Colouring Probe H-Free Graphs

Daniël Paulusma
Department of Computer Science, Durham University, UK

Johannes Rauch
Institute of Optimization and Operations Research, Ulm University, Germany

Erik Jan van Leeuwen
Department of Information and Computing Sciences, Utrecht University, The Netherlands

Abstract

The NP-complete problems Colouring and k-Colouring ($k \geq 3$) are well studied on H-free graphs, i.e., graphs that do not contain some fixed graph H as an induced subgraph. We research to what extent the known polynomial-time algorithms for H-free graphs can be generalized if we only know some of the edges of the input graph. We do this by considering the classical probe graph model introduced in the early nineties. For a graph H, a partitioned probe H-free graph (G, P, N) consists of a graph $G = (V, E)$, together with a set $P \subseteq V$ of probes and an independent set $N = V \setminus P$ of non-probes, such that $G + F$ is H-free for some edge set $F \subseteq \binom{N}{2}$. We show the following:

- We fully classify Colouring on partitioned probe H-free graphs and show that the obtained complexity dichotomy differs from the known dichotomy of Colouring for H-free graphs.
- We fully classify 3-Colouring on partitioned probe P_t-free graphs: we prove polynomial-time solvability for $t \leq 5$ and NP-completeness for $t \geq 6$. In contrast, 3-Colouring on P_t-free graphs is known to be polynomial-time solvable for $t \leq 7$ and quasi-polynomial-time solvable for $t \geq 8$.

Our main result is our polynomial-time algorithm for 3-Colouring on partitioned P_5-free graphs. For this result, and also for all our other polynomial-time results, we do not need to know the edge set F; we only need to know its existence. Moreover, the class of probe P_5-free graphs includes not only paths of arbitrary length but even all bipartite graphs and is much richer than the class of P_5-free graphs. The latter is also evidenced by the fact that there exist graph problems, such as Matching Cut, that are known to be polynomial-time solvable for P_5-free graphs but NP-complete for partitioned probe P_5-free graphs. In particular, unlike the class of 3-colourable P_5-free graphs, the class of 3-colourable probe P_5-free graphs has unbounded mim-width. Hence, our polynomial-time result for 3-Colouring for probe P_5-free graphs suggests that there may be another, deeper overarching reason why 3-Colouring is polynomial-time solvable for P_5-free graphs.

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1 Introduction

Colouring is a classical graph problem. Given a graph G and a positive integer k, it asks whether it is possible to colour the vertices of G with k colours such that any two adjacent vertices receive different colours. The variant where k is fixed beforehand, and not part of the input anymore, is known as k-Colouring. It is well known that 3-Colouring, and thus Colouring, are NP-complete problems [39]. This led to a rich body of literature that tries to understand what graph structure causes the computational hardness in Colouring. In our paper we extend this body of work by researching the computational complexity of Colouring and k-Colouring on classes of graphs that generalize the well-known H-free graphs (a graph G is *H-free* if G does not contain H as an induced subgraph) but for which we do not know all the edges. Before discussing our model of incomplete information, we first briefly survey some relevant known results for Colouring and k-Colouring.

H-Free Graphs. Král et al. [54] showed that if H is a (not necessarily proper) induced subgraph of P_4 or P_3+P_1, where P_t denotes the path on t vertices, then Colouring on H-free graphs is solvable in polynomial time; otherwise, it is NP-complete. For k-Colouring, the complexity status on H-free graphs has not been resolved yet. For every $k \geq 3$, k-Colouring for H-free graphs is NP-complete if H has a cycle [37] or an induced claw [50, 56]. However, the remaining case where H is a linear forest (disjoint union of paths) has not been settled yet. For P_t-free graphs, the cases $k \leq 2$, $t \geq 1$ (trivial), $k \geq 3$, $t \leq 5$ [49], $k = 3$, $6 \leq t \leq 7$ [9] and $k = 4$, $t = 6$ [28, 29] are polynomial-time solvable and the cases $k = 4$, $t \geq 7$ [51] and $k \geq 5$, $t \geq 6$ [51] are NP-complete. The cases $k = 3$ and $t \geq 8$ are still open, despite some evidence that these cases are polynomial-time solvable due to a quasi-polynomial-time algorithm [60]. We refer to the survey [41] and some later articles [25, 26, 48, 53] for results on k-Colouring for H-free graphs if H is a disconnected linear forest, and to [52] for the most recent results on Colouring for (H_1, H_2)-free graphs, for which also still many cases remain open.

Probe H-Free Graphs. In this article, we aim to further our understanding of the complexity of Colouring and k-Colouring by studying *probe graphs*. Probe graphs are used to model graphs for which the global structure is known (e.g. H-freeness). However, we only know the complete set of neighbours for *some* vertices of a probe graph G. These vertices are the *probes* of G. The other vertices are the *non-probes* of G and form an independent set in G, as we do not know which of them are adjacent to each other. We only know that there exists a "certifying" set F of edges on the non-probes such that $G + F$ exhibits the global structure (e.g. being H-free). In particular, the subgraph of G induced by the set of probes already has this global structure (e.g. is H-free). The notion of probe graphs was introduced by Zhang et al. [64] in genome research to make a genome mapping process more efficient.

Formally, for a graph class $\mathcal{G}$, the class $\mathcal{G}_p$ consists of all graphs G that can be modified into a graph from $\mathcal{G}$ by adding some edges between an independent set N of G. If for a graph in $\mathcal{G}_p$, the sets P and $N = V \setminus P$ are given, then we speak of a *partitioned* probe graph. Hence, a *partitioned probe H-free* graph (G, P, N) consists of a graph $G = (V, E)$, together with a set $P \subseteq V$ of probes and an independent set $N = V \setminus P$ of non-probes, such that $G + F$ is H-free for some edge set $F \subseteq \binom{N}{2}$. We note that an H-free graph itself is also a (partitioned) probe H-free graph, namely with $P = V$ and $N = \emptyset$. Hence, for every graph H, the class of (partitioned) probe H-free graphs contains the class of H-free graphs. Consequently, any NP-completeness results for H-free graphs immediately carry over to partitioned probe H-free graphs. However, it also leads to the following research question:

If an NP-complete problem Π is polynomial-time solvable on H-free graphs for some graph H, is Π also polynomial-time solvable on (partitioned) probe H-free graphs?

Our Focus. To investigate our research question, we consider Colouring and k-Colouring for (partitioned) probe H-free graphs. For some graphs H, such as $H = P_4$ [22], probe H-free graphs can be recognized in polynomial time. However, for most graphs H, including $H = P_5$, the recognition of probe H-free graphs and the distinction between probes and non-probes are open problems. Hence, we usually require the sets of probes P and non-probes N to be part of the input, i.e., we must consider partitioned probe H-free graphs. Note that we can colour a probe H-free graph G with one extra colour (assigned to each vertex in N) than the number of colours used for $G[P]$. The challenge is to determine if we need that extra colour.

Related Work. So far, most previous work on probe graphs focused on characterising and recognising classes of probe graphs [4,5,20,22,42,43,55]. However, recently, the first systematic studies of optimisation problems on partitioned probe H-free graphs were undertaken. Brettell et al. [15] considered Vertex Cover on partitioned probe H-free graphs and Dabrowski et al. [31] did the same for Matching Cut and some of its variants. A takeaway from [15, 31] is that determining the complexity of Vertex Cover for (partitioned) probe P_5-free graphs seems challenging, whereas Matching Cut is NP-complete on partitioned probe P_5-free graphs, even though they are both polynomial-time solvable on P_5-free graphs [38, 58].

Helpful for algorithmic studies as [15,31] is that probe graphs inherit some properties from the graph class they are based on. This is also true when studying Colouring. For example, Golumbic and Lipshteyn [42] proved that probe chordal graphs are perfect. Hence, we obtain that Colouring is polynomial-time solvable for probe chordal graphs, as Colouring is so for perfect graphs [45, 46]. In 2012, Chandler et al. [21] conjectured that this holds even for Colouring on partitioned probe perfect graphs. Moreover, the following is known:

▶ **Proposition 1** ([15, 22]). *Let $\mathcal{G}$ be a class of graphs and let w be a fixed integer.*
(i) *If $\mathcal{G}$ has clique-width at most w, then $\mathcal{G}_p$ has clique-width at most $2w$.*
(ii) *If $\mathcal{G}$ has mim-width at most w, then $\mathcal{G}_p$ has mim-width at most $2w$.*

Hence, as $\mathcal{G} \subseteq \mathcal{G}_p$ holds for every graph class $\mathcal{G}$, we obtain that a graph class $\mathcal{G}$ has bounded mim-width (clique-width) if and only if $\mathcal{G}_p$ has bounded mim-width (clique-width). However, if the yes-instances for some problem Π in $\mathcal{G}$ have bounded width, this may no longer hold for the yes-instances for Π in $\mathcal{G}_p$. If we can still solve Π on $\mathcal{G}_p$, this means there might be a deeper reason for the polynomial-time behaviour of Π on $\mathcal{G}$. We will also research this.

Our Results. As mentioned, we focus on Colouring and k-Colouring for probe H-free graphs. Our first result is a full dichotomy of Colouring on partitioned probe H-free graphs (for two graphs G_1 and G_2, we write $G_1 \subseteq_i G_2$ if G_1 is an induced subgraph of G_2).

▶ **Theorem 2.** *For a graph H,* Colouring *is polynomial-time solvable for probe H-free graphs if $H \subseteq_i P_4$, and else it is* NP*-complete even for partitioned probe H-free graphs.*

By Theorem 2, Colouring is NP-complete for partitioned probe $3P_1$-free graphs, while Colouring is even polynomial-time solvable on $(P_3 + P_1)$-free graphs [54]. It is known that the class of H-free graphs has bounded mim-width [13] if and and only if it has bounded clique-width (see e.g. [33]) if and only if H is an induced subgraph of P_4. Hence, Theorem 2 also implies, together with Proposition 1, that Colouring on (not necessarily partitioned) probe H-free graphs is solvable in polynomial time exactly when the mim-width or clique-width is bounded. In fact, we apply Proposition 1 to show the first part of Theorem 2, while for the second part we modify a known hardness reduction for Colouring [6]; see Section 3.

Our second result is a full dichotomy for 3-Colouring on partitioned probe P_t-free graphs:

▶ **Theorem 3.** *For an integer $t \geq 1$,* 3-Colouring *on partitioned probe P_t-free graphs is polynomial-time solvable if $t \leq 5$ and* NP*-complete if $t \geq 6$.*

In Section 4 we prove the polynomial part of Theorem 3 by giving our **main result**: a polynomial-time algorithm for 3-Colouring for partitioned probe P_5-free graphs. The class of P_5-free graphs has been extensively studied for many well-known graph problems, yielding polynomial-time algorithms not only for k-Colouring for any $k \geq 1$ [49], but also Vertex Cover [58], Feedback Vertex Set [1], Independent Feedback Vertex Set [8] and very recently, Odd Cycle Transversal [2], or more generally, Maximum Partial List H-Coloring for every fixed graph H [57]. Our polynomial-time result for 3-Colouring on partitioned probe P_5-free graphs is the *first* result that generalizes a known polynomial-time result for a (classical) graph problem on P_5-free graphs to partitioned probe P_5-free graphs.

In Section 5 we prove the second part of Theorem 3. In fact, we show that 3-Colouring is NP-complete even on partitioned probe $(P_6, 2P_3, 3P_2)$-free graphs. In contrast, 3-Colouring is polynomial-time solvable even on P_7-free graphs [9] and sP_2-free graphs for all $s \geq 1$ [32].

In Section 6, we point out *many* natural directions for future research. In particular, we determine all (disconnected) graphs H for which 3-Colouring on probe partitioned H-free graphs is still open and solve one such open case, namely when $H = P_3 + sP_1$. Moreover, we consider k-Colouring for $k \geq 4$ and solve one open case, namely when $H = P_2 + sP_1$ for $s \geq 1$, by proving that for every $s \geq 0$, all probe $(P_2 + sP_1)$-free graphs are $(s + 1)P_2$-free.

Proof Ideas behind Our Main Result. As evidenced by the C_5, there are P_5-free graphs that are not perfect, and there exist three different proofs for showing that 3-Colouring is polynomial-time solvable on P_5-free graphs. As discussed below, these proofs are not applicable to probe P_5-free graphs, even though two of them provide good starting points.

First, in the proof of Brettell et al. [14], it was shown that k-colourable P_5-free graphs have *bounded mim-width* for all $k \geq 1$, directly implying that k-Colouring becomes polynomial-time solvable [17]. However, bipartite graphs are even 2-colourable and readily seen to be even probe $2P_2$-free (as shown below explicitly for paths) while already chordal bipartite graphs have not only unbounded mim-width [12] but even *unbounded sim-width* [10]. This also shows that probe 3-colourable P_5-free graphs, which have bounded mim-width due to Proposition 1, are only a small subclass of the class of 3-colourable probe P_5-free graphs.

Second, in the proof of Randerath, Schiermeyer and Tewes [62], a connected 3-colourable P_5-free graph is shown to have a dominating set D of constant size. By precolouring D in every possible way, polynomially many instances of 2-List Colouring (i.e., where all lists of admissible colours have size 2) are obtained. As 2-List Colouring can be formulated as 2-SAT, it is polynomial-time solvable [36]. This approach fails for probe P_5-free graphs. To see this, let G be a path $u_1u_2 \ldots u_{2n}$. Let $P = \{u_1, u_3, \ldots, u_{2n-1}\}$ and $N = \{u_2, u_4, \ldots, u_{2n}\}$. Note that P and N are independent. Hence, making N a clique yields a split graph, which is $2P_2$-free, so G is even probe $2P_2$-free. However, for large n, G has no small dominating set.

Finally, in the proof of Woeginger and Sgall [63], it is first shown that every (C_3, P_5)-free graph G is 3-colourable. Next, the case where G contains at least one triangle C (which may not be dominating) is considered. After colouring C, the following rule is applied exhaustively: whenever a vertex v has two neighbours not coloured alike, give v the third colour. Afterwards, it is again shown that an instance of 2-List Colouring is obtained. Also this approach does not work for probe P_5-free graphs due to their more intricate structure. By extending the arguments as in [63], we can show that even all C_3-free probe P_5-free graphs, which include all (probe) (C_3, P_5)-free graphs, are 3-colourable (see Section 4). However, C_3-free probe P_5-free graphs form only a small subclass of 3-colourable probe P_5-free graphs.

As we explain below, our proof for partitioned probe P_5-free graphs turns out to be substantially more involved than the second and third proofs for P_5-free graphs and does not rely on the boundedness of some width parameter.

First, if all connected components of the subgraph induced by the probes are bipartite, we can colour the probes with colours 1 and 2 and use colour 3 for the non-probes (as the non-probes form an independent set). Next, we show that at most one connected component K of the subgraph induced by the set of probes can be non-bipartite. Then, by P_5-freeness, we find that K has a short odd cycle C. The fact that in general, C is not a dominating cycle of the graph causes several complications. To overcome these complications, we branch on the colours of C and then just like [63], we try to extend the colouring as much as possible using propagation rules, with the aim to reduce eventually to polynomially many instances of 2-List Colouring. We show that the latter is possible via (i) a new decomposition of probe P_5-free graphs, which carefully takes into account the fact that we do not know the missing edges between the non-probes that make the graph P_5-free (as illustrated in Figure 3) and (ii) an adaptation of the standard 2-SAT formula for 2-List Colouring.

2 Preliminaries

Let G be a graph, and k be a positive integer. The *order* of G is its number of vertices, and the *size* of G is its number of edges. For a vertex v of G, we denote its *(open) neighbourhood* by $N_G(v)$, and its *closed neighbourhood* by $N_G[v] = N_G(v) \cup \{v\}$. For a set S of vertices of G, let $N_G[S] = \bigcup_{v \in S} N_G[v]$, and $N_G(S) = N_G[S] \setminus S$. A vertex $v \notin S$ is *complete* to a set of vertices S if v is adjacent to every vertex of S, and v is *anticomplete* to S if v is not adjacent to any vertex of S. Let S' be another set of vertices of G that is disjoint to S. If every vertex of S is complete (anticomplete) to S', then S is *complete* (*anticomplete*) to S'. We write $G[S]$ for the subgraph of G induced by S. For two vertex-disjoint graphs G_1 and G_2, we let $G_1 + G_2$ denote their disjoint union, which is the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. For a graph G and integer $s \geq 1$, sG denotes the disjoint union of s copies of G.

For a graph H, we say that G is *H-free* if there is no set of vertices S such that $G[S]$ is isomorphic to H. We say that G is *probe H-free* if there is a partition of the vertices of G into a set of probes P and a set of non-probes N, such that N is independent in G, and there is a set of edges $F \subseteq \binom{N}{2}$ such that $G + F$ is H-free. Note that $G[P]$ is H-free if G is probe H-free. A *partitioned probe H-free graph* is a triple (G, P, N), where G is a probe H-free graph with P as the probes and N as the non-probes, that is, the sets of probes and non-probes are given. For a set $\{H_1, \dots, H_r\}$ of graphs, a graph G is *$(H_1, \dots, H_r)$-free* if G is H_i-free for every $i \in \{1, \dots, r\}$. A graph G is *probe $(H_1, H_2, \dots)$-free* if there is an independent set N of non-probes in G and a set of edges $F \subseteq \binom{N}{2}$ such that $G + F$ is $(H_1, H_2, \dots)$-free. In a partitioned probe $(H_1, H_2, \dots)$-free graph (G, P, N), the graph G is probe $(H_1, H_2, \dots)$-free with set of probes P and set of non-probes N.

We define $[k] = \{1, \dots, k\}$. A *partial k-colouring* of G is a function $\psi : V(G) \to [k] \cup \{\perp\}$ such that, if $uv \in E(G)$ with $\psi(u), \psi(v) \in [k]$, then $\psi(u) \neq \psi(v)$. If v is a vertex of G with $\psi(v) \in [k]$, then v is *coloured* (under ψ). Let ψ' be another partial k-colouring of G. Then ψ' is an *extension* of ψ if $\psi(v) \in [k]$ implies that $\psi'(v) = \psi(v)$. A *k-colouring* of G is a partial k-colouring under which every vertex of G is coloured. For $S \subseteq V(G)$, we define $\psi(S) = \{\psi(v) : v \in S\}$.

Algorithm 1 is a simple colour propagation algorithm that is essential to the proof of Theorem 3. The following properties of Algorithm 1 are easy and their proofs are omitted:

Algorithm 1 Simple colour propagation.

```
Input: A graph G, and a partial k-colouring ψ.
Output: An extension of ψ, or an error.

// Propagation Rule
while there is an uncoloured vertex v ∈ V(G) and i ∈ [k] such that v has a neighbour
 of every colour except colour i, that is, [k] \ {i} ⊆ ψ(N_G(v)) ⊆ ([k] \ {i}) ∪ {⊥} do
    set ψ(v) ← i
forall v ∈ V(G) do
    if v has a neighbour of every colour, that is, [k] ⊆ ψ(N_G(v)) then
        return an error
return ψ
```

▶ **Lemma 4.** *Let ψ be a partial k-colouring of a graph G.*

(i) *If Algorithm 1 on (G, ψ) returns an extension ψ' of ψ and $v \in V(G)$ is coloured under ψ', then v has the same colour under any k-colouring of G that is an extension of ψ (if any exist).*

(ii) *If Algorithm 1 on (G, ψ) returns an error, then there is no k-colouring of G that is an extension of ψ.*

(iii) *Algorithm 1 runs in polynomial time.*

We use the following well-known lemma, which is due to Edwards [36]. We provide a proof to adapt it later.

▶ **Lemma 5.** *Given a graph G and a partial k-colouring ψ of G, for every uncoloured vertex $v \in V(G)$, define the set of available colours of v as $L(v) = [k] \setminus \psi(N_G(v))$. If $|L(v)| \leq 2$ for every uncoloured vertex $v \in V(G)$, then deciding if there is a k-colouring that is an extension of ψ is possible in polynomial time.*

Proof. Let the SAT formula $\mathcal{F}$ in conjunctive normal form have variables x_v^i for every uncoloured vertex $v \in V(G)$ and every $i \in L(v)$, and clauses

- $\bigvee_{i \in L(v)} x_v^i$ for every uncoloured vertex v (i.e., if $L(v) = \emptyset$, then $\mathcal{F}$ is not satisfiable) and
- $\bar{x}_u^i \vee \bar{x}_v^i$ for every $uv \in E(G)$ with uncoloured vertices u and v and $i \in L(u) \cap L(v)$.

According to the assumptions $\mathcal{F}$ is a 2-SAT formula. By construction, there is a k-colouring of G that is an extension of ψ if and only if $\mathcal{F}$ is satisfiable. This completes the proof since deciding the satisfiability of a 2-SAT formula is possible in polynomial time [3]. ◀

3 The Proof of Theorem 2

In this section we show Theorem 2. The proof of our next result is based on an existing construction from [6], and we omit the proof details.

▶ **Proposition 6.** COLOURING *is* NP*-complete on partitioned probe $3P_1$-free graphs.*

We combine Proposition 6 with the NP-completeness part of the dichotomy of Král et al. [54], the fact that P_4-free graphs have clique-width 2 [30] and Proposition 1 to obtain Theorem 2 (proof details omitted):

▶ **Theorem 2** (restated)**.** *For a graph H,* COLOURING *is polynomial-time solvable for probe H-free graphs if $H \subseteq_i P_4$, and else it is* NP*-complete even for partitioned probe H-free graphs.*

4 The Proof of the Polynomial Part of Theorem 3

In this section we prove the main result of our paper, namely that 3-Colouring is polynomial-time solvable for partitioned probe P_5-free graphs.

We first show the following independent result (Proposition 7) in more or less the same way as done in [63] for showing that (C_3, P_5)-free graphs are 3-colourable. The main difference is that we must take into account the probes and non-probes. As such, the proof of this result serves as warm-up exercise illustrating some of the arguments we will use in a more involved way in the proof of our main result. Note that probe (C_3, P_5)-free graphs form a subclass of C_3-free probe P_5-free graphs. This containment is proper, as

(i) probe (C_3, P_5)-free graphs have bounded mim-width, due to (C_3, P_5)-free graphs having bounded mim-width [14] and Proposition 1, and

(ii) chordal bipartite graphs, which are C_3-free probe P_5-free, have unbounded mim-width [12].

▶ **Proposition 7.** *All C_3-free probe P_5-free graphs, and thus all probe (C_3, P_5)-free graphs, are 3-colourable.*

Proof. Let $G = (V, E)$ be a C_3-free probe P_5-free graph. Let P be the set of probes and $N = V \setminus P$ be the set of non-probes, so N is an independent set. By definition, there exists a set F of edges with both end-vertices in N such that $G + F$ is P_5-free. We may assume without loss of generality that G is connected, as otherwise we consider every connected component of G separately.

If $G[P]$ is bipartite, then we colour $G[P]$ with colours 1 and 2, and as N is independent, we can use colour 3 for the vertices of N. Now suppose that $G[P]$ is not bipartite. This means that $G[P]$ has an odd cycle C. As G is C_3-free and probe P_5-free, $G[P]$ must be (C_3, P_5)-free. Hence, C has length 5. Let $V(C) = \{v_1, \dots, v_5\}$ in that order. We will now use, with a little bit of extra care, the same arguments as in [63].

We first claim that every vertex not on C has a neighbour on C. Else, as G is connected, there exists a vertex $u \in V \setminus V(C)$ that is anti-complete to C and that has a neighbour v that is adjacent to at least one vertex on C, say $vv_1 \in E$. If $uvv_1v_2v_3$ is an induced P_5 in G, then $uvv_1v_2v_3$ is also an induced P_5 in $G + F$. The reason is that F contains no edge that is incident with a vertex from $\{v_1, v_2, v_3\}$, because v_1, v_2, v_3 all belong to P. As G is C_3-free and $G + F$ is P_5-free, this means that v must be adjacent to v_3 in G. By applying the same arguments on the path $uvv_3v_4v_5$, we find that v must be adjacent to v_5 in G as well. However, now v, v_1, v_5 form a triangle in G, contradicting the C_3-freeness of G.

We now claim that every vertex not on C has exactly two neighbours v_i and v_{i+2} on C for some $i \in \{1, \dots, 5\}$, where we write $v_6 := v_1$ and $v_7 := v_2$. Let $v \in V \setminus V(C)$. By the above, v has a neighbour on C, say v is adjacent to v_1. Recall that F does not contain any edges incident to vertices of C, as $V(C) \subseteq P$. Hence, if $vv_1v_2v_3v_4$ is an induced P_5 in G, then $vv_1v_2v_3v_4$ is also an induced P_5 in $G + F$. As G is C_3-free and $G + F$ is P_5-free, this means that v is either adjacent to v_3 (take $i = 1$) or to v_4 (take $i = 4$). Due to the above, we can decompose V as

$$V = V(C) \cup V_{1,3} \cup V_{2,4} \cup V_{3,5} \cup V_{4,1} \cup V_{5,2},$$

where for $i \in \{1, \dots, 5\}$, the set $V_{i,i+2}$ consist of all vertices of $V \setminus C$ whose neighbours on C are exactly v_i and v_{i+2}. We give v_1, v_2, v_3, v_4, v_5 colours $1, 2, 1, 2, 3$, respectively. Moreover, we colour all the vertices of $V_{1,3}$, $V_{2,4}$, $V_{3,5}$, $V_{4,1}$ and $V_{5,2}$ with colours $2, 1, 2, 3, 1$, respectively. As G is C_3-free, the five sets $V_{i,i+1}$ are all independent. Therefore, the only potential conflicts

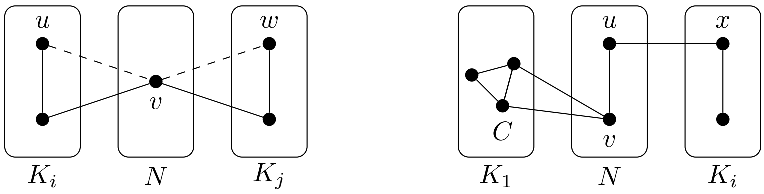

Figure 1 Left: Proof of Claim 9. The dashed lines indicate non-existing edges. Right: Proof of Claim 10. Note that $uv \in F$.

could be between vertices from $V_{1,3}$ and $V_{3,5}$, which are all coloured 2, or between vertices from $V_{2,4}$ and $V_{5,2}$, which are all coloured 1. However, the former vertices are all incident to v_3, and thus form an independent set in G, and similarly, the latter vertices are all adjacent to v_2, and thus als form an independent set in G. We conclude that we have indeed constructed a 3-colouring of G, completing the proof. ◀

We are now ready to show the main result of our paper, which we prove in the way as outlined at the end of Section 1.

▶ **Theorem 8.** 3-Colouring *is polynomial-time solvable for partitioned probe P_5-free graphs.*

Proof. Let (G, P, N) be a partitioned probe P_5-free graph. We may assume that G is connected; otherwise, we execute the given algorithm for every connected component of G. Let $F \subseteq \binom{N}{2}$ be such that $G + F$ is P_5-free. We define F only for verifying correctness; the polynomial-time algorithm does not use F. If G is P_5-free, then it is possible in polynomial time to determine whether G is 3-colourable [49]. Therefore, we may assume that G is not P_5-free and, in particular, $|N| \geq 2$ and $|F| \geq 1$. We may also assume that G does not contain a clique of order at least 4; otherwise, G is not 3-colourable. Let $K_1, \ldots, K_t$ be the connected components of $G[P]$ that contain at least one edge. We may assume at least one such connected component exists; else G is bipartite with partite sets P and N, and thus clearly 3-colourable in polynomial time.

Getting initial structure. We begin by proving two claims that describe the structure of edges between $K_1, \ldots, K_t$ and N.

▷ Claim 9. Every vertex of N that is neither complete nor anticomplete to K_i for some $i \in [t]$ is complete or anticomplete to K_j for every $j \in [t]$ with $j \neq i$.

Proof. Let $v \in N$ be neither complete nor anticomplete to K_i. Suppose that v has a neighbour in K_j, where $j \neq i$. It suffices to prove that v is complete to K_j. Assume, for a contradiction, that $w \in V(K_j)$ is not adjacent to v. By assumption, there exists $u \in V(K_i)$ that is not adjacent to v. A shortest u-v-path with internal vertices in K_i followed by a shortest v-w-path with internal vertices in K_j is induced in $G + F$ and has length at least 4; see Figure 1. Since such a path exists, there is an induced P_5 in $G + F$, a contradiction. ◁

If $K_1, \ldots, K_t$ are all bipartite, then G is clearly 3-colourable, since N is independent in G. Therefore, we may assume that $t \geq 1$ and K_1 is not bipartite. This implies that K_1 contains an induced odd cycle and, since K_1 is P_5-free because of $V(K_1) \subseteq P$, such a cycle has length 3 or length 5. We now pick an induced odd cycle C in K_1 as follows. If K_1 contains an induced C_5, then let C be any such C_5. If K_1 does not contain an induced C_5, but contains an induced C_3 that dominates K_1, then let C be any such C_3. Otherwise, we pick C to be an arbitrary C_3. Note that computing C is possible in polynomial time.

If a single vertex of $V(G) \setminus C$ dominates C, then G is clearly not 3-colourable. Hence, we may assume from here that this is not the case. This fact (that we often use implicitly) has important implications. In particular, no vertex of N dominates K_1. But also:

▷ Claim 10. Let $u \in N$ be a vertex with no neighbour in K_1. If u has a neighbour in K_i with $i \geq 2$, then a vertex of N with a neighbour in K_1 is complete to K_i.

Proof. Consider a shortest u-C-path Q in $G + F$. As u has no neighbour in K_1, Q has length at least 2. Let w be the vertex of C where Q ends and let v be the vertex on Q preceding w. Using the observation preceding the claim, v is not complete to C. We may thus assume that Q was chosen such that there exists a vertex $z \in N_C(w) \setminus N_{G+F}(v)$. If $v \in K_1$, then as u does not neighbour K_1, the path Qz has length at least 4, a contradiction to the fact that $G + F$ is P_5-free. Hence, $v \in N \setminus \{u\}$ and v is neither complete nor anticomplete to K_1. If v is not a neighbour of u in $G + F$, then Qz is an induced path in $G + F$ of length at least 4, a contradiction. Let x be a neighbour of u in K_i. If x is not a neighbour of v in $G + F$, then the path $xuvwz$ is an induced P_5 in $G + F$, a contradiction; see Figure 1 right. Hence, v has a neighbour in K_i, and the claim follows from Claim 9. ◁

▷ Claim 11. The connected components $K_2, \ldots, K_t$ are all bipartite or G is not 3-colourable.

Proof. Assume (without loss of generality) K_2 is not bipartite and G is 3-colourable. From our earlier observation, if some vertex is complete to K_1 or to K_2, then G is not 3-colourable, a contradiction. As G is connected, K_2 has a neighbour $u \in N$. Hence, u is neither complete or anticomplete to K_2. As u cannot be complete to K_1, by Claim 9, u is anticomplete to K_1. Then, by Claim 10, there is a vertex in N that is complete to K_2, a contradiction. ◁

We can check in linear time whether $K_2, \ldots, K_t$ are all indeed bipartite.

Colouring C. Let $K = K_1$ for brevity and $I = P \setminus V(K)$. Note that $G[I]$ consists only of isolated vertices and bipartite connected components. We branch on all partial 3-colourings ψ that only colour every vertex of C. There are constantly many branches, as there are only constantly many such partial 3-colourings. We propagate the colours through K by executing Algorithm 1 on (K, ψ). If an error occurred, then there is no 3-colouring of G that is an extension of ψ by Lemma 4 (ii), and we backtrack. So we may assume that no error occurred, and for simplicity we denote the returned extension of ψ by ψ again.

We explicitly only propagated the colours through K. We now partition of $V(K)$. Let

- K_c^i be the set of vertices of K with colour i for $i \in [3]$,
- $K_c = \bigcup_{i\in[3]} K_c^i$,
- K_u^i be the set of uncoloured vertices of K with a neighbour of colour i for $i \in [3]$,
- $K_u = \bigcup_{i\in[3]} K_u^i$, and
- $K_r = V(K) \setminus (K_c \cup K_u)$ consist of the remaining vertices of K.

Note that $G[K_c]$ is connected, because C is connected, and we assign colours to uncoloured vertices only with the Propagation Rule in Algorithm 1. Also note that the vertices of K_u^i have only neighbours of colour $i \in [3]$ since they are uncoloured.

Our ultimate goal is to apply Lemma 5. So far we are not in a position to apply it, since there may be vertices (for example in K_r) that do not have a coloured neighbour. In the remaining proof, we distinguish two cases, depending on the length of C.

Case 1: C has length 5. We show that all vertices already have a coloured neighbour.

▷ Claim 12. Every vertex of $V(G) \setminus K_c$ has a neighbour in K_c.

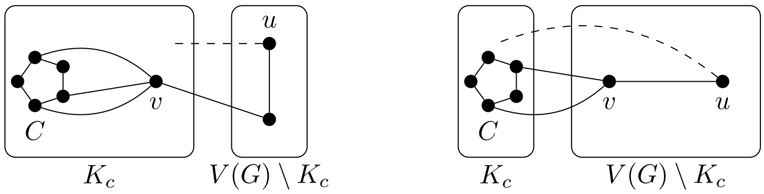

Figure 2 Proof of Claim 12. Dashed lines indicate non-existing edges.

Proof. Assume, for a contradiction, that $u \in V(G) \setminus K_c$ has no neighbour in K_c. Consider a shortest u-C-path Q in $G + F$. Let v be the vertex of Q that has a neighbour in C. Note that v is not complete to C; otherwise, we would have concluded that G is not 3-colourable. If v is in K_c itself, then Q has length at least 3, and there would be an induced P_5 in $G + F$ with vertices in $V(Q) \cup V(C)$; see Figure 2 left. Hence, v is not in K_c. Then v has at most two neighbours in C, and Q has length at least 2, and there would be an induced P_5 in $G + F$ with vertices in $V(Q) \cup V(C)$, a contradiction; see Figure 2 right. ◁

Claim 12 implies that Lemma 5 is applicable in this case. Therefore, deciding if there is a 3-colouring of G that is an extension of ψ is possible in polynomial time. If there is no such 3-colouring of G, then we backtrack.

Case 2: C has length 3. First, note that for every vertex $v \in K_c$, we have that v has two neighbours with two distinct colours in $[3] \setminus \{\psi(v)\}$, since C is a clique and we assign colours to uncoloured vertices only through the Propagation Rule in Algorithm 1. We now give a more precise partition of N; see Figure 3. Let $M = N_G(I)$ and $L = N \setminus M$. Let

- M_c and L_c be the set of vertices of M and L, respectively, that have two neighbours in K_c with two distinct colours,
- M_u^i and L_u^i be the set of vertices of $M \setminus M_c$ and $L \setminus L_c$, respectively, with a neighbour in K_c^i for $i \in [3]$,
- $M_u = \bigcup_{i \in [3]} M_u^i$, $L_u = \bigcup_{i \in [3]} L_u^i$,
- $M_r = M \setminus (M_c \cup M_u)$, and $L_r = L \setminus (L_c \cup L_u)$.

Let J be the set of vertices of I with no neighbour in M_c. Note that no vertex of K_r, L_r, M_r, and J has a coloured neighbour. We now show how in the end we can apply Lemma 5.

Handling K_r and L_r. Since $L \subseteq N$ is independent in G and G is connected, every vertex of L has a neighbour in K. If, in G, a vertex $v \in L_r$ has only neighbours in K_u^i for one $i \in [3]$, then a 3-colouring of $G - v$ that extends ψ can be extended to a 3-colouring of G by assigning colour i to v. We remove any such v from G and continue. Now, every vertex of L_r has neighbours in K_r^i for at least two distinct $i \in [3]$, or has a neighbour in K_r. We prove two claims, one for each of the two described types of vertices in L_r.

▷ Claim 13. For $i, j \in [3]$ with $i \neq j$, if $u \in L_r$ has a neighbour in K_u^i, and $v \in L_r$ has a neighbour in K_u^j, then u and v have the same neighbours in $K_u^i \cup K_u^j$.

Proof. Assume, for a contradiction, that $x \in K_u^i$ is a neighbour of u, but not a neighbour of v. Let $w \in K_u^j$ be a neighbour of v. Consider a shortest x-w-path Q in G with internal vertices in K_c. As Qv is not an induced P_5 in $G + F$, we must have $xw \in E(G)$. Let y be the neighbour of x in Q, and let z be a neighbour of y that is adjacent to neither x nor w. Note that z exists since every vertex of K_c has two neighbours of two distinct colours. Now, $vwxyz$ is an induced P_5 in $G + F$, a contradiction; see Figure 4 left. ◁

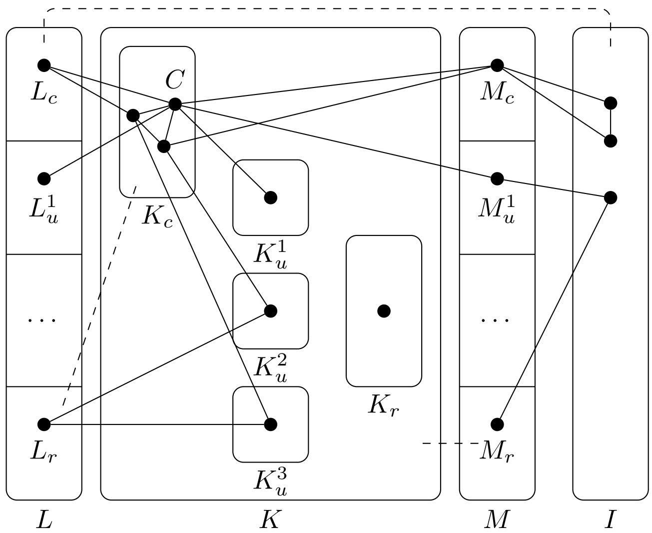

Figure 3 An illustration of the partition of K, L, and M. Note that $P = V(K) \cup I$ and $N = M \cup L$. Dashed lines indicate some of the non-existing edges.

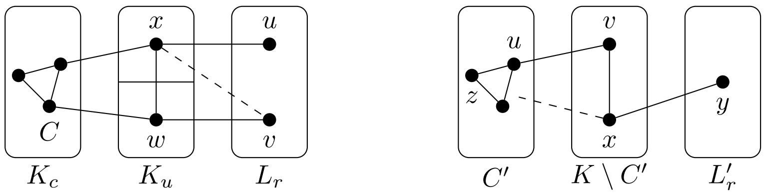

Figure 4 Left: Proof of Claim 13. Right: Proof of Claim 14. Dashed lines indicate non-existing edges.

Let L'_r be the set of vertices of L_r with a neighbour in K_r.

▷ Claim 14. A single vertex of K dominates the vertices of $K_r \cup L'_r$.

Proof. If $K_r = \emptyset$, then $L'_r = \emptyset$ and the statement is trivial. Hence, $K_r \neq \emptyset$. As K is a connected P_5-free graph, K contains a connected dominating set D that induces a P_3-free graph or a C_5 [19]. As we are in Case 2, D cannot be a C_5. Hence, D is a clique.

If $|D| \geq 4$, then G contains a clique of order at least 4, which we already excluded. If $|D| = 3$, then K contains a C_3 that dominates K. By the choice of C and the fact that we are in Case 2, C dominates K. Hence, our application of the Propagation Rule ensures that $K_r = \emptyset$, a contradiction. If $|D| = 1$ and the vertex of D is in C, then we arrive at a contradiction as before. If $|D| = 1$ and the vertex of D is not in C, then this vertex and C form a clique of order at least 4, which we already excluded. It remains that $|D| = 2$. In other words, K contains a dominating edge uv.

We must have that $N_K(u)$ and $N_K(v)$ are disjoint; otherwise, there would be a dominating triangle in K, which we can exclude as before. Without loss of generality, let $N_G(u)$ contain at least two vertices of C. This implies $u \in K_c$. As there is no edge between K_c and K_r by definition of K_r, v dominates K_r.

It remains to show that v is complete to L'_r. Suppose $y \in L'_r \setminus N_G(v)$ exists. Let $x \in K_r$ be a neighbour of y. As u neighbours two vertices of C and $u \in K_c$, vertex u is in a cycle C' of length 3, which is contained in K_c (possibly $C = C'$). Let $z \in V(C') \setminus \{u\}$. As $N_K(u)$ and

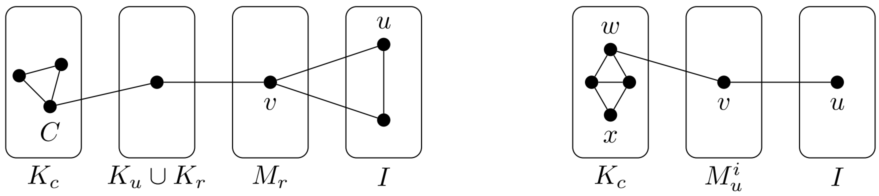

Figure 5 Left: Proof of Claim 16 (i). Right: Proof of Claim 16 (ii).

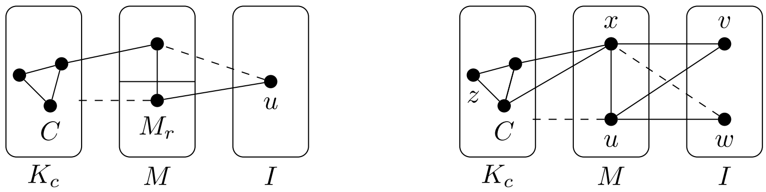

Figure 6 Left: Proof of Claim 17. Right: Proof of Claim 18. Dashed lines indicate non-existing edges.

$N_K(v)$ are disjoint, v is not adjacent to z. Also, z is not adjacent to x as $x \in K_r$ and $z \in K_c$, and y is not adjacent to u, as $y \in L_r$ and $u \in K_c$. As $zuvxy$ is not an induced P_5 in $G + F$, we obtain $vy \in E(G)$, a contradiction; see Figure 4 right. Hence, v dominates L'_r. ◁

Claim 13 and Claim 14 together imply that:

▷ Claim 15. $K_r \cup L_r$ is dominated by a set D of at most two vertices of K.

Handling M_r and J. We now describe the structure of the edges between K and $M = N_G(I)$.

▷ Claim 16. (i) Every vertex of M_r has no neighbour in K, and (ii) For every $i \in [3]$, M_u^i is complete to K_c^i.

Proof. We first prove (i). Suppose, for the sake of contradiction, that there exists a vertex $v \in M_r$ that has a neighbour in K. Since $v \in M_r \subseteq M$, v has a neighbour $u \in I$. By definition of M_r, v has no neighbour in K_c. Let Q be a shortest v-C-path in G with internal vertices in K. The path Q must contain a vertex of K_u^i for some $i \in [3]$ by assumption and therefore has length at least 2. Then there is an induced P_5 in $G + F$ with vertices in $\{u\} \cup V(Q) \cup V(C)$, a contradiction; see Figure 5 left.

We continue with (ii). For some $i \in [3]$, let $v \in M_u^i$ such that v is not complete to K_c^i. Since $v \in M$, v has a neighbour $u \in I$. Since $v \in M_u^i$, v has a neighbour in K_c^i. Let $x \in K_c^i$ be a non-neighbour of v. Let $w \in K_c^i$ be a neighbour of v that is closest to x in $G[K_c]$. Let Q be a shortest w-x-path in $G[K_c]$, which exists since $G[K_c]$ is connected. As $w, x \in K_c^i$, they are not adjacent. Thus, Q has length at least 2, and uvQ contains an induced P_5 in $G + F$, a contradiction; see Figure 5 right. Hence, for every $i \in [3]$, M_u^i is complete to K_c^i. ◁

We continue with two claims describing the structure of the edges between I and $M_c \cup M_u$.

▷ Claim 17. Every vertex of I has a neighbour in $M_c \cup M_u$.

Proof. Assume, for a contradiction, that the vertex $u \in I$ only has neighbours in M_r. Since every vertex of M_r has no neighbours in K by Claim 16 (i), a shortest u-C-path Q in $G + F$ has length at least 3. This implies that there is an induced P_5 in $G + F$ with vertices in $V(Q) \cup V(C)$, a contradiction; see Figure 6 left. The claim follows. ◁

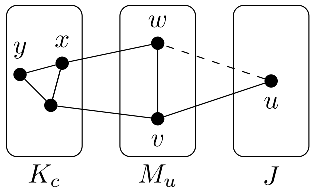

Figure 7 Proof of Claim 20. Dashed lines indicate non-existing edges.

▷ Claim 18. If $u \in M_r$, then every vertex of $N_G(u)$ has the same neighbours in $M_c \cup M_u$.

Proof. Note that $N_G(u) \subseteq I$ by Claim 16 (i) and since $M_r \subseteq N$ is independent. Let $v, w \in N_G(u)$. Assume, for a contradiction, that the vertex $x \in M_c \cup M_u$ is a neighbour of v, but not a neighbour of w. By considering a shortest u-C-path in $G + F$ containing the vertices v and x, we see that $ux \in F$. Let $z \in K_c$ be a vertex that is not adjacent to x in G, which exists, or G would not be 3-colourable. Therefore, wux together with a shortest x-z-path with internal vertices in K_c contains an induced P_5 in $G + F$, a contradiction; see Figure 6 right. As $v, w \in N_G(u)$ were arbitrary, the proof is complete. ◁

Claim 19 is an important consequence of Claims 17 and 18.

▷ Claim 19. If there is 3-colouring ψ' of $G - M_r$ that is an extension of ψ, then there is a 3-colouring of G that is an extension of ψ'.

Proof. Assume, for a contradiction, that for a vertex $u \in M_r$, there exist vertices $v_i \in N_G(u)$ with $\psi'(v_i) = i$ for every $i \in [3]$. Note that $v_1, v_2, v_3 \in I$ by Claim 16 (i). By Claim 17 and Claim 18, there exists a vertex $w \in M_c \cup M_u$ that is adjacent to v_1, v_2, and v_3, a contradiction to the fact that ψ' is a 3-colouring of $G - M_r$. Therefore, for every vertex $u \in M_r$, there is a colour $i \in [3]$ such that no neighbour of u in G has colour i under ψ'. At this point, choosing any such colour for every vertex of M_r gives a 3-colouring of G that is an extension of ψ'. ◁

Claim 19 implies that it suffices to decide if there is a 3-colouring $G - M_r$ that is an extension of ψ. Hence, from now on, assume that $M_r = \emptyset$. Recall that J is the set of vertices of I with no neighbour in M_c. Consequently, by Claim 17, every vertex of J has a neighbour in M_u. Claim 9 implies that every vertex of $M_c \cup M_u$ is either complete or anticomplete to each connected component of $G[I]$. It follows that, if $u \in J$, then J contains all vertices of the connected component of u in $G[I]$. We prove one more claim about the structure of the edges between M_u and J.

▷ Claim 20. If M_u^i is nonempty for at least two $i \in [3]$, then the bipartite subgraph of G spanned by the edges of G with one end in M_u and the other end in J is complete.

Proof. Let K' be an arbitrary connected component of $G[J]$. Keep in mind that K' is a connected component of $G[I]$ too. Let $i, j \in [3]$ with $i \neq j$ be such that M_u^i is nonempty, and K' has a neighbour v in M_u^j. Note that such i and j exist by assumption and Claim 17, and v is complete to K' by Claim 9. We prove that M_u^i is complete to K'.

Assume, for a contradiction, that $w \in M_u^i$ has no neighbour in K'. Let u be an arbitrary neighbour of v in K'. Consider a shortest v-w-path Q with internal vertices in K_c, which exists since $G[K_c]$ is connected. As $i \neq j$, the path Q has length at least 3, and, by Claim 16 (ii), the path Q has length exactly 3. Since uQ is not an induced P_5 in $G + F$, we have $vw \in F$. Let x be the neighbour of w in Q, let $k \in [3] \setminus \{i, j\}$, and let y be a neighbour of x in K_c with colour k. Note that y exists since every vertex of K_c has two neighbours in K_c with two distinct colours. Now $uvwxy$ is an induced P_5 in $G+F$, a contradiction; see Figure 7. So w has a neighbour in K'. Claim 9 implies that w is complete to K'. Since $w \in M_u^i$ was chosen arbitrarily, this proves that M_u^i is complete to K'.

A similar argument shows that for $k \in [3] \setminus \{i,j\}$, if M_u^k is nonempty, then M_u^k is complete to K' too. By interchanging the roles of i and j, we see that M_u^j is complete to K'. Since K' was chosen arbitrarily, and since every such connected component of $G[J]$ has a neighbour in M_u by Claim 17, this completes the proof. ◁

Colouring G. At this point, we are in a position to decide if there is a 3-colouring of G that is an extension of ψ. First, Claim 15 implies that $K_r \cup L_r$ is dominated by a set D of at most two vertices of K. We branch on the (constantly many) consistent extensions of ψ into 3-colourings that additionally colour every vertex of D, which we call ψ again for simplicity.

Observe that every vertex of $K_r \cup L_r$ has a coloured neighbour now. As $M_r = \emptyset$, we now only need to achieve the same for J in order to apply Lemma 5. If $J = \emptyset$, then Lemma 5 is directly applicable. Therefore, we decide in polynomial time if there is a 3-colouring of G that is an extension of ψ. If there is no such 3-colouring, then we backtrack.

We now assume that $J \neq \emptyset$. Since vertices in J are not adjacent to M_c by definition and $M_r = \emptyset$, $M_u \neq \emptyset$. If M_u^i is nonempty for at least two $i \in [3]$, then we choose a vertex $v \in M_u$. We branch on the extensions ψ' of ψ that additionally colour v. Observe that now every vertex of I has a coloured neighbour under ψ' by the definition of J and Claim 20. Now, Lemma 5 is applicable. Therefore, we decide in polynomial time if there is a 3-colouring of G that is an extension of ψ'. If there is no such 3-colouring, then we backtrack.

It remains the case that there is exactly one $i \in [3]$ such that M_u^i is nonempty. Every vertex in J has neighbours only in $J \cup M_u^i$. In particular, for each connected component K' of $G[J]$, which is a connected component of $G[I]$, the colour i may be used without creating conflicts outside of K'. Recall that K' is bipartite by Claim 11. Hence, we wish to extend ψ by, for each connected component K' of $G[J]$, colouring one of its partite set by colour i. However, we cannot immediately decide which partite set, and make a small detour.

Let K' be a connected component of $G[J]$ that contains an edge. Let u be a neighbour of K' in M_u^i. Since $u \in M_u^i$, it is adjacent to K, and thus neither complete nor anticomplete to K. Hence, u is complete to K' by Claim 9. Thus, $N_G(K')$ is complete to K'. Therefore, all vertices of $N_G(K')$ must receive the same colour in any 3-colouring of G that extends ψ. We ensure this first, for each such connected component K', and then extend the colouring to J.

We apply the formula $\mathcal{F}$ of Lemma 5 to $G - J$, adapted as follows. For every connected component K' in $G[J]$ that contains an edge, and for every two distinct vertices $u, v \in N_G(K')$, we add the clauses $(\bar{x}_u^k \vee x_v^k) \wedge (x_u^k \vee \bar{x}_v^k)$ to $\mathcal{F}$ for every $k \in [3] \setminus \{i\}$. These clauses ensure that two such vertices u and v receive the same colour. (Alternatively we could identify these vertices. At this point it does not matter that this does not preserve probe P_5-freeness.) After that, we resolve the satisfiability of the 2-SAT formula $\mathcal{F}$ in polynomial time [3]. If $\mathcal{F}$ is not satisfiable, then there is no 3-colouring of G that is an extension of ψ, and we backtrack. Otherwise, let ψ' be a 3-colouring of $G - J$ obtained from a satisfying assignment of $\mathcal{F}$. We can extend ψ' to a 3-colouring of G by assigning colour i to isolated vertices in $G[J]$, and by assigning the remaining two colours to the nontrivial bipartite connected components of $G[J]$, which is possible due to the extra clauses we added to $\mathcal{F}$. This completes the proof of Theorem 8. ◀

5 The Proof of the NP-Completeness Part of Theorem 3

Theorem 3 states that for $t \geq 1$, 3-Colouring on partitioned probe P_t-free graphs is polynomial-time solvable if $t \leq 5$ and NP-complete if $t \geq 6$. In Section 4 we showed the polynomial part of Theorem 3. We now show the NP-completeness part by reducing from 1-Precolouring Extension, which has as input: an integer $k \geq 3$, a graph G with at least k vertices, and a partial k-colouring ψ of G that assigns k vertices $v_1, \dots, v_k$ colours $1, \dots, k$, respectively. Can ψ be extended to a k-colouring of G? Bodlaender et al. [7] proved that this problem is NP-complete, even if $k = 3$, G is bipartite and the precoloured vertices all belong to the same partition set of G. Cai [18] used this result to prove that 3-Colouring is NP-complete for graphs that become bipartite by deleting three edges. We use the gadget of [18] to show the following, which implies the NP-completeness part of Theorem 3.

▶ **Theorem 21.** 3-Colouring *is* NP*-complete on partitioned probe* $(P_6, 3P_2, 2P_3)$*-free graphs.*

Proof. We reduce an instance $(3, G, \psi, \{v_1, v_2, v_3\})$ of 1-Precolouring Extension, where G is a bipartite graph with bipartition A and B, and the precoloured vertices v_1, v_2, v_3 belong to A without loss of generality, to an instance of 3-Colouring. As mentioned, this variant of 1-Precolouring Extension is still NP-complete [7]. The bipartition of G can be computed in polynomial time. Let G' be the graph from [18], which is obtained in polynomial-time from G by turning $\{v_1, v_2, v_3\}$ into a clique. The graph G' is probe $(P_6, 2P_3, 3P_2)$-free, which is witnessed by the fact that the graph obtained from G' by turning the independent set B into a clique is $(P_6, 2P_3, 3P_2)$-free. It is easy to see that $(3, G, \psi, \{v_1, v_2, v_3\})$ is a yes-instance of 1-Precolouring Extension if and only if (G', A, B) is a yes-instance of 3-Colouring. This proves that 3-Colouring is NP-hard on partitioned probe $(P_6, 2P_3, 3P_2)$-free graphs. ◀

6 Additional Results and Concluding Remarks

In our paper, we considered the probe graph model introduced by Zhang et al. [64]. Our aim was to research *to what extent* polynomial-time results for H-free graphs can be extended to probe H-free graphs. We first gave a dichotomy for Colouring restricted to (partitioned) probe H-free graphs and then showed our main result, which states that the known polynomial-time result for 3-Colouring for P_5-free graphs (whose yes-instances all have bounded mim-width) can be extended to partitioned probe P_5-free graphs (whose yes-instances even have unbounded sim-width). We also proved that this result cannot be generalized to partitioned probe P_6-free graphs unless P = NP by showing NP-completeness even for partitioned $(P_6, 3P_2, 2P_3)$-free graphs. As 3-Colouring is polynomial-time solvable even for P_7-free graphs [9] and sP_2-free graphs for all $s \geq 1$ [32], our results give a clear indication of the difference in computational complexity if not all edges of the input graph are known, under the probe graph model. They also lead to a range of natural directions for future work, as we discuss below.

First, the dichotomy for 3-Colouring for partitioned probe H-free graphs has not been fully settled. We are able to prove the following additional result (proof omitted):

▶ **Theorem 22.** *For every* $s \geq 0$*,* 3-Colouring *is polynomial-time solvable on partitioned probe* $(P_3 + sP_1)$*-free graphs.*

Theorem 3, Theorems 21–22 and the result that 3-Colouring is NP-complete on H-free graphs if H is not a linear forest [37, 50] leave only the following open cases:

▶ **Problem 23.** *Determine the complexity of* 3-Colouring *on partitioned probe H-free graphs when H is $2P_2 + sP_1$ ($s \geq 1$), $P_3 + P_2 + sP_1$ ($s \geq 0$), $P_4 + sP_1$ ($s \geq 1$), $P_4 + P_2 + sP_1$ ($s \geq 0$), or $P_5 + sP_1$ ($s \geq 1$).*

Second, since k-Colouring is polynomial on P_5-free graphs [49] even for all $k \geq 3$, we ask:

▶ **Problem 24.** *For $k \geq 4$, determine the complexity of k-*Colouring *on partitioned probe P_5-free graphs.*

Crucial properties in our proof for 3-Colouring on partitioned probe P_5-free graphs, such as the fact that there is a single non-bipartite connected component and that no vertex is complete to the cycle C we pick in it, no longer hold if $k \geq 4$. We do note that probe (K_s, P_5)-free graphs have bounded mim-width for every $s \geq 1$, due to (K_s, P_5)-free graphs having bounded mim-width for every $k \geq 1$ [14] and Proposition 1. Hence, as we may assume that an input graph for 4-Colouring is K_5-free, a good starting point is to consider 4-Colouring for K_5-free partitioned probe P_5-free graphs, or even for K_5-free partitioned probe $2P_2$-free graphs.

We recall that for solving 3-Colouring on probe P_5-free graphs, we only need the partition into P and N. To solve Problem 24, it would also be interesting to research if the problem becomes easier under the assumption that we also know the set of edges F.

As another starting point for solving Problem 24, we can show the following result (proof omitted):

▶ **Theorem 25.** *For every $s \geq 0$ and $k \geq 1$, k-*Colouring *is polynomial-time solvable on (not necessarily partitioned) probe $(P_2 + sP_1)$-free graphs.*

We note that for $k = 3$, Theorem 25 does not need the partition $V = P \cup N$, whereas Theorem 22 does, so the two theorems are not comparable.

We could prove Theorem 25 by showing that for all $s \geq 1$, every probe $(P_2 + sP_1)$-free graphs is $(s + 1)P_2$-free, and we ask:

▶ **Problem 26.** *Are there other sets $\mathcal{H}$, for which there exists a finite set $\mathcal{H}'$ such that every probe $\mathcal{H}$-free graph is $\mathcal{H}'$-free?*

An inclusion as in Problem 26 is in general strict, e.g., probe P_5-free graphs form a hereditary graph class that is not finitely defined. To explain this, as probe P_5-free graphs are closed under vertex deletion, there is a unique minimal set of graphs $\mathcal{F}_{P_5}$ such that a graph is probe P_5-free if and only if it is $\mathcal{F}_{P_5}$-free. We can show that $\{C_7, C_9, C_{11}, \ldots\} \subsetneq \mathcal{F}_{P_5}$ (proof omitted). So, in particular, probe P_5-free graphs form a proper subclass of $(C_7, C_9, C_{11}, \ldots)$-free graphs, that is, graphs with no odd hole of length at least 7. This leads to the following natural question:

▶ **Problem 27.** *Determine the complexity of* 3-Colouring *for $(C_7, C_9, C_{11}, \ldots)$-free graphs.*

It is known that 3-Colouring is polynomial-time solvable for odd-hole free graphs, i.e., $(C_5, C_7, C_9, \ldots)$-free graphs (see [59]) and $(K_4, C_7, C_9, C_{11}, \ldots)$-free graphs are χ-bounded [27].

Knowing more about $\mathcal{F}_{P_5}$ would help solving the following open problem, which is solved for $H = P_4$ in polynomial time [22] and which in turn might help solving 3-Colouring on probe P_5-free graphs without a given partition (P, N) of their vertex set.

▶ **Problem 28.** *Determine the complexity of recognizing probe P_5-free graphs.*

We also ask for which other graph classes $\mathcal{G}$, are Colouring and k-Colouring polynomially solvable on the class of (partitioned) probe graphs $\mathcal{G}_p$? Recall from Section 1 that Colouring is polynomial-time solvable for probe chordal graphs (as these graphs are perfect) and that in 2012, Chandler et al. [21] conjectured the same for partitioned probe perfect graphs.

We finish with two other, *broader* directions for future work. First, we recall our result (Proposition 7) that C_3-free probe P_5-free graphs (and thus all probe (C_3, P_5)-free graphs) are 3-colourable, which generalizes the same result for (C_3, P_5)-free graphs [63]. There is a long history for obtaining constant (tight) bounds on the chromatic number of (H_1, H_2)-free graphs; see also the recent survey [24]. In particular, for all $r, t \geq 1$, every (K_r, P_t)-free graph can be coloured with at most $(t-2)^{r-2}$ colours [44], improving an older result in [47]. It is also known e.g. that every (C_3, sP_2)-free graph is $(2s-2)$-colourable for all $s \geq 3$ [11], and that every $(C_3, 2P_3)$-free graph [61], every $(C_3, P_2 + P_4)$-free graph [16] and every $(K_4, 2P_2)$-free graph [40] is 4-colourable. In particular, subclasses of P_5-free graphs are well studied; see [23, 34, 35] for several of such results. *To what extent can all these results be extended to the probe version?* Proposition 7 illustrates there exist graph classes with a positive answer.

Second, our results indicate that the reason for polynomial-time solvability of 3-Colouring for P_5-free graphs goes beyond boundedness of mim-width. The question, which may also be asked for other problems, is if we can push further in this direction. So far, we assumed that the set of non-probes N is an independent set. This is an extreme case in a more general model in which we assume that we *do* know some of the edges in $G[N]$.

▶ **Problem 29.** *Determine if* 3-Colouring *is polynomially solvable on graphs* (G, P, N) *that can be made* P_5*-free by adding new edges to* $G[N]$*, which may have some initial edges.*

If we allow an arbitrary initial set of edges in $G[N]$, then the problem is already NP-complete on graphs that can be made $2P_1$-free: take the graph $G = (V, E)$ used in any NP-hardness reduction for 3-Colouring; set $N := V$; and let F be the set of all edges that are not already in G (so $G + F$ is complete, that is, $G + F$ is $2P_1$-free).

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