of the sampling problem is widely open. On general graphs, the problem is #BIS-hard for any fixed $\beta > 0$ [24]; for graphs of max degree d (where d is a fixed integer), the problem becomes #BIS-hard when $\beta > \beta_c$ [20], where $\beta_c = \ln \frac{q-2}{(q-1)^{1-2/d}-1}$ is known as the ordered/disordered threshold (we will discuss this in more detail shortly). It is also conjectured that the problem admits a polynomial time algorithm when $\beta < \beta_c$ but this is open in general; in fact, many of the standard tools that are used to analyse Markov chains provably fail well below the β_c threshold. The quintessential example here is the random d -regular graph, which underpins all the relevant phenomena behind this picture. For dn even, we let $\mathbb{G} = \mathbb{G}_{n,d}$ be a graph chosen uniformly at random from the set of all d -regular graphs with n vertices. We use “whp” as a shorthand for “with probability $1 - o(1)$ as n grows large”.

For a typical random graph \mathbb{G} , it is known that a sample from the Potts distribution $\mu_{\mathbb{G}}$ is “disordered” for $\beta < \beta_c$, and “ordered” for $\beta > \beta_c$; roughly, this means that the colours in the former case appear equally often whereas in the second case one of the colours strictly dominates over the rest (see Lemma 3 for the formal statement). The intricate feature of the Potts model on the random d -regular graph is that these competing modes are both present as “local maxima” throughout a window (β_u, β'_u) containing β_c even though only one of them has the vast majority of the mass (except at $\beta = \beta_c$ itself, where both modes appear with constant probability). This already poses problems to standard MCMC algorithms such as Glauber and Swendsen-Wang dynamics since the simultaneous presence of the modes causes exponentially slow mixing from worst-case initialisations, see [26, 14]. At a more conceptual level, any sampling algorithm has to take into account the presence of the other mode which obliterates from the very start standard analysis tools (e.g., correlation decay/spectral independence).

Helmuth, Janssen and Perkins [26] introduced a cluster-expansion technique that allowed them to control more precisely how much a typical sample differs from the corresponding mode, and obtained an algorithm based on the interpolation method for all $\beta > 0$ when $d \geq 5$ and $q \geq d^{\Omega(d)}$; their algorithm applies more generally to expander graphs (under some mild conditions). See also the works [16, 11, 12] for various refinements of their method. Following a series of developments [21, 22], this cluster expansion argument has been converted into an MCMC algorithm (for large d and q) [19] using the random-cluster dynamics with appropriate initialisation that avoids the bottlenecks in the distribution (see Section 2.2 for details). See also [10, 8, 23] for closely related results on the random d -regular graph (and lattices) that apply for large β .

Despite these positive algorithmic developments, the cluster expansion technique that underpins the analysis of these algorithms is inherently “perturbative”, relying roughly on controlling how much a typical configuration differs from the max-energy configurations and

arguing that the difference is relatively small. In the case of the Potts model, the resulting estimates are accurate only when q is quite large relative to d . By contrast, it is conjectured that sampling on the random regular graph should be possible for all $q, d \geq 3$ (and all $\beta > 0$), and hence more precise tools are needed to settle the picture for small q, d . The only result so far that works for general $q, d \geq 3$ is by Blanca and Gheissari [9] who showed that random-cluster dynamics mixes in $O(n \log n)$ time when $\beta < \beta_u$, for arbitrary initialisation. Note that for $\beta > \beta_u$ the results of [14] give worst-case initialisations that slow down the mixing time of the Glauber dynamics to $e^{\Omega(n)}$; more generally, for $\beta > \beta_u$, the presence of the ordered mode imposes correlation phenomena that are hard to handle, especially for small values of q, d .

1.1 Results

Our main contribution is to do a delicate analysis of the Potts distribution and the random-cluster dynamics that goes beyond the threshold β_u , using planting. Planting is a technique used in the analysis of random CSPs to capture how the space of solutions is correlated with the structure of the random instance, see [1, 15, 4] for some applications. In terms of sampling however, there is no recipe for converting planting methods to efficient sampling algorithms, and typically some extra sampling step is needed on top, see for example [2, 17] for such approaches; even so, the resulting sample usually has some accuracy limitation, i.e., the TV-distance from the target distribution cannot be made arbitrarily small (typically it is required to be at least an inverse polynomial in n). Here, we instead use planting as a means to obtain sharp estimates that are used in the analysis of random-cluster dynamics; we thus show fast convergence of the latter, yielding automatically strong sampling guarantees. A different sampling idea that uses planting in a somewhat related manner has recently been considered in [27], in the context of sampling from spherical spin glasses.

Using this planting framework, we give an algorithm to sample from the Potts distribution on the random d -regular graph \mathbb{G} that covers the optimal range of “high temperatures” $\beta < \beta_c$ for all integers $q, d \geq 3$; as we will explain in more detail later the algorithm is running random-cluster dynamics for $O(n \log n)$ steps, initialised appropriately (from the “empty” configuration).

► **Theorem 1.** *For integers $d, q \geq 3$ and real $\beta \in (0, \beta_c)$, the following holds whp over $\mathbb{G} = \mathbb{G}_{n,d}$.*

There is an MCMC algorithm that, on input \mathbb{G} and $\varepsilon \geq e^{-\Omega(n)}$, outputs in $O(n \log n \log(\frac{1}{\varepsilon}))$ steps a sample $\hat{\sigma} \in [q]^n$ whose distribution $\hat{\mu}$ satisfies $\|\hat{\mu} - \mu_{\mathbb{G},q,\beta}\|_{\text{TV}} \leq \varepsilon$.

Our analysis framework also applies to the ordered regime $\beta > \beta_c$ where we improve significantly upon various restrictions from previous works [11, 16, 26, 19]. For example, [26, 22] applied when $q \geq d^{\Omega(d)}$ (and $d \geq 5$), whereas the recent work by Carlson et al [11] (see also [16]) applied when $d \geq d_0$ and $q \geq d^c$ for some large constants d_0, c . Using planting, we obtain a sampling result for all $d \geq 3$ when q is roughly larger than d^5 . We show the following.

► **Theorem 2.** *For integers $d \geq 3, q \geq (5d)^5$ and real $\beta > \beta_c$, the following holds whp over $\mathbb{G} = \mathbb{G}_{n,d}$.*

There is an MCMC algorithm that, on input \mathbb{G} and $\varepsilon \geq e^{-\Omega(n)}$, outputs in $O(n \log n \log(\frac{1}{\varepsilon}))$ steps a sample $\hat{\sigma} \in [q]^n$ whose distribution $\hat{\mu}$ satisfies $\|\hat{\mu} - \mu_{\mathbb{G},q,\beta}\|_{\text{TV}} \leq \varepsilon$.

1.2 Proof outline

Our algorithms are based on a well-known edge representation of the Potts model, the random-cluster (RC) model, where configurations are edge subsets weighted by the number of edges present and the number of induced components. We run the random-cluster dynamics (the analogue of Glauber dynamics in this edge setting), starting from the empty set of edges in the disordered regime $\beta < \beta_c$ and the full set of edges in the ordered regime $\beta > \beta_c$. The reason for working in the RC representation is that, in contrast to the Potts model, the RC model enjoys certain monotonicity properties that facilitate tracking the evolution of Glauber dynamics. In the next Preliminaries section we define the RC model and the relevant random-cluster chain. We also give a more detailed overview of the phases of the Potts model, and how planting gives a handle for the disordered and ordered settings. Most of these parts are largely imported from [14, 20]. In Section 3, we convert the planting for Potts into suitable planting for the RC model, giving the analogue of the ordered and disordered phases in that setting. Then, we state the main contribution of this work which is establishing weak spatial mixing within the disordered and ordered phases (a notion that originated in [21]). Sections 4 and 5 give the technical core of our arguments.

Roughly, in the disordered regime (Section 4), the key ingredient is to show that the dynamics converges to a set of configurations that do not form large components. For $\beta < \beta_u$, Blanca and Gheissari [9] accomplished this by a very careful random-graph revealing process coupled with the evolution of random-cluster dynamics. Using planting, we are able to capture precisely the correlation between the random graph and a sample from stationarity for $\beta < \beta_c$, reducing the study of the component structure to showing that the planted random graph is subcritical. Using the monotonicity of the RC model, this implies that the RC dynamics initialised from the empty set (“all-out”) enjoys the same component structure as the stationary distribution (for exponentially many steps). Having this in place (Lemma 14), the rest of the proof is more streamlined after some small reworking of certain correlation decay estimates up to the ordered/disordered threshold β_c (Lemma 15).

In the ordered regime (Section 5), the planted model is instead supercritical and a typical configuration has a “giant” component, whose presence complicates the arguments significantly. The goal now is roughly to show that bichromatic edges do not typically form large components; more precisely, the key ingredient is to show that every vertex v is surrounded by close-by vertices (each at some distance $\ell < \frac{1}{2} \log_{d-1} n$) that belong to the giant component, a so-called “wired” boundary. Planting here allows us to do a head-on analysis of the probability that, in a typical sample from stationarity, there exists a path of length ℓ that does not include any vertex in the giant. We get a tight bound for this using careful first moment arguments and percolation bounds; then, for $q \geq (5d)^5$, a union bound over the paths starting at v gives the existence of the desired wired boundary (in a typical configuration). Relative to the earlier works [26, 16, 11, 19], bichromatic components roughly correspond to so-called polymers; our techniques show that a smaller lower bound on the size of the giant component suffices in order to ensure that the polymers are not large.

1.3 Further Discussion

The planting arguments that we use in this paper follow by careful first and second moment considerations in [20], which have been carried out for the ferromagnetic Potts model, see also [5] for similar-flavored “silent” planting results. The moment analysis has previously been used in [13] for sampling independent sets and proper colourings in random bipartite graphs, though there the approach ultimately relies on cluster-expansion type of arguments

[29, 12]. Our more direct planting approach can be carried out for sampling independent sets on random bipartite graphs (that shares similar monotonicity and planting properties) and obtain sharper bounds for sampling using Glauber dynamics. Note however that, as pointed out in [13, Lemma 10], it is unlikely that current polymer-based techniques will achieve better asymptotic estimates in terms of the degree d . Our technique should also apply to the $G(n, d/n)$ setting (for arbitrary $q, d > 1$) provided that the analogous “silent” planting is in place; this seems more easily feasible in the disordered regime but more difficult for the ordered regime where the Poisson-tree neighbourhoods are expected to cause bigger variance.

It is also relevant to note that the results we stated earlier for the cluster-expansion technique and fast mixing of the random-cluster dynamics in the literature apply to non-integer q , in the random-cluster representation. It is conceivable that a similar planting to the one we consider here for the Potts model can be carried out for non-integer q in the random-cluster representation, though the related first and second moment computations will likely require careful arguments for the distribution of the number of components in random graph percolation; see [6] for some related developments.

2 Preliminaries

2.1 The configuration model

To obtain a handle on the random d -regular $\mathbb{G} = \mathbb{G}_{n,d}$, we work throughout in the configuration model. Formally, for integers $n, d \geq 1$ with dn even, let $[n] \times [d]$ be a set of “half-edges”; we denote by $G \sim \mathcal{G}_{n,d}$ a (multi)graph on vertex set $[n]$ whose edges are obtained by taking a uniformly random perfect matching of the half-edges; we add an edge between $u, v \in [n]$ in G whenever the half-edges (u, i) and (v, j) are matched together for some $i, j \in [d]$. For brevity, instead of $G \sim \mathcal{G}_{n,d}$ we sometimes use \mathbf{G} to denote the random (multi)graph chosen. Any property that holds whp over \mathbf{G} also holds whp over \mathbb{G} (see, e.g., [28]), so henceforth we focus on the configuration model \mathbf{G} .

2.2 The random-cluster dynamics

Given a graph $G = (V, E)$ and real parameters $q > 0$ and $p \in [0, 1]$, the random-cluster model is a probability distribution $\pi = \pi_{G,q,p}$ on the subsets of E . Let Ω_G be the set of all configurations. Each configuration $F \subseteq E$ has probability $\pi(F) = q^{c(F)} p^{|F|} (1-p)^{|E \setminus F|} / Z_{RC,G}$ where $c(F)$ is the number of connected components in the graph (V, F) , and $Z_{RC,G}$ is a normalisation factor. For integer q and $p = 1 - e^{-\beta}$, it is well known that sampling from this probability distribution is equivalent to sampling from the q -state Potts model with parameter β , see Section 3 for details.

Let $\hat{p} := \frac{p}{(1-p)q+p}$. The random-cluster (RC) dynamics initialised from a configuration X_0 is a Markov chain $(X_t)_{t \geq 0}$ whose transition X_t to X_{t+1} is as follows:

1. First choose an edge $e \in E$ uniformly at random.
2. If e is a cut edge of the graph $(V, X_t \cup \{e\})$, then with probability \hat{p} , set $X_{t+1} := X_t \cup \{e\}$.
With all remaining probability set $X_{t+1} := X_t \setminus \{e\}$
3. Otherwise, with probability p , set $X_{t+1} := X_t \cup \{e\}$. With all remaining probability set $X_{t+1} := X_t \setminus \{e\}$.

It is a standard fact that the distribution of X_t converges to π_G , regardless of the initialisation. To work around the slow convergence in the metastability window when $\beta \in (\beta_u, \beta'_u)$, we will later consider the RC dynamics starting from the two extreme configurations, the all-out ($X_0 = \emptyset$) and the all-in ($X_0 = E$).

2.3 Phases of the Potts model

Let $\nu = (\nu_1, \dots, \nu_q)$ be a q -dimensional probability vector. For an n -vertex graph G , let

$$S_\nu(\theta) = \left\{ \sigma \in [q]^{V(G)} : \sum_{i \in [q]} \left| |\sigma^{-1}(i)| - n\nu_i \right| \leq \theta n \right\},$$

where $\theta = \theta(q, d, \beta) > 0$ is a small constant which we will suppress from notation. Hence, $S_\nu(\theta)$ consists of these configurations where the colour frequencies are given by the vector ν . Let also

$$Z_\nu(G) = \sum_{\sigma \in S_\nu} w_G(\sigma) \text{ and } \Psi = \Psi_{d,q,\beta}(\nu) = \lim_{\theta \downarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}[Z_\nu(\mathbf{G})],$$

where $w_G(\sigma) = e^{\beta m(\sigma)}$ is the weight of σ . In short, the function $\Psi(\nu)$ captures the (logarithm of the) expected contribution to the partition function $Z(\mathbf{G})$ of the random graph \mathbf{G} given by the configurations in S_ν . As we shall see shortly, the local/global maximisers of Ψ play a key role in determining the modes of the Gibbs distribution.

There are three relevant thresholds that come into play

$$\beta_u(q, d) < \beta_c(q, d) < \beta'_u(q, d).$$

For high temperatures ($\beta < \beta_u$), there is a unique global maximiser of Ψ , given by the uniform vector

$$\nu_d = \left(\frac{1}{q}, \dots, \frac{1}{q} \right)$$

which corresponds to the “disordered” configurations. As the temperature decreases and β crosses the threshold β_u , another local maximum of Ψ emerges at an “ordered” vector ν_o where one of the q colours dominates over the others (which are roughly balanced). More precisely, ν_o is given by¹

$$\nu_o = \left(a, \frac{1-a}{q-1}, \dots, \frac{1-a}{q-1} \right), \text{ and } a \in \left(\frac{1}{q}, 1 \right) \text{ satisfies } a = \frac{t^d}{t^d + q - 1} \text{ where } e^\beta - 1 = \frac{(t-1)(t^{d-1} + q - 1)}{t^{d-1} - t}. \tag{2.1}$$

Interestingly, the local maximum at ν_o does not become global until $\beta \geq \beta_c$ where $\beta_c = \ln \frac{q-2}{(q-1)^{1-2/d} - 1}$ is known as the ordered/disordered threshold; notably, the only point where both ν_o and ν_d are global maxima is at $\beta = \beta_c$. Note also that ν_d continues being a local maximum till $\beta < \beta'_u$ where $\beta'_u = \ln(1 + \frac{q}{d-1})$; so, in the interval (β_u, β'_u) both ν_d and ν_o are local maxima of Ψ which causes the metastability phenomena mentioned in the introduction, see [14] for details.

For convenience, let S_d and S_o denote the sets of configurations S_ν when $\nu = \nu_d$ and $\nu = \nu_o$, respectively. For a graph G , define $Z_d(G)$ and $Z_o(G)$ analogously, and let $\mu_{G,d} = \mu_G(\cdot \mid S_d)$ and $\mu_{G,o} = \mu_G(\cdot \mid S_o)$ denoted the conditional Gibbs distributions. The following lemma from [20] gives some sharp concentration estimates for $Z_d(\mathbf{G})$ and $Z_o(\mathbf{G})$ (derived using moments and the small subgraph conditioning method) and are the key behind our planting approach.

¹ The value of β_u is defined as the infimum value of $\beta > 0$ such that there exists $a \in (1/q, 1)$ which satisfies (2.1); then it is simple to check that such an a exists for all $\beta \geq \beta_u$. For certain values of β (namely when $\beta \in (\beta_u, \beta'_u)$), there may be more than one values of $a \in (1/q, 1)$ that satisfy the equation; when this is the case, the value of a that is relevant for the ordered phase is the largest such value.

► **Lemma 3** ([20, Theorem 5 & Lemma 16]). *Let $d, q \geq 3$ be integers, and $\beta \geq 0$ be real.² Let $f(n) > 0$ be any function with $f(n) = o(1)$. Whp over \mathbf{G} , we have:*

$$Z_d(\mathbf{G}) \geq f(n) \mathbb{E}[Z_d(\mathbf{G})] \text{ if } \beta \leq \beta_c \quad \text{and} \quad Z_o(\mathbf{G}) \geq f(n) \mathbb{E}[Z_o(\mathbf{G})] \text{ if } \beta \geq \beta_c.$$

Moreover, $\mu_{\mathbf{G}}(S_d) = 1 - e^{-\Omega(n)}$ when $\beta < \beta_c$, and $\mu_{\mathbf{G}}(\tilde{S}_o) = 1 - e^{-\Omega(n)}$ when $\beta > \beta_c$, where \tilde{S}_o is the union of S_o with its $q - 1$ permutations. For $\beta = \beta_c$, it holds that $\mu_{\mathbf{G}}(S_d \cup \tilde{S}_o) = 1 - e^{-\Omega(n)}$ and $\mu_{\mathbf{G}}(S_d), \mu_{\mathbf{G}}(S_o) \geq f(n)$.

2.4 Planting disordered and ordered configurations

We now introduce the planting distribution which will be the key in obtaining our results. Recall that \mathbf{G} denotes a graph chosen according to the configuration model $\mathcal{G}_{n,d}$; in the expressions below, expectations are taken with respect to the choice of \mathbf{G} . For $\sigma \in [q]^n$, define a random graph $\hat{\mathbf{G}}(\sigma)$ by letting

$$\mathbb{P}[\hat{\mathbf{G}}(\sigma) = G] = \frac{\mathbb{P}[\mathbf{G} = G] e^{\beta m_{\mathbf{G}}(\sigma)}}{\mathbb{E}[e^{\beta m_{\mathbf{G}}(\sigma)}]}. \quad (2.2)$$

Furthermore, for $\theta > 0$, recalling the disordered and ordered partition functions Z_d, Z_o from the previous section, we define random configurations $\hat{\sigma}_d = \hat{\sigma}_d(\theta) \in [q]^n$ and $\hat{\sigma}_o = \hat{\sigma}_o(\theta) \in [q]^n$ with distributions

$$\mathbb{P}[\hat{\sigma}_d = \sigma] = \frac{\mathbf{1}\{\sigma \in S_d\} \mathbb{E}[e^{\beta m_{\mathbf{G}}(\sigma)}]}{\mathbb{E}[Z_d(\mathbf{G})]}, \quad \mathbb{P}[\hat{\sigma}_o = \sigma] = \frac{\mathbf{1}\{\sigma \in S_o\} \mathbb{E}[e^{\beta m_{\mathbf{G}}(\sigma)}]}{\mathbb{E}[Z_o(\mathbf{G})]}. \quad (2.3)$$

We refer to $(\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d)$ and $(\hat{\mathbf{G}}(\hat{\sigma}_o), \hat{\sigma}_o)$ as the disordered and ordered planted distributions respectively.

The following lemma allows us to relate the planted distributions with the distributions of the (conditional) disordered $\sigma_{\mathbf{G},d} \sim \mu_{\mathbf{G},d}$ and ordered $\sigma_{\mathbf{G},o} \sim \mu_{\mathbf{G},o}$ configurations of the random graph \mathbf{G} . Recall that $\mathcal{G}_{n,d}$ denotes the set of all n -vertex regular (multi)graphs with degree d .

► **Lemma 4.** *Let $d, q \geq 3$ be integers and $\beta > 0$ be real. Let $f(n), g(n) > 0$ be any functions with $f(n) = o(1)$. Let $\mathcal{E} \subseteq \mathcal{G}_{n,d} \times [q]^n$ be an arbitrary event.*

- (1) $\beta \leq \beta_c$: *if $\mathbb{P}[(\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d) \in \mathcal{E}] \leq g(n)$, then, whp over \mathbf{G} , $\mathbb{P}[(\mathbf{G}, \sigma_{\mathbf{G},d}) \in \mathcal{E} \mid \mathbf{G}] \leq \frac{g(n)}{f(n)}$.*
- (2) $\beta \geq \beta_c$: *if $\mathbb{P}[(\hat{\mathbf{G}}(\hat{\sigma}_o), \hat{\sigma}_o) \in \mathcal{E}] \leq g(n)$, then, whp over \mathbf{G} , $\mathbb{P}[(\mathbf{G}, \sigma_{\mathbf{G},o}) \in \mathcal{E} \mid \mathbf{G}] \leq \frac{g(n)}{f(n)}$.*

The proof is given in the full version [18]. To utilise Lemma 4, the key observation is that, for the planted models, once we condition on the vertex and edge statistics, the models become uniformly distributed. Formally, for a graph $G \in \mathcal{G}_{n,d}$ and a configuration $\sigma \in [q]^n$, let ν^σ be vertex colour statistics under σ , i.e.,

$$\text{for a colour } i \in [q], \nu_i^\sigma = |\sigma^{-1}(i)|/n.$$

Similarly, let $\rho^{G,\sigma}$ be the $[q] \times [q]$ matrix with the following half-edge colour statistics under σ , i.e.,

$$\text{for colours } i, j \in [q], \rho_{ij}^{G,\sigma} = \frac{1}{2|E(G)|} \sum_{u,v \in V(G)} \mathbf{1}\{\{u, v\} \in E(G), \sigma(u) = i, \sigma(v) = j\}. \quad (2.4)$$

² Here, we assume that the constant $\theta > 0$, used in the definition of Z_d and Z_o , is a sufficiently small constant depending only on d, q, β .

Recall that $2|E(G)| = dn$. So, for a colour class $i \in [q]$, out of the $dn\nu_i^\sigma$ half-edges incident to it, there are $dn\rho_{ij}^{G,\sigma}$ “going” to a colour class $j \in [q]$ (note, for $i = j$, there are $\frac{dn}{2}\rho_{ii}^{G,\sigma}$ edges in G that lie inside the colour class i). We have that $\rho^{G,\sigma}$ is symmetric with $\sum_{i,j} \rho_{ij}^{G,\sigma} = 1$. Observe further that, conditioned on $\nu^{\hat{\sigma}_d} = \nu$ and $\rho^{\hat{G}(\hat{\sigma}_d), \hat{\sigma}_d} = \rho$ the pair $(\hat{G}(\hat{\sigma}_d), \hat{\sigma}_d)$ is uniformly distributed over the pairs $(G, \sigma) \in \mathcal{G}_{n,d} \times [q]^n$ with $\nu^\sigma = \nu$ and $\rho^{G,\sigma} = \rho$. Similarly for the pair $(\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o)$ conditioned on $\nu^{\hat{\sigma}_o}$ and $\rho^{\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o}$.

To put this observation into use, we will need more quantitative information on the vertex and edge colour statistics in the planted models. By definition of $\hat{\sigma}_d, \hat{\sigma}_o$, we already know that $\|\nu^{\hat{\sigma}_d} - \nu_d\|_1 \leq \theta$ and $\|\nu^{\hat{\sigma}_o} - \nu_o\|_1 \leq \theta$. Analogously, $\rho^{\hat{G}(\hat{\sigma}_d), \hat{\sigma}_d}$ and $\rho^{\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o}$ are concentrated around the vectors ρ_d and ρ_o where

$$\text{for } i, j \in [q], \quad \rho_{d,ij} = \frac{e^{\beta \mathbf{1}\{i=j\}}}{\sum_{i',j' \in [q]} e^{\beta \mathbf{1}\{i'=j'\}}}, \quad \rho_{o,ij} = \frac{e^{\beta \mathbf{1}\{i=j\}} (\nu_{o,i} \nu_{o,j})^{(d-1)/d}}{\sum_{i',j' \in [q]} e^{\beta \mathbf{1}\{i'=j'\}} (\nu_{o,i'} \nu_{o,j'})^{(d-1)/d}}. \quad (2.5)$$

We have the following concentration statement.

► **Lemma 5** ([14, Lemmas 3.4 & 3.5]). *Let $d, q \geq 3$ be integers and $\beta > 0$ be real. There exists $c > 0$ such that for any $t = \omega(n^{-1/2})$ it holds that*

- (i) For $\beta < \beta'_u$: $\mathbb{P}\left[\|\nu^{\hat{\sigma}_d} - \nu_d\|_1 + \|\rho^{\hat{G}(\hat{\sigma}_d), \hat{\sigma}_d} - \rho_d\|_1 \geq t\right] \leq e^{-ct^2n}$.
- (ii) For $\beta > \beta_u$: $\mathbb{P}\left[\|\nu^{\hat{\sigma}_o} - \nu_o\|_1 + \|\rho^{\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o} - \rho_o\|_1 \geq t\right] \leq e^{-ct^2n}$.

3 Mixing within the disordered and ordered phases for the RC model

3.1 Phases for the random cluster model via Potts

For a graph $G \in \mathcal{G}_{n,d}$ and $\sigma \in [q]^n$, let $\mathbf{F}(G, \sigma)$ denote the random subset of edges obtained by keeping each monochromatic edge of G under σ with probability p . It is a standard fact (see, e.g., [25]) that, for integer $q \geq 3$, we can obtain a sample $\mathbf{F} \sim \pi_G$ by first sampling $\sigma \sim \mu_G$ and then keeping each monochromatic edge of G under σ with probability $p = 1 - e^{-\beta}$ (the so-called percolation step).

We use this for the random graph \mathbf{G} to get a handle on the random cluster model on \mathbf{G} , i.e., $\mathbf{F}(\mathbf{G}, \sigma)$ has distribution $\pi_{\mathbf{G}}$. From Lemma 5, we can figure out in particular how many monochromatic edges we should expect from each phase, so we can translate the phases for the Potts model to the random cluster model. More precisely, set

$$\mathbf{m}_d(\beta) = p \frac{d}{2} \sum_{i \in [q]} \rho_{d,ii} \quad \text{and} \quad \mathbf{m}_o(\beta) = p \frac{d}{2} \sum_{i \in [q]} \rho_{o,ii}.$$

With a bit of algebra, we can derive the following expressions for \mathbf{m}_d and \mathbf{m}_o from (2.5): for $\beta < \beta'_u$, we have $\mathbf{m}_d(\beta) = \frac{d}{2} \frac{e^\beta - 1}{e^\beta + q - 1}$, while, for $\beta > \beta_u$, we have $\mathbf{m}_o(\beta) = \frac{d}{2} \frac{(e^\beta - 1)(x^2 + \frac{(1-x)^2}{q-1})}{1 + (e^\beta - 1)(x^2 + \frac{(1-x)^2}{q-1})}$ with $x = \frac{t^{d-1}}{t^{d-1} + q - 1}$ and $t > 1$ defined from $e^\beta - 1 = \frac{(t-1)(t^{d-1} + q - 1)}{t^{d-1} - t}$, as in (2.1).

We are now ready to define the ordered/disordered phases for the random-cluster model. Due to monotonicity properties (and in contrast to the phases of the Potts model), we can just use the values of \mathbf{m}_d and \mathbf{m}_o at the critical temperature $\beta = \beta_c$; this will be convenient later on.

► **Definition 6** (The ordered/disordered phases). *Let $q, d \geq 3$ be integers. For a graph $G \in \mathcal{G}_{n,d}$ and $\rho := m_o(\beta_c) - m_d(\beta_c) > 0$, define*

$$\Omega_{G,d} = \{F \subseteq E(G) : |F| \leq nm_d(\beta_c) + n\rho/4\} \text{ and } \Omega_{G,o} = \{F \subseteq E(G) : |F| \geq nm_o(\beta_c) - n\rho/4\}.$$

Let also $\pi_{G,d} = \pi_G(\cdot \mid \Omega_{G,d})$ and $\pi_{G,o} = \pi_G(\cdot \mid \Omega_{G,o})$.

Using the translation between the Potts and random-cluster models, we have the following.

► **Corollary 7.** *Let $d, q \geq 3$ be integers and $\beta > 0$ be real. Then, whp over $G \sim \mathcal{G}_{n,d}$, we have $\pi_G(\Omega_{G,d}) = 1 - e^{-\Omega(n)}$ when $\beta < \beta_c$ and $\pi_G(\Omega_{G,o}) = 1 - e^{-\Omega(n)}$ when $\beta > \beta_c$.*

For $\beta = \beta_c$, $\pi_G(\Omega_{G,d} \cup \Omega_{G,o}) = 1 - e^{-\Omega(n)}$ and $\pi_G(\Omega_{G,d}), \pi_G(\Omega_{G,o}) \geq 1/\ln \ln n$.

Proof of Corollary 7. This follows from Lemma 3, and the correspondence between $\sigma \sim \mu_G$ and $F \sim \pi_G$ stated at the start of the section, using $f(n) = 1/\ln \ln n$ for the lower bounds at $\beta = \beta_c$. ◀

To analyse the ordered/disordered phases of the random cluster model using the planted distributions, we will need the following intermediate distributions. Namely, for a graph G , let $\hat{\pi}_{G,d}$ be the distribution on $F \subseteq E(G)$ obtained by first sampling $\sigma_{G,d} \sim \mu_{G,d}$ and then keeping each monochromatic edge with probability $p = 1 - e^{-\beta}$. Define similarly $\hat{\pi}_{G,o}$. The proof of the following is given in the full version [18].

► **Lemma 8.** *Let $d, q \geq 3$ be integers and $\beta > 0$ be real. Then, whp over $G \sim \mathcal{G}_{n,d}$, for any event $\mathcal{A} \subseteq 2^{E(G)}$, the following holds.*

- (i) *if $\beta \leq \beta_c$, then $\pi_{G,d}(\mathcal{A}) \leq \hat{\pi}_{G,d}(\mathcal{A}) + e^{-\Omega(n)}$.*
- (ii) *if $\beta \geq \beta_c$, then $\pi_{G,o}(\mathcal{A}) \leq \hat{\pi}_{G,o}(\mathcal{A}) + e^{-\Omega(n)}$.*

3.2 Mixing and Weak Spatial Mixing (WSM) within a phase

Treading a path initiated in [21], to prove Theorems 1 and Theorem 2 it suffices to show the following results for the disordered and ordered regimes, respectively.

► **Theorem 9.** *Let $d, q \geq 3$ be integers and $\beta \in (0, \beta_c]$. Then, whp over $G \sim \mathcal{G}_{n,d}$, for every $\varepsilon \geq e^{-\Omega(n)}$ the RC dynamics $(X_t)_{t \geq 0}$ initialised from all-out satisfies $\|X_T - \pi_{G,d}\|_{\text{TV}} \leq \varepsilon$ for $T = O(n \log n \log \frac{1}{\varepsilon})$.*

► **Theorem 10.** *Let $d \geq 3, q \geq (5d)^5$ be integers and $\beta \geq \beta_c$ be a real. Then, whp over $G \sim \mathcal{G}_{n,d}$, for every $\varepsilon \geq e^{-\Omega(n)}$ the RC dynamics $(X_t)_{t \geq 0}$ initialised from all-in satisfies $\|X_T - \pi_{G,o}\|_{\text{TV}} \leq \varepsilon$ for $T = O(n \log n \log \frac{1}{\varepsilon})$.*

Proof of Theorems 1 and 2. We first do the proof of Theorem 1. By Corollary 7, it holds that $\|\pi_G - \pi_{G,d}\|_{\text{TV}} = e^{-\Omega(n)}$ and by Theorem 9 we can obtain a sample from $\pi_{G,d}$ in T steps using random-cluster dynamics. Then, using the translation between the Potts and RC models [25], this gives a sample $\hat{\sigma}$ whose distribution is within TV-distance $\leq \varepsilon$ from μ_G , as wanted. The proof of Theorem 2 is analogous using Corollary 7 and Theorem 10. ◀

It was shown in [21] that, for the ferromagnetic Ising model, a sufficient condition for fast convergence to equilibrium within the phase is the property of *weak spatial mixing (WSM) within a phase*, paired with some mixing-time estimates in local neighbourhoods. The same condition applies for the random-cluster model and, more generally, for monotone models. In the case of the random regular graph, the local neighbourhoods resemble the d -regular tree and the required mixing-time results have been established in [7]. So, the main piece

missing in order to obtain Theorems 9 and 10 is showing WSM within the corresponding phase. Roughly, the notion captures that, when we condition on the phase, the influence of a boundary configuration at distance ℓ is small.

More formally, for a graph $G = (V, E)$, an integer ℓ and a vertex v , let $B_\ell(v)$ be the graph induced by vertices at distance $\leq \ell$ from v , whose vertex and edge sets are going to be denoted by $V(B_\ell(v))$ and $E(B_\ell(v))$, respectively. A boundary condition τ on $B_\ell(v)$ is a subset of the edges in $E \setminus E(B_\ell(v))$; then $\pi_{G,v,\ell}^\tau$ denotes the RC distribution induced on $B_\ell(v)$ conditioned on τ , i.e., for $S \subseteq E(B_\ell(v))$ it holds that $\pi_{G,v,\ell}^\tau(S) = \frac{\pi_G(S \cup \tau)}{\sum_{S' \subseteq E(B_\ell(v))} \pi_G(S' \cup \tau)}$. There are two extreme boundary conditions on $B_\ell(v)$ which will be most relevant: (i) the *all-in* or *wired* condition $+$ where all edges in $E \setminus E(B_\ell(v))$ are included in the boundary condition, and (ii) the *all-out* or *free* boundary condition $-$ where none of the edges in $E \setminus E(B_\ell(v))$ are included. We use $\pi_{G,v,\ell}^+$ and $\pi_{G,v,\ell}^-$ to denote the corresponding conditional RC distributions.

For an edge $e \in E$, let 1_e denote the subsets of E that include e . Following [19], we say that we have WSM within the disordered phase (respectively, ordered) at distance ℓ if for each vertex v and each edge e incident to v it holds that $|\pi_{G,v,\ell}^-(1_e) - \pi_{G,d}(1_e)| \leq \frac{1}{100m}$ (respectively, $|\pi_{G,v,\ell}^+(1_e) - \pi_{G,o}(1_e)| \leq \frac{1}{100m}$) with $m = |E|$.

Our main technical results for establishing Theorems 9 and 10 are the following.

► **Theorem 11.** *Let $d, q \geq 3$ be integers and $\beta \leq \beta_c$ be a positive real. Then, there exists $\delta > 0$ such that, whp over $G \sim \mathcal{G}_{n,d}$, $G = (V, E)$ has WSM within the disordered phase at a distance $\ell = \lfloor (\frac{1}{2} - \delta) \log_{d-1} n \rfloor$. That is, with $m = |E|$, for every $v \in V$ and every edge e incident to v , it holds that $|\pi_{G,v,\ell}^-(1_e) - \pi_{G,d}(1_e)| \leq \frac{1}{100m}$.*

► **Theorem 12.** *Let $d \geq 3, q \geq (5d)^5$ be integers and $\beta \geq \beta_c$ be a real. Then, there exists $\delta > 0$ such that, whp over $G \sim \mathcal{G}_{n,d}$, G has WSM within the ordered phase at a distance $\ell = \lfloor (\frac{1}{2} - \delta) \log_{d-1} n \rfloor$. That is, with $m = |E|$, for every $v \in V$ and every edge e incident to v , it holds that $|\pi_{G,v,\ell}^+(1_e) - \pi_{G,o}(1_e)| \leq \frac{1}{100m}$.*

Using these, the proofs of Theorems 9 and 10 can be done using by now standard arguments; we give the details for completeness in the full version [18].

We therefore focus on the WSM proofs. The proof for the disordered regime $\beta \leq \beta_c$ is given in Section 4, and the proof for the ordered regime $\beta \geq \beta_c$ is given in Section 5.1.

4 Proof for WSM in the disordered regime

To show WSM in the disordered regime, we follow the strategy outlined in Section 1.2. We begin with some relevant definitions. Let $G = (V, E)$ be a graph. Recall, that for a vertex v and $\ell > 0$, $B_\ell(v)$ denotes the radius- ℓ ball around v in G . Also, denote by $S_\ell(v)$ the set of vertices at distance exactly ℓ from v . For a set of vertices S , we let $E(S)$ the set of edges with both endpoints in S .

► **Definition 13.** *Let $G = (V, E)$ be a graph and v be a vertex. For $K, \ell > 0$, we say that a subset $F \subseteq E$ is K -shattered at distance ℓ from v if all but K vertices of $S_\ell(v)$ belong to distinct components of the graph $(V, F \setminus E(B_\ell(v)))$.*

► **Lemma 14.** *Let $d, q \geq 3$ be integers and $\beta \leq \beta_c$ be a positive real. For any constant $\epsilon > 0$, there is $K > 0$ so that the following holds whp over \mathbf{G} . For every vertex $v \in V$, define the event*

$$\mathcal{A}_v = \{F \subseteq E(\mathbf{G}) \mid F \text{ is } K\text{-shattered at distance } \ell = \lfloor (\frac{1}{2} - 2\epsilon) \log_{d-1} n \rfloor \text{ from } v\}.$$

Then, for the RC disordered phase of \mathbf{G} , we have $\pi_{\mathbf{G},d}(\bigcap_v \mathcal{A}_v) \geq 1 - 1/n^3$.

Proof. For a graph $G \in \mathcal{G}_{n,d}$ and $\sigma \in [q]^n$, let $\mathbf{F}(G, \sigma)$ denote the random subset of edges obtained by keeping each monochromatic edge of G under σ with probability p . Fix $\epsilon > 0$ and let $K = \lceil 10/\epsilon \rceil$. We will consider the disordered planted graph $(\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d)$ and show that for an arbitrary vertex v it holds that

$$\mathbb{P}[\mathbf{F}(\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d) \notin \mathcal{A}_v] \leq 2/n^5. \tag{4.1}$$

Assuming this for now, we obtain by Lemma 4 and a union bound over the vertices that $\mathbb{P}[\mathbf{F}(\mathbf{G}, \sigma_{\mathbf{G},d}) \notin \bigcap_{v \in V} \mathcal{A}_v \mid \mathbf{G}] \leq \ln n/n^4$ as well. Note that $\mathbf{F}(\mathbf{G}, \sigma_{\mathbf{G},d})$ has distribution $\hat{\pi}_{\mathbf{G},d}$, so Lemma 8 yields that $\pi_{\mathbf{G},d}(\bigcap_v \mathcal{A}_v) \geq \hat{\pi}_{\mathbf{G},d}(\bigcap_v \mathcal{A}_v) - e^{-\Omega(n)} \geq 1 - 1/n^3$.

It remains to prove (4.1). By Lemma 5, we have $\|\nu^{\hat{\sigma}_d} - \nu_d\|_1 + \|\rho^{\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d} - \rho_d\|_1 = O(n^{-1/3})$ w.p. $1 - e^{-\Omega(n^{1/3})}$. Consider arbitrary ν and ρ with $\|\nu - \nu_d\|_1 + \|\rho - \rho_d\|_1 = O(n^{-1/3})$. Conditioned on $\nu^{\hat{\sigma}_d} = \nu$ and $\rho^{\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d} = \rho$, we have that $(\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d)$ is uniformly random among all pairs (G, σ) with $\nu^\sigma = \nu$ and $\rho^{G,\sigma} = \rho$.

To proceed, fix σ with $\nu^\sigma = \nu$ and condition on $\hat{\sigma}_d = \sigma$. Condition further that $\rho^{\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d} = \rho$ and on the induced graph on $B_\ell(v)$. For convenience, let $\tilde{\mathbf{F}}$ denote the (random) set of edges outside $B_\ell(v)$ after the percolation step, i.e., the set $\mathbf{F}(\hat{\mathbf{G}}(\hat{\sigma}_d), \hat{\sigma}_d) \setminus E(B_\ell(v))$.

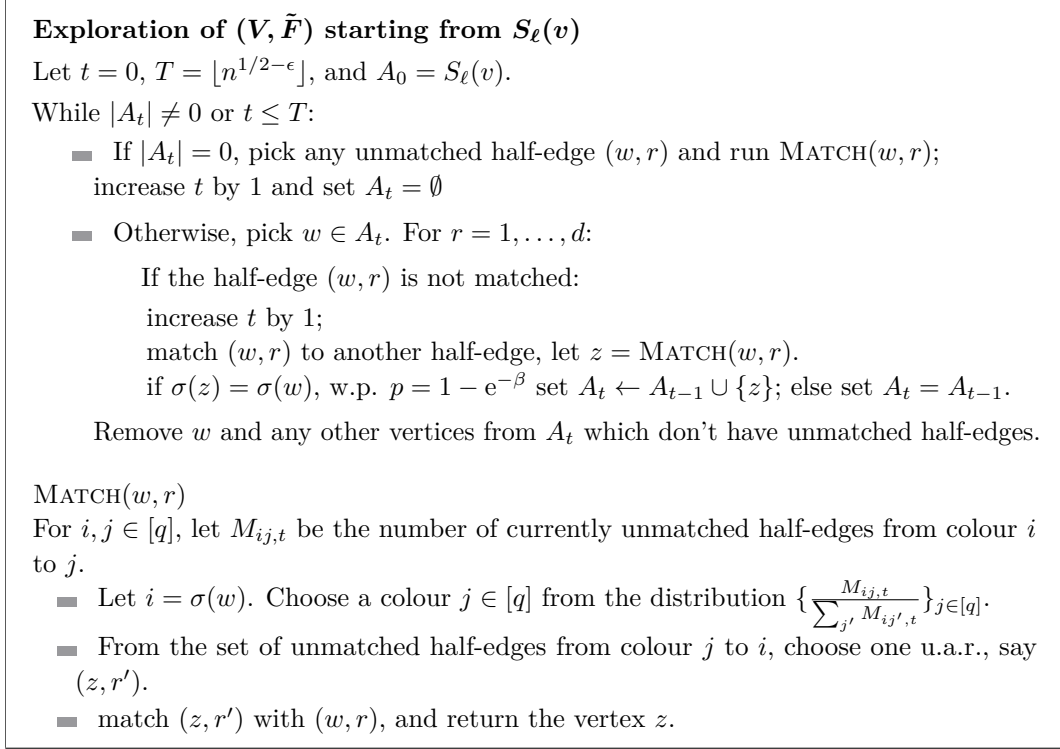
Let $T := \lfloor n^{1/2-\epsilon} \rfloor$. For $t = 1, \dots, T$, we explore the (union of the) components of vertices $u \in S_\ell(v)$ in the graph $(V, \tilde{\mathbf{F}})$ by revealing at each step an edge incident to a vertex belonging to any of these (chosen arbitrarily). We denote by A_t the set of ‘‘active vertices’’ in these components that still have unmatched half-edges, so that $|A_t| = 0$ implies that all the components have been fully explored. Let $A_{\leq t} = \cup_{t' \leq t} A_{t'}$ denote the set of all vertices that have been active up to time t . The process is described precisely in Figure 1.

Note that at any time t , for $i, j \in [q]$, there is a number $M_{ij,t}$ of prescribed half-edges with one endpoint of colour i that needs to be matched to a vertex of colour j (note, this goes down when we reveal such an edge connecting colours i and j). The revealing of an edge at time t adjacent to an active vertex w with colour $\sigma(w) = i$ has therefore two stages: we first reveal the other endpoint z in the graph $\hat{\mathbf{G}}(\sigma)$, chosen proportionally to the counts $M_{ij,t}$; if $\sigma(w) = \sigma(z)$, we further toss a coin with probability $p = 1 - e^{-\beta}$ to determine whether the edge belongs to $\tilde{\mathbf{F}}$, and add z to the set of active vertices accordingly. The key point here is that we only need to reveal the components involving $S_\ell(v)$ in the graph $(V, \tilde{\mathbf{F}})$, rather than the components in $\hat{\mathbf{G}}(\hat{\sigma}_d)$ (which are much bigger). Indeed, as we will see shortly the former is dominated by a subcritical branching process and therefore at most $O(\log n)$ vertices are revealed for each of the vertices in $S_\ell(v)$.

To make this precise, for a time $t = 1, \dots, T$, let $\mathbf{1}_t$ denote the indicator of the event that the t -th half edge was matched to a vertex in the set $A_{\leq t} := \cup_{t' \leq t} A_{t'}$ consisting of vertices that have been active at some point. Let also $\mathbf{1}'_t$ be the indicator that the edge revealed was monochromatic and that it survived the percolation step. Note that the number of the remaining unmatched half-edges $M_{ij,t}$ from colour i to j satisfies $0 \leq dn\rho_{ij} - M_{ij,t} \leq 2T \leq 2n^{1/2-\epsilon}$, so $M_{ij,t} = dn\rho_{ij} + o(n)$ for all the relevant t . Similarly, $|A_{\leq t}| \leq |S_\ell(v)| + T \leq 2n^{1/2-\epsilon}$. So, with \mathcal{E}_t denoting the first t edges revealed, we have the crude bounds

$$\mathbb{P}(\mathbf{1}_t \mid \mathcal{E}_{t-1}) \leq \frac{d|A_{\leq t}|}{\min_i \sum_j M_{ij,t}} \leq \frac{1}{n^{1/2+\epsilon/2}} \quad \text{and} \quad \mathbb{P}(\mathbf{1}'_t \mid \mathcal{E}_{t-1}) \leq p \max_{i \in [q]} \frac{M_{ii,t}}{\sum_j M_{ij,t}} \leq \frac{e^\beta - 1}{e^\beta + q - 1} + o(1).$$

It follows that the sum $\sum_{t=1}^T \mathbf{1}_t$ is dominated above by a sum of T independent coin tosses with probability $\frac{1}{n^{1/2+\epsilon/2}}$, whereas the sum $\sum_{t=1}^T \mathbf{1}'_t$ is dominated above by a sum of T independent coin tosses with probability $\frac{e^\beta - 1}{e^\beta + q - 1} + o(1) < \frac{1-2\kappa}{d-1}$ for some small constant $\kappa = \kappa(q, d, \beta) > 0$ (using that $e^\beta - 1 < q/(d-2)$ for $\beta < \beta'_u$). Therefore, recalling that



■ **Figure 1** The exploration process of the percolated planted graph used in the proof of Lemma 14. This is a combination of exposing first an edge in the planted configuration model, captured by the routine $\text{MATCH}(\cdot, \cdot)$, followed by a percolation step with probability $p = 1 - e^{-\beta}$ whenever the edge is monochromatic.

$K = \lceil 10/\epsilon \rceil$ and $T = \lfloor n^{1/2-\epsilon} \rfloor$, we have

$$\mathbb{P}\left(\sum_{t=1}^T \mathbf{1}_t > K\right) \leq \binom{T}{K} \frac{1}{n^{K(1/2+\epsilon/2)}} \leq 1/n^5 \quad \text{and} \quad \mathbb{P}\left(\sum_{t=1}^T \mathbf{1}'_t \geq \frac{T(1-\kappa)}{d-1}\right) = e^{-\Omega(n^{1/3})},$$

where the second probability bound follows by standard Chernoff bounds (using that $T = \Omega(n^{1/3})$).

Note that the event $|A_T| > 0$ implies that $|A_t| > 0$ for all $t = 0, 1, \dots, T$ which in turn implies that at least one vertex gets removed from the set of active vertices every (at most) $d-1$ time steps; indeed, when exploring the neighbours of a vertex $w \in A_t$ at a particular time t , there are at most $d-1$ half-edges incident to w remaining unmatched since one of the d half-edges of w has been previously matched (to activate w). We therefore have the inequality $0 \leq |A_T| \leq |S_\ell(v)| + \sum_{t=1}^T \mathbf{1}'_t - \frac{T-(d-1)}{d-1}$. Since $|S_\ell(v)| \leq d(d-1)^\ell \leq dn^{1/2-2\epsilon} \leq \frac{\kappa T}{d-1}$, it follows that

$$\mathbb{P}(|A_T| > 0) \leq \mathbb{P}\left(|S_\ell(v)| - \frac{T-(d-1)}{d-1} + \sum_{t=1}^T \mathbf{1}'_t \geq 0\right) \leq \mathbb{P}\left(\sum_{t=1}^T \mathbf{1}'_t \geq \frac{T(1-\kappa)}{d-1}\right) = e^{-\Omega(n^{1/3})}.$$

Combining these, we obtain that with probability $\geq 1 - 2/n^5$ both of $|A_T| = 0$ and $\sum_{t=1}^T \mathbf{1}_t < K$ hold. The first implies that by time T all the components of $S_\ell(v)$ in the graph (V, \tilde{F}) have been explored, and the second implies that all but (at most) K of these vertices belong to distinct components. This finishes the proof of (4.1) and hence completes the proof of the lemma. ◀

We next use an argument of Blanca and Gheissari [9] which shows a correlation decay bound for K -shattered configurations in a treelike graph in terms of the existence of two root-to-leaf paths. The argument in [9] is technically proved for $\beta < \beta_u$ using some uniqueness estimates, but it can be extended to the regime $\beta \leq \beta_c$ by observing that in a treelike ball, a K -shattered configuration induces a distribution that is comparable to percolation on the tree with parameter \hat{p} (this coincides with the cut-edge update probability in the definition of the random cluster dynamics). A key feature of the correlation decay bound in [9] is the existence of two paths which gives the $\hat{p}^{2\ell}$ term in the lemma below, and which ultimately allows us to make the probability less than $\frac{1}{100m}$ that is required for WSM, while keeping the distance ℓ less than $\frac{1}{2} \log_{d-1} n$ that is the threshold for having treelike neighbourhoods. We give the proof of the following lemma in the full version [18].

► **Lemma 15.** *Let $d, q \geq 3$ be integers, and $\beta \leq \beta_c$, $\delta \in (0, 1/2)$ positive reals.*

There exists a constant $C = C(d, q, \beta, \delta) > 0$ such that whp over $G \sim \mathcal{G}_{n,d}$ the following holds. For all $v \in V(G)$ and an edge e incident to it, and $\ell := \lfloor (\frac{1}{2} - \delta) \log_{d-1} n \rfloor$,

$$|\pi_{G,v,\ell}^-(1_e) - \pi_{G,d}(1_e)| \leq C\hat{p}^{2\ell} + \frac{1}{n^3} \text{ where } \hat{p} := \frac{e^\beta - 1}{q + e^\beta - 1}.$$

We can now give the proof of Theorem 11.

► **Theorem 11.** *Let $d, q \geq 3$ be integers and $\beta \leq \beta_c$ be a positive real. Then, there exists $\delta > 0$ such that, whp over $G \sim \mathcal{G}_{n,d}$, $G = (V, E)$ has WSM within the disordered phase at a distance $\ell = \lfloor (\frac{1}{2} - \delta) \log_{d-1} n \rfloor$. That is, with $m = |E|$, for every $v \in V$ and every edge e incident to v , it holds that $|\pi_{G,v,\ell}^-(1_e) - \pi_{G,d}(1_e)| \leq \frac{1}{100m}$.*

Proof. Consider $\beta \leq \beta_c$. Since $\beta_c < \beta'_u = \ln(1 + \frac{q}{d-2})$, we have that $\hat{p}(\beta) < \hat{p}(\beta'_u) = \frac{1}{d-1}$. Hence there is an $\epsilon(q, d, \beta) > 0$ such that $\hat{p}(\beta) = \frac{1}{(d-1)^{1+\epsilon}}$. Let $\delta = \frac{\epsilon}{4(1+\epsilon)}$ and $\ell = \lfloor (\frac{1}{2} - \delta) \log_{d-1} n \rfloor$.

Lemma 15 gives that, whp over $G \sim \mathcal{G}_{n,d}$, for any edge e and vertex v incident to it, it holds that $|\pi_{G,v,\ell}^-(1_e) - \pi_{G,d}(1_e)| \leq C\hat{p}^{2\ell} + \frac{1}{n^3}$. By the choice of δ, ϵ and ℓ , we get that $\hat{p}^{2\ell} \leq n^{-(1+\epsilon/2)}/(\hat{p}^2)$, so for large enough n we have $C\hat{p}^{2\ell} + \frac{1}{n^3} \leq \frac{1}{100m}$, concluding the proof of WSM within the disordered phase. ◀

5 Proof for WSM within the ordered regime

To show WSM within the ordered phase for the random-cluster model on $G \sim \mathcal{G}_{n,d}$, we follow the strategy outlined in Section 1.2. The core of the argument will be to show that there are no paths of length $\ell \geq 1/2 \log_{d-1} n$ that avoid the largest component in a typical configuration from $\pi_{G,o}$. We will do this by considering percolation on the planted random graph model $(\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o)$ for $\mu_{G,o}$. It will thus be relevant to consider random graph percolation on (almost) d -regular random graphs where we have removed a path of length ℓ .

For a graph G , let $\mathcal{C}_1(G)$ denote the largest component of G (breaking ties arbitrarily), and more generally, $\mathcal{C}_i(G)$ denote the i -th largest component of G . It is a standard fact [3, 30, 14] that the size of the largest component in a percolated random d -regular graph is controlled by the extinction probability φ of a Galton-Watson process with distribution $\text{Bin}(d-1, p)$ on the tree. For $p \in (\frac{1}{d-1}, 1)$, define $\varphi = \varphi(p)$, $\chi = \chi(p)$ and $\hat{\varphi} = \hat{\varphi}(p)$ to be the unique solutions on the interval $(0, 1)$ to the following

$$\varphi = (1 - \varphi + p\varphi)^{d-1}, \quad \chi := 1 - (p\varphi + 1 - p)^d, \quad \hat{\varphi} := (1 - \varphi + p\varphi)^{d-2}. \tag{5.1}$$

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In particular, the size of the largest component is roughly equal to $\chi n + o(n)$; see [3, 30] for more details. We also remark that the quantity $\hat{\varphi}$ is the extinction probability of a slightly modified GW-tree where every vertex has degree d apart from the root which has degree $d - 2$; this will be relevant later for our path considerations.

We show a similar result for the size of the giant component for slightly non-regular degree sequences \mathbf{d} , where we allow a small set S of vertices to have degree less than d ; in our WSM setting, we will further be interested in the probability that a particular subset of the vertices S' does not belong to the giant. A variant of percolation we will use is the *exact edge model* for random graphs with a given degree sequence $\mathbf{d} = (d_1, \dots, d_n)$ and edge count $m \leq \frac{1}{2} \sum d_i$. Let $\tilde{\mathcal{G}}_{n,\mathbf{d},m}$ be the random graph model where each $i \in [n]$ is associated to d_i half-edges; then we choose a subset of $2m$ half-edges and a matching of these uniformly at random; the final (multi)graph is obtained by projecting the edges on the vertex set $[n]$.

► **Lemma 16.** *Let $d \geq 3$ be an integer and $p \in (0, 1)$. Let $\delta, \varepsilon > 0$ be arbitrarily small constants.*

Suppose that $S \subseteq [n]$ satisfies $|S| = O(n^{1/6})$ and $\mathbf{d} = (d_1, \dots, d_n)$ is a degree sequence such that $d_i = d$ for $i \notin S$ and $d_i \in [0, d]$ for $i \in S$. Then, there exists a constant $M(d, p) > 0$, such that for $\tilde{\mathcal{G}}_{n,\mathbf{d},m}$ with $m = \frac{1}{2}dpn + o(n)$, the following hold:

1. *If $p < \frac{1}{d-1}$, $\mathbb{P}[\mathcal{C}_1(\tilde{\mathcal{G}}_{n,\mathbf{d},m}) \geq Mn^{1/2}] \leq e^{-\Omega(n)}$.*
2. *If $p > \frac{1}{d-1}$, $\mathbb{P}[\mathcal{C}_1(\tilde{\mathcal{G}}_{n,\mathbf{d},m}) \leq (\chi - \delta)n] \leq e^{-\Omega(n)}$ and $\mathbb{P}[\mathcal{C}_2(\tilde{\mathcal{G}}_{n,\mathbf{d},m}) \geq Mn^{2/3}] \leq e^{-\Omega(n)}$. Moreover, with $\varphi = \varphi(p)$ as in (5.1), for any subset $S' \subseteq S$,*

$$\mathbb{P}[S' \cap \mathcal{C}_1(\tilde{\mathcal{G}}_{n,\mathbf{d},m}) = \emptyset] \leq (1 + \varepsilon)^{|S'|} \prod_{i \in S'} (p\varphi + 1 - p)^{d_i}.$$

Intuitively, the bound in the second item follows by revealing the R -neighbourhoods of vertices in S' for a suitably large constant R (these are whp trees and disjoint from each other); then, for a vertex $i \in S'$, the factor $(p\varphi + 1 - p)^{d_i}$ captures the probability that the percolation dies out within the neighbourhood of the vertex i . The full proof of Lemma 16 is given in the full version [18].

Next, we consider the density of the edges inside the color classes. The following lemma says that in the ordered regime one colour class is supercritical, while the others subcritical. Recall that ν_o and ρ_o are the vertex and edge statistics for the ordered phase respectively, defined in (2.1) and (2.5).

► **Lemma 17** ([14, Lemma 5.6]). *Let $q, d \geq 3$ be integers. For $\beta > \beta_u$ and $p = 1 - e^{-\beta}$, the ordered vectors ν_o and ρ_o satisfy $p \frac{\rho_o, ii}{\nu_o, i} > \frac{1}{d-1}$ for $i = 1$, and $p \frac{\rho_o, ii}{\nu_o, i} < \frac{1}{d-1}$ for $i \in \{2, \dots, q\}$.*

Let $G = (V, E)$ be a graph. For a subset of edges $F \subseteq E$ and a vertex v , we let F_v be the edges in F incident to v . We will also consider the largest component \mathcal{C} of the graph (V, F) , i.e. the component with the largest number of vertices (breaking ties arbitrarily). We say that \mathcal{C} avoids a path P if there is no vertex $v \in \mathcal{C}$ that belongs to P . The next lemma is the key technical step in our argument and will be used to show that, in the RC ordered phase, the largest component (which has size $\Omega(n)$ due to the supercriticality of colour 1, cf. Lemma 17) is “very close” to every vertex v .

► **Lemma 18.** *Let $d, q \geq 3$ be integers, $\beta \geq \beta_c$ be real and $p = 1 - e^{-\beta}$. For any constant $\varepsilon > 0$, the following holds for any integer ℓ satisfying $\ln \ln n \leq \ell \leq n^{1/6}$, whp over $G \sim \mathcal{G}_{n,d}$.*

For a vertex $v \in V(G)$, consider the event

$$\mathcal{A}_v = \left\{ F \subseteq E(G) \mid \begin{array}{l} \text{there exists a path of length } \ell \text{ from } v \text{ in } G \text{ which} \\ \text{avoids the largest component of } (V \setminus \{v\}, F \setminus F_v) \end{array} \right\}.$$

Then, for the RC ordered phase of G , $\pi_{G,o}(\bigcup_{v \in V(G)} \mathcal{A}_v) \leq nd(d-1)^{\ell-1}(A + \varepsilon)^\ell$ where $A = \max_{i \in [q]} \{1 - (1 - \sqrt{\hat{\varphi}_1}) \frac{\rho_{o,i1}}{\nu_{o,i}}\}$ with ρ_o, ν_o as in (2.1) and $\hat{\varphi}_1 = \hat{\varphi}(p_1)$ as in (5.1) for $p_1 = p \frac{\rho_{o,11}}{\nu_{o,1}}$.

To parse the bound on $\pi_{G,o}(\bigcup_{v \in V(G)} \mathcal{A}_v)$ note that the factor $nd(d-1)^{\ell-1}$ is just an upper bound on the total number of paths of length ℓ . The factor A^ℓ controls the probability that a particular path P avoids the largest component. The term $\hat{\varphi}_1$ in A is the probability that the Galton-Watson process dies out from a vertex v of the path that belongs to the supercritical color class 1 (note that v has “remaining” degree $d-2$ other than its neighbours on the path, and that the relevant percolation parameter is given by p_1). The term $\frac{\rho_{o,i1}}{\nu_{o,i}}$ accounts for the frequency that we encounter vertices in the supercritical color class 1 along the path P , which follows from the definition of the planting model. The exact quantitative dependence of A to $\hat{\varphi}_1$ and $\frac{\rho_{o,i1}}{\nu_{o,i}}$ follows by a more technical calculation involving eigenvalues. Later, in Lemma 19, we will show that A can be made less than $1/d^5$ by taking q sufficiently large.

Proof of Lemma 18 (Sketch). From Lemmas 4 and 8, it suffices to show for the ordered planted graph $(\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o)$ that (for all n sufficiently large)

$$\mathbb{P}\left[\mathbf{F}(\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o) \in \bigcup_{v \in V(G)} \mathcal{A}_v\right] \leq nd(d-1)^{\ell-1}(A + \frac{\varepsilon}{2})^\ell. \tag{5.2}$$

From this, we obtain by Lemma 4 that $\mathbb{P}[\mathbf{F}(\mathbf{G}, \sigma_{\mathbf{G},o}) \in \bigcup_{v \in V} \mathcal{A}_v \mid \mathbf{G}] \leq n \ln(n) d(d-1)^{\ell-1}(A + \frac{\varepsilon}{2})^\ell$; since $\mathbf{F}(\mathbf{G}, \sigma_{\mathbf{G},o})$ has distribution $\hat{\pi}_{\mathbf{G},o}$, Lemma 8 yields that, whp over \mathbf{G} , it holds that $\pi_{\mathbf{G},o}(\bigcup_v \mathcal{A}_v) \leq \hat{\pi}_{\mathbf{G},o}(\bigcup_v \mathcal{A}_v) + e^{-\Omega(n)} \leq nd(d-1)^{\ell-1}(A + \varepsilon)^\ell$.

By Lemma 5, we have $\|\nu^{\hat{\sigma}_o} - \nu_o\|_1 + \|\rho^{\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o} - \rho_o\|_1 = O(n^{-1/3})$ w.p. $1 - e^{-\Omega(n^{1/3})}$. Condition on $\nu^{\hat{\sigma}_o} = \nu$ and $\rho^{\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o} = \rho$ where ν and ρ satisfy $\|\nu - \nu_o\|_1 + \|\rho - \rho_o\|_1 = O(n^{-1/3})$, and note that the pair $(\hat{G}(\hat{\sigma}_o), \hat{\sigma}_o)$ is uniformly random from the pairs having these statistics.

Let P be an arbitrary sequence of vertices $(v_0, v_1, \dots, v_\ell)$, together with a q -colouring $\tau = (\tau_0, \dots, \tau_\ell)$. Let also ξ be a set of paired half-edges that turn v_0, \dots, v_ℓ into a path in that order. Let \mathbf{E} be the set of random edges in the graph before percolation. Denote by

$$\mathcal{E}_{P,\tau,\xi} = \{\xi \subseteq \mathbf{E}, \hat{\sigma}_o(v_0) = \tau_0, \dots, \hat{\sigma}_o(v_\ell) = \tau_\ell\}$$

the event that v_0, v_1, \dots, v_ℓ form a path in \hat{G} (in this order) using the half-edges prescribed by ξ and that they belong to the colour classes that τ prescribes. In the full version [18], we give an exact expression for $\mathbb{P}(\mathcal{E}_{P,\tau,\xi})$ in terms of ν and ρ which yields that

$$\mathbb{P}(\mathcal{E}_{P,\tau,\xi}) \sim (dn)^{-\ell} \nu_{o,\tau_0} \prod_{k=0}^{\ell-1} \frac{\rho_{o,\tau_k \tau_{k+1}}}{\nu_{o,\tau_k}}, \tag{5.3}$$

where $f(n) \sim g(n)$ denotes $\frac{f(n)}{g(n)} \rightarrow 1$ as $n \rightarrow \infty$.

Fix now an arbitrary vertex v and let P be an arbitrary sequence $(v_0, v_1, \dots, v_\ell)$ with $v_0 = v$, together with a q -colouring $\tau = (\tau_0, \dots, \tau_\ell)$ and pairing ξ (as above). Condition on the event $\mathcal{E}_{P,\tau,\xi}$. Let \mathbf{F} be the edge set obtained by keeping each monochromatic edge with probability p , and \mathbf{F}_v be the set of these edges that are incident to v . Then, for the graph $\hat{G}_v(\mathbf{F}) := (V \setminus \{v\}, \mathbf{F} \setminus \mathbf{F}_v)$, we show in the full version [18] that color 1 is supercritical and the others are subcritical (using Lemma 17); by invoking then Lemma 16 to the supercritical color class 1, we further obtain that

$$\mathbb{P}\left[\begin{array}{l} \text{the largest component of } \hat{G}_v(\mathbf{F}) \\ \text{avoids the } \tau\text{-coloured path } P \end{array} \mid \mathcal{E}_{P,\tau,\xi}\right] \leq (1 + \frac{\varepsilon}{3})^\ell \hat{\varphi}_1^{\ell_1(\tau)}, \tag{5.4}$$

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where $\ell_1(\tau)$ is the number of vertices with colour 1 under τ and $\hat{\varphi}_1 := (p_1\varphi_1 + 1 - p_1)^{d-2}$ where $p_1 = p \frac{\rho_{o,1}}{\nu_{o,1}}$ and $\varphi_1 = \varphi(p_1)$ (as in (5.1)). Combining (5.3) and (5.4) via a union bound (note that there are $\sim n^\ell$ sequences P with $\ell + 1$ vertices starting from v and $d^2(d(d-1))^{\ell-1}$ ways to select half-edges ξ to turn each of these into a path) yields that

$$\mathbb{P} \left[\begin{array}{l} \text{the largest component of } \hat{\mathbf{G}}_v(\mathbf{F}) \text{ avoids} \\ \text{some } \tau\text{-coloured path from } v \text{ in } G \end{array} \right] \leq (1 + \frac{\varepsilon}{3})^\ell d(d-1)^{\ell-1} \nu_{o,\tau_0} \prod_{k=0}^{\ell-1} \frac{\rho_{o,\tau_k \tau_{k+1}}}{\nu_{o,\tau_k}} \hat{\varphi}_1^{\mathbf{1}\{\tau_k=1\}}. \quad (5.5)$$

At this stage, it remains to take a union bound over all possible colourings τ on $\ell + 1$ vertices, i.e., sum the right-hand-side of (5.5) over all τ . The resulting expression can be written more succinctly in the form $(1 + \frac{\varepsilon}{3})^\ell d(d-1)^{\ell-1} \cdot \nu_o^T M^\ell \mathbf{1}$ where M is the $q \times q$ matrix whose (i, j) -entry is given by $\frac{\rho_{o,ij}}{\nu_{o,i}}$ and $\hat{\nu}_o$ is the vector whose i -th entry is equal to $\nu_{o,1}/\hat{\varphi}_1$ if $i = 1$ and $\nu_{o,i}$ if $i \in \{2, \dots, q\}$. Note that $M = D^{-1/2} \hat{M} D^{1/2}$ where D is the $q \times q$ diagonal matrix with $\hat{\nu}_o$ on the diagonal and \hat{M} is the symmetric matrix $\frac{\rho_{o,ij}}{(\hat{\nu}_{o,i} \hat{\nu}_{o,j})^{1/2}}$. In particular, we have that \hat{M} and M have the same eigenvalues, which are all real. Let $\sqrt{\hat{\nu}_o}$ denote the vector whose entries are the square roots of the entries of $\hat{\nu}_o$. Then, we can write

$$\hat{\nu}_o^T M^\ell \mathbf{1} = \sqrt{\hat{\nu}_o}^T \hat{M}^\ell \sqrt{\hat{\nu}_o} \leq \frac{1}{\hat{\varphi}_1} \sqrt{\nu_o}^T \hat{M}^\ell \sqrt{\nu_o}.$$

Now, we will show that entry-wise it holds that $\hat{M} \sqrt{\nu_o} \leq A \sqrt{\nu_o}$ where $A := \max_{i \in [q]} \{1 - \frac{(1 - \sqrt{\hat{\varphi}_1}) \rho_{o,i1}}{\nu_{o,i}}\}$ is as in the lemma statement. Note first that $(\hat{M} \sqrt{\nu_o})_i = \sum_j \frac{\rho_{o,ij}}{(\hat{\nu}_{o,i} \hat{\nu}_{o,j})^{1/2}} (\nu_{o,j})^{1/2}$. Since $\hat{\nu}_{o,j} = \nu_{o,j}$ for $j \neq 1$ and $\sum_j \rho_{o,ij} = \nu_{o,i}$ for $i \in [q]$, we have that

$$(\hat{M} \sqrt{\nu_o})_1 = \frac{\hat{\varphi}_1 \rho_{o,11}}{(\nu_{o,1})^{1/2}} + \sum_{j \neq 1} \frac{\sqrt{\hat{\varphi}_1} \rho_{o,1j}}{(\nu_{o,1})^{1/2}} = \frac{(\hat{\varphi}_1 - \sqrt{\hat{\varphi}_1}) \rho_{o,11}}{(\nu_{o,1})^{1/2}} + \sqrt{\hat{\varphi}_1} (\nu_{o,1})^{1/2} \leq \sqrt{\hat{\varphi}_1} A (\nu_{o,1})^{1/2}.$$

For $i \neq 1$, we have

$$(\hat{M} \sqrt{\nu_o})_i = \frac{\sqrt{\hat{\varphi}_1} \rho_{o,i1}}{(\nu_{o,i})^{1/2}} + \sum_{j \neq 1} \frac{\rho_{o,ij}}{(\nu_{o,i})^{1/2}} = (\nu_{o,i})^{1/2} - \frac{(1 - \sqrt{\hat{\varphi}_1}) \rho_{o,i1}}{(\nu_{o,i})^{1/2}} \leq A (\nu_{o,i})^{1/2}.$$

This finishes the proof of $\hat{M} \sqrt{\nu_o} \leq A \sqrt{\nu_o}$, and therefore we can bound the left-hand-side of (5.5) with $d(d-1)^{\ell-1} (A + \frac{\varepsilon}{2})^\ell$. Noting that the event in (5.5) is just $\mathbb{P} \left[\mathbf{F}(\hat{\mathbf{G}}(\hat{\sigma}_o), \hat{\sigma}_o) \in \bigcup_{v \in V(G)} \mathcal{A}_v \right]$, (5.2) follows by a union bound over v . \blacktriangleleft

We next bound the quantity A appearing in Lemma 18 by a suitably small constant so that we can take a union bound over paths; it is for this step that we require $q \geq (5d)^5$. The proof is in the full version [18]

► Lemma 19. *Let $d \geq 3$, $q \geq (5d)^5$ and $\beta \geq \beta_c$. Then, for all $i \in [q]$, it holds that $(1 - \sqrt{\hat{\varphi}_1}) \frac{\rho_{o,i1}}{\nu_{o,i}} > 1 - \frac{1}{(d-1)^5}$.*

5.1 WSM within the ordered phase

Next we prove WSM within the ordered phase. We first define a *wired boundary* and show how it implies the WSM within the ordered phase, and then prove that it occurs with a good probability.

► **Definition 20.** Let $G = (V, E)$ be a graph, $v \in V$, $\ell \geq 1$ and let $F \subseteq E$ be a RC configuration. Consider the ℓ -neighbourhood $B_\ell(v)$ of v in G . We say that $S \subseteq V(B_\ell(v)) \setminus \{v\}$ is a wired boundary around v in $B_\ell(v)$, if

- S is a cut set separating v from the outside of the ball. That is, C_v , the component of v in $G \setminus S$, is disjoint from $G \setminus B_\ell(v)$; and
- in the graph $(V \setminus V(C_v), \{e \in F : e \cap V(C_v) = \emptyset\})$, all vertices of S are in the same component.

► **Lemma 21.** Let $d, q \geq 3$ be integers, $\beta \geq \beta_c$ a real. Then whp over $G \sim \mathcal{G}_{n,d}$ the following holds. For any $v \in V$, any edge e incident to v , and any $\ell \leq \frac{1}{2} \log_{d-1} n$,

$$|\pi_{G,v,\ell}^+(1_e) - \pi_{G,o}(1_e)| \leq \pi_{G,o}(\text{no wired boundary around } v \text{ in } B_\ell(v)) + e^{-\Omega(n)}.$$

To prove the lemma, we compare with π_G . When $\beta > \beta_c$, we obtain, from Corollary 7, that $\|\pi_{G,o} - \pi_G\|_{TV} = e^{-\Omega(n)}$. Then we can bound $|\pi_{G,v,\ell}^+(1_e) - \pi_G(1_e)|$ by the probability (under π_G) that there is no wired boundary around v in $B_\ell(v)$: informally, this is because probability of 1_e conditional on the wired boundary is the same under both π_G and $\pi_{G,v,\ell}^+$, and due to the monotonicity of the RC distribution, the probability of the wired boundary under $\pi_{G,v,\ell}^+$ is at least that under π_G . For more details as well as the case when $\beta = \beta_c$ see the full proof of the Lemma is in the full version [18].

By Lemma 18 all simple length- ℓ paths from v intersect the largest component of $G_{v,F}$, so we can construct a wired boundary S by taking, for each length- ℓ path from v in G , the first vertex on the path belonging to the largest component of $G_{v,F}$. We thus show the following lemma in the full version [18].

► **Lemma 22.** Let $G = (V, E)$ be a graph, let $v \in V$ be an arbitrary vertex, and let $\ell \geq 1$ an integer. For any $F \subseteq E$, let F_v be the set of edges in F that are incident to v and let $G_{v,F} := (V \setminus \{v\}, F \setminus F_v)$. Suppose that all simple length- ℓ paths from v in G intersect the largest component of $G_{v,F}$. Then F has a wired boundary around v in $B_\ell(v)$.

From these two lemmas we can now obtain WSM within the ordered phase.

► **Theorem 12.** Let $d \geq 3, q \geq (5d)^5$ be integers and $\beta \geq \beta_c$ be a real. Then, there exists $\delta > 0$ such that, whp over $G \sim \mathcal{G}_{n,d}$, G has WSM within the ordered phase at a distance $\ell = \lfloor (\frac{1}{2} - \delta) \log_{d-1} n \rfloor$. That is, with $m = |E|$, for every $v \in V$ and every edge e incident to v , it holds that $|\pi_{G,v,\ell}^+(1_e) - \pi_{G,o}(1_e)| \leq \frac{1}{100m}$.

Proof. By Lemma 19, we have $A := \{\max_i (1 - (1 - \sqrt{\hat{\varphi}_1})^{\frac{\rho_{o,i}}{\nu_{o,i}}}) < (d-1)^{-5}$. Let $\varepsilon > 0$ be a small constant (depending on d) such that $A + \varepsilon \leq (d-1)^{-(5+\varepsilon)}$. Let $\delta := \frac{\varepsilon}{4(4+\varepsilon)}$. By Lemma 21, $|\pi_{G,v,\ell}^+(1_e) - \pi_{G,o}(1_e)|$ is at most

$$e^{-\Omega(n)} + 1 - \pi_{G,o}(\exists \text{ a wired boundary around } v \text{ in } B_\ell(v)).$$

By Lemma 22, this is at most

$$e^{-\Omega(n)} + 1 - \pi_{G,o}(\text{ all simple length-}\ell \text{ paths from } v \text{ in } G \text{ intersect the largest component of } G_{v,F}).$$

By Lemma 18 this is at most $e^{-\Omega(n)} + nd(d-1)^{\ell-1}(A + \varepsilon/2)^\ell$. So whp over $G \sim \mathcal{G}_{n,d}$ it holds for every vertex v and edge e incident to it that

$$|\pi_{G,o}(1_e) - \pi_{G,v,\ell}^+(1_e)| \leq e^{-\Omega(n)} + \frac{nd}{d-1}(d-1)^{-(4+\varepsilon)\ell} \leq e^{-\Omega(n)} + \frac{1}{n^{1+\varepsilon/4}}$$

which is at most $\frac{1}{100m}$ for all n large enough. ◀

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