Demystifying Codensity Monads via Duality

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Abstract

Codensity monads provide a universal method to generate complex monads from simple functors. Recently, a wide range of important monads in logic, denotational semantics, and probabilistic computation, such as several incarnations of the ultrafilter monad, the Vietoris monad, and the Giry monad, have been presented as codensity monads, using complex arguments. We propose a unifying categorical approach to codensity presentations of monads, based on the idea of relating the presenting functor to a *dense* functor via a suitable *duality* between categories. We prove a general presentation result applying to every such situation and demonstrate that most codensity presentations known in the literature emerge from this strikingly simple duality-based setup, drastically alleviating the complexity of their proofs and in many cases completely reducing them to standard duality results. Additionally, we derive a number of novel codensity presentations using our framework, including the first non-trivial codensity presentations for the filter monads on sets and topological spaces, the lower Vietoris monad on topological spaces, and the expectation monad on sets.

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1 Introduction

Monads are among the most fundamental concepts of category theory. They provide a common abstraction of algebraic theories [31] and notions of computation [33], and the tight interplay between both viewpoints has inspired decades of fruitful research in theoretical computer science. While monads and their underlying structure can be defined from scratch,

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it is often simpler and more informative to *generate* monads via a universal construction. A powerful method to do so has been discovered by Kock [28] in the 1960s: Every functor $F\colon \mathbf{C}_0 \to \mathbf{C}$ canonically induces a monad $\mathsf{Cody}(F)$ on the category $\mathbf{C}$, the *codensity monad* of F, provided only that certain limits in $\mathbf{C}$ exist. It is constructed via the right Kan extension of F along itself and generalizes the familiar construction of monads from adjunctions. The prototypical example is the ultrafilter monad on the category of sets (whose algebras are compact Hausdorff spaces [31]), which has been characterized as the codensity monad of the inclusion $\mathbf{Set}_{\mathsf{f}} \hookrightarrow \mathbf{Set}$ of the category of finite sets into sets [27]. Generally, while every monad can be presented as the codensity monad of *some* functor (e.g. the forgetful functor of its Eilenberg-Moore category), the goal is to come up with the *simplest* possible presentation.

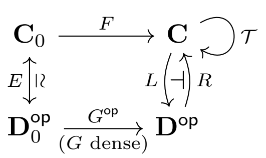
In recent years, codensity monads have found a growing number of applications in computer science; in particular, they have been identified as an elegant categorical underpinning of profinite methods in automata theory [1, 7, 15, 41], program optimizations in functional languages [18], and the construction of liftings of monads along fibrations relevant in type theory [26]. Furthermore, a number of key monads appearing in logic, denotational semantics, and probabilistic computation have been presented as codensity monads, most notably generalizations of the ultrafilter monad to algebraic and topological categories [3, 29], the Vietoris hyperspace monad on Stone spaces [15], and probability monads such as the all-important Giry monad on measurable spaces and many of its variants [5, 36, 37, 42]. The importance of such codensity presentations is that they relate the given (fairly complex) monads to structurally much simpler generating functors and endow those monads with a universal property. This provides new insights into the structure of the monads themselves, facilitating their use in theory and practice. For example, desirable properties of probability monads such as commutativity or affinity, which are instrumental in the synthetic approach to probability theory based on Markov categories [8, 11] or effectuses [21], and somewhat tedious to prove directly, can be derived in a principled manner from their codensity presentation [37].

All the above codensity presentations are non-trivial results. Their proofs in the literature rely on a careful analysis of the structure of the respective monads and on domain-specific knowledge, such as advanced results from measure theory for probability monads. Overall, this leads to technically challenging and largely ad hoc proofs of codensity presentations.

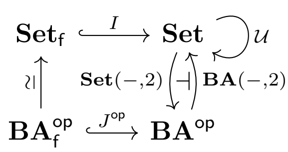
In the present paper, we address this issue by developing a *simple*, *general* and *uniform* method to synthesize codensity presentations of monads, putting a common umbrella over most known instances, and many more. The core insight underlying our contribution is that

Codensity Monads = **Density** + **Duality**.

More specifically, our approach to codensity monads rests on a simple but effective observation: all "interesting" codensity presentations (where $\mathcal{T}$ is the codensity monad of some functor F) follow the pattern shown in diagram (1.1), that is, the functor F decomposes as $F \cong RG^{\mathsf{op}}E$ into a *dual equivalence* E, the opposite of a *dense* functor G, and a *contravariant right adjoint* R generating the monad $\mathcal{T}$. For instance, the presentation of the ultrafilter monad as the codensity monad of the inclusion functor $I\colon \mathbf{Set}_{\mathsf{f}} \hookrightarrow \mathbf{Set}$ is captured by diagram (1.2). Here we use the familiar duality $\mathbf{BA}_{\mathsf{f}}^{\mathsf{op}} \simeq \mathbf{Set}_{\mathsf{f}}$ between finite Boolean algebras and finite sets, and density of the inclusion functor $J\colon \mathbf{BA}_{\mathsf{f}} \hookrightarrow \mathbf{BA}$ is the standard fact that every Boolean algebra is a canonical colimit of finite Boolean algebras.



(1.1)



(1.2)

Such a decomposition, which we call a *codensity setting*, is all that is needed to get a codensity presentation of the monad $\mathcal{T}$. Our main result (Theorem 3.2) states that:

In every codensity setting (1.1)*, the monad* $\mathcal{T}$ *is the codensity monad of the functor* F*.*

While not technically difficult, this theorem is the conceptual gist behind most known codensity presentations of monads. In fact, it can be used in two different ways: to characterize the codensity monad $\mathcal{T}$ of a given functor F, and conversely, to discover a simple functor F presenting a given monad $\mathcal{T}$ as a codensity monad. In both cases, it simplifies the proofs of codensity presentations by reducing a question regarding *codensity* to one about *density*. Formally these are of course dual concepts, but in practice density is well-understood, especially in algebraic contexts, and usually follows from general results, while codensity rarely occurs when working with "everyday" categories of algebras or spaces.

We demonstrate the strength and scope of our main theorem by deriving codensity presentations for a wide variety of different monads. Our applications not only cover most codensity presentations known in the literature, including all of the monads mentioned above [3, 5, 15, 27, 29, 36, 37, 42], but also several new examples, notably the first non-trivial codensity presentations for the filter monads on sets and topological spaces, for the (lower) Vietoris hyperspace monad on topological spaces, and for the expectation monad on sets. In all these cases, establishing the codensity presentation boils down to simply instantiating our theorem to a suitable codensity setting (1.1). Much like in the setting (1.2) for the ultrafilter monad, the choice of the duality and of the dense functor is usually fairly obvious and suggested by standard results on the corresponding algebraic categories, and the chosen dualities are very basic ones between categories of finite algebras or categories of relations.

For all monads with a known codensity presentation covered in our paper, the principled and modular nature of our approach leads to proofs that are dramatically shorter and more transparent compared to those found in the original literature. In fact, a lot of the complexity of the latter can be attributed to the fact that the authors (implicitly) rediscover the arguments underlying the duality and density results appearing in the respective instantiation of (1.1). By applying our categorical framework, we can instead appeal to well-known properties of the respective categories and get this work entirely for free. In this way, all our codensity presentations of (ultra)filter and Vietoris-type monads become an essentially straightforward instance of our main theorem. For probability monads like the Giry monad, the (only) non-straightforward part lies in identifying a suitable dual adjunction inducing the given monad, which requires representation theorems relating linear functionals to probability measures. This identifies precisely the part of the reasoning where measure theory is needed.

2 Categorical Background

We assume some familiarity with basic category theory [30], such as equivalence functors, (co)limits, adjunctions, and monads. In the following we introduce the notation used in the paper, and review the key concepts of (co)dense functors and codensity monads.

Monads. Recall that a *monad* $\mathcal{T} = (T, \eta, \mu)$ on category $\mathbf{C}$ is given by an endofunctor T on $\mathbf{C}$ and two natural transformations, the *unit* $\eta\colon \mathsf{Id}_{\mathbf{C}} \to T$ and the *multiplication* $\mu\colon TT \to T$, satisfying the usual unit and associative laws. We denote by $\mathbf{Kl}(\mathcal{T})$ the Kleisli category for $\mathcal{T}$; its objects are those of $\mathbf{C}$, and morphisms from X to Y are morphisms $f\colon X \to TY$ with the usual Kleisli composition. Moreover, we write $\mathbf{EM}(\mathcal{T})$ for the *Eilenberg-Moore* category for $\mathcal{T}$, the category of $\mathcal{T}$-algebras and their morphisms. The category $\mathbf{Kl}(\mathcal{T})$ comes with a forgetful functor $U_{\mathcal{T}}\colon \mathbf{Kl}(\mathcal{T}) \to \mathbf{C}$ mapping $X \in \mathbf{C}$ to TX and a Kleisli morphism $f\colon X \to TY$ to

$f^\# = \mu_X \circ Tf\colon TX \to TY$. Moreover, there is a full embedding $I_\mathcal{T}\colon \mathbf{Kl}(\mathcal{T}) \hookrightarrow \mathbf{EM}(\mathcal{T})$ given by $X \mapsto (TX, \mu_X)$ and $f \mapsto f^\#$ that identifies the Kleisli category $\mathbf{Kl}(\mathcal{T})$ with the full subcategory of $\mathbf{EM}(\mathcal{T})$ given by free $\mathcal{T}$-algebras. For a monad $\mathcal{T}$ on $\mathbf{Set}$, the category of sets and functions, we write $\mathbf{Kl}_\mathsf{f}(\mathcal{T})$ for the full subcategory of $\mathbf{Kl}(\mathcal{T})$ given by finite sets, or equivalently, the category of finitely generated free $\mathcal{T}$-algebras. We denote the domain restrictions of the above functors also by $U_\mathcal{T}\colon \mathbf{Kl}_\mathsf{f}(\mathcal{T}) \to \mathbf{Set}$ and $I_\mathcal{T}\colon \mathbf{Kl}_\mathsf{f}(\mathcal{T}) \hookrightarrow \mathbf{EM}(\mathcal{T})$.

Algebraic Categories. One important class of monads are *free-algebra monads* on **Set**. Every algebraic theory (Σ, E), specified by signature a Σ of finitary operation symbols and a set E of equations between Σ-terms, induces a monad $\mathcal{T}$ on **Set** which maps to each set X the free (Σ, E)-algebra generated by X, carried by the set of Σ-terms modulo equations. The category $\mathbf{EM}(\mathcal{T})$ is isomorphic to the category of all (Σ, E)-algebras. The monad $\mathcal{T}$ is *finitary* (i.e. preserves directed colimits), and conversely, every finitary monad on **Set** is induced by some algebraic theory. In fact, *all* monads on **Set** have an algebraic presentation by operations and equations, provided that large signatures and infinitary operations are admitted. For example, the category of algebras for the finite power set monad $\mathcal{P}_\mathsf{f}$ is isomorphic the category **JSL** of join-semilattices with bottom (equivalently, the category **MSL** of meet-semilattices with top). The full power set monad $\mathcal{P}$ yields the category **CJSL** of complete (semi)lattices and join-preserving maps. We tacitly identify isomorphic categories; for instance, the functor $I_\mathcal{P}\colon \mathbf{Kl}(\mathcal{P}) \hookrightarrow \mathbf{EM}(\mathcal{P})$ is identified with the functor $I_\mathcal{P}\colon \mathbf{Kl}(\mathcal{P}) \hookrightarrow \mathbf{CJSL}$ sending a set X to the free complete semilattice $\mathcal{P}X$. See Manes [31] for more background on the relation between monads and algebraic theories.

Dual Adjunctions. Another natural source of monads are adjunctions. We use the notation $L \dashv R\colon \mathbf{D} \to \mathbf{C}$, or simply $L \dashv R$, for an adjunction where $L\colon \mathbf{C} \to \mathbf{D}$ has a right adjoint $R\colon \mathbf{D} \to \mathbf{C}$. Every adjunction with unit $\eta\colon \mathsf{Id}_\mathbf{C} \to RL$ and counit $\varepsilon\colon LR \to \mathsf{Id}_\mathbf{D}$ induces a monad $(RL, \eta, R\varepsilon L)$ on $\mathbf{C}$. We denote this monad by RL, leaving the structure implicit.

A *dual adjunction* is an adjunction of type $L \dashv R\colon \mathbf{D}^\mathsf{op} \to \mathbf{C}$. Many dual adjunctions of interest are given by *dualizing objects* [24, Sec. VI.4], that is, $\mathbf{C}$ and $\mathbf{D}$ are concrete categories (with respective forgetful functors $|{-}|$ to **Set**), and there are objects $S \in \mathbf{C}$ and $T \in \mathbf{D}$ with

$$|S| = |T|, \qquad |LX| \cong \mathbf{C}(X, S), \qquad \text{and} \qquad |RY| \cong \mathbf{D}(Y, T).$$

Then the induced monad RL is given by a "double hom-set" construction. For example, the dual adjunction $\mathbf{Set}(-,2) \dashv \mathbf{Set}(-,2)\colon \mathbf{Set}^\mathsf{op} \to \mathbf{Set}$ where $2 = \{0,1\}$ yields the *neighbourhood monad* on **Set**. Similarly, by reading 2 as a Boolean algebra, we obtain a dual adjunction $\mathbf{Set}(-,2) \dashv \mathbf{BA}(-,2)\colon \mathbf{BA}^\mathsf{op} \to \mathbf{Set}$ between the categories of sets and Boolean algebras that induces the *ultrafilter monad*. Both monads are studied in detail in Section 4.

Dense Functors. Given a functor $F\colon \mathbf{A} \to \mathbf{B}$ and $B \in \mathbf{B}$, the *slice category* $F \downarrow B$ has as objects all morphisms $f\colon FA \to B$ for $A \in \mathbf{A}$, and a morphism from $(f\colon FA \to B)$ to $(g\colon FA' \to B)$ is a morphism $h\colon A \to A'$ of $\mathbf{A}$ with $f = g \cdot Fh$. There is an obvious projection

$$\pi_B\colon F \downarrow B \to \mathbf{A}, \qquad (f\colon FA \to B) \mapsto A.$$

The functor F is *dense* [14] if each $B \in \mathbf{B}$ is the colimit of the (possibly large) diagram $F \circ \pi_B$, with cocone f ($f \in F{\downarrow}B$). Equivalently, the functor $\mathbf{B} \to [\mathbf{A}^\mathsf{op}, \mathbf{Set}]$ given by $B \mapsto \mathbf{A}(F-, B)$ is fully faithful. (Here $[\mathbf{C}, \mathbf{D}]$ is the possibly superlarge category of functors between categories $\mathbf{C}$ and $\mathbf{D}$ and natural transformations.) A subcategory $\mathbf{A} \hookrightarrow \mathbf{B}$ is *dense* if its inclusion is dense. Informally, this says that each object of $\mathbf{B}$ can be canonically built from objects of $\mathbf{A}$.

▶ **Example 2.1.** In most of our applications, we will consider dense full subcategories of $\mathbf{EM}(\mathcal{T})$ for a monad $\mathcal{T}$ on **Set**. There are several generic choices of such subcategories:

1. The full subcategory $I_{\mathcal{T}} \colon \mathbf{Kl}(\mathcal{T}) \hookrightarrow \mathbf{EM}(\mathcal{T})$ of free algebras is dense.
2. If $\mathcal{T}$ is finitary, the full subcategory $I_{\mathcal{T}} \colon \mathbf{Kl}_{\mathsf{f}}(\mathcal{T}) \hookrightarrow \mathbf{EM}(\mathcal{T})$ of finitely generated free algebras is also dense. Moreover, if there is an algebraic theory presenting the monad $\mathcal{T}$ with an upper bound $n > 0$ to the arities of operations, the one-object full subcategory $\{Tn\} \hookrightarrow \mathbf{EM}(\mathcal{T})$ given by the free algebra on n generators is dense [19, Sec. 2.2].
3. If $\mathcal{T}$ is finitary, another dense full subcategory $(\mathbf{EM}(\mathcal{T}))_{\mathsf{fp}} \hookrightarrow \mathbf{EM}(\mathcal{T})$ is given by the finitely presentable algebras (i.e. algebras presentable by finitely many generators and relations). This follows from the fact that $\mathbf{EM}(\mathcal{T})$ is locally finitely presentable [2, Cor. 3.7] and that in any such category the finitely presentable objects form a dense subcategory [2, Ex. 1.24.1]. Note that in the case where the monad $\mathcal{T}$ preserves finite sets, $(\mathbf{EM}(\mathcal{T}))_{\mathsf{fp}}$ coincides with the full subcategory $(\mathbf{EM}(\mathcal{T}))_{\mathsf{f}} \hookrightarrow \mathbf{EM}(\mathcal{T})$ of finite algebras.

▶ **Example 2.2.** For every small category $\mathbf{C}$, the Yoneda embedding $\tilde{y} \colon \mathbf{C}^{\mathsf{op}} \hookrightarrow [\mathbf{C}, \mathbf{Set}]$ given by $C \mapsto \mathbf{C}(C, -)$ is dense [30, Sec. III.7, Thm. 1].

Dual to density, we have the notion of a *codense* functor $F \colon \mathbf{A} \to \mathbf{B}$: every object $B \in \mathbf{B}$ is the limit of the canonical diagram of all morphisms $f \colon B \to FA$ with $A \in \mathbf{A}$.

▶ **Notation 2.3.** For F and B as above and $G \colon \mathbf{A} \to \mathbf{C}$, we write

$$\operatorname*{colim}_{\substack{f\colon FA \to B\\ A \in \mathbf{A}}} GA \qquad \text{or simply} \qquad \operatorname*{colim}_{f\colon FA \to B} GA$$

for the colimit of the diagram $G \circ \pi_F \colon (F \downarrow B) \to \mathbf{A} \to \mathbf{C}$; similarly for limits.

Kan Extensions. The *right Kan extension* of a functor $F \colon \mathbf{A} \to \mathbf{C}$ along a functor $J \colon \mathbf{A} \to \mathbf{B}$ is given by a functor $\mathsf{Ran}_J\, F \colon \mathbf{B} \to \mathbf{C}$ and a bijection

$$[\mathbf{B}, \mathbf{C}](G, \mathsf{Ran}_J\, F) \cong [\mathbf{A}, \mathbf{C}](GJ, F) \tag{2.1}$$

natural in G. Concretely, this means that there is natural transformation $\varphi \colon (\mathsf{Ran}_J\, F)J \to F$ (called the *counit*) such that every natural transformation $\alpha \colon GJ \to F$ factorizes as

$$\alpha = (\, GJ \xrightarrow{\hat{\alpha}J} (\mathsf{Ran}_J\, F)J \xrightarrow{\varphi} F\,) \qquad \text{for some unique } \hat{\alpha} \colon G \to \mathsf{Ran}_J\, F. \tag{2.2}$$

If the limit below exists for all $X \in \mathbf{C}$, the right Kan extension is given by

$$(\mathsf{Ran}_J\, F)X = \lim_{f\colon X \to JA} FA. \tag{2.3}$$

This holds, for instance, if the category $\mathbf{A}$ is small and $\mathbf{C}$ is complete. Right Kan extensions of this type are called *pointwise*. All Kan extensions emerging in our applications are pointwise.

Codensity Monads. The *codensity monad* [28] of a functor $F : \mathbf{A} \to \mathbf{C}$ is the monad

$$\mathsf{Cody}(F) = (\mathsf{Ran}_F\, F, \eta, \mu)$$

where $\mathsf{Ran}_F\, F$ is the right Kan extension of F along itself (if it exists), and the unit $\eta \colon \mathsf{Id} \to \mathsf{Ran}_F\, F$ and multiplication $\mu \colon (\mathsf{Ran}_F\, F)(\mathsf{Ran}_F\, F) \to \mathsf{Ran}_F\, F$ are the natural transformations corresponding to the natural transformations below, where φ is the counit of $\mathsf{Ran}_F\, F$:

$$F \xrightarrow{\mathsf{id}} F \qquad \text{and} \qquad (\mathsf{Ran}_F\, F)(\mathsf{Ran}_F\, F)F \xrightarrow{(\mathsf{Ran}_F\, F)\varphi} (\mathsf{Ran}_F\, F)F \xrightarrow{\varphi} \mathsf{Ran}_F\, F.$$

If the Kan extension $\mathsf{Ran}_F\, F$ is pointwise, then $\mathsf{Cody}(F)X = \lim_{f\colon X \to FA} FA$. Moreover, the action of $\mathsf{Cody}(F)$ on a morphism $h\colon X' \to X$ and the unit and multiplication at $X \in \mathbf{C}$ are uniquely determined by the commutative diagrams below, where f ranges over all $f\colon X \to FA$ with $A \in \mathbf{A}$ and $\pi_f\colon \mathsf{Cody}(F)X \to FA$ is the corresponding limit projection:

$$
\begin{array}{ccc}
\mathsf{Cody}(F)X' & \xrightarrow{\mathsf{Cody}(F)h} & \mathsf{Cody}(F)X \\
& \searrow^{\pi_{f\circ h}} & \downarrow{\scriptstyle \pi_f} \\
& & FA
\end{array}
\qquad\qquad
\begin{array}{ccccc}
X & \xrightarrow{\eta_X} & \mathsf{Cody}(F)X & \xleftarrow{\mu_X} & \mathsf{Cody}(F)\mathsf{Cody}(F)X \\
& \searrow^{f} & \downarrow{\scriptstyle \pi_f} & \swarrow_{\pi_{\pi_f}} & \\
& & FA & &
\end{array}
$$

A *codensity presentation* of a monad $\mathcal{T}$ on $\mathbf{C}$ is a functor $F\colon \mathbf{A} \to \mathbf{C}$ such that the monad $\mathcal{T}$ is isomorphic to the monad $\mathsf{Cody}(F)$ (in the usual category of monads and monad morphisms).

▶ **Remark 2.4.**

1. The codensity monad may be seen as a measure of codensity of the functor F; indeed, F is codense if and only if its codensity monad is the identity monad.
2. The construction of codensity monads generalizes the construction of monads from adjunctions: If R has a left adjoint L, the codensity monad of R exists and is isomorphic to the induced monad RL [29]. In particular, since every monad arises from an adjunction, every monad is a codensity monad. This means that the challenge is not to find *some* codensity presentation of a given monad, but rather to find a simple and natural one.

3 Codensity Monads = Density + Duality

In this section we present our core technical result, a simple and general criterion for a monad induced by a given dual adjunction to be the codensity monad of a given functor. The following setting generalizes a common pattern: a dual adjunction between two categories restricting to an adjoint equivalence of certain subcategories.

▶ **Definition 3.1** (Codensity Setting). A *codensity setting* is given by categories and functors as in diagram (3.1), where (1) L is left adjoint to R, (2) E is an equivalence, (3) G is dense, and (4) the outside commutes up to natural isomorphism ($RG^{\mathsf{op}} \cong FE$).

$$
\begin{array}{ccc}
\mathbf{C}_0 & \xrightarrow{F} & \mathbf{C} \\
{\scriptstyle E}\uparrow\wr & & {\scriptstyle L}\downarrow \dashv \uparrow{\scriptstyle R} \\
\mathbf{D}_0^{\mathsf{op}} & \xrightarrow{G^{\mathsf{op}}} & \mathbf{D}^{\mathsf{op}}
\end{array}
\tag{3.1}
$$

The adjunction $L \dashv R$ of a codensity setting induces a monad, and the decomposition of the functor F is enough to deduce that F is a codensity presentation of that monad:

▶ **Theorem 3.2.** *In every codensity setting* (3.1)*, the codensity monad of the functor F exists, is pointwise, and is isomorphic to the monad induced by the adjunction $L \dashv R$:*

$$RL \cong \mathsf{Cody}(F).$$

Below we sketch an elementary proof of this theorem via computations of (co)limits. Two more conceptual arguments based on general properties of Kan extensions and on string diagrams, respectively, can be found in the full arXiv paper.

Proof sketch. We have the following isomorphisms natural in $X \in \mathbf{C}$:

$$\begin{aligned}
RLX &\cong R(\operatorname*{colim}_{f\colon GD\to LX} GD) && G \text{ dense} \\
&\cong \lim_{f\colon GD\to LX} RGD && \text{right adjoints preserve limits} \\
&\cong \lim_{g\colon X\to RGD} RGD && L \dashv R\colon \mathbf{D}^{\mathsf{op}} \to \mathbf{C} \\
&\cong \lim_{g\colon X\to FED} FED && RG^{\mathsf{op}} \cong FE \\
&\cong \lim_{g\colon X\to FC} FC && E \text{ equivalence functor} \\
&\cong (\mathsf{Ran}_F\, F)X && \text{limit formula for } \mathsf{Ran}.
\end{aligned}$$

This proves that $\mathsf{Cody}(F)$ is pointwise and that $RL \cong \mathsf{Cody}(F)$ as functors. A routine verification shows that this is an isomorphism of monads, i.e. the unit and multiplication are preserved. ◀

▶ **Remark 3.3.** Theorem 3.2 is related to a recent result by Doña Mateo [32, Prop. 2.15] who independently proved that for every composite RH of a codense functor H and a right adjoint R, the codensity monad of RH is the monad induced by R. Theorem 3.2 corresponds to the setting $H = G^{\mathsf{op}}E^{-1}$ where E^{-1} is an inverse equivalence of E. Thus, it adds to Doña Mateo's abstract result the crucial insight that the codense functor H should in practice be decomposed into a dual equivalence and the opposite of a dense functor.

▶ **Remark 3.4.**

1. Every codensity monad has a codensity setting: for any functor $F\colon \mathbf{C}_0 \to \mathbf{C}$ with $\mathbf{C}_0$ small its codensity monad is induced by the *conerve-totalization adjunction*

$$\begin{array}{ccc}
\mathbf{C}_0 & \xrightarrow{\quad F \quad} & \mathbf{C} \circlearrowleft \mathsf{Cody}(F) \\
\| & & N \big\downarrow \dashv \big\uparrow R \\
(\mathbf{C}_0^{\mathsf{op}})^{\mathsf{op}} & \xrightarrow{\; y^{\mathsf{op}} \;} & [\mathbf{C}_0, \mathbf{Set}]^{\mathsf{op}},
\end{array} \tag{3.2}$$

   where $NC = \mathbf{C}_0(C, F(-))$ is the *conerve* and its right adjoint R is the right Kan extension $\mathsf{Ran}_{\tilde{y}}\, F$ of F along the contravariant Yoneda embedding $\tilde{y}\colon \mathbf{C}_0 \to [\mathbf{C}_0, \mathbf{Set}]^{\mathsf{op}}$. The Yoneda embedding $y\colon \mathbf{C}_0 \to [\mathbf{C}_0^{\mathsf{op}}, \mathbf{Set}]$ is dense, and the adjunction induces $\mathsf{Cody}(F)$ by definition [29, Section 2].
   Combining Item 1 with Remark 2.4.2 we know that every monad admits *some* codensity setting, so the goal is again to find a simple and natural one, for example by corestricting the adjunction from (3.2) to a good subcategory of $[\mathbf{C}_0, \mathbf{Set}]$.
2. Di Liberti [10] showed that if one additionally assumes density of F and cocompleteness of $\mathbf{C}$ the adjunction (3.2) factorizes through Isbell's dual adjunction between presheaves and copresheaves over $\mathbf{C}_0$. While this result is conceptually interesting (especially for the algebras of such codensity monads) it does not simplify finding concrete codensity presentations without substantial additional effort. Besides, for a wide class of codensity presentations we consider, for example for probability monads (see Section 6), the density assumption is not satisfied.

In the following sections, we will illustrate the wide scope of Theorem 3.2 and use it to derive codensity presentations for a number of popular monads. To show that a given monad $\mathcal{T}$ on $\mathbf{C}$ is the codensity monad of a functor $F\colon \mathbf{C}_0 \to \mathbf{C}$, we employ a uniform recipe:

1. Identify a suitable dual adjunction $L \dashv R\colon \mathbf{D}^{\mathsf{op}} \to \mathbf{C}$ inducing the monad $\mathcal{T}$.
2. Extend the functor F and the adjunction $L \dashv R$ to a codensity setting (3.1).
3. Apply Theorem 3.2 to conclude $\mathcal{T} \cong \mathsf{Cody}(F)$.

In all our applications, the crucial (and sometimes non-trivial) step is the choice of the dual adjunction $L \dashv R$ in Step 1. Then Step 2 is typically straightforward: The functor E in (3.1) is given by some simple dual equivalence $\mathbf{D}_0^{\mathsf{op}} \simeq \mathbf{C}_0$ known in the literature, and likewise the density of the functor G amounts to some standard property of the category $\mathbf{D}$.

4 Ultrafilter and Double Dualization Monads

The first class of codensity monads that we cover as instances of Theorem 3.2 are (ultra)filter monads and their close relatives, double dualization monads [3, 10, 27, 29].

4.1 Ultrafilter Monads

We start with what is probably the best known instance of a codensity monad: the characterization of the ultrafilter monad on **Set** as the codensity monad of the inclusion $\mathbf{Set}_{\mathsf{f}} \hookrightarrow \mathbf{Set}$ of finite sets into sets. This classical result goes back to Kennison and Gildenhuys [27]; see also Leinster [29] for a streamlined exposition.

Let **BA** denote the category of Boolean algebras. An *ultrafilter* on a Boolean algebra B is a subset $U \subseteq B$ such that (1) U is upwards closed, (2) U is closed under meet, and (3) for every $b \in B$, either $b \in U$ or $\neg b \in U$. Equivalently, an ultrafilter is given by a morphism $\chi \in \mathbf{BA}(B, 2)$, where $2 = \{0, 1\}$ is the two-element Boolean algebra, by identifying χ with the preimage $\chi^{-1}(1) \subseteq B$. An *ultrafilter* on a set X is an ultrafilter on the Boolean algebra $\mathbf{Set}(X, 2) \cong \mathcal{P}X$ of predicates (equivalently subsets) of X. The *ultrafilter monad* $\mathcal{U}$ on **Set** sends a set X to the set of $\mathcal{U}X = \mathbf{BA}(\mathbf{Set}(X, 2), 2)$ of its ultrafilters; more precisely, $\mathcal{U}$ is the monad induced by the adjunction in (4.1) below. The category $\mathbf{EM}(\mathcal{U})$ of algebras for $\mathcal{U}$ is isomorphic to the category of compact Hausdorff spaces and continuous maps [31, 1.5.24–33]; hence the ultrafilter monad provides a bridge between algebra, Boolean logic, and topology. The ultrafilter monad $\mathcal{U}$ is captured by codensity setting (4.1) shown below on the left:

$$\begin{array}{ccc} \mathbf{Set}_{\mathsf{f}} & \overset{I}{\hookrightarrow} & \mathbf{Set} \circlearrowright \mathcal{U} \\ {\scriptstyle \mathbf{BA}(-,2)}\uparrow \wr & & {\scriptstyle \mathbf{Set}(-,2)} \downarrow \dashv \uparrow {\scriptstyle \mathbf{BA}(-,2)} \\ \mathbf{BA}_{\mathsf{f}}^{\mathsf{op}} & \overset{J^{\mathsf{op}}}{\hookrightarrow} & \mathbf{BA}^{\mathsf{op}} \end{array} \quad (4.1) \qquad \begin{array}{ccc} \mathbf{Set}_{\mathsf{f}} & \overset{I}{\hookrightarrow} & \mathbf{Top} \circlearrowright \overline{\mathcal{U}} \\ {\scriptstyle \mathbf{BA}(-,2)}\uparrow \wr & & {\scriptstyle \mathbf{Top}(-,2)} \downarrow \dashv \uparrow {\scriptstyle \mathbf{BA}(-,2)} \\ \mathbf{BA}_{\mathsf{f}}^{\mathsf{op}} & \overset{J^{\mathsf{op}}}{\hookrightarrow} & \mathbf{BA}^{\mathsf{op}} \end{array} \quad (4.2)$$

Here I and J are the inclusions of the full subcategories of finite sets and finite Boolean algebras, respectively, and the equivalence functor on the left is the restriction of *Birkhoff duality* [6] between distributive lattices and finite posets to finite sets (discrete posets). The functor J is dense by Example 2.1.3; note that since the free Boolean algebra 2^{2^X} on a finite set X is finite, we have $\mathbf{BA}_{\mathsf{fp}} = \mathbf{BA}_{\mathsf{f}}$. From Theorem 3.2 we obtain:

▶ **Theorem 4.1** (Kennison and Gildenhuys [27]). *The ultrafilter monad $\mathcal{U}$ on* **Set** *is the codensity monad of the inclusion* $\mathbf{Set}_{\mathsf{f}} \hookrightarrow \mathbf{Set}$.

A similar result was given by Sipoş [38] for the ultrafilter monad $\overline{\mathcal{U}}$ on the category **Top** of topological spaces and continuous maps assigning to a space X the space $\overline{\mathcal{U}}X = \mathbf{BA}(\mathbf{Top}(X, 2), 2)$ of ultrafilters of clopens; that is, $\overline{\mathcal{U}}$ is the monad given by the adjunction in (4.2). Here 2 is the two-element discrete space, $\mathbf{Top}(X, 2) \cong \mathrm{Cl}\, X$ is the Boolean algebra of clopen subsets of a space X, and $\mathbf{BA}(B,2)$ is the space of ultrafilters of a Boolean algebra B, viewed as subspace of the space $2^{|B|}$ equipped with the product topology. Algebras for $\overline{\mathcal{U}}$ correspond to Stone spaces (compact Hausdorff spaces with a basis of clopens) [38].

To capture the topological ultrafilter monad in our setting, we just use **Top** in lieu of **Set** in (4.1), which leads to the codensity setting (4.2). Here I identifies a finite set with a discrete topological space. From Theorem 3.2 we obtain:

▶ **Theorem 4.2** (Sipoș [38]). *The ultrafilter monad* $\overline{\mathcal{U}}$ *on* **Top** *is the codensity monad of the inclusion* $\mathbf{Set}_\mathsf{f} \hookrightarrow \mathbf{Top}$.

4.2 Filter Monads

A natural generalization of ultrafilter monads are *filter monads*. They are captured in our setting as well by working with semilattices in lieu of Boolean algebras.

In more detail, let **MSL** denote the category of meet-semilattices with a top element (equivalently, algebras for the finite power set monad $\mathcal{P}_\mathsf{f}$). A *filter* on a meet-semilattice M is a non-empty subset $F \subseteq M$ that is both upwards closed and closed under meets. A filter corresponds to morphism $\chi \in \mathbf{MSL}(M,2)$, where $2 = \{0,1\}$ is the two-element semilattice with the meet given by minimum, by identifying χ with the preimage $\chi^{-1}(1) \subseteq M$. A *filter* on a set X is a filter on the semilattice $\mathbf{Set}(X,2) \cong \mathcal{P}X$. The *filter monad* $\mathcal{F}$ on **Set** sends a set X to the set of $\mathcal{F}X = \mathbf{MSL}(\mathbf{Set}(X,2),2)$ of its filters; that is, $\mathcal{F}$ is the monad induced by the adjunction in (4.3). Algebras for $\mathcal{F}$ correspond to complete lattices where finite meets distribute over directed joins [9]. Two suitable codensity settings for $\mathcal{F}$ are given below:

$$\begin{array}{ccc} \mathbf{MSL}_\mathsf{f} & \xrightarrow{\;U\;} & \mathbf{Set} \circlearrowleft \mathcal{F} \\ {\scriptstyle \mathbf{MSL}(-,2)}\uparrow\wr & & {\scriptstyle \mathbf{Set}(-,2)}\downarrow \dashv \uparrow{\scriptstyle \mathbf{MSL}(-,2)} \\ \mathbf{MSL}_\mathsf{f}^\mathsf{op} & \xhookrightarrow{\;J^\mathsf{op}\;} & \mathbf{MSL}^\mathsf{op} \end{array} \qquad (4.3) \qquad \begin{array}{ccc} \mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f}) & \xrightarrow{\;U_{\mathcal{P}_\mathsf{f}}\;} & \mathbf{Set} \circlearrowleft \mathcal{F} \\ {\scriptstyle (-)^\mathsf{op}}\uparrow\wr & & {\scriptstyle \mathbf{Set}(-,2)}\downarrow \dashv \uparrow{\scriptstyle \mathbf{MSL}(-,2)} \\ \mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f})^\mathsf{op} & \xhookrightarrow{\;I_{\mathcal{P}_\mathsf{f}}^\mathsf{op}\;} & \mathbf{MSL}^\mathsf{op} \end{array} \qquad (4.4)$$

In setting (4.3), U is the forgetful functor, and J is the dense inclusion (Example 2.1.3), where we use that $\mathbf{EM}(\mathcal{P}_\mathsf{f}) \cong \mathbf{MSL}$ (viewing **MSL** as the algebraic category of commutative idempotent monoids) and that finitely presentable semilattices coincide with the finite ones. Moreover, we make use of the self-duality $\mathbf{MSL}_\mathsf{f}^\mathsf{op} \simeq \mathbf{MSL}_\mathsf{f}$ of finite meet-semilattices [24, Sec. VI.3.6], which maps a finite meet-semilattice M to the semilattice $\mathbf{MSL}(M,2)$ of its filters, with the semilattice structure defined pointwise. This codensity setting restricts to (4.4); to see this, first recall the functors $U_{\mathcal{P}_\mathsf{f}}$ and $I_{\mathcal{P}_\mathsf{f}}$ from Section 2 (p. 3, "Monads") and that $I_{\mathcal{P}_\mathsf{f}}$ is dense (Example 2.1.2). (Note that it maps a set X to $\mathcal{P}_\mathsf{f}X$ equipped with $\cup$ as the binary operation.) Here $\mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f})^\mathsf{op} \simeq \mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f})$ is the standard identity-on-objects self-duality of the category of finite sets and relations, sending $f\colon X \to \mathcal{P}_\mathsf{f}Y$ to $f^\mathsf{op}\colon Y \to \mathcal{P}_\mathsf{f}X$ given by $f^\mathsf{op}(y) = \{x : y \in f(x)\}$. The outer square clearly commutes since both $(f^\mathsf{op})^\# = 2^{f^\#}\colon \mathcal{P}_\mathsf{f}Y \to \mathcal{P}_\mathsf{f}X$ take the preimage of $S \subseteq Y$ under $f^\#$.

From Theorem 3.2 we obtain two novel codensity presentations of the filter monad:

▶ **Theorem 4.3.** *The filter monad* $\mathcal{F}$ *on* **Set** *is the codensity monad of the forgetful functors* $U\colon \mathbf{MSL}_\mathsf{f} \to \mathbf{Set}$ *and* $U_{\mathcal{P}_\mathsf{f}}\colon \mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f}) \to \mathbf{Set}$.

Similar to the ultrafilter monad, this reasoning carries over from **Set** to suitable topological categories. For example, the filter monad $\overline{\mathcal{F}}$ on **Top** is given by $\overline{\mathcal{F}}X = \mathbf{MSL}(\mathbf{Top}(X,\mathbb{S}),2)$, that is, $\overline{\mathcal{F}}$ is induced by the adjunction in the diagram below. Here $\mathbb{S}$ is the *Sierpinski space* carried by $2 = \{0,1\}$ with open sets $\emptyset$, $\{1\}$, $\{0,1\}$, and the set $\mathbf{MSL}(M,2)$ is topologized as a subspace of the product space $\mathbb{S}^{|M|}$. As open subsets of a space X correspond to continuous maps from X to $\mathbb{S}$, the monad $\overline{\mathcal{F}}$ sends X to the space $\overline{\mathcal{F}}X$ of filters of open sets. Algebras for $\overline{\mathcal{F}}$ carried by T_0 spaces correspond to continuous lattices [9]. To capture $\overline{\mathcal{F}}$ as a codensity monad, we modify the codensity setting (4.4) to the one shown below. Here we regard the functor $U_{\mathcal{P}_\mathsf{f}}\colon \mathbf{Kl}_f(\mathcal{P}_\mathsf{f}) \to \mathbf{Set}$ from (4.3) as a functor to $\mathbf{Top}$ by identifying $U_{\mathcal{P}_\mathsf{f}}X = \mathcal{P}_\mathsf{f}X$ with the topological space $\mathbb{S}^X$.

$$\begin{array}{ccc} \mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f}) & \xrightarrow{U_{\mathcal{P}_\mathsf{f}}} & \mathbf{Top} \circlearrowleft \overline{\mathcal{F}} \\ {\scriptstyle (-)^{\mathsf{op}}}\uparrow\wr & & {\scriptstyle \mathbf{Top}(-,\mathbb{S})}\downarrow \dashv \uparrow{\scriptstyle \mathbf{MSL}(-,2)} \\ \mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f})^{\mathsf{op}} & \xhookrightarrow{I^{\mathsf{op}}_{\mathcal{P}_\mathsf{f}}} & \mathbf{MSL}^{\mathsf{op}} \end{array}$$

From Theorem 3.2 we obtain a novel codensity presentation of $\overline{\mathcal{F}}$:

▶ **Theorem 4.4.** *The filter monad* $\overline{\mathcal{F}}$ *on* $\mathbf{Top}$ *is the codensity monad of the functor* $U_{\mathcal{P}_\mathsf{f}}\colon \mathbf{Kl}_\mathsf{f}(\mathcal{P}_\mathsf{f}) \to \mathbf{Top}$ *sending* X *to* $\mathbb{S}^X$*.*

4.3 Double Dualization Monads

Another interesting class of codensity monads that naturally emerge in our framework are certain *double dualization monads*, which are monads given by dual adjunctions as shown in the first diagram below, where $\mathbf{C}$ is a symmetric monoidal closed category, D is a fixed object of $\mathbf{C}$, and $\mathbf{C}(C, D) \in \mathbf{C}$ denotes the internal hom object of $C, D \in \mathbf{C}$.

$$\mathbf{C} \underset{\mathbf{C}(-,D)}{\overset{\mathbf{C}(-,D)}{\rightleftarrows}}_{\top} \mathbf{C}^{\mathsf{op}} \qquad \mathbf{Set} \underset{\mathbf{Set}(-,2)}{\overset{\mathbf{Set}(-,2)}{\rightleftarrows}}_{\top} \mathbf{Set}^{\mathsf{op}} \qquad \mathbf{MSL} \underset{\mathbf{MSL}(-,2)}{\overset{\mathbf{MSL}(-,2)}{\rightleftarrows}}_{\top} \mathbf{MSL}^{\mathsf{op}} \qquad \mathbf{Vect} \underset{\mathbf{Vect}(-,K)}{\overset{\mathbf{Vect}(-,K)}{\rightleftarrows}}_{\top} \mathbf{Vect}^{\mathsf{op}}$$

The other three adjunctions above are concrete instances. Here $\mathbf{Vect}$ is the category of vector spaces over a field K and linear maps, and $X^* = \mathbf{Vect}(X, K)$ is the dual space of a space X with pointwise structure. The induced monads are the *neighbourhood monad* on $\mathbf{Set}$ (whose algebras are complete atomic Boolean algebras [31, 1.5.17–23]), the monad on $\mathbf{MSL}$ sending a semilattice to its semilattice of filters of filters, and the monad sending a vector space X to its double dual space X^{**} (whose algebras are *linearly compact vector spaces* [29, Thm. 7.8]). Suitable codensity settings for these three monads are given as follows:

$$\begin{array}{ccc} \mathbf{BA}_\mathsf{f} & \xrightarrow{U} & \mathbf{Set} \\ \wr\uparrow & & {\scriptstyle \mathbf{Set}(-,2)}\downarrow\dashv\uparrow{\scriptstyle \mathbf{Set}(-,2)} \\ \mathbf{Set}_\mathsf{f}^{\mathsf{op}} & \xhookrightarrow{I^{\mathsf{op}}} & \mathbf{Set}^{\mathsf{op}} \end{array} \qquad \begin{array}{ccc} \mathbf{MSL}_\mathsf{f} & \xhookrightarrow{I} & \mathbf{MSL} \\ \wr\uparrow & & {\scriptstyle \mathbf{MSL}(-,2)}\downarrow\dashv\uparrow{\scriptstyle \mathbf{MSL}(-,2)} \\ \mathbf{MSL}_\mathsf{f}^{\mathsf{op}} & \xhookrightarrow{I^{\mathsf{op}}} & \mathbf{MSL}^{\mathsf{op}} \end{array} \qquad \begin{array}{ccc} \mathbf{Vect}_\mathsf{fd} & \xhookrightarrow{I} & \mathbf{Vect} \\ \wr\uparrow & & {\scriptstyle \mathbf{Vect}(-,K)}\downarrow\dashv\uparrow{\scriptstyle \mathbf{Vect}(-,K)} \\ \mathbf{Vect}_\mathsf{fd}^{\mathsf{op}} & \xhookrightarrow{I^{\mathsf{op}}} & \mathbf{Vect}^{\mathsf{op}} \end{array}$$

Here $U\colon \mathbf{BA}_\mathsf{f} \to \mathbf{Set}$ denotes the forgetful functor, $\mathbf{Vect}_\mathsf{fd}$ is the full subcategory of $\mathbf{Vect}$ given by finite dimensional vector spaces, and the functors I are inclusion functors, which are dense (Example 2.1.3). In the first two diagrams we use the dualities $\mathbf{BA}_\mathsf{f}^{\mathsf{op}} \simeq \mathbf{Set}_\mathsf{f}$ and $\mathbf{MSL}_\mathsf{f}^{\mathsf{op}} \simeq \mathbf{MSL}_\mathsf{f}$ encountered before, and in the third one we use the familiar self-duality of the category of finite dimensional vector spaces, which amounts to the fact that every such space is naturally isomorphic to its double dual. All three diagrams clearly commute.

From Theorem 3.2 we obtain the following codensity presentations; the first appears to be new, the second is due to Adámek and Sousa [3], and the third due to Leinster [29].

▶ **Theorem 4.5.**

1. *The neighbourhood monad on* $\mathbf{Set}$ *is the codensity monad of* $\mathbf{BA}_\mathsf{f} \to \mathbf{Set}$.
2. *The double dualization monad on* $\mathbf{MSL}$ *is the codensity monad of* $\mathbf{MSL}_\mathsf{f} \hookrightarrow \mathbf{MSL}$.
3. *The double dualization monad on* $\mathbf{Vect}$ *is the codensity monad of* $\mathbf{Vect}_\mathsf{fd} \hookrightarrow \mathbf{Vect}$.

The last item readily generalizes to modules over a commutative semiring S, where in lieu of $\mathbf{Vect}_\mathsf{fd}$ one takes finitely generated free S-modules. Note that all vector spaces are free.

For a slightly more intricate example, we consider the category $\mathbf{Ab}$ of abelian groups and as the dualizing object the circle group $S^1 = \{x \in \mathbb{C} \mid |x| = 1\}$, a subgroup of the multiplicative group of non-zero complex numbers. It forms a (compact Hausdorff) topological group w.r.t. the Euclidian topology on $\mathbb{C}$. A suitable codensity setting is given by:

$$\begin{array}{ccc} \{S^1 \times S^1\} & \xrightarrow{\;U\;} & \mathbf{Ab} \\ \cong\uparrow & & \mathbf{Ab}(-,S^1)\left(\dashv\right)\mathbf{Ab}(-,S^1) \\ \{\mathbb{Z}\times\mathbb{Z}\}^\mathsf{op} & \xhookrightarrow{\;J^\mathsf{op}\;} & \mathbf{Ab}^\mathsf{op} \end{array}$$

While all previous examples are based on simple dualities, we now use a corollary of a more advanced duality result: (discrete) *Pontryagin duality* [34, 35], the dual equivalence $\mathbf{Ab}^\mathsf{op} \simeq \mathbf{CHAb}$ between $\mathbf{Ab}$ and the category $\mathbf{CHAb}$ of compact Hausdorff abelian groups and continuous group morphisms. The equivalence sends an abelian group A to the topological group $\mathbf{Ab}(A, S^1)$ of all group morphisms into S^1, topologized as a subspace of $(S^1)^{|A|}$. Under this duality, the additive group $\mathbb{Z}\times\mathbb{Z}$ (the free abelian group on two generators) corresponds to the torus $S^1 \times S^1$, since

$$S^1 \times S^1 \cong \mathbf{Ab}(\mathbb{Z}, S^1) \times \mathbf{Ab}(\mathbb{Z}, S^1) \cong \mathbf{Ab}(\mathbb{Z}\times\mathbb{Z}, S^1).$$

Thus, we see that Pontryagin duality restricts to a dual equivalence between the one-object full subcategories $\{\mathbb{Z}\times\mathbb{Z}\} \hookrightarrow \mathbf{Ab}$ and $\{S^1 \times S^1\} \hookrightarrow \mathbf{CHAb}$. The functor U is the composition of the inclusion $\{S^1 \times S^1\} \hookrightarrow \mathbf{CHAb}$ with the forgetful functor to $\mathbf{Ab}$. The inclusion $J\colon \{\mathbb{Z}\times\mathbb{Z}\} \hookrightarrow \mathbf{Ab}$ is dense (Example 2.1.2), and the outside of the diagram commutes, since Pontryagin duality is given by the functor $\mathbf{Ab}(-, S^1)\colon \mathbf{Ab}^\mathsf{op} \to \mathbf{CHAb}$. From Theorem 3.2 we obtain the following new result:

▶ **Theorem 4.6.** *The double dualization monad on* $\mathbf{Ab}$ *(with respect to the dualizing object* S^1*) is the codensity monad of the above forgetful functor* $U\colon \{S^1 \times S^1\} \to \mathbf{Ab}$.

4.4 Isbell Duality

We conclude this section with an example of a more abstract nature: for every small category $\mathbf{C}$, the codensity monad of the Yoneda embedding $\mathbf{C} \hookrightarrow [\mathbf{C}^\mathsf{op}, \mathbf{Set}]$ is the monad given by *Isbell duality* [19]. This result due to Kock [28] originally motivated the introduction of codensity monads. Recall that Isbell duality is the dual adjunction in (4.5) between the categories of contravariant and covariant presheaves on $\mathbf{C}$ defined by

$$\mathcal{O}(X) = (\, C \mapsto [\mathbf{C}^\mathsf{op}, \mathbf{Set}](X, \mathbf{C}(-, C))\,) \quad \text{and} \quad \mathsf{Spec}(A) = (\, C \mapsto [\mathbf{C}, \mathbf{Set}](A, \mathbf{C}(C, -))\,).$$

The two Yoneda embeddings y and $\tilde{y}$ give rise to the codensity setting shown below:

$$\begin{aligned} y\colon \mathbf{C} &\hookrightarrow [\mathbf{C}^\mathsf{op}, \mathbf{Set}], & C &\mapsto \mathbf{C}(-, C), \\ \tilde{y}\colon \mathbf{C}^\mathsf{op} &\hookrightarrow [\mathbf{C}, \mathbf{Set}], & C &\mapsto \mathbf{C}(C, -). \end{aligned}$$

$$\begin{array}{ccc} \mathbf{C} & \xhookrightarrow{\;y\;} & [\mathbf{C}^\mathsf{op}, \mathbf{Set}] \\ \Vert & & \mathcal{O}\left(\dashv\right)\mathsf{Spec} \\ (\mathbf{C}^\mathsf{op})^\mathsf{op} & \xhookrightarrow{\;(\tilde{y})^\mathsf{op}\;} & [\mathbf{C}, \mathbf{Set}]^\mathsf{op} \end{array} \qquad (4.5)$$

Indeed, the embedding $\tilde{y}$ is dense (Example 2.2), and the outside of the diagram commutes up to isomorphism by the Yoneda lemma: for all objects $C, D \in \mathbf{C}$, we have

$$\mathsf{Spec}(\tilde{y}(D))(C) = [\mathbf{C}, \mathbf{Set}](\mathbf{C}(D,-), \mathbf{C}(C,-)) \cong \mathbf{C}(C,D) = y(D)(C).$$

From Theorem 3.2 we obtain:

▶ **Theorem 4.7** (Kock [28], Di Liberti [10]). *For every small category* $\mathbf{C}$*, the monad on* $[\mathbf{C}^{\mathsf{op}}, \mathbf{Set}]$ *induced by Isbell duality is the codensity monad of* $y \colon \mathbf{C} \hookrightarrow [\mathbf{C}^{\mathsf{op}}, \mathbf{Set}]$.

5 Vietoris Monads

As another important class of codensity monads, we study monads on categories of topological spaces that associate to a given space a suitably topologized space of interesting subsets.

5.1 The Vietoris Monad on Stone Spaces

We start with the Vietoris monad on the category **Stone** of Stone spaces and continuous maps. The *Vietoris space* $\mathbb{V}X$ of a Stone space X is given by the set of all closed subsets of X equipped with the *hit-or-miss topology*, which has the following subbasic open sets:

$$\Diamond U = \{C \subseteq X \text{ closed} \mid C \cap U \neq \emptyset\}, \; \Box U = \{C \subseteq X \text{ closed} \mid C \cap U = \emptyset\}, \text{ for } U \subseteq X \text{ clopen.}$$

We can represent $\mathbb{V}X$ as a double dual space as follows. Let **JSL** be the category of join-semilattices with bottom. Morphisms from J to the join-semilattice $2 = \{0, 1\}$ (with join given by maximum) correspond to ideals: non-empty subsets of J that are downwards closed and closed under join. The set $\mathbf{JSL}(J, 2)$ of ideals is topologized as a subspace of the product space $2^{|J|}$, where 2 carries the discrete topology. We then have the isomorphism $\mathbb{V}X \cong \mathbf{JSL}(\mathbf{Stone}(X, 2), 2)$ which identifies a closed set $C \subseteq X$ with the ideal of all clopen sets $U \subseteq X$ with $C \cap U = \emptyset$ (i.e. $C \in \Box U$). The object mapping $X \mapsto \mathbb{V}X$ thus naturally extends to a monad, for which the appropriate codensity setting is given by (5.1):

$$\begin{array}{ccc} \mathbf{Kl}_{\mathsf{f}}(\mathcal{P}_{\mathsf{f}}) & \xrightarrow{U_{\mathcal{P}_{\mathsf{f}}}} & \mathbf{Stone} \circlearrowleft \mathbb{V} \\ {\scriptstyle (-)^{\mathsf{op}}} \uparrow \wr & & {\scriptstyle \mathbf{Stone}(-,2)} \downarrow \dashv \uparrow {\scriptstyle \mathbf{JSL}(-,2)} \\ \mathbf{Kl}_{\mathsf{f}}(\mathcal{P}_{\mathsf{f}})^{\mathsf{op}} & \xhookrightarrow{I^{\mathsf{op}}_{\mathcal{P}_{\mathsf{f}}}} & \mathbf{JSL}^{\mathsf{op}} \end{array} \qquad (5.1)$$

$$\begin{array}{ccc} \mathbf{Kl}(\mathcal{P}) & \xrightarrow{U_{\mathcal{P}}} & \mathbf{Top} \circlearrowleft \mathbb{V}_{\downarrow} \\ {\scriptstyle (-)^{\mathsf{op}}} \uparrow \wr & & {\scriptstyle \mathbf{Top}(-,\mathbb{S})} \downarrow \dashv \uparrow {\scriptstyle \mathbf{JSL}(-,2)} \\ \mathbf{Kl}(\mathcal{P})^{\mathsf{op}} & \xhookrightarrow{I^{\mathsf{op}}_{\mathcal{P}}} & \mathbf{CJSL}^{\mathsf{op}} \end{array} \qquad (5.2)$$

Here $U_{\mathcal{P}_{\mathsf{f}}} \colon \mathbf{Kl}_f(\mathcal{P}_{\mathsf{f}}) \to \mathbf{Set}$ from (4.4) is lifted to **Stone**, identifying $U_{\mathcal{P}_{\mathsf{f}}}X = \mathcal{P}_{\mathsf{f}}X$ with the discrete space 2^X. The functor $I_{\mathcal{P}_{\mathsf{f}}}$ is the dense inclusion (Example 2.1.2), using $\mathbf{EM}(\mathcal{P}_{\mathsf{f}}) \cong \mathbf{JSL}$. Commutativity of (5.1) is shown like for (4.4). From Theorem 3.2 we obtain:

▶ **Theorem 5.1** (Gehrke, Petrișan, Reggio [15]). *The Vietoris monad* $\mathbb{V}$ *on* **Stone** *is the codensity monad of the functor* $U_{\mathcal{P}_{\mathsf{f}}} \colon \mathbf{Kl}_{\mathsf{f}}(\mathcal{P}_{\mathsf{f}}) \to \mathbf{Stone}$.

5.2 The Lower Vietoris Monad on Topological Spaces

By varying the ingredients of the above setting a bit, codensity presentations for other Vietoris-type monads emerge easily. Here we consider the *lower Vietoris monad*, a.k.a. *Hoare hyperspace monad* [12], on the category **Top** of topological spaces. Given a topological space X, the *lower Vietoris space* is the topological space $\mathbb{V}_{\downarrow}X$ of all closed subsets of X

with the topology generated by the subbasic open sets $\Diamond U$ (as defined above) for $U \subseteq X$ *open*. Similar to $\mathbb{V}X$, we can represent $\mathbb{V}_{\downarrow}X$ as a double dual space. Let **CJSL** be the category of complete join-semilattices and join-preserving maps (i.e. algebras for the power set monad $\mathcal{P}$). Morphisms in **CJSL** from J to $2 = \{0,1\}$ correspond to complete ideals: subsets of J that are downwards closed and closed under arbitrary joins. The set $\mathbf{CJSL}(J,2)$ of all complete ideals is topologized as a subspace of the product space $\mathbb{S}^{|J|}$. We then have $\mathbb{V}_{\downarrow}X \cong \mathbf{CJSL}(\mathbf{Top}(X,\mathbb{S}),2)$, where the isomorphism identifies a closed set $C \subseteq X$ with the complete ideal of all open sets $U \subseteq X$ with $C \cap U = \emptyset$. Thus, the object mapping $X \mapsto \mathbb{V}_{\downarrow}X$ extends to a monad, whose codensity setting is given by (5.2). Note that this setup is very similar to the filter monad on **Top** from Section 4.2, except that we now work with complete join-semilattices in lieu of (finitary) meet-semilattices and with the full Kleisli category for the monad $\mathcal{P}$ in lieu of its finite restriction. The canonical functor $U_{\mathcal{P}} \colon \mathbf{Kl}(\mathcal{P}) \to \mathbf{Set}$ is taken as a functor to **Top** by identifying $U_{\mathcal{P}_\mathsf{f}}X = \mathcal{P}_\mathsf{f}X$ with the topological space $\mathbb{S}^X$, and $I_{\mathcal{P}}$ is the dense inclusion (Example 2.1.1), using that $\mathbf{EM}(\mathcal{P}) \cong \mathbf{CJSL}$. The outside commutes analogously to (5.1). From Theorem 3.2 we obtain the following new result:

▶ **Theorem 5.2.** *The lower Vietoris monad $\mathbb{V}_{\downarrow}$ on* **Top** *is the codensity monad of $U_{\mathcal{P}}$.*

5.3 The Sobrification Monad on Topological Spaces

Just like Stone spaces correspond to algebras for the filter monad on **Top** (Section 4.2), the well-studied subcategory of *sober spaces* [20, 24] is captured by the *sobrification monad*, introduced by Sipoş [38]. It emerges in our framework by working with frames in lieu of meet-semilattices. Recall that a *frame* is a poset X which has all finite meets and all joins and which satisfies the infinite distributive law $x \wedge \bigvee_{i\in I} x_i = \bigvee_{i \in I} x \wedge x_i$ for all $x, x_i \in X$, $i \in I$. We denote by **Frm** the category of frames and maps preserving finite meets and all joins. It is isomorphic to the category of algebras $\mathbf{EM}(\mathcal{L})$ for the *free frame monad* $\mathcal{L}$ on **Set** sending a set X to the set $\mathcal{L}X$ of upwards closed subsets of $\mathcal{P}_\mathsf{f}X$ [25, C1.1, Lem. 1.1.3].

The *sobrification* monad on **Top** is given by $\mathcal{S}X = \mathbf{Frm}(\mathbf{Top}(X,\mathbb{S}),2)$; more precisely, $\mathcal{S}$ is the monad induced by the dual adjunction in (5.3). Sipoş [38] has shown that algebras for $\mathcal{S}$ correspond to sober spaces and has given a codensity presentation of this monad, which in our duality-based framework corresponds to the codensity setting given by:

$$\begin{array}{ccc} \mathbf{Sier} & \xhookrightarrow{\;J\;} & \mathbf{Top} \circlearrowleft \mathcal{S} \\ {\scriptstyle E}\uparrow\wr & & {\scriptstyle \mathbf{Top}(-,\mathbb{S})}\downarrow \dashv \uparrow{\scriptstyle \mathbf{Frm}(-,2)} \\ \mathbf{Kl}(\mathcal{L})^{\mathsf{op}} & \xhookrightarrow{\;I_{\mathcal{L}}^{\mathsf{op}}\;} & \mathbf{Frm}^{\mathsf{op}} \end{array} \tag{5.3}$$

Here J is the inclusion of the full subcategory **Sier** of **Top** consisting of powers $\mathbb{S}^X$, where X is any set, and $I_{\mathcal{L}}$ is the canonical dense inclusion functor from Example 2.1.1. The equivalence functor E and commutation of the diagram are given by the following lemma.

▶ **Lemma 5.3.** *The object mapping $X \mapsto \mathbb{S}^X \cong \mathbf{Frm}(\mathcal{L}X, 2)$ defines a dual equivalence E between the Kleisli category $\mathbf{Kl}(\mathcal{L})$ and the full subcategory* **Sier** $\hookrightarrow$ **Top**.

From Theorem 3.2 we obtain:

▶ **Theorem 5.4** (Sipoş [38]). *The sobrification monad $\mathcal{S}$ on* **Top** *is the codensity monad of the embedding* **Sier** $\hookrightarrow$ **Top**.

6 Probability Monads

Finally, we investigate monads of interest in probabilistic and quantum computation, and give a uniform treatment of their codensity presentations based on restrictions of the Kleisli category of the (countable) distribution monad as suggested in the work of Shirazi [37]. As noted by van Belle [42], integral representation theorems play a key role in such presentations, and we show that in order to apply our framework, they are in fact the only non-trivial part.

6.1 The Expectation Monad

We start with the *expectation monad* [23, 44], a probability monad that has recieved a lot of attention in recent years, e.g. in quantum foundations. A *finitely additive probability measure* on a set X is a finitely additive probability measure on the discrete measurable space $(X, \Sigma_X = \mathcal{P}X)$, that is, a map $p\colon \mathcal{P}X \to [0,1]$ with $p(X) = 1$ and $p(A + B) = p(A) + p(B)$ for disjoint $A, B \in \mathcal{P}X$. If X is finite, then every finitely additive probability measure on X is *discrete*: recall that a *discrete finite probability measure* on X is a map $p\colon X \to [0,1]$ with $\sum_{x\in X} p(x) = 1$ and $p(x) = 0$ for all but finitely many $x \in X$. We denote the set of finitely additive probability measures on a set X by $\mathcal{E}X$, and the set of discrete finite probability measures by $\mathcal{D}_\mathsf{f}X$. For the codensity presentation of the expectation monad we embed these notions into an algebraic setting given by *effect algebras* and *effect modules* [17, 22].

An *effect algebra* is a partial commutative monoid $(A, \oplus, 0)$ together with an operator $(-)^\perp\colon A \to A$ satisfying that $x^\perp$ is the unique element in A with $x \oplus x^\perp = 1$, where $1 = 0^\perp$, and $x \oplus 1$ is defined iff $x = 0$. Morphisms of effect algebras preserve all this structure, forming a category $\mathbf{EA}$. The unit interval $[0,1]$ forms an effect algebra with $r \oplus s = r + s$ defined if $r + s \le 1$, and $r^\perp = 1 - r$. Boolean algebras form a full subcategory of effect algebras, with $a \oplus b = a \vee b$ defined whenever $a \wedge b = 0$, and $a^\perp = \neg a$. The following non-trivial density result, which is not covered by Example 2.1, is due to Staton and Uijlen [39, Cor. 10]:

▶ **Proposition 6.1.** *The inclusion* $J\colon \mathbf{BA}_\mathsf{f} \hookrightarrow \mathbf{EA}$ *is dense.*

Effect algebras carry a monoidal structure $\otimes$ such that $A \otimes B$ represents bilinear morphisms [22], analogous to commutative monoids, with tensor unit $2 = \{0, 1\}$. The unit interval $[0,1]$ with multiplication forms a monoid $[0,1] \otimes [0,1] \to [0,1]$ for this monoidal structure.

An *effect module* is an effect algebra E that is a $[0,1]$-module for this tensor product, that is, E is equipped with a bilinear map $[0,1] \times E \to E$ of effect algebras. Effect module morphisms are effect algebra morphisms preserving the action, forming a category $\mathbf{EMod}$. Its forgetful functor to $\mathbf{EA}$ has a left adjoint $A \mapsto [0,1] \otimes A$. This adjunction yields a monad $\mathcal{M}$ on $\mathbf{EA}$ given by $\mathcal{M}A = [0,1] \otimes A$. On finite Boolean algebras we have $\mathcal{M}(2^X) \cong [0,1]^X$. We extend the density result from Proposition 6.1 to effect modules, where we denote the full subcategory of $\mathbf{Kl}(\mathcal{M})$ on finite Boolean algebras by $\mathbf{Kl}_{\mathbf{BA}_\mathsf{f}}(\mathcal{M})$:

▶ **Proposition 6.2.** *The inclusion* $I_\mathcal{M}\colon \mathbf{Kl}_{\mathbf{BA}_\mathsf{f}}(\mathcal{M}) \hookrightarrow \mathbf{EMod}$*,* $2^X \mapsto [0,1]^X$*, is dense.*

Finitely additive probability measures on a set X correspond to effect algebra morphisms from 2^X to $[0,1]$, that is, $\mathcal{E}X \cong \mathbf{EA}(2^X, [0,1])$. They also correspond to effect *module* morphisms from $[0,1]^X$ to $[0,1]$, which is the content of the following discrete integral representation theorem [23, Prop. 33]:

▶ **Theorem 6.3.** *For every set* X*, there is a natural bijection*

$$\mathbf{EMod}([0,1]^X, [0,1]) \cong \mathbf{EA}(2^X, [0,1]).$$

The bijection sends $f \in \mathbf{EMod}([0,1]^X, [0,1])$ to the measure p with $p(A) = f(\chi_A)$, where χ_A is the characteristic map of $A \subseteq X$ given by $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$.

We thus define the *expectation monad* $\mathcal{E}$ on $\mathbf{Set}$ by $\mathcal{E}X = \mathbf{EMod}([0,1]^X, [0,1])$; more precisely, $\mathcal{E}$ is given by the adjunction in the codensity setting below:

$$\begin{array}{ccc} \mathbf{Kl}_{\mathsf{f}}(\mathcal{D}_{\mathsf{f}}) & \xrightarrow{U_{\mathcal{D}_{\mathsf{f}}}} & \mathbf{Set} \circlearrowright \mathcal{E} \\ {\scriptstyle \mathbf{Kl}_{\mathbf{BA}_{\mathsf{f}}}(\mathcal{M})(-,2)} \uparrow \wr & & {\scriptstyle \mathbf{Set}(-,[0,1])} \downarrow \dashv \uparrow {\scriptstyle \mathbf{EMod}(-,[0,1])} \\ \mathbf{Kl}_{\mathbf{BA}_{\mathsf{f}}}(\mathcal{M})^{\mathsf{op}} & \xhookrightarrow{I_{\mathcal{M}}^{\mathsf{op}}} & \mathbf{EMod}^{\mathsf{op}} \end{array}$$

The inclusion $I_{\mathcal{M}}$ is dense by Proposition 6.2 and the outer square commutes since $\mathcal{M}2 = [0,1] \otimes 2^1 \cong [0,1]$. The duality of the left extends the duality $\mathbf{BA}_{\mathsf{f}}^{\mathsf{op}} \simeq \mathbf{Set}_{\mathsf{f}}$. Indeed, for $A, B \in \mathbf{Set}_{\mathsf{f}}$ we have the following natural bijection:

$$\mathbf{Kl}_{\mathsf{f}}(\mathcal{D}_{\mathsf{f}})(A,B) = \mathbf{Set}(A, \mathcal{D}_{\mathsf{f}}B) \cong \mathbf{EA}(2^B, [0,1]^A) \cong \mathbf{Kl}_{\mathbf{BA}_{\mathsf{f}}}(\mathcal{M})(2^B, 2^A).$$

From Theorem 3.2 we obtain a novel codensity presentation of $\mathcal{E}$:

▶ **Theorem 6.4.** *The expectation monad $\mathcal{E}$ is the codensity monad of $U_{\mathcal{D}_{\mathsf{f}}} \colon \mathbf{Kl}_{\mathsf{f}}(\mathcal{D}_{\mathsf{f}}) \to \mathbf{Set}$.*

▶ **Remark 6.5.** The expectation *functor* $\mathcal{E}X \cong \mathbf{EA}(2^X, [0,1])$ has a much simpler presentation: One can show, by (co)limit manipulation very similar to our proof of Theorem 3.2, that $\mathcal{E}$ is the right Kan extension of $\mathcal{D}_{\mathsf{f}} \colon \mathbf{Set}_{\mathsf{f}} \to \mathbf{Set}$ along $I \colon \mathbf{Set}_{\mathsf{f}} \hookrightarrow \mathbf{Set}$. This only requires density of $\mathbf{BA}_{\mathsf{f}} \hookrightarrow \mathbf{EA}$ (Proposition 6.1), not the representation result for $\mathcal{E}$ (Theorem 6.3).

6.2 The Giry Monad

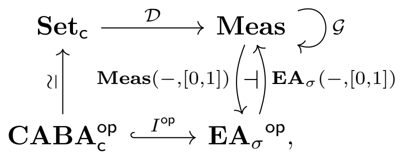
To capture general probability theory, we move from finitely additive to countably additive probability measures. A *probability measure* on a measurable space (X, Σ_X) is a map $p \colon \Sigma_X \to [0,1]$ such that $p(X) = 1$ and $p(\sum_i A_i) = \sum_i p(A_i)$ for every countable family $(A_i)_i$ of pairwise disjoint sets. We denote the set of probability measures on X by $\mathcal{G}X$. The assignment $X \mapsto \mathcal{G}X$ forms a monad on the category $\mathbf{Meas}$ of measurable spaces and measurable maps called the *Giry monad* [16]. Here, $\mathcal{G}X$ is equipped with the Σ-subalgebra of the power $[0,1]^{\Sigma_X}$ of $[0,1]$ in $\mathbf{Meas}$, where $[0,1]$ carries the usual Borel σ-algebra. On a countable discrete space $(X, \Sigma_X = 2^X)$, a probability measure corresponds to a map $p \colon X \to [0,1]$ with $\sum_x p(x) = 1$. We denote the set of all such discrete probability measures by $\mathcal{D}X$. If we equip $\mathcal{D}X$ with the Σ-subalgebra of $[0,1]^X$, then $\mathcal{D} \colon \mathbf{Set}_{\mathsf{c}} \to \mathbf{Meas}$ forms a functor, where $\mathbf{Set}_{\mathsf{c}} \hookrightarrow \mathbf{Set}$ denotes the full subcategory of countable sets.

Again, these constructions can be expressed in the language of effect algebras, though we have to use a countably infinitary sum operation. A *σ-effect algebra* [4] is an effect algebra A with a partial commutative associative operation $\oplus_{n<\omega} \colon A^\omega \rightharpoonup A$ that behaves as expected with unit 0 and addition $\oplus$ of A (here commutativity means that the operation is independent of permutation of the arguments). Morphisms of σ-effect algebras preserve all this structure, forming a category $\mathbf{EA}_\sigma$. It contains the category of σ-algebras and the category of countably complete Boolean algebras as subcategories. The familiar dual equivalence between $\mathbf{Set}$ and the category $\mathbf{CABA}$ of complete atomic Boolean algebras [24, Sec. VI.4.6] restricts to countable sets, yielding a dual equivalence between $\mathbf{Set}_{\mathsf{c}}$ and the category $\mathbf{CABA}_{\mathsf{c}}$ of countably complete Boolean algebras of the form 2^A for a countable set A.

The codensity presentation of $\mathcal{G}$ again rests on a representation theorem [42, Cor. 3.8]:

▶ **Theorem 6.6.** *For every $X \in \mathbf{Meas}$ we have $\mathcal{G}X \cong \mathbf{EA}_\sigma(\mathbf{Meas}(X,[0,1]),[0,1])$.*

The isomorphism is analogous to that of Theorem 6.3. This means that the Giry monad is given by the adjunction in the diagram below, which also establishes a codensity setting:



Here I is the inclusion, and we have:

▶ **Lemma 6.7.** *The inclusion* $I\colon \mathbf{CABA}_{\mathsf{c}} \hookrightarrow \mathbf{EA}_\sigma$ *is dense.*

From Theorem 3.2 we obtain the codensity presentation of $\mathcal{G}$ due to van Belle [42].

▶ **Theorem 6.8** (van Belle [42])**.** *The Giry monad* $\mathcal{G}$ *on measurable spaces is the codensity monad of* $\mathcal{D}\colon \mathbf{Set}_{\mathsf{c}} \to \mathbf{Meas}$.

Let us mention that in much the same way, Theorem 3.2 yields codensity presentations of
1. the Giry monad by the forgetful functor $U_{\mathcal{D}}\colon \mathbf{Kl}_{\mathsf{c}}(\mathcal{D}) \to \mathbf{Meas}$, analogous to Theorem 6.4;
2. the *countable expectation monad* $\mathcal{E}_\sigma X = \mathbf{EMod}_\sigma([0,1]^X, [0,1])$ on $\mathbf{Set}$ by the functor $\mathcal{D}\colon \mathbf{Set}_{\mathsf{c}} \to \mathbf{Set}$, analogous to Theorem 6.8.

Here $\mathbf{EMod}_\sigma$ is the category of *σ-effect modules* [4], the infinitary version of effect modules. Both results rest on the fact that every $\mathbf{EA}_\sigma$-morphism of type $[0,1]^X \to [0,1]$ is an $\mathbf{EMod}_\sigma$-morphism, i.e. linear [42, Lem. A.3]. More details are given in the full arXiv version.

▶ **Remark 6.9.** As already noted in [42, Rem. 4.3], the Giry *functor* $\mathcal{G}$ can also be presented as the right Kan extension of $\mathcal{D}\colon \mathbf{Set}_{\mathsf{c}} \to \mathbf{Meas}$ along the inclusion $I\colon \mathbf{Set}_{\mathsf{c}} \hookrightarrow \mathbf{Meas}$ equipping a countable set with the discrete Σ-algebra. Similar to the case of $\mathcal{E}$ (Remark 6.5), this follows via an easy computation as in the proof of Theorem 3.2, and does not need an integral representation theorem.

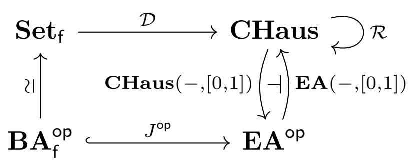
6.3 The Radon Monad and the Kantorovich Monad

Variants of the Giry monad on subcategories of measurable spaces are treated similarly in our framework. Here we consider the *Radon monad* [13], which captures an important subclass of probability measures. A *Radon* probability measure on a compact Hausdorff space X is a probability measure μ on Bo_X (the Borel σ-algebra generated by the open sets of X) satisfying the condition $\mu(A) = \sup\{\, \mu(K) \mid K \subseteq A \text{ and } K \text{ compact} \,\}$ for all $A \in \mathsf{Bo}_X$. The set of all Radon probability measures on X is denoted by $\mathcal{R}X$; it forms a compact Hausdorff space when topologized as a subspace of the product space $[0,1]^{\mathsf{Bo}_X}$ in $\mathbf{CHaus}$.

Similar to Theorem 6.6, Radon measures come with a representation theorem, which yields a particularly simple codensity presentation. It is a variant of the Riesz-Markov representation theorem and proved via the Daniell-Stone representation theorem in [42]:

▶ **Theorem 6.10.** *For every* $X \in \mathbf{CHaus}$*, we have* $\mathcal{R}X \cong \mathbf{EA}(\mathbf{CHaus}(X,[0,1]),[0,1])$.

This representation theorem reduces the Radon monad to the codensity setting shown below:



Here $\mathcal{D}X$ is topologized as a subspace of the hypercube $[0,1]^X$, and J is the inclusion functor, which is dense by Proposition 6.1. From Theorem 3.2 we obtain the following codensity presentation of the Radon monad, which is due to van Belle [42]:

▶ **Theorem 6.11** (van Belle [42]). *The Radon monad on compact Hausdorff spaces is the codensity monad of* $\mathcal{D}\colon \mathbf{Set}_\mathsf{f} \to \mathbf{CHaus}$.

▶ **Remark 6.12.** The Radon monad restricts to the *Kantorovich* [43] or *bounded Lipschitz monad* [42] on the category **CMet** of compact metric spaces and nonexpansive maps by equipping it with the *Kantorovich metric*. Similarly, the above functors $\mathcal{D}$ and $\mathbf{EA}(-,[0,1])$ corestrict to **CMet** by equipping them with the Kantorovich and supremum metric, respectively. Replacing in the above codensity setting **CHaus** by **CMet** and the functors by their metric counterparts yields the codensity presentation of the Kantorovich monad by $\mathcal{D}$ [42].

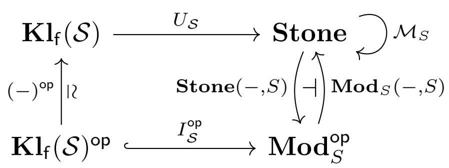
6.4 Finitely Additive Measures in a Finite Semiring

As our last example, we depart from classical probability measures and generalize to measures valued in a semiring [36]. Fix a finite semiring S, viewed as a discrete topological space. An *S-valued measure* on a Stone space X is a map $p\colon \mathrm{Cl}\,X \to S$ satisfying $p(\emptyset) = 0$ and $p(A+B) = p(A)+p(B)$ for disjoint clopens A, B. Let $\mathcal{M}_S X$ denote the set of such measures.

The set $\mathcal{M}_S X \subseteq S^{\mathrm{Cl}\,X}$ is a closed subspace [36, Lem. 3.4] and hence a Stone space. The integral representation theorem in this case is particularly simple: Recall that an *S-module* is an abelian group A with an action $S \times A \to A$ satisfying the usual compatibility conditions with respect to the operations of S and A. For example, the set $\mathbf{Stone}(X,S)$ is an S-module under pointwise operations. A morphism of S-modules is a morphism of the underlying abelian groups commuting with the action, giving a category $\mathbf{Mod}_S$. It is isomorphic to the category of algebras for the free-semimodule monad $\mathcal{S}$ on **Set**, where $\mathcal{S}X = \{f\colon X \to S \mid f(x) = 0 \text{ for all but finitely many } x\}$. Given $X \in \mathbf{Stone}$ we topologize $\mathbf{Mod}_S(\mathbf{Stone}(X,S),S)$ as a subspace of $S^{\mathbf{Stone}(X,S)}$. Then the following is easy to see:

▶ **Lemma 6.13.** *For every Stone space X we have* $\mathcal{M}_S X \cong \mathbf{Mod}_S(\mathbf{Stone}(X,S),S)$.

This implies that a suitable codensity setting for $\mathcal{M}_S$ is given by the diagram below:



The inclusion $I_\mathcal{S}$ is dense by Example 2.1.2, using that $\mathbf{EM}(\mathcal{S}) \cong \mathbf{Mod}_S$, and the outer square commutes similar to (4.4). From Theorem 3.2 we obtain the following result:

▶ **Theorem 6.14** (Reggio [36]). *The monad $\mathcal{M}_S$ on* **Stone** *is the codensity monad of the functor* $U_\mathcal{S}\colon \mathbf{Kl}_\mathsf{f}(\mathcal{S}) \to \mathbf{Stone}$.

7 Conclusion and Future Work

We have introduced a general, unifying approach to codensity presentations of monads, based on the simple core principle of relating codensity to density via duality. We have shown that numerous known and new codensity presentations for important monads, e.g. (ultra)filter, Vietoris, and probability monads, emerge as instances of our framework, in many cases almost for free.

One interesting direction for future work is developing our Theorem 3.2 in the formal theory of monads (i.e. monads in 2-categories) and generalize it to *monad extensions* [40], also known as *pushforward monads* [32]. The latter yield a general method to extend a monad along a 1-cell; codensity monads correspond to extensions of the identity monad.

A previous approach towards a general understanding of codensity presentations is due to Adámek and Sousa [3]. These authors introduce a notion of ultrafilter monad on a general symmetric monoidal closed category that is locally finitely presentable, and prove that this monad is the codensity monad of the full subcategory of finite presentable objects provided that the category has a nice cogenerator. Most of their concrete instances of ultrafilter monads are easily captured by our duality framework (along the lines of Section 4.1); whether their general presentation result is an instance of our Theorem 3.2 is an open problem.

Finally, on the more applied side, we aim to use our framework to study codensity presentations of additional monads. We are particularly interested in variants of the Vietoris monad related to compact spaces and probabilistic powerdomains [12]. Here, the choice of the corresponding dual adjunction (and even of the dualizing objects) is far from obvious.

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