


Refining the Complexity Landscape of Speed Scaling: Hardness and Algorithms

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Abstract

We study the computational complexity of scheduling jobs on a single speed-scalable processor with the objective of capturing the trade-off between the (weighted) flow time and the energy consumption. This trade-off has been extensively explored in the literature through a number of problem formulations that differ in the specific job characteristics and the precise objective function. Nevertheless, the computational complexity of four important problem variants has remained unresolved and was explicitly identified as an open question in prior work (see [12]). In this paper, we settle the complexity of these variants.

More specifically, we prove that the problem of minimizing the objective of total (weighted) flow time plus energy is NP-hard for the cases of (i) unit-weight jobs with arbitrary sizes, and (ii) arbitrary-weight jobs with unit sizes. These results extend to the objective of minimizing the total (weighted) flow time subject to an energy budget and hold even when the schedule is required to adhere to a given priority ordering.

In contrast, we show that when a completion-time ordering is provided, the same problem variants become polynomial-time solvable. The latter result highlights the subtle differences between priority and completion orderings for the problem.

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1 Introduction

Energy is a fundamental and scarce resource that powers every aspect of society. Data centers are among the largest energy consumers globally [1]. As the demand for digital services continues to grow, so does the need for more sustainable computing, which in turn calls for the integration of energy efficiency considerations into the design and analysis of algorithms, alongside classical computational resources such as time and space. Within this context, energy management techniques play a central role in optimizing the trade-off between energy consumption and quality of service. One of the most extensively studied such techniques in the algorithmic literature is *speed scaling*. Speed scaling grants the operating system the



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ability to dynamically adapt the processor speed, allowing higher speeds when performance is critical at the expense of a higher energy consumption, and lower speeds when energy savings are prioritized, for example during off-peak times.

To formalize this trade-off, consider the following underlying scheduling problem: Given a set of n preemptable jobs, each job j is associated with a *release time* r_j denoting the earliest time the job can begin being processed, a *processing volume* v_j denoting the number of CPU cycles required for processing the job, and a *weight* w_j denoting the relative importance of the job. The jobs are to be processed on a speed-scalable processor and the power-consumption corresponding to each allowable speed is given by a power function P . The set of different allowable speeds can be discrete or continuous (in the latter case, P is usually considered to be a continuous convex function of the speed). When the processor runs at a *speed* s , it can process s units of volume per unit of time. One can think of the speed as the CPU frequency. Naturally, the energy consumption of the processor is defined by the integral over time of $P(s(t))$, where $s(t)$ denotes the speed of the processor at time t . A schedule determines which job is processed at what speed at each time t , so that each job j is fully processed after its respective release time r_j . As a QoS objective, we consider the total (weighted) *flow time* (also known as *response time*) of a schedule, that is, the (weighted) sum over all jobs of the difference between the time their processing is completed and the time they were released. In other words, the total (weighted) flow time of a schedule is the total accumulated time that jobs spend in the system. We are interested in the tradeoff between the energy consumption and the total (weighted) flow time, which we capture by considering as an objective function the sum of energy and total (weighted) flow time.¹

We adopt the quintuple notation $\star\star\star\star$ from [12] to describe different variants of the problem related to this work: The first entry denotes whether the objective is the total (weighted) *Flow* plus *Energy* or the total (weighted) flow time subject to an energy *Budget* (think, e.g., of a battery-powered device). The second entry distinguishes whether the flow is *Integral* or *Fractional*², the third whether the speed is *Continuous* or *Discrete*, the fourth whether jobs are *Weighted* or *Unweighted* (i.e., all weights are one), and the fifth and final entry whether the sizes are *Arbitrary* or *Unit*.

The combination of speed scaling with a flow time objective was first explored by Pruhs et al. [19] who presented a polynomial-time algorithm for the B-ICUU variant of the problem. The combined objective, which is also the focus of this work, was introduced by Albers and Fujiwara [3]. They studied the unit-size job setting and proposed a polynomial-time algorithm based on dynamic programming for FE-ICUU, which can also be extended to the energy-budget variant. With respect to the fractional flow objective, Antoniadis et al. [5] showed that FE-FDWA is solvable in polynomial time by an incremental algorithm. On the other hand, the variants \star -I \star WA are known to be NP-hard already in the fixed-speed setting [16], while Megow and Verschae [18] show the NP-hardness of B-IDWA. Finally, Barcelo et al. [12] settle the computational complexity of variants B-IDUA, B-IDWU, FE-IDUU, and FE-FCWA: The first two are NP-hard whereas the latter two are solvable in polynomial time. Four variants remained open: FE-IDUA, FE-ICUA, FE-IDWU, FE-ICWU (see also [12] and the open problem discussion in [11]). Our first main contribution is to resolve those cases by proving that all four variants are NP-hard.

¹ Note that the desired trade-off between energy and flow can be represented by scaling the power function P accordingly.

² In this work we only consider the, more common, integral flow – but keep the tuple entry nevertheless for reasons of consistency.

Our NP-hardness results further suggest that simultaneously optimizing the speed(s) at which each job is processed and the order at which jobs are processed is inherently difficult. For the unweighted variants, if the average speed at which each job is processed is given, then an optimal schedule is easily constructed by prioritizing the jobs according to the shortest remaining processing time rule (SRPT). However, we show that determining the optimal speeds is NP-hard even when given a priority ordering – for most variants of the problem even if this priority ordering corresponds to an optimal schedule. In contrast, as our second main contribution, we also show that a completion ordering is sufficient for efficiently computing optimal speed schedules. We note that such a result for FE-IDUA was claimed before by Barcelo et al. [12] as an extension of an algorithm for FE-IDUU, which was only analyzed for the case of two speeds. They claim that, for this 2-speed algorithm, “it is straightforward to generalize it to k speeds”, but we provide counterexamples for the two, in our opinion, most natural generalizations (see Section 5 for more details). Our algorithm on the other hand uses a different, LP-based approach. Furthermore, our results can be extended to the budget version of each variant as well.

Our contribution

Our first main result is showing in Section 3 that FE-IDUA and FE-IDWU are NP-hard.

► **Theorem 1.** *FE-IDUA is NP-hard.*

► **Theorem 2.** *FE-IDWU is NP-hard.*

The proof of Theorem 1 consists of a reduction from the budget variant of the problem B-IDUA with two available speeds, which is known to be NP-hard [12]. Via an intricate construction, we are able to simulate an energy-budget constraint within the flow-plus-energy setting for any such two-speed B-IDUA instance. At a high level, the idea is to create an additional job of comparatively large volume along with a substantial number of smaller jobs which are released significantly after all other jobs. In an optimal schedule, these latter small jobs must be processed immediately upon release so as to keep the total flow time low. This forces the large-volume job to complete before the release of the smaller jobs, and in turn, a specific amount of volume from the original B-IDUA instance needs to be run at speed s_2 thereby simulating the original budget constraint. Our reduction gives further insight into the relationship between the corresponding budget and flow-energy trade-off variants of the problem.

Note that the above reduction crucially depends on a job of comparatively large volume, something that the FE-IDWU variant does not allow for. As a consequence, the proof of Theorem 2 requires a more technically involved argument. The reduction in this case is directly from SUBSETSUM: instead of introducing a large-volume job, we introduce a large number of low-weight unit-size jobs. However, this causes the challenge that the completion-time ordering of an optimal schedule becomes difficult to predict: small changes in speed might affect the order significantly. We overcome this challenge by exploiting the gap between YES and NO instances in the original proof of the NP hardness of B-IDWU in [12]: By carefully setting the weight of these additional jobs, we are able to simulate an energy-budget constraint while at the same time not affecting the total (weighted) flow of the original jobs by too much.

As a corollary, both NP-hardness results can be extended to the respective continuous-speed problems. Therefore, Theorems 1 and 2 resolve the complexity of the last four unresolved variants and complete the computational complexity landscape of flow-plus-energy and flow under an energy-budget problems.

For each of the considered problem variants, any algorithm must address two decisions for each interval of time: determine which job to run and select a speed at which to run it. We dive deeper into the complexity of the problem by exploring how the availability of different types of information regarding which job to run affects the computational complexity of the problem. We note that given an optimal speed profile (or equivalently, the average speed of each job under an optimal schedule), two types of such information are sufficient to efficiently compute an optimal schedule: a *priority ordering* (specifying which job among the available ones should be processed) and a *completion-time ordering* (specifying the exact order in which the jobs should complete).

We show that our NP-hardness reductions can be extended to show that all considered problem variants remain NP-hard, even if schedules are required to adhere to a given priority ordering. We discuss in more detail and prove these extensions in Section 3.3.

At first glance, being given a completion-time ordering might seem as informative as being given a priority ordering. However, somewhat surprisingly, we show that all considered variants become solvable in polynomial time when a completion-time ordering is given.

We extend the quintuple notation by the suffix -P or -C, indicating that feasible schedules must adhere to a given priority ordering or completion-time ordering, respectively. In Section 4, we obtain the following result regarding completion-time orderings.

► **Theorem 3.** *There is a polynomial-time algorithm that computes optimal schedules for all \star -ID \star -C variants.*

In addition to providing efficient algorithms, Theorem 3 allows for a better understanding of the interplay between deciding on the job order and on the processor speed at each time. As a consequence, our results provide more insight into where the complexity of the problem stems from. In particular, when combined with our previous NP-hardness results, Theorem 3 directly implies that for \star -IDUA, \star -IDWU and \star -IDWA it is NP-hard to compute an optimal completion-time ordering.

The algorithm implied by Theorem 3 is based on a novel LP formulation for FE-ID- \star -C that effectively makes use of the structure imposed by the completion-time ordering. The LP formulation can be adapted to solve the respective budget variants B-ID \star -C as well, with a simple modification in the objective function and the addition of one constraint.

Further related work

Speed scaling was introduced to the algorithmic literature thirty years ago by Yao, Demers and Shenker [21]. Their seminal work focused on minimizing the total energy consumption of a schedule subject to deadline constraints and considered both the offline and the online settings. Since then, a plethora of different speed scaling settings have been considered, one of which is that of flow times as a QoS – motivated by the fact that, in many practical situations, jobs are not labeled with deadlines.

In addition to the results already discussed, the problem has been extended to multiprocessors (see [13], which also shows that exact optimal B-ICWU schedules cannot be computed even on a machine that can do real arithmetic operations and root computations). The online setting of different variations has also been widely explored; see [3, 10, 14, 17, 8] for some examples.

For a more extensive overview of the literature on energy-efficient scheduling, we refer the reader to the surveys by Albers [2] and Irani and Pruhs [15].

2 Formal Problem Description and Notation

Consider n jobs $1, \dots, n$ that have to be scheduled on a single speed-scalable processor. Each job j is associated with a corresponding *release time* $r_j \in \mathbb{Q}_{\geq 0}$, a *processing volume* $v_j \in \mathbb{Q}_{\geq 0}$, and *weight* $w_j \in \mathbb{Q}_{\geq 0}$. The processor is equipped with $k + 1$ distinct nonnegative rational *speeds*: $s_0 < \dots < s_k$, with corresponding nonnegative rational *power consumptions* $P_0 < \dots < P_k$. If the processor runs at speed s_i for τ time units, it processes a volume of $\tau \cdot s_i$ and consumes an amount $\tau \cdot P_i$ of energy. We assume that the processor idles when it uses the lowest speed (that is $s_0 = 0$), that $P_0 = 0$, and w.l.o.g., that $\min_j \{r_j\} = 0$.

A *schedule* is defined as $\sigma = (J, S)$. Here, $J(t) \in \{\emptyset, 1, \dots, n\}$ specifies which job is being processed at time t , where \emptyset indicates the processor is idle. Any job j can only be processed no later than its release time r_j . In other words for all j , and $t \in [0, r_j)$, $J(t) \neq j$. Preemption of jobs is allowed. Similarly, $S(t) \in \{0, 1, \dots, k\}$ specifies at what speed the server runs at time t . For $i \in \{1, \dots, k\}$, $S(t) = i$ indicates that the processor runs at speed s_i . We define

$$X_j(\sigma) := \{t : J(t) = j \wedge S(t) > 0\}.$$

Note that $X_j(\sigma)$ is the union of disjoint intervals; it consists of exactly the times during which job j is processed under schedule σ . We use $x_j(\sigma)$ to denote the total duration during which j runs under σ and $\chi(\sigma) = \sum_{j=1}^n x_j(\sigma)$ to denote the total amount of time during which σ is executing some job. We refer to $x_j(\sigma)$ as the *processing time* of j in σ . We say that a schedule is *feasible* if every job is fully processed no earlier than its release time. That is if it satisfies the following for all $j \in \{1, \dots, n\}$:

1. $\int_{t \in X_j(\sigma)} s_{S(t)} dt = v_j$, and
2. we have $t \geq r_j$, for all $t \in X_j(\sigma)$.

The *completion time* of j in σ , denoted by $C_j(\sigma)$, is the maximum t in $X_j(\sigma)$. The *flow time* of j in σ is defined as $F_j(\sigma) := C_j(\sigma) - r_j$ and the total (weighted) flow time of σ is

$$F(\sigma) := \sum_{j=1}^n w_j F_j(\sigma).$$

The total energy consumption of σ is naturally defined as the power of σ integrated over time: $E(\sigma) := \int_{t=0}^{\infty} P_{S(t)} dt$.

For a schedule σ , we say that j runs at speed $s_j(\sigma) := \frac{v_j}{x_j(\sigma)}$ (with slight abuse of notation). Note that $s_j(\sigma)$ refers to the average processing speed of j under σ . To distinguish this from the speeds s_1, \dots, s_k that the processor is allowed to run at as per the problem statement, we will often refer to the latter as the given or allowed speeds. For any $i \in \{1, \dots, k\}$, we denote by $x_j^i = v_j/s_i$ and $E_j^i = x_j^i P_i$, the processing time and energy consumption of j respectively under the assumption that j is processed solely at the given speed s_i . It is an easy observation that for any j and any feasible schedule σ , $x_j(\sigma)$ and $E_j(\sigma)$ can be expressed as a convex combination of the x_j^i 's and E_j^i 's:

► **Observation 4.** For any job j and $\lambda_j^1, \dots, \lambda_j^k \geq 0$ with $\sum_{i=1}^k \lambda_j^i = 1$, there is a schedule σ with $x_j(\sigma) = \sum_{i=1}^k \lambda_j^i x_j^i$ and $E_j(\sigma) = \sum_{i=1}^k \lambda_j^i E_j^i$.

Proof. Processing an amount v_j of volume at a speed of s_i , requires x_j^i time and E_j^i energy. Processing job j at speed s_i for $\lambda_j^i x_j^i$ time results in $\lambda_j^i v_j$ volume being processed at an energy of $\lambda_j^i E_j^i$. Let $y_j^i(\sigma)$ be the amount of time σ processes job j at speed s_i . To be more precise: $y_j^i(\sigma) = |\{t : J(t) = j, S(t) = i\}|$. The result is obtained by setting $\lambda_j^i = y_j^i(\sigma)/x_j^i$. ◀

In general, one can obtain any convex combination $x_j(\sigma) = \sum_{i=1}^k \lambda_j^i x_j^i$ with $E_j(\sigma) = \sum_{i=1}^k \lambda_j^i E_j^i$ by using each given speed s_i for $\lambda_j^i x_j^i$ time during $X_j(\sigma)$. Suppose that for a processing time of x_i , rather than running at speed s_i , we use a combination of two different given speeds s_q and s_r . If this results in a lower energy consumption, then we would always prefer to use this combination of s_q and s_r and never use s_i . Hence, to avoid any given speeds being superfluous, we have the following.

► **Observation 5.** *Let $i \in \{2, \dots, k-1\}$, $q \in \{1, i-1\}$, and $r \in \{i+1, k\}$. For a job j , let $\lambda \in (0, 1)$ be such that $x_j^i = \lambda x_j^q + (1-\lambda)x_j^r$. Then $E_j^i < \lambda E_j^q + (1-\lambda)E_j^r$.*

Proof. If $E_j^i \geq \lambda E_j^q + (1-\lambda)E_j^r$, then rather than ever using speed s_i , we can use the corresponding combination of s_q and s_r , which gives the same processing time, but with the same or lower energy consumption. Since $x_j^i = \lambda x_j^q + (1-\lambda)x_j^r$ implies $\frac{1}{s_i} = \lambda \frac{1}{s_q} + (1-\lambda) \frac{1}{s_r}$, this then holds for every job, hence s_i will be superfluous. ◀

The following observation follows from Observation 5 and shows us that while processing a job, we either use one given speed, or two consecutive speeds s_i and s_{i+1} .

► **Observation 6.** *Consider a schedule σ and job j with $s_i \leq s_j(\sigma) \leq s_{i+1}$. Let $\lambda \in [0, 1]$ be such that $x_j(\sigma) = \lambda x_j^i + (1-\lambda)x_j^{i+1}$, then $E_j(\sigma) = \lambda E_j^i + (1-\lambda)E_j^{i+1}$.*

Proof. The proof can be found in the full version [6]. ◀

Problem Variants

Recall that we adopt the quintuple notation $\star\star\star\star$ from [12]. For completeness, we formally restate the relevant specific entries with the notation introduced above. For the first entry, we consider two different objectives: either minimize $F(\sigma) + E(\sigma)$, or minimize only $F(\sigma)$ subject to an energy constraint $E(\sigma) \leq B$ for some given budget B . We refer to the former as the *flow + energy* variant (FE), and the latter as the *budget* variant (B). For the second entry, we only consider *integral flow* (I) $F(\sigma)$ as defined above, but the related concept of a *fractional flow* (F) has been considered in the literature. For the third entry, the allowed speeds of the processor could either be *discrete* (D) as defined above, or *continuous* (C), where the set of allowable speeds is $[0, H]$ for some $H > 0$, or $[0, \infty)$ and the power consumption is defined by a function P of the speed with $P(0) = 0$. Note that Observation 6 implies that the discrete speed setting can be modeled by the continuous one with a piecewise linear power function P and the discrete speed setting is therefore a special case of the continuous setting (see also Theorem 7 in [12]). For the fourth entry, in the *unweighted* variant (U), we have $w_1 = \dots = w_n = 1$, whereas for the *weighted* variant (W) we only require that $w_j \in \mathbb{R}_{>0}$ for all $j \in \{1, \dots, n\}$. Finally, for the fifth entry, in the *unit size* variant (U), we have $v_1 = \dots = v_n = 1$, whereas in the *arbitrary size* variant (A) we only require that $v_j \in \mathbb{R}_{>0}$ for all $j \in \{1, \dots, n\}$. Recall that we also consider each variant with a fixed priority (P) or completion (C) ordering and extend the quintuple notation by a suffix accordingly.

Note that the above definition of a schedule $\sigma = (J, S)$ applies to all described variants of the problem.

2.1 Affection and Shrinking Energy

In this section we introduce the notions of *affection* and *shrinking energy*. Intuitively, the affection of a job measures the improvement in total flow resulting from processing that job in less time (i.e., at higher speed), while its shrinkage energy measures the energy needed to

achieve this reduction in the processing time of the job. The affection and shrinking energy combined, provide information on which jobs (if any) one could accelerate in order to obtain an improved schedule. Similar definitions are used in [12], however, we do deviate from their notation, so as to be able to use it in a more general setting.

► **Definition 7 (Affection).** Consider a schedule σ .

- If $C_j(\sigma) \leq C_{j'}(\sigma)$ and $C_j(\sigma) > r_{j'}$, then j affects j' .
- If j affects j' and j' affects j'' , then j affects j'' .
- The affection set of j in σ , denoted by $K_j(\sigma)$, is the set of jobs affected by j in σ . We define the (weighted) affection as $\kappa_j(\sigma) := \sum_{j' \in K_j(\sigma)} w_{j'}$.

The lower affection, denoted $\kappa^+(\sigma)$, is defined analogously to affection, but additionally allows $C_j(\sigma) = r_j$. Note that a job always (lower) affects itself. Intuitively, the affection of job j captures how much the flow will improve if we reduce the processing time of j .

► **Observation 8.** Let σ be a schedule and consider the processing time $x_j(\sigma)$ of j in σ . Then there is an $\varepsilon' > 0$ such that for all $0 < \varepsilon \leq \varepsilon'$, the following holds.

- Let $\sigma^{-\varepsilon}$ be a schedule with the same completion order as σ , such that $x_{j'}(\sigma^{-\varepsilon}) = x_{j'}(\sigma)$ for all $j' \neq j$, and $x_j(\sigma^{-\varepsilon}) = x_j(\sigma) - \varepsilon$. Then $F(\sigma^{-\varepsilon}) = F(\sigma) - \varepsilon \kappa_j(\sigma)$.
- Let $\sigma^{+\varepsilon}$ be a schedule with the same completion order as σ , such that $x_{j'}(\sigma^{+\varepsilon}) = x_{j'}(\sigma)$ for all $j' \neq j$, and $x_j(\sigma^{+\varepsilon}) = x_j(\sigma) + \varepsilon$. Then $F(\sigma^{+\varepsilon}) = F(\sigma) + \varepsilon \kappa_j^+(\sigma)$.

Proof. The first statement follows by observing that when $C_j(\sigma) \leq C_{j'}(\sigma)$ and $C_j(\sigma) > r_{j'}$, reducing the processing time of j allows to also reduce the completion time of j' (without changing $x_{j'}$), and by transitivity the same holds for any j'' with $C_{j'}(\sigma) \leq C_{j''}(\sigma)$ and $C_{j'}(\sigma) > r_{j''}$. The second statement can be argued analogously. ◀

► **Definition 9.** Let s_1, \dots, s_k be given speeds. We define $\Delta_i := (P_{i+1}s_i - P_i s_{i+1}) / (s_{i+1} - s_i)$ for $i \in \{1, \dots, k-1\}$, and $\Delta_0 = -\infty$, and $\Delta_k := \infty$.

► **Definition 10 (Shrinking/expanding energy).** Let σ be a schedule. For any job j ,

- if $s_j(\sigma) \in [s_i, s_{i+1})$, the shrinking energy of j in σ is $\Delta_j(\sigma) := \Delta_i$.
- if $s_j(\sigma) \in (s_i, s_{i+1}]$, the expanding energy of j in σ is $\Delta_j^+(\sigma) := \Delta_i$.

If $s_j(\sigma) = s_k$, the shrinking energy is $\Delta_j(\sigma) := \Delta_k = \infty$, while if $s_j(\sigma) = s_1$ the expanding energy of j in σ is $\Delta_j^+(\sigma) := \Delta_0 = -\infty$.

Note that if $s_j(\sigma) \in (s_i, s_{i+1})$, then $\Delta_j(\sigma) = \Delta_j^+(\sigma)$.

By Observation 8, the (lower) affection of j quantifies the change in flow for small changes in the processing time of j . The following lemma gives us an analogous result for the shrinking/expanding energy and the change in energy consumption.

► **Lemma 11.** Let σ be a schedule where job j has processing time $x_j(\sigma) \in [x_j^i, x_j^{i+1}]$, and let $\varepsilon > 0$. Then,

- if $\varepsilon \leq x_j(\sigma) - x_j^{i+1}$, the schedule $\sigma^{-\varepsilon}$ with $x_j(\sigma^{-\varepsilon}) = x_j(\sigma) - \varepsilon$ and $x_{j'}(\sigma^{-\varepsilon}) = x_{j'}(\sigma)$ for all $j' \neq j$ has total energy consumption equal to $E(\sigma^{-\varepsilon}) = E(\sigma) + \varepsilon \Delta_j(\sigma)$.
- if $\varepsilon \leq x_j^i - x_j(\sigma)$, the schedule $\sigma^{+\varepsilon}$ with $x_j(\sigma^{+\varepsilon}) = x_j(\sigma) + \varepsilon$ and $x_{j'}(\sigma^{+\varepsilon}) = x_{j'}(\sigma)$ for all $j' \neq j$ has total energy consumption equal to $E(\sigma^{+\varepsilon}) = E(\sigma) - \varepsilon \Delta_j^+(\sigma)$.

Proof. We only prove the first statement, as the second can be proven with an analogous argument.

Suppose $\varepsilon \leq x_j(\sigma) - x_j^{i+1}$ and let $0 < \lambda \leq 1$ be such that $x_j(\sigma) = \lambda x_j^i + (1 - \lambda) x_j^{i+1}$. By Observation 6, we have $E_j(\sigma) = \lambda E_j^i + (1 - \lambda) E_j^{i+1}$. Since $x_j(\sigma^{-\varepsilon}) \geq x_j^{i+1}$, there exists a $\tilde{\lambda} \in [0, \lambda)$ such that $x_j(\sigma^{-\varepsilon}) = \tilde{\lambda} x_j^i + (1 - \tilde{\lambda}) x_j^{i+1}$, which gives us $E_j(\sigma^{-\varepsilon}) = \tilde{\lambda} E_j^i + (1 - \tilde{\lambda}) E_j^{i+1}$.

Then

$$\varepsilon = (\lambda - \tilde{\lambda})(x_j^i - x_j^{i+1}) = (\lambda - \tilde{\lambda}) \left(\frac{v_j}{s_i} - \frac{v_j}{s_{i+1}} \right) = \frac{v_j(\lambda - \tilde{\lambda})}{s_i s_{i+1}} (s_{i+1} - s_i),$$

and we obtain that

$$\begin{aligned} E_j(\sigma^{-\varepsilon}) - E_j(\sigma) &= (\tilde{\lambda} - \lambda)(E_j^i - E_j^{i+1}) = v_j(\lambda - \tilde{\lambda}) \left(\frac{P_i}{s_i} - \frac{P_{i+1}}{s_{i+1}} \right) \\ &= \frac{v_j(\lambda - \tilde{\lambda})}{s_i s_{i+1}} (P_{i+1} s_i - P_i s_{i+1}) = \varepsilon \frac{P_{i+1} s_i - P_i s_{i+1}}{s_{i+1} - s_i} = \varepsilon \Delta_j(\sigma). \end{aligned} \quad \blacktriangleleft$$

► **Corollary 12.** *Let σ^* be an optimal schedule for FE-ID**. Then for any job j*

- $\kappa_j(\sigma^*) - \Delta_j(\sigma^*) \leq 0$, and
- $\Delta_j^+(\sigma^*) - \kappa_j^+(\sigma^*) \leq 0$.

Proof. The result follows directly from Observation 8 and Lemma 11. ◀

3 Hardness Results

3.1 Hardness of FE-IDUA

We show that FE-IDUA is NP-hard through a reduction from B-IDUA with two speeds, a problem shown to be NP-hard by Barcelo et al. [12, proof of Theorem 1].

Consider an instance \mathcal{I}^B of B-IDUA with given speeds $s_1 < s_2$, power consumptions $P_1 < P_2$, budget B , and n jobs with volumes v_1, \dots, v_n and release times r_1, \dots, r_n . We assume w.l.o.g. that $s_1 = 1$. The following notation is used in the construction of an instance \mathcal{I}^{FE} , based on \mathcal{I}^B , that establishes the reduction from B-IDUA to FE-IDUA. Let $V = \sum_{j \in \{1, \dots, n\}} v_j$ be the total volume of jobs in \mathcal{I}^B , and let $Y := \frac{B - P_1 V}{\Delta_1}$. Intuitively, a schedule that only utilizes s_1 already consumes an amount $P_1 V$ of energy. Thus, Y denotes the maximum decrease of the total processing time that the budget allows, compared to such a schedule. Note that, w.l.o.g.,

$$P_1 V < B < P_2 \frac{V}{s_2}, \quad (1)$$

since, if this is not the case, then \mathcal{I}^B is either infeasible or trivial.

Let σ^* be an optimal schedule for \mathcal{I}^B , let σ_1 be the SRPT schedule for \mathcal{I}^B that processes every job at a speed of exactly s_1 , let $C_{\max}(\sigma_1)$ be the makespan of σ_1 , and let $I(\sigma_1) = C_{\max}(\sigma_1) - V$ the total idle time (recall that $\min\{r_j\} = 0$).

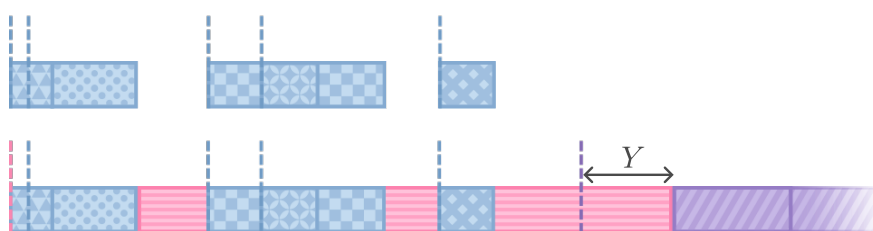
The following observation follows directly from (1) and Lemma 11.

► **Observation 13.** *For any optimal schedule σ^* for \mathcal{I}^B it holds that $\chi(\sigma^*) = V - Y$.*

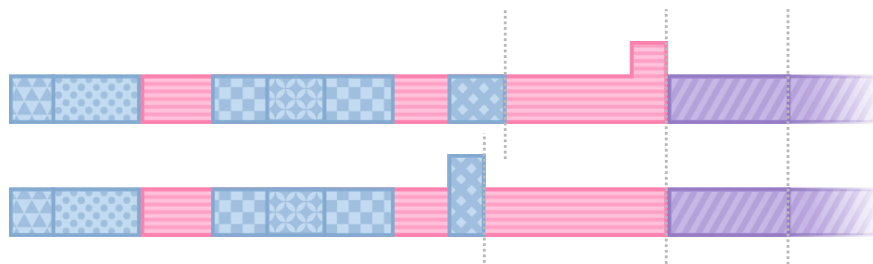
That is, an optimal schedule for \mathcal{I}^B uses the complete budget B to shrink the total processing time to $V - Y$.

We construct \mathcal{I}^{FE} as follows.

- **Jobs.** \mathcal{I}^{FE} has $2n + 1$ jobs: Jobs 1 to n are identical to the ones of \mathcal{I}^B . I.e., $\tilde{r}_i = r_i$ and $\tilde{v}_i = v_i$, for $i = 1, \dots, n$. Job $n + 1$ has volume $\tilde{v}_{n+1} = I(\sigma_1) + (s_2 + 1)V + Y$ and $\tilde{r}_{n+1} = 0$ and jobs $n + 2, \dots, 2n + 1$ all have a volume of $Y + 1$ and a release time of $C_{\max}(\sigma_1) + (s_2 + 1)V$.
- **Processor.** The processor has the same two speeds $\tilde{s}_1 = s_1$ and $\tilde{s}_2 = s_2$ as in \mathcal{I}^B . The corresponding power consumptions are given by $\tilde{P}_1 = P_1$, and $\tilde{P}_2 = (n + 3/2)(s_2 - 1) + P_1 s_2$, and the associated shrinking energy $\tilde{\Delta}_1 = \frac{\tilde{P}_2 \tilde{s}_1 - \tilde{P}_1 \tilde{s}_2}{\tilde{s}_2 - \tilde{s}_1}$.



■ **Figure 1** An instance of B-IDUA (depicted at the top) is transformed into an instance of FE-IDUA (depicted at the bottom).



■ **Figure 2** A comparison between speeding up job $n + 1$ and job n . The dashed lines indicate the completion times and show that speeding up job $n + 1$ improves the flow of at most $n + 1$ jobs, while speeding up job n improves the flow of at least $n + 2$ jobs.

Figure 1 depicts how \mathcal{I}^{FE} is constructed from \mathcal{I}^B . Note that \mathcal{I}^{FE} can be computed in polynomial time and that for the shrinking energy in \mathcal{I}^{FE} , we have $n + 1 < \tilde{\Delta}_1 < n + 2$.

The intuition behind the reduction is that in an optimal schedule $\tilde{\sigma}^*$ for \mathcal{I}^{FE} exactly B energy is consumed to process jobs $1, \dots, n$, and in turn these are processed “independently” of the other jobs. Therefore, $\tilde{\sigma}^*$ produces an optimal schedule for \mathcal{I}^B . The schedule that $\tilde{\sigma}^*$ produces for \mathcal{I}^B is simply $\tilde{\sigma}^*$ restricted to jobs $\{1, \dots, n\}$, which we define as $\tilde{\sigma}_{\{1, \dots, n\}}^* = (\tilde{J}_{\{1, \dots, n\}}^*, \tilde{S}_{\{1, \dots, n\}}^*)$, with

$$(\tilde{J}_{\{1, \dots, n\}}^*(t), \tilde{S}_{\{1, \dots, n\}}^*(t)) = \begin{cases} (\tilde{J}^*(t), \tilde{S}^*(t)), & \text{if } \tilde{J}^*(t) \in \{1, \dots, n\} \\ (\emptyset, \emptyset), & \text{otherwise} \end{cases}.$$

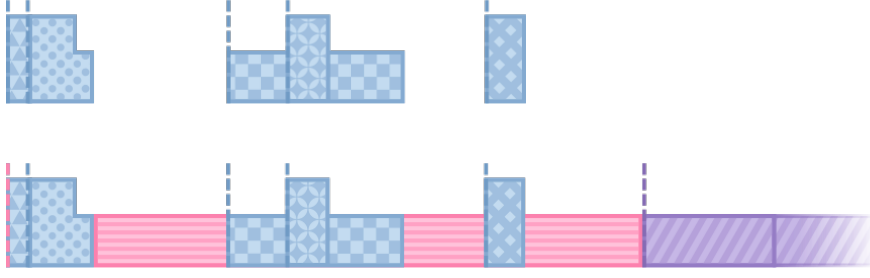
► **Lemma 14.** *In an optimal solution for \mathcal{I}^{FE} , jobs $n + 1, \dots, 2n + 1$ complete last.*

Proof. First consider the schedule where job $n + 1$ is processed at speed s_2 , while all other jobs are scheduled at speed s_1 . From the fixed-speed version of the problem – and ignoring energy cost – we know that the optimal order to process the jobs in that case is by using the SRPT rule [20]. By construction, job $n + 1$ has larger processing time than any job j in $\{1, \dots, n\}$ and, therefore at time r_j , completes later than each. For any schedule, we can argue the same. Note also that, in turn, r_j for $j \in \{n + 2, \dots, 2n + 1\}$ is larger than the last completion time of the jobs in $\{1, \dots, n\}$ in any schedule. Thus, the lemma holds. ◀

► **Lemma 15.** *In an optimal solution for \mathcal{I}^{FE} , jobs $n + 1, \dots, 2n + 1$ have average speed s_1 .*

Proof. Since $\tilde{\Delta}_1 > n + 1$, and jobs $n + 1, \dots, 2n + 1$ complete last in an optimal schedule, by Corollary 12, these jobs run at speed s_1 . ◀

Figure 2 depicts the difference between speeding up a job in $\{1, \dots, n\}$ and a jobs in $\{n + 1, \dots, 2n + 1\}$.



■ **Figure 3** Jobs $n + 1, \dots, 2n + 1$ are added to an optimal schedule for \mathcal{I}^B and run at speed s_1 . This gives a schedule for \mathcal{I}^{FE} where jobs $1, \dots, n$ have a total processing time of $V - Y$, and are the only jobs that run at a speed higher than s_1 .

▶ **Lemma 16.** For \mathcal{I}^{FE} , any optimal solution uses a total amount of time of $V - Y$ for jobs $1, \dots, n$.

Proof. Consider an optimal schedule $\tilde{\sigma}^*$ for \mathcal{I}^{FE} . Assume first, for the sake of contradiction, that $\tilde{\sigma}^*$ uses strictly less time than $V - Y$ for jobs $1, \dots, n$. Then, until r_{n+2} the amount of time that we do not process a job in $\{1, \dots, n\}$ is strictly greater than $r_{n+2} - V + Y = v_{n+1}$. Hence, job $n + 1$ must have a completion time strictly smaller than r_{n+2} . In other words, for any $j \in \{1, \dots, n + 1\}$ we have $\kappa_j^+(\tilde{\sigma}^*) \leq n + 1 < \tilde{\Delta}_1^+(\tilde{\sigma}^*)$. This contradicts Corollary 12.

Assume next, for the sake of contradiction, that $\tilde{\sigma}^*$ uses strictly more time than $V - Y$ for jobs $1, \dots, n$. Then, by Lemma 15 it must be the case that $C_{n+1}(\tilde{\sigma}^*) > r_{n+2}$ and therefore any job $j \in \{1, \dots, n\}$ must have $\kappa_j \geq n + 2$. This again contradicts Corollary 12. ◀

By Lemma 15 and Lemma 16, we have that in an optimal schedule for \mathcal{I}^{FE} , only jobs in $\{1, \dots, n\}$ are sped up, and they have a total processing time of $V - Y$. Note that by Observation 13, this also holds for an optimal schedule for \mathcal{I}^B . Hence, it follows that if we take an optimal schedule for \mathcal{I}^B and add the jobs $n + 1, \dots, 2n + 1$ at speed s_1 , then we obtain a schedule for \mathcal{I}^{FE} that satisfies these conditions, as depicted in Figure 3. This naturally leads to the following question: If we take an optimal schedule for \mathcal{I}^{FE} and remove jobs $n + 1, \dots, 2n + 1$, do we obtain an optimal schedule for \mathcal{I}^B ? The following lemma shows that this indeed the case.

▶ **Lemma 17.** The schedule $\tilde{\sigma}_{\{1, \dots, n\}}^*$ is optimal for \mathcal{I}^B .

Proof. We first argue that $\tilde{\sigma}_{\{1, \dots, n\}}^*$ is a feasible schedule. Indeed, by definition jobs $\{1, \dots, n\}$ are scheduled identically in $\tilde{\sigma}^*$ and $\tilde{\sigma}_{\{1, \dots, n\}}^*$, and each such job has the same volume and release time in each of the corresponding schedules. It directly follows that each such job is completely processed after its respective release time. By Lemma 16, it also follows that the budget B is not exceeded.

Assume that $\tilde{\sigma}_{\{1, \dots, n\}}^*$ is not optimal for \mathcal{I}^B , in other words that there exists some other feasible schedule σ' for \mathcal{I}^B with strictly less flow than $\tilde{\sigma}_{\{1, \dots, n\}}^*$. We argue that this contradicts the optimality of $\tilde{\sigma}^*$ for \mathcal{I}^{FE} . Indeed consider schedule $\tilde{\sigma}'$ for \mathcal{I}^{FE} defined as follows:

$$(\tilde{J}'(t), \tilde{S}'(t)) = \begin{cases} (\tilde{J}^*(t), \tilde{S}^*(t)) & \text{if } t \geq r_{n+2} \\ (J'(t), S'(t)) & \text{if } J'(t) \neq \emptyset \\ (n + 1, \tilde{s}_1) & \text{at any other time } t. \end{cases}$$

First note that $\tilde{\sigma}'$ is defined for all times t , and by construction produces feasible schedules. Also note that it uses time $V - Y$ for jobs $1, \dots, n$ and all other jobs are processed at a speed of \tilde{s}_1 . Thus, the overall energy consumption of $\tilde{\sigma}'$ is the same as $\tilde{\sigma}^*$. The same holds for the flow times of jobs $n + 1, \dots, 2n + 1$. However, the total flow time of jobs $1, \dots, n$ is strictly smaller under $\tilde{\sigma}'$ than under $\tilde{\sigma}^*$, leading to a contradiction. ◀

► **Theorem 1.** *FE-IDUA is NP-hard.*

Proof. Given Lemma 17, it remains to argue that \mathcal{I}^{FE} can be constructed in polynomial time from \mathcal{I}^B and $\tilde{\sigma}_{\{1, \dots, n\}}^*$ in polynomial time from $\tilde{\sigma}^*$. The first one clearly follows by construction. The second one is less obvious, given that a schedule is defined as two functions of t . However, we note that for any optimal schedule it is without loss of generality to assume that each job starts processing at a speed of s_2 (respectively \tilde{s}_2), and only switches to s_1 (respectively \tilde{s}_1) at most once. Thus, the respective schedules and the associated transformation can be described in polynomial time. ◀

► **Corollary 18.** *FE-ICUA is NP-hard.*

3.2 Hardness of FE-IDWU

We show that FE-IDWU is NP-hard through a reduction from SUBSETSUM. In [11, Section 3.2.2], Barcelo et al. give a reduction from SUBSETSUM to B-IDWU where they show that in the B-IDWU instance, there is a gap between the optimal flows corresponding to a YES-instance and a NO-instance of SUBSETSUM. To adapt this idea for a reduction to FE-IDWU, we simulate a budget in a similar way to the reduction from B-IDUA to FE-IDUA from Section 3.1. Rather than using a job with a comparatively large volume, we use multiple lightweight unit-sized jobs. This does introduce a new challenge, as the order of the jobs becomes harder to predict (i.e. a lightweight job could still be prioritized over a heavier job). We circumvent this hurdle by making the weight of the new jobs so lightweight that the increase in the optimal flow due to their addition cannot be too large. To be more specific: even with the inclusion of the lightweight jobs, there remains a gap between the optimal flows corresponding to a YES-instance and a NO-instance of SUBSETSUM.

Consider an instance \mathcal{I}^S of SUBSETSUM, where we are given m elements $a_1 \geq \dots \geq a_m$ with $a_i \in \mathbb{N}$ and a value $A \in \mathbb{N}$ with $a_1 < A < \sum_{i=1}^m a_i$. The instance is a YES-instance if and only if there is a subset $L \subseteq \{1, \dots, m\}$ such that $\sum_{i \in L} a_i = A$. SUBSETSUM is known to be an NP-hard problem. We assume that $a_i \leq 2a_{i'}$ for all $i, i' \in \{1, \dots, m\}$ as Barcelo et al. show that even with this assumption, SUBSETSUM remains NP-hard [11, proof of Theorem 38].

For our reduction to FE-IDWU, we define the instance \mathcal{I}^{FE} as follows. We first construct a *job package* \mathcal{J}_i for each of the elements a_i . These are identical to the packages from the reduction to B-IDWU from [11, Section 3.2.2] and defined as follows. Each \mathcal{J}_i consists of $m + 1$ jobs: $\mathcal{J}_i = \{(i, 0), \dots, (i, m)\}$. For $i \in \{1, \dots, m\}$, job $(i, 0)$ has weight $w_{i,0} = \frac{a_i}{m}$, and jobs (i, j) with $j \geq 1$ have weight $w_{i,j} = 2ma_i^3$. Job $(1, 0)$ releases at $r_{1,0} = 0$ and jobs $(i, 0)$ with $i \in \{2, \dots, m\}$ have release time $r_{i,0} = r_{i-1,0} + m + 1$. The remaining jobs in \mathcal{J}_i release at $r_{i,1} = \dots = r_{i,m} = r_{i,0} + 1 - \alpha_i$ with $\alpha_i = \frac{a_i}{2a_1^2}$. For ease of notation, we denote $r_i = r_{i,1} = \dots = r_{i,m}$ and $w_i = w_{i,1} = \dots = w_{i,m}$.

Since we want to utilize results from the reduction to B-IDWU by Barcelo et al., we want to construct an FE-IDWU instance that simulates a budget. To do this, we add two new job packages: \mathcal{J}_0 and \mathcal{J}_{m+1} , which serve similar roles to job $n + 1$, and jobs $n + 1, \dots, 2n + 1$ from the reduction in Section 3.1.

7:12 Refining the Complexity Landscape of Speed Scaling: Hardness and Algorithms

Let $\mathcal{J}_0 = \{(0, 1), \dots, (0, K)\}$ and $\mathcal{J}_{m+1} = \{(m+1, 1), \dots, (m+1, \tilde{K})\}$ with $K = \left\lceil \frac{m^2}{2} + \frac{A}{2a_1^2} \right\rceil$ and $\tilde{K} = 99m^8 a_1^3$. All jobs in \mathcal{J}_0 have release time $r_0 = r_{0,i} = 0$ and weight $w_0 = w_{0,i} = \frac{1}{32m^5}$. All jobs in \mathcal{J}_{m+1} release at $r_{m+1} = r_{m+1,i} = m(m+1) + \left\lceil \frac{m^2}{2} + \frac{A}{2a_1^2} \right\rceil - \frac{m^2}{2} - \frac{A}{2a_1^2}$, and have weight $w_{m+1} = w_{m+1,i} = \frac{1}{33m^5}$. Note that although the number of jobs depends on A , all jobs in \mathcal{J}_0 and \mathcal{J}_{m+1} are respectively identical and therefore the encoding length remains polynomial.

For the processor, we have two speeds: $s_1 = 1$ and $s_2 = 2$, with respective power consumptions $P_1 = 1$ and $P_2 = 3m^3 a_1^3 + \frac{1}{33m^5} + 2$. This give us that $\Delta_1 = 3m^3 a_1^3 + \frac{1}{33m^5}$.

► **Observation 19.** *Let σ^* be an optimal schedule for \mathcal{I}^{FE} . Then for any $j \in \mathcal{J}_{m+1}$, we have that $s_j(\sigma^*) = s_1$ and for all $j' \in \bigcup_{i=0}^m \mathcal{J}_i$, we have $C_j(\sigma^*) < C_{j'}(\sigma^*)$.*

Proof. Consider a job $j \in \mathcal{J}_{m+1}$. For any $j' \in \bigcup_{i=0}^m \mathcal{J}_i$, we have $w_j < w_{j'}$ and $r_j > r_{j'}$. Hence, it can never be optimal to complete j before j' . It follows that j can also not affect any of the jobs in $\bigcup_{i=0}^m \mathcal{J}_i$, from which we obtain that $\kappa_j(\sigma^*) \leq \tilde{K}w_{m+1} = 3m^3 a_1^3 < \Delta_1$. By Corollary 12, we must have $s_j(\sigma^*) = s_1$. ◀

For a schedule σ , let $F_i(\sigma) = \sum_{j \in \mathcal{J}_i} F_j(\sigma)$ and $\chi_i = \sum_{j \in \mathcal{J}_i} x_j(\sigma)$. Furthermore, let $\tilde{\chi}(\sigma) = \sum_{i=0}^m \chi_i$ and $\tilde{F}(\sigma) = \sum_{i=0}^m F_i(\sigma)$.

Let $Y := \frac{m^2}{2} + \frac{A}{2a_1^2}$. The following observation shows that for finding an optimal schedule, it is sufficient to only consider schedules with $\tilde{\chi}(\sigma) = m(m+1) + K - Y$.

► **Lemma 20.** *In any optimal schedule σ^* for \mathcal{I}^{FE} , we have that $\tilde{\chi}(\sigma^*) = m(m+1) + K - Y$.*

Proof sketch. By Observation 19, having $\tilde{\chi}(\sigma^*) = m(m+1) + K - Y$ is equivalent to finishing the processing of all jobs in $\bigcup_{i=0}^m \mathcal{J}_i$ exactly at time

$$m(m+1) + K - \left(\frac{m^2}{2} + \frac{A}{2a_1^2} \right) = r_{m+1}.$$

First, suppose that $\tilde{\chi}(\sigma^*) > r_{m+1}$. The shrinkage among the jobs in $\bigcup_{i=0}^m \mathcal{J}_i$ is strictly less than $\frac{m^2}{2} + \frac{A}{2a_1^2} < \frac{m(m+1)}{2}$. Hence, there must be a job j in $\bigcup_{i=0}^m \mathcal{J}_i$ that does not fully run at speed s_2 and thus has $\Delta_j(\sigma^*) = \Delta_1$. Furthermore, since $\tilde{\chi}(\sigma^*) > r_{m+1}$, job j affects all jobs in \mathcal{J}_{m+1} and we have

$$\kappa_j(\sigma^*) = w_0 + \tilde{K}w_{m+1} \geq \frac{1}{32m^5} + \frac{99m^8 a_1^3}{33m^5} = \frac{1}{32m^5} + 3m^3 a_1^3 > \Delta_1 = \Delta_j(\sigma^*).$$

By Corollary 12, this contradicts that σ^* is optimal.

Now suppose that $\tilde{\chi}(\sigma^*) < r_{m+1}$. Then there must be some job j in $\bigcup_{i=0}^m \mathcal{J}_i$ that runs faster than s_1 . In this case j does affect any of the jobs in \mathcal{J}_{m+1} , and we have

$$\kappa_j^+(\sigma^*) \leq Kw_0 + mw_{1,0} + m^2 w_i \leq 3m^3 a_1^3 < \Delta_1 = \Delta_j^+(\sigma^*).$$

By Corollary 12, this contradicts that σ^* is optimal. ◀

Let σ_1 be a schedule where all jobs in $\bigcup_{i=0}^m \mathcal{J}_i$ run at speed $s_1 = 1$. Then $\tilde{\chi}(\sigma_1) = m(m+1) + K$. Compared to σ_1 , in an optimal schedule the total processing time of $\bigcup_{i=0}^m \mathcal{J}_i$ shrinks by a total of Y . We can consider the problem as optimally “distributing” the total shrinkage of Y among all jobs in $\bigcup_{i=0}^m \mathcal{J}_i$. Note that by Observation 19, if $\tilde{\chi}(\sigma^*) = m(m+1) + K - Y$ then finding an optimal schedule for \mathcal{J}_{m+1} is trivial. Thus, we turn our focus to optimizing the flow of $\bigcup_{i=0}^m \mathcal{J}_i$.

For a schedule σ and $i \in \{1, \dots, m\}$, let $y_i(\sigma) = m + 1 - \chi_i(\sigma)$. In other words, compared to its processing time at speed s_1 , the processing time of \mathcal{J}_i shrinks by $y_i(\sigma)$ in σ . By Lemma 20, we must have that $\sum_{i=1}^m y_i(\sigma) \leq Y$.

Now consider a problem where among all schedules with a shrinkage of Y , we want to find one that minimizes the flow of $\bigcup_{i=1}^m \mathcal{J}_i$. It is not trivial whether an optimal schedule σ^* for \mathcal{I}^{FE} is also optimal for this problem. This would require proving that for \mathcal{I}^{FE} , it is never optimal to shrink a job in \mathcal{J}_0 or to give it priority over a job in $\bigcup_{i=1}^m \mathcal{J}_i$. Hence, we take a different approach and show that for σ^* , the flow of $\bigcup_{i=0}^m \mathcal{J}_i$ is not much larger than the optimal flow of $\bigcup_{i=1}^m \mathcal{J}_i$ for a shrinkage of Y .

► **Lemma 21.** *For \mathcal{I}^{FE} , let σ^* be an optimal schedule and let σ^{**} be a schedule that, among all schedules σ with $\sum_{i=1}^m y_i(\sigma) = Y$, minimizes the flow of $\bigcup_{i=1}^m \mathcal{J}_i$. Then*

$$\sum_{i=1}^m F_i(\sigma^{**}) \leq \tilde{F}(\sigma^*) \leq \sum_{i=1}^m F_i(\sigma^{**}) + \frac{1}{16m}.$$

Proof. Since σ^{**} minimizes $\sum_{i=1}^m F_i(\sigma^{**})$, we have $\sum_{i=1}^m F_i(\sigma^{**}) \leq \sum_{i=1}^m F_i(\sigma^*) \leq \tilde{F}(\sigma^*)$.

Note that σ^{**} will always prioritize jobs in $\bigcup_{i=1}^m \mathcal{J}_i$ over jobs in \mathcal{J}_0 . It follows that w.l.o.g., we may assume all jobs in \mathcal{J}_0 run at speed s_1 and σ^{**} satisfies $\tilde{\chi}(\sigma^{**}) = m(m+1) + K - Y$. Thus, each job $j \in \mathcal{J}_0$ has $C_j(\sigma^{**}) \leq r_{m+1}$, from which it follows that

$$\tilde{F}(\sigma^*) \leq \tilde{F}(\sigma^{**}) \leq \sum_{i=1}^m F_i(\sigma^{**}) + Kr_{m+1}w_0.$$

Since $A < ma_1 \leq m^2 a_1^2$ we have that $K = \left\lceil \frac{m^2}{2} + \frac{A}{2a_1^2} \right\rceil \leq \left\lceil \frac{m^2}{2} + \frac{m^2}{2} \right\rceil = m^2$. W.l.o.g., we assume that $m \geq 2$. Using that $r_{m+1} = m(m+1) + K - Y$, we obtain

$$\tilde{F}(\sigma^*) \leq \sum_{i=1}^m F_i(\sigma^{**}) + m^2 \frac{m(m+1) + 1}{32m^5} \leq \sum_{i=1}^m F_i(\sigma^{**}) + \frac{1}{16m}. \quad \blacktriangleleft$$

We can now use Lemma 21 to relate the optimal flow \mathcal{I}^{FE} to the budget instance from the reduction to B-IDWU by Barcelo et al. [12]. Consider an instance \mathcal{I}^B of B-IDWU with the same two speeds: $\tilde{s}_1 = 1$ and $\tilde{s}_2 = 2$, job packages $\tilde{\mathcal{J}}_i = \mathcal{J}_i$ for $i \in \{1, \dots, m\}$ (with identical weights and release times), power consumptions $\tilde{P}_1 = 1$ and $\tilde{P}_2 = 4$, and a budget $B = 2Y$. Let $\tilde{\sigma}^B$ denote the schedule where all jobs (i, j) with $j \geq 1$ run fully at speed s_2 , and for all $i \in \{1, \dots, m\}$ $(i, 0)$ runs at speed s_1 and finishes after all other jobs in \mathcal{J}_i . Let $F^B = \sum_{i=1}^m F_i(\tilde{\sigma}^B)$.

► **Lemma 22** (Barcelo et al. [11, Theorem 38]). *Let $\tilde{\sigma}^*$ be an optimal schedule for \mathcal{I}^B .*

- *If \mathcal{I}^S is a YES-instance, then $\sum_{i=1}^m F_i(\tilde{\sigma}^*) \leq F^B - \frac{A}{2} - \frac{1}{8m}$.*
- *If \mathcal{I}^S is a NO-instance, then $\sum_{i=1}^m F_i(\tilde{\sigma}^*) \geq F^B - \frac{A}{2}$.*

► **Lemma 23.** *Let σ^* be an optimal schedule for \mathcal{I}^{FE} . Then $\tilde{F}(\sigma^*) < F^B - \frac{A}{2}$ if and only if \mathcal{I}^S is a YES-instance of SUBSETSUM.*

Proof. For \mathcal{I}^S , we have $\tilde{\Delta}_1 = 2$, and thus a budget of $2Y$ is equivalent to a total shrinkage of Y among the jobs in $\bigcup_{i=1}^m \tilde{\mathcal{J}}_i$. It follows that $\sum_{i=1}^m y_i(\tilde{\sigma}^*) = Y$. Since $\tilde{\mathcal{J}}_i = \mathcal{J}_i$ for all $\{i = 1, \dots, m\}$, by Lemma 21 we have that

$$\sum_{i=1}^m F_i(\tilde{\sigma}^*) \leq \tilde{F}(\sigma^*) \leq \sum_{i=1}^m F_i(\tilde{\sigma}^*) + \frac{1}{16m}.$$

From Lemma 22, it follows that if \mathcal{I}^S is a YES-instance, then

$$\tilde{F}(\sigma^*) \leq \sum_{i=1}^m F_i(\tilde{\sigma}^*) + \frac{1}{8m} \leq F^B - \frac{A}{2} - \frac{1}{16m},$$

and if \mathcal{I}^S is a NO-instance, then

$$\tilde{F}(\sigma^*) \geq \sum_{i=1}^m F_i(\tilde{\sigma}^*) \geq F^B - \frac{A}{2}. \quad \blacktriangleleft$$

From Lemma 21, it follows directly that FE-IDWU is NP-hard, proving **Theorem 2**.

► **Corollary 24.** *FE-ICWU is NP-hard.*

3.3 Fixed Priority Ordering

Suppose that we are given a fixed priority ordering of the jobs $1 \prec \dots \prec n$. This ordering specifies that, at any point in time and among all jobs that are released and not yet completed, the one processed is the one that comes first in the ordering. Thus, together with a speed profile $S(t)$, a priority ordering fully specifies a schedule. This leads to the following question: *Given a priority ordering such that there is an optimal schedule that adheres to this ordering, can we efficiently find an optimal schedule?*

Recall that *-***-P denotes the problem variants with a fixed priority ordering. A priority ordering is called *optimal* if there is an optimal schedule that adheres to the ordering. Below we show that B-IDUA-P, FE-IDUA-P, B-IDWU, and FE-IDWU-P are NP-hard. For B-IDUA-P, FE-IDUA-P, and B-IDWU-P, we actually prove a stronger statement: the problems are NP-hard, even when the priority ordering is known to be optimal.

► **Theorem 25.** *B-IDUA-P is NP-hard, even if the given ordering is known to be optimal.*

Proof sketch. The result follows from extending the analysis of the reduction from SUBSET-SUM to B-IDUA by Barcelo et al. [11]. Further details can be found in the full version [6]. ◀

► **Corollary 26.** *FE-IDUA-P is NP-hard, even if the given ordering is known to be optimal.*

Proof. Consider an instance \mathcal{I}^B of B-IDUA-P with priority ordering $1 \prec \dots \prec n$, and assume this ordering is known to be optimal. Let \mathcal{I}^{FE} be the FE-IDUA instance obtained by applying the reduction from Section 3.1 to \mathcal{I}^B . Let $n+1, \dots, 2n+1$ be the newly added jobs, as described in the reduction. By Lemma 14 and the proof of Lemma 16, any optimal schedule for \mathcal{I}^{FE} adheres to the priority ordering $1 \prec \dots \prec n \prec n+1 \prec \dots \prec 2n+1$. From Lemma 17 and Theorem 25, it follows that FE-IDUA-P is NP-hard, even when the priority ordering is known to be optimal. ◀

► **Theorem 27.** *B-IDWU-P is NP-hard, even if the given ordering is known to be optimal.*

Proof. For the reduction from SUBSETSUM to B-IDWU [11, Observation 36] (also described in the proof of Lemma 23), Barcelo shows that there is always an optimal schedule that satisfies any priority ordering with $(i, 1) \prec \dots \prec (i, m) \prec (i, 0)$, for all $i \in \{1, \dots, m\}$. We can extend this to a full priority ordering by setting $(i, j) \prec (i', j')$ for $i < i'$. The job packages are released far enough apart such that in any (reasonable) schedule, all jobs in a package finish before the first job in the next package is released. Hence, this full priority ordering is always optimal. ◀

► **Theorem 28.** *FE-IDWU-P is NP-hard.*

Proof. Consider an FE-IDWU-P instance \mathcal{I}^{FE} where we have the instance of FE-IDWU as described in Section 3.2 with a priority ordering that satisfies the following. For $i \in \{1, \dots, m\}$, we have $(i, 1) \prec \dots \prec (i, m) \prec (i, 0)$, and for any jobs $j \in \bigcup_{i=1}^m \mathcal{J}_i$, $j' \in \mathcal{J}_0$, and $j'' \in \mathcal{J}_{m+1}$, we have $j \prec j' \prec j''$.

Observation 19 shows that adhering to $j \prec j''$ and $j' \prec j''$ for $j \in \bigcup_{i=1}^m \mathcal{J}_i$, $j' \in \mathcal{J}_0$, and $j'' \in \mathcal{J}_{m+1}$ is trivial. Since the jobs in $\bigcup_{i=1}^m \mathcal{J}_i$ are prioritized over jobs in \mathcal{J}_0 , an optimal schedule for \mathcal{I}^{FE} must be as σ^{**} from Lemma 21. Hence, Lemma 21 still holds, and by the same argument as for Theorem 27, FE-IDWU with a fixed priority ordering is also NP-hard. ◀

4 An LP for Fixed Completion Ordering

In this section, we investigate different variants of the problem with fixed completion-time ordering. We are given an ordering $1 \prec \dots \prec n$, of the jobs and require that $\hat{C}_1(\sigma) \leq \dots \leq \hat{C}_n(\sigma)$, where the *extended completion time* $\hat{C}_j(\sigma)$ of job j under schedule σ is defined by

$$\begin{aligned} \hat{C}_1(\sigma) &:= C_1(\sigma) \\ \hat{C}_j(\sigma) &:= \max\{C_j(\sigma), \hat{C}_{j-1}(\sigma)\}, \quad \forall j \in \{2, \dots, n\}, \end{aligned}$$

where $C_j(\sigma)$ is given by the maximum t in $X_j(\sigma)$ (i.e., the “classic” definition of a completion time). The extended flow time of each job j is given as $\hat{F}_j(\sigma) = \hat{C}_j(\sigma) - r_j$. An intuition behind the extended completion times is although the actual processing of a job j might finish at $C_j(\sigma)$, j can only be returned to the user once $1, \dots, j-1$ are returned.³ In this case it is natural that the flow time also accounts for the time interval $[C_j(\sigma), \hat{C}_j(\sigma))$.

We define $Q_j := \{j' : j' \preceq j\}$, $R_t^- := \{j : r_j \leq t\}$ and $R_t^+ := \{j : r_j \geq t\}$. Intuitively, for time t , $R_t^+ \cap Q_j$ gives a subset of the jobs that need to be fully processed before we can complete j . We will show that the following LP computes the optimal schedule for FE-ID***-C* with given completion ordering $1 \prec \dots \prec n$ ⁴.

$$\min \sum_j w_j \hat{C}_j + \sum_{i,j} \lambda_j^i E_j^i \tag{2}$$

s.t.

$$\hat{C}_j \geq \hat{C}_{j-1} \quad \forall j > 1 \tag{3}$$

$$\hat{C}_j \geq r_{j'} + \sum_{j'' \in Q_j \cap R_{r_{j'}}^+} \sum_i \lambda_{j''}^i x_{j''}^i \quad \forall j, \forall j' \in Q_j \cap R_{r_j}^- \tag{4}$$

$$\sum_i \lambda_j^i = 1 \quad \forall j \tag{5}$$

$$\lambda_j^i \geq 0 \quad \forall i, j. \tag{6}$$

We denote a feasible solution $(\hat{C}_1, \dots, \hat{C}_n, \lambda_1^1, \dots, \lambda_n^k)$ as (\hat{C}, λ) . Recall that by Observation 4, for any schedule σ and job j , $x_j(\sigma)$ is a convex combination of the x_j^i .

► **Theorem 29.** *The LP computes in polynomial time an optimal schedule for FE-ID***-C*.*

³ One can only serve desert after serving the main course, even if the desert was prepared earlier.

⁴ Since $\sum_j w_j \hat{F}_j + \sum_{i,j} \lambda_j^i E_j^i = \sum_j w_j \hat{C}_j + \sum_{i,j} \lambda_j^i E_j^i - \sum_j w_j r_j$, and $\sum_j w_j r_j$ is given and thus constant, optimizing over either objective is equivalent.

We start by showing that any feasible schedule for a FE-ID**⁻C instance implies a feasible solution to the LP with the same objective value.

► **Lemma 30.** *Let σ be a feasible schedule for FE-ID**⁻C with given completion ordering $1 \prec \dots \prec n$. Then there is a feasible solution (\hat{C}, λ) to the LP such that for every j : $\hat{C}_j = \hat{C}_j(\sigma)$, $x_j(\sigma) = \sum_i \lambda_j^i x_j^i$ and $E_j(\sigma) = \sum_{i,j} \lambda_j^i E_j^i$.*

Proof. Note that by Observation 4, we can find a λ which satisfies constraints (5) and (6), and furthermore $x_j(\sigma) = \sum_i \lambda_j^i x_j^i$ and $E_j(\sigma) = \sum_{i,j} \lambda_j^i E_j^i$. By the definition of $\hat{C}_j(\sigma)$ and the feasibility of σ , constraint (3) is also satisfied.

In order to show that constraint (4) is satisfied, consider arbitrary $j, j' \in Q_j \cap R_{r_j}^-$ and $j'' \in Q_j \cap R_{r_{j'}}^+$. Since $j'' \prec j$ and $r_{j''} \geq r_{j'}$, by the feasibility of σ , it must work on job j'' for at least $x_{j''} = \sum_i \lambda_{j''}^i x_{j''}^i$ during $[r_{j'}, \hat{C}_j(\sigma))$. Summing up over all such j'' , we obtain

$$\hat{C}_j = \hat{C}_j(\sigma) \geq r_{j'} + \sum_{j'' \in Q_j \cap R_{r_{j'}}^+} \sum_i \lambda_{j''}^i x_{j''}^i. \quad \blacktriangleleft$$

We next show that an optimal solution to the LP implies a feasible schedule with the same objective value.

► **Lemma 31.** *Let (\hat{C}, λ) be an optimal solution to the LP. Then, for FE-ID**⁻C there is a schedule σ with $\hat{C}_j(\sigma) = \hat{C}_j$, $x_j(\sigma) = \sum_i \lambda_j^i x_j^i$, and $E_j(\sigma) = \sum_{i,j} \lambda_j^i E_j^i$.*

Proof. Given (\hat{C}, λ) , we construct a corresponding schedule σ as follows: we run each job j for a total of $\sum_i \lambda_j^i x_j^i$ time (by employing the two relevant consecutive speed levels and at most one speed switch). At every time t , we process the active job with the earliest \hat{C}_j (note that this is also the earliest job in the completion-time ordering). It is straightforward to express the resulting schedule in terms of $\sigma = (J(t), S(t))$.

Schedule σ satisfies the last two equalities in the statement by construction. Again by construction, every job is fully processed after its release time. It remains to argue that $\hat{C}_j(\sigma) = \hat{C}_j$, as well as the completion-time ordering is satisfied. Let $\hat{C}_j(\sigma)$ be the extended completion time of job j in σ . By definition, the completion-time ordering holds for $\hat{C}_j(\sigma)$ and it only remains to prove that $\hat{C}_j(\sigma) = \hat{C}_j$ holds for every job j .

We first argue that for each job j there holds $\hat{C}_j(\sigma) \geq \hat{C}_j$. Note that for any $t \leq r_j$, any feasible schedule must fully process all jobs in $Q_j \cap R_t^+$ within the interval $[t, \hat{C}_j)$. In other words, for σ , any job j and any $t \leq r_j$ there must hold that $\hat{C}_j(\sigma) - t \geq \sum_{j'' \in Q_j \cap R_t^+} x_{j''}(\sigma) = \sum_{j'' \in Q_j \cap R_t^+} \sum_i \lambda_{j''}^i x_{j''}^i$. So consider an arbitrary job j , and let $t' \leq r_j$ be the largest time such that σ is either idling or processing a job j^* with $j^* \succ j$ just before t' . Note that t' must be the release time of some $j' \in Q_j$. By the above observation, it must be the case that $\hat{C}_j(\sigma) \geq r_{j'} + \sum_{j'' \in Q_j \cap R_{r_{j'}}^+} \sum_i \lambda_{j''}^i x_{j''}^i$. Furthermore, by construction, σ never unnecessarily idles and does not process any job not in $Q_{j''} \cap R_{r_{j'}}^+$ throughout $[r_{j'}, \hat{C}_j)$. So overall it must be the case that $\hat{C}_j(\sigma) = r_{j'} + \sum_{j'' \in Q_j \cap R_{r_{j'}}^+} \sum_i \lambda_{j''}^i x_{j''}^i$.

If for job j constraint (4) of the LP is tight, then $\hat{C}_j(\sigma) = \hat{C}_j$ directly follows. Otherwise, by optimality of (\hat{C}, λ) , constraint (3) must be tight. In this case it is sufficient to show the existence of a job $j' \prec j$ with $\hat{C}_j = \hat{C}_{j'}$, such that for j' constraint (4) is tight, since by the above argument $\hat{C}_{j'}(\sigma) = \hat{C}_{j'}$ holds, and by the definition of $\hat{C}_j(\sigma)$. The existence of such a j' follows by constraint (3) and the fact that for job 1 constraint (4) must be tight. ◀

Finally, we prove the main theorem of this section.

Proof of Theorem 29. Given an instance \mathcal{I}^{FE} of FE-ID**-C, construct the LP, solve it, construct corresponding schedule σ for \mathcal{I}^{FE} as in the proof of Lemma 31.

Schedule σ is optimal, since it has the same objective value as that of an optimal solution to the LP, and by Lemma 30 the objective value of an optimal solution to the LP cannot be higher than the objective value of an optimal schedule for the corresponding input instance.

Solving the LP and building the respective schedule can be done in polynomial time. ◀

Now suppose we change the objective of the LP to $\min \sum_j w_j \hat{C}_j$ and add the constraint $\sum_{i,j} \lambda_j^i E_j^i \leq B$. By using the same proof, we can show that this new LP solves any B-ID** -C variant. Together with Theorem 29, this shows us that any *-ID** -C variant can be solved in polynomial time, proving **Theorem 3**.

5 Towards a Combinatorial Algorithm for FE-ID** -C

While the LP allows us to solve any *-ID** -C variant in polynomial time, there are still benefits in finding a combinatorial algorithm. Barcelo et al.[12] claim such an algorithm for FE-IDUA-C by a straightforward generalization of their algorithm for FE-IDUU with only two speeds. To be more precise, they suggest that a straightforward generalization of their algorithm solves FE-IDUU with an arbitrary number of speeds, and a slightly more general result yields an algorithm for FE-IDUA-C. Unfortunately, we found that the two, in our opinion, most natural generalizations of their algorithm do not work for an arbitrary number of speeds. For more details, see the full version [6].

Here, we provide a different and simple combinatorial algorithm for FE-IDUU with an arbitrary number of speeds. For unit-size and unit-weight jobs, ordering the jobs from earliest to latest release date (FIFO) gives an optimal completion ordering for any speed profile $S(t)$. This follows from a simple exchange argument. Hence, the problem is equivalent to finding optimal speeds. Algorithm 1 is a greedy algorithm that increases the speed of a job that leads to the biggest improvement in the objective function. Furthermore, we conjecture that a slight adaptation of this algorithm also solves any *-ID** -C variant.

■ **Algorithm 1** The $(\kappa - \Delta)$ -Algorithm.

while *There is a job j with $\kappa_j \geq \Delta_j$* **do**
 Increase the speed of job $j^* := \arg \max_j (\kappa_j - \Delta_j)$ (break ties arbitrarily), until
 either $\kappa_{j^*} - \Delta_{j^*} < 0$, or $\kappa_{j^*} - \Delta_{j^*} \neq \max_j (\kappa_j - \Delta_j)$.

► **Theorem 32.** *Algorithm 1 computes an optimal schedule for FE-IDUU in polynomial time.*

Proof. The proof can be found in the full version [6]. ◀

Recall the definition of affection and note that Observation 8 does not hold for $\hat{F}(\sigma) = \sum_j (\hat{C}_j(\sigma) - r_j)$ instead of $F(\sigma)$. For a fixed completion ordering we define the *extended affection* as follows.

► **Definition 33** (Extended affection). *Consider a schedule σ for a fixed completion ordering $1 \prec \dots \prec n$. To simplify notation, define $\hat{C}_0 := -\infty$.*

- *For $j \leq j'$, we have that $j' \in \hat{K}_j$ if either:*
 - $C_j(\sigma) > r_{j'}$ and $\hat{C}_{j'} > \hat{C}_{j'-1}$, or
 - $\hat{C}_j > \hat{C}_{j-1}$ and $\hat{C}_j = \hat{C}_{j'}$.
- *If $j' \in \hat{K}_j$ and $j'' \in \hat{K}_{j'}$, then $j'' \in \hat{K}_j$.*
- *The extended affection of job j is defined as $\hat{\kappa}_j(\sigma) := \sum_{j' \in \hat{K}_j(\sigma)} w_{j'}$.*

► **Conjecture 34.** *The algorithm obtained by replacing κ with $\hat{\kappa}$ in Algorithm 1 computes an optimal schedule for any FE-ID^{**}-C variant in polynomial time.*

6 Discussion and Future Work

We showed the NP-hardness of FE-IDUA, FE-ICUA, FE-IDWU and FE-ICWU, and furthermore that all these problem variants are solvable in polynomial time when given a completion-time ordering.

Recently, there has been renewed interest in approximation algorithms for the fixed-speed case of minimizing weighted flow for arbitrary-size jobs. This line of work has culminated in a Polynomial-Time Approximation Scheme (PTAS) [7] (recall that \star -I \star WA is NP-hard [16]). In light of this, it would be interesting to study the exact approximability of the NP-hard variants on speed-scalable processors. Our linear program for the problem given a completion time order might be a useful starting point, as it implies that it suffices to – in some sense – approximate the optimal completion-time ordering.

In particular, for FE-IDUA, there exists an online $(2 + \epsilon)$ -competitive algorithm (see [4, 9]). While this naturally implies an offline $(2 + \epsilon)$ -approximation algorithm, it seems plausible that full knowledge of all jobs and their characteristics could lead to better approximation guarantees.

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