Covering a Polyomino-Shaped Stain with Non-Overlapping Identical Stickers

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Abstract

You find a stain on the wall and decide to cover it with non-overlapping stickers of a single identical shape (rotation and reflection are allowed). Is it possible to find a sticker shape that fails to cover the stain? In this paper, we consider this problem under polyomino constraints and complete the classification of always-coverable stain shapes (polyominoes). We provide proofs for the maximal always-coverable polyominoes and construct concrete counterexamples for the minimal not always-coverable ones, demonstrating that such cases exist even among hole-free polyominoes. This classification consequently yields an algorithm to determine the always-coverability of any given stain. We also show that the problem of determining whether a given sticker can cover a given stain is NP-complete, even though exact cover is not demanded. This result extends to the 1D case where the connectivity requirement is removed. As an illustration of the problem complexity, for a specific hexomino (6-cell) stain, the smallest sticker found in our search that avoids covering it has, although not proven minimum, a bounding box of 325×325.

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Supplementary Material *Software (Source Code)*: `https://github.com/ogiekako/pub_polycover` archived at `swh:1:dir:fc95d1b4d48db39869693c172544b125d185ab56`

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1 Introduction

In this paper, we consider the problem of covering a shape with non-overlapping identical shapes. Let U be a set, $\mathcal{P}, \mathcal{Q} \subset 2^U$ be families of subsets of U, and let an equivalence relation $\sim$ be fixed on $\mathcal{P}$. For a given $P \in \mathcal{P}$ (sticker) and $Q \in \mathcal{Q}$ (stain), if there are $P_1, \cdots, P_k \in \mathcal{P}$ (where k is a non-negative integer) such that each of them is equivalent to P, $P_i \cap P_j \neq \emptyset \Rightarrow i = j$, and $Q \subset P_1 \cup \cdots \cup P_k$, we say that P *flatly covers* Q. Hereafter, unless otherwise noted, "*cover*" will mean "flatly cover". We say that Q is *always-coverable* if for any $P \in \mathcal{P}$, P covers Q. In this work, we primarily focus on the case where shapes are polyominoes and the equivalence relation is by isometries (translations, rotations, and reflections).

The main contribution of this paper is a complete characterization of always-coverable polyominoes. We provide a necessary and sufficient condition for a polyomino to be always-coverable. Surprisingly, no polyomino of size 7 or larger is always-coverable. For every polyomino that is always-coverable, we analytically prove that it can be covered by any polyomino sticker, and for every polyomino that is not always-coverable, we provide a concrete sticker that cannot cover it. These were found through comprehensive computational search and rigorous verification.

Based on this, we introduce the decision problem TROUBLESOME STICKER.

> TROUBLESOME STICKER $(\mathcal{P}, \sim, \mathcal{Q})$
>
> Input: $Q \in \mathcal{Q}$
>
> Question: Is Q *always-coverable*? That is, is it true that for any $P \in \mathcal{P}$, P covers Q?

In this paper, (1) we provide a necessary and sufficient condition for a polyomino to be always-coverable. Surprisingly, no polyomino of size 7 or larger is always-coverable. For every polyomino that is always-coverable, we prove that it can be covered by any polyomino sticker, and for every polyomino that is not always-coverable, we provide a concrete sticker that cannot cover it. (2) We prove NP-completeness of FLAT COVER for polyominoes, which asks if a given stain can be covered by a given sticker, and NP-completeness of a 1D variant without connectivity requirement.

There are various studies on the problem of covering a plane with one type of 2D shape (TILING). The discovery of shapes which only admit aperiodic tiling [19] is well known. Restricting the domain from arbitrary 2D shapes to polyominoes, the necessary and sufficient condition for one type of polyomino to cover a plane by translation only, and an efficient way to determine the condition, are known [2, 7]. The restriction on the number of types is essential because if we allow the use of 11 different types of polyominoes, the decision problem becomes undecidable [17]. Even if the connectivity requirement is dropped, it is known that it is still decidable whether one type can cover a plane [3]. If the region to be (exactly) covered is not a plane but a given polyomino, then the problem of tiling a given region with one type of tromino (whether of type L or I), allowing rotations, is known to be NP-complete and ASP-complete [15, 11], and the corresponding counting problem is $\sharp$P-complete [15].

While TILING requires that the region be exactly covered, some studies discuss problems that allow the covering to exceed the region to be covered. One such problem is whether a set of arbitrary k points can be covered by circular coins of the same size. This problem was popularized by Peter Winkler's column [20] which discussed the problem for $k = 10$ proposed by Naoki Inaba [12], and while it is known that there is an arrangement of $k = 53$ points that cannot be covered [16], and it is proven that there is no such arrangement for $k \leq 12$ [20], the problem is unsolved for intermediate values of k to the best of our knowledge.

Common to the problems we have seen so far is the requirement to cover a region with non-overlapping congruent shapes. We refer to the target region as the *stain* and the covering shape as the *sticker*. In the problem of covering points with coins, the shape of the stickers (disk of a given size) was already given, and the shape of the stain that cannot be covered by them was of our interest. Our study is motivated by the idea of swapping the roles of the input: instead of fixing the sticker, we fix the stain. That is, we consider the problem of finding a sticker that cannot cover a given stain (TROUBLESOME STICKER). We will see this in detail in Section 3, but we give here an example. If the stain is the pentomino I (Figure 1), is there a polyomino that cannot cover it (allowing rotations and reflections)? The answer is yes, and Figure 2 shows the smallest such polyomino we know of [13]. It is interesting that the complex shape emerges from a simple question. The main result we present in this paper is that for every polyomino-shaped stain, we have either shown a polyomino that cannot cover it, or given a proof that any polyomino can cover it. Polyominoes were chosen

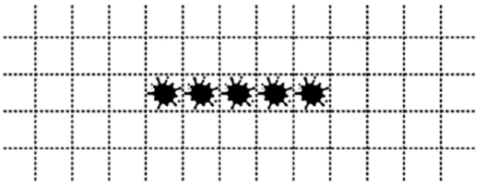

Figure 1 Pentomino I (1×5) shaped stain.

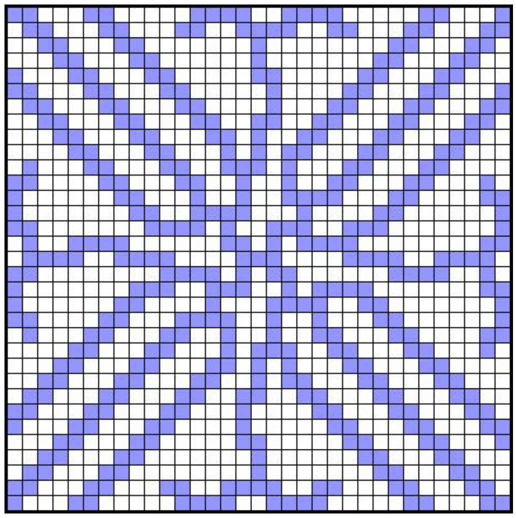

Figure 2 The smallest known polyomino (33×33) that never covers pentomino I (Found manually by Toshihiro Shirakawa on Feb. 13, 2010[*]).

as our domain because they are a popular subject of study (popularized by the work of Solomon Golomb [10] and Martin Gardner [8]), and are easy to handle by computer programs thanks to their discrete nature. Although not treated in this paper, studying the problem on continuous objects will also be fun.

The last section presents open problems that arose during our research process.

2 Preliminaries

A non-empty finite set $P \subset \mathbb{Z}^2$ is called a *polyomino* if and only if it induces a connected subgraph on the integer grid (where edges connect points at distance 1).

Let P be a polyomino. A polyomino obtained by *translating* P is any polyomino that is equal to $P + d$ for some $d \in \mathbb{Z}^2$. A polyomino obtained by *rotating* P is any polyomino obtained by rotating P around the origin by a multiple of 90 degrees. A polyomino obtained by *reflecting* P is any polyomino obtained by reflecting P about the x or y axis. We consider the positive direction of the x-axis as the *right direction* and the positive direction of the y-axis as the *up direction*.

$P + \left[-\frac{1}{2}, \frac{1}{2}\right]^2$, which is a subset of $\mathbb{R}^2$, is commonly called *the animal of P*. Since there is a one-to-one correspondence between a polyomino and its animal, it is common practice to draw its animal when illustrating a polyomino. If specific coordinates are omitted in the illustration, it means the origin can be anywhere. Sometimes $p \in \mathbb{Z}^2$ is illustrated by the animal of $\{p\}$ and is called a *square*. *The upper right point of a square p* is the upper rightmost point of the animal of $\{p\}$, and the same applies to upper left, lower left, and lower right. A square contained in the polyomino under consideration is sometimes called its *cell*.

The bounding box of P is the smallest rectangle with both sides parallel to the x and y axes that includes the animal of P. *The top side of P* is the top side of the bounding box of P. Note that reflecting P about its top side yields another polyomino, which has no

[*] Toshihiro Shirakawa, personal communication, Jan. 20, 2026.

element in common with P. The bottom, left, and right sides of P are defined in the same way. *The width (resp. height) of* P is the width (resp. height) of the bounding box of P. For $p \in P$, *the row index of* p *in* P is $\frac{1}{2}$ plus the difference between the y coordinate of the top side of P and the y coordinate of p (the minimum possible row index is 1). *The* i*-th row of* P is the set of elements in P whose row index in P is i.

Two polyominoes are defined to *overlap* if their intersection is not empty. Note that since polyominoes are defined as subsets of $\mathbb{Z}^2$, the condition of being non-overlapping naturally avoids geometric boundary contact issues that occur when considering regions in $\mathbb{R}^2$. The operation of taking the union of two polyominoes that do not overlap is specifically denoted by $\oplus$.

In Section 3 and Appendix A, we use the term *include* in a specific sense: for a set to *include* a polyomino Q means that the set contains a congruent copy of Q (i.e., at least one of the polyominoes obtained by arbitrarily translating, rotating, or reflecting Q).

3 Troublesome Sticker on Polyominoes

Let $\mathcal{P}$ be the set of all polyominoes. We write $\sim_E$ to denote the equivalence relation that equates two polyominoes that can become the same after applying arbitrary translations, rotations, and reflections. In this section we show the following theorem.

▶ **Theorem 1.** *There exists an algorithm that solves* Troublesome Sticker $(\mathcal{P}, \sim_E, \mathcal{P})$ *in* $O(1)$ *time.*

For a polyomino Q, we say that Q is *always-coverable* if any polyomino can cover Q. Theorem 1 in other words means we can determine in $O(1)$ time whether a given polyomino is always-coverable or not. This theorem is obtained as a direct corollary of the following characterization theorem:

▶ **Theorem 2** (Characterization theorem). *A polyomino is always-coverable if and only if it does not include any polyomino in* $\mathcal{I}$ *(Figure 3). Specifically, polyominoes in* $\mathcal{I}$ *are not always-coverable, while polyominoes in* $\mathcal{J}$ *are always-coverable.*

It is clear that the following remarks hold.

▶ Remark 3.
- If Q is always-coverable, any polyomino included in Q is also always-coverable.
- If Q is not always-coverable, any polyomino that includes Q is also not always-coverable.

▶ Remark 4. Suppose that a polyomino Q is not always-coverable. Then, for any positive integer n, there exists a polyomino whose height, width, and number of cells are all greater than or equal to n that cannot cover Q.

Proof. Given a sticker P_0 that fails to cover Q, define P'_i as the union of P_i and the reflection of P_i about its right side, and P_{i+1} as the union of P'_i and the reflection of P'_i about its top side, then $P_{\lceil \log_2 n \rceil}$ is a polyomino whose height, width, and number of cells are all $\geq n$ that cannot cover Q. ◀

3.1 Not always-coverable Polyominoes

From Remark 11 (see Section 4), determining whether a specific sticker covers a specific small stain is computationally tractable. Thus, verifying a potential counterexample (a sticker that fails to cover the stain) is straightforward using a computer. We computationally found and verified counterexamples for all polyominoes in the set $\mathcal{I}$ (Figure 3). The exact shapes of these counterexamples are detailed in Appendix B, establishing the following theorem.

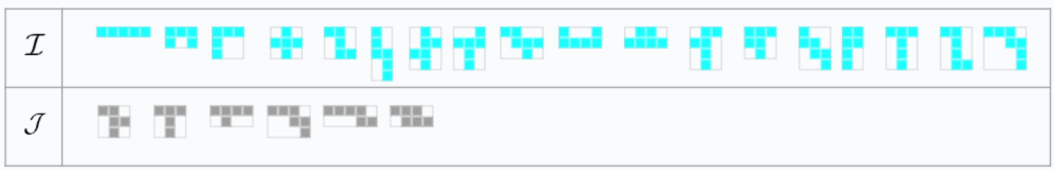

Figure 3 Not always-coverable polyominoes $\mathcal{I}$ and always-coverable polyominoes $\mathcal{J}$.

Figure 4 Pentomino "Y".

Figure 5 Pentomino "T".

▶ **Lemma 5.** *Polyominoes in $\mathcal{I}$ are not always-coverable.*

Every polyomino with 7 cells includes at least one polyomino in $\mathcal{I}$. (This can be easily shown by a computational search). From this and Remark 3, we obtain the following.

▶ **Corollary 6.** *Polyominoes with 7 or more cells are not always-coverable.*

The algorithm used to search for counterexamples is based on the simulated annealing method [14]. Simulated annealing is a probabilistic technique for approximating the global optimum of a given function, operating by gradually decreasing the probability of accepting worse solutions as it explores the search space. For computational efficiency, the shape of the stickers examined by the algorithm was restricted to rotationally and reflectionally symmetric trees (only the central part may be asymmetric so that the candidate itself does not include the stain). Neighborhoods for the annealing method include adding/removing a cell, random state flipping in a 2×2 and 3×3 region, and swapping the states of two random squares. The penalty is basically the number of ways to cover the stain with two stickers, but it is made higher if the stain can be covered with the cells near the side of the sticker or the outermost diagonal line that intersects the sticker. Also, for each way to cover the stain with two stickers, the smaller the number of ways to prevent the covering by adding a cell, the higher the penalty. The resulting behavior we see is that, over time, the average position of the cells contributing to the covering moves away from the side or diagonal line towards the center, and eventually a counterexample is found. We applied speed-up techniques such as delta-updating the penalty, and limiting the depth of interference of two stickers at the expense of the accuracy of the coverability check, and after finding candidates for counterexamples, these were fully checked using a naive algorithm. The program used is available for download at `https://github.com/ogiekako/pub_polycover`.

3.2 Always-coverable Polyominoes

For each polyomino in $\mathcal{J}$ in Figure 3, we prove that it is always-coverable by contradiction. That is, we assume that there exists a polyomino (counterexample) which cannot cover the stain and derive a contradiction. In this section we give a proof that pentomino Y (Figure 4) and pentomino T (Figure 5) are always-coverable, and in Appendix A, we give a proof for other polyominoes. From Remark 4, we can assume that the counterexamples we assume in our proof are sufficiently large.

▶ **Lemma 7.** *Pentomino Y is always-coverable.*

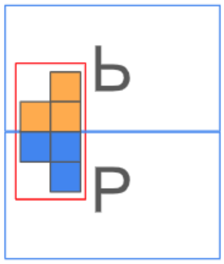

Figure 6 Consecutive cells in the first row create a contradiction.

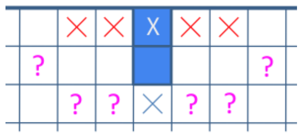

Figure 7 A contradiction occurs if X is connected to any ? .

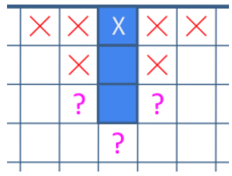

Figure 8 A contradiction occurs if P contains any ? .

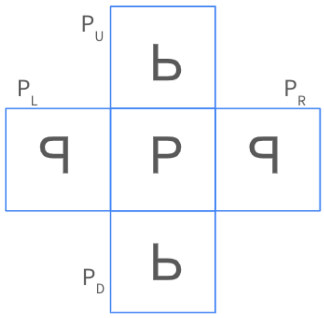

Figure 9 Definitions of $P_{\mathrm{U}}, P_{\mathrm{D}}, P_{\mathrm{L}}, P_{\mathrm{R}}$.

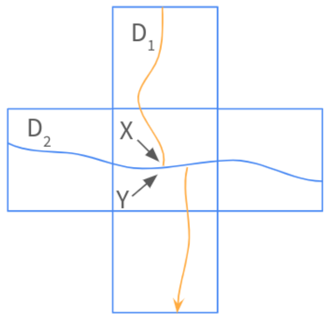

Figure 10 Definitions of D_1, D_2, X, Y.

Proof. Let Q be a pentomino Y and assume that Q cannot be covered by a sufficiently large polyomino P. In particular, we assume the width and height of P are both greater than 6. Let P_{rev} be the reflection of P about its top side. There are no two consecutive cells (cells with a distance of 1) in the first row of P, because otherwise there should be two consecutive cells in the first row one of which is adjacent to a cell in the second row, and thus $P_{\mathrm{rev}} \oplus P$ includes Q (Figure 6), which is a contradiction. Hence, the polyomino obtained by translating P_{rev} by one square to the left (resp. right) and then by one square down, which we call P_{L} (resp. P_{R}), does not overlap P. This means that there are also no two cells in the first row of P with a distance of 2 (otherwise, $P_{\mathrm{L}} \oplus P$ would include Q). Hence, the translation of P_{L} (resp. P_{R}) by one square to the left (resp. right), which we call P_{L2} (resp. P_{R2}), also does not overlap P. We now turn our attention to an arbitrary cell X in the first row of P. We can say that there is a cell on the square two below X (otherwise, there would be a cell at one of the positions marked by ? in Figure 7 that connects to X without passing through any other square marked with ? , but in any case, one of $P_{\mathrm{L2}} \oplus P, P_{\mathrm{L}} \oplus P, P_{\mathrm{R}} \oplus P, P_{\mathrm{R2}} \oplus P$ would include Q, which is a contradiction). Therefore, P does not contain the squares lower left and lower right of X (otherwise, $P_{\mathrm{rev}} \oplus P$ would include Q). Hence, the translation of P_{L} (resp. P_{R}) by one square down, which we call P_{LD} (resp. P_{RD}), also does not overlap P. There is a cell in one of the positions marked by ? in Figure 8, but in any case it leads to a contradiction because $P_{\mathrm{LD}} \oplus P$ or $P_{\mathrm{RD}} \oplus P$ would include Q. ◀

▶ **Lemma 8.** *Pentomino T (Figure 5) is always-coverable.*

Proof. Let Q be a pentomino T and assume that Q cannot be covered by a sufficiently large polyomino P. Consider the polyomino $P_{\mathrm{U}} \oplus P_{\mathrm{D}} \oplus P_{\mathrm{L}} \oplus P_{\mathrm{R}} \oplus P$ where $P_{\mathrm{U}}, P_{\mathrm{D}}, P_{\mathrm{L}}, P_{\mathrm{R}}$ are the polyominoes obtained by reflecting P about its top, bottom, left, and right side

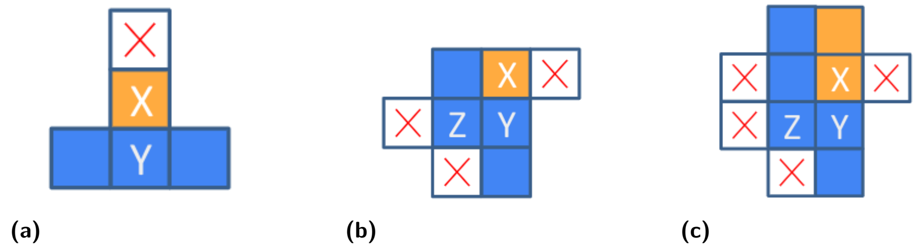

Figure 11 Case analysis for Lemma 8.

respectively (Figure 9). From the assumption this does not include Q. Consider two cells with a distance of 1 as adjacent, and look at the oriented shortest path D_1 from any top-side cell to a bottom-side cell in $P_\mathrm{U} \oplus P_\mathrm{D} \oplus P$ and the shortest path D_2 from any left-side cell to a right-side cell in $P_\mathrm{L} \oplus P_\mathrm{R} \oplus P$. Note that D_1 and D_2 must have an intersection in P. Since $D_1 \cup D_2 \subset P_\mathrm{U} \oplus P_\mathrm{D} \oplus P_\mathrm{L} \oplus P_\mathrm{R} \oplus P$, $D_1 \cup D_2$ does not include Q. Let X be the first cell in D_1 that is adjacent to any cell in D_2, and Y be any cell in D_2 adjacent to X, and without loss of generality, assume that X is the cell above Y (Figure 10). From the way X is taken, $X \notin D_2$. Since Y is a cell on the shortest path D_2, D_2 contains exactly two cells adjacent to Y. It is not possible for D_2 to contain both the left and right squares of Y (otherwise since $D_1 \cup D_2$ does not include Q, D_1 would not contain the square above X, and then the cell before X in D_1 must be adjacent to a cell in D_2, which is a contradiction (Figure 11a)). Hence, D_2 contains the square below Y. It also contains either the square to the left or to the right of Y. We assume that it contains the left of Y and call it Z (if it contains the right of Y, the same argument holds). From its shortestness, D_2 does not contain the square under Z. In order not to cover Q, it also does not contain the left of Z and thus contains the square above Z. In order not to cover Q, the right of X is not contained in D_1 (Figure 11b). From the way X is taken, D_1 contains the square above X as the cell before X in D_1. In order not to cover Q, D_2 does not contain the square upper left of Z, and thus contains the square two above Z. Then the cell above X becomes adjacent to a cell in D_2, which is a contradiction (Figure 11c). ◀

▶ **Lemma 9.** *Polyominoes in $\mathcal{J}$ are always-coverable.*

Proof. From Lemma 7 and 8, pentomino Y and T are always-coverable. Any other polyominoes in $\mathcal{J}$ are proven to be always-coverable in Appendix A. ◀

3.3 Proof of Theorem 1

Proof. Every polyomino either includes a polyomino in $\mathcal{I}$ or is included in a polyomino in $\mathcal{J}$. For polyominoes with 7 or more cells, this has already been shown by Corollary 6. For polyominoes with fewer than 7 cells, this is easily confirmed by a computerized exhaustive search.

Let us outline an algorithm that decides whether a given stain Q is always-coverable in $O(1)$ time. By Corollary 6, if the number of cells in Q is 7 or more, we can immediately reject it as not always-coverable. Thus, we may assume the input Q has at most 6 cells. The total number of polyominoes of size up to 6, up to translations, is finite and bounded by a constant. For such a Q, we can test in $O(1)$ time whether Q includes any polyomino in $\mathcal{I}$ (by checking all $|Q| \cdot O(1)$ possible alignments of the constant-sized elements of $\mathcal{I}$ inside Q). If it includes an element of $\mathcal{I}$, Q is not always-coverable; otherwise, by our characterization, it is always-coverable. ◀

It is also possible, in linear time in the input size, to output a polyomino that cannot cover a given polyomino Q if it is not always-coverable. We can find a polyomino in $\mathcal{I}$ that Q includes (which takes linear time with respect to the size of Q by naive pattern matching, as elements of $\mathcal{I}$ are of constant size) and output the known counterexample corresponding to that polyomino in $\mathcal{I}$. Since the size of $\mathcal{I}$ is $O(1)$, this entire procedure works in linear time.

Finally, it is notable that all the counterexamples we have given in Appendix B have no holes, i.e., all the animals induced from them are simply connected. This means that even if we restrict the domain from all polyominoes (Theorem 1) to only simply connected polyominoes (polyominoes without holes), we still obtain an analogous result:

▶ **Theorem 10.** *Let $\mathcal{P}'$ be the set of all simply connected polyominoes, and $\sim_E$ be the equivalence relation under translation, rotation, and reflection.* Troublesome Sticker*$(\mathcal{P}', \sim_E, \mathcal{P}')$ is solvable in $O(1)$ time, and if the stain is not always-coverable, a sticker that cannot cover it can be found in linear time.*

4 NP-completeness of Flat Cover

For any Troublesome Sticker$(\mathcal{P}, \sim, \mathcal{Q})$, the following decision problem, which we name Flat Cover, can be considered.

Flat Cover $(\mathcal{P}, \sim, \mathcal{Q})$
Input: $P \in \mathcal{P}, Q \in \mathcal{Q}$
Question: Can P (flatly) cover Q?

▶ Remark 11. Determining if a polyomino P of size n can cover a polyomino Q of size k is solvable in $O(n^k)$ time via depth-first search.

In this section, we show that Flat Cover for polyominoes is NP-complete.

▶ **Theorem 12.** *Let $\mathcal{P}$ be the set of all polyominoes, and $\sim_E$ be the equivalence relation under translation, rotation, and reflection. Then,* Flat Cover*$(\mathcal{P}, \sim_E, \mathcal{P})$ is* NP*-complete.*

We show this by a reduction from 3-Precoloring Extension on induced subgrids.

▶ **Theorem 13** (3-Precoloring Extension on induced subgrids [4])**.** *Let $G = (V, E)$ be an induced subgrid, i.e., an induced subgraph of the infinite integer grid $\mathbb{Z}^2$ where $E = \{\{u, v\} \subset V \mid \|u - v\|_1 = 1\}$. Given a subset $S \subset V$ and a partial coloring $\varphi : S \to \{1, 2, 3\}$ (a precoloring), the decision problem of whether there exists a proper 3-coloring $c : V \to \{1, 2, 3\}$ such that $c(v) = \varphi(v)$ for all $v \in S$ and $c(u) \neq c(v)$ for all $\{u, v\} \in E$ is* NP*-complete.*

▶ **Lemma 14.** Flat Cover *for polyominoes belongs to* NP*.*

Proof. To see that this problem belongs to NP, observe that a certificate consists of the placement (coordinates and orientations) of a bounded number of stickers (at most the number of cells in Q). These can be verified in polynomial time to ensure they completely cover Q and do not overlap. ◀

Proof of Theorem 12. From Lemma 14, Flat Cover is in NP. We now focus on showing NP-hardness. We reduce 3-Precoloring Extension on induced subgrids to Flat Cover. Let an instance of 3-Precoloring Extension be given by an induced subgrid $G = (V, E)$ and a partial coloring $\varphi : S \to \{1, 2, 3\}$. W.l.o.g. we can assume G is connected. We construct a sticker P and a stain Q in polynomial time.

```
......#...
.##...#.#.
.###.####.
..######..
..####....
##########
..######..
.#######..
.#....###.
......#...
```

Figure 12 The sticker P. Each "#" represents a cell.

```
##.....#
.##.####
.##.###.
.####...
..#####.
.###.##.
####.##.
#.....##
```

Figure 13 The gadget Q_0, defined as $P_1 \cap P_2 \cap P_3$. Each "#" represents a cell.

Let P be the 10×10 polyomino defined in Figure 12. Let P_1, P_2, P_3 denote P, $\mathrm{rot}(P, 90°)$, and $\mathrm{rot}(P, -90°)$ respectively, corresponding to colors $1, 2, 3$.

For each vertex $v \in V$ of the induced subgrid, we associate an anchor square $B_v := 8v + [0, 7]^2$ (scaling by factor 8), and denote its center by $m(v) := 8v + (7/2, 7/2)$. The sticker P is designed to have the following properties, which we computationally verified by an exhaustive search over all isometries (translations, rotations, and reflections). Here, a cover of a region is called *minimal* if every placed sticker intersects that region (equivalently, no placed sticker is completely disjoint from it).

1. There exists a connected 8×8 polyomino Q_0 derived from P (Figure 13) such that, for each $\delta \in \{(\pm 8, 0), (0, \pm 8)\}$, $Q_0 \cup (Q_0 + \delta)$ is connected. Moreover, Q_0 has exactly three minimal covers by P, where either P_1, P_2, or P_3 is placed locating its center at the center of Q_0.
2. Each P_i ($i \in \{1, 2, 3\}$) has exactly one minimal cover by P, where P_i (sticker) is placed locating its center at the center of P_i (stain).
3. When two stickers are placed at adjacent grid positions (distance 8 apart) each having the orientation of either P_1, P_2, or P_3, the two stickers do not overlap if and only if they have distinct colors. Specifically, the protrusions of two stickers with the same orientation placed at distance 8 overlap, preventing such placement.

In the reduction, we place the following for each $v \in V$: If $v \notin S$ (uncolored), place Q_0 so that it occupies B_v. If $v \in S$ with $\varphi(v) = i$, place a copy of P_i whose center is at $m(v)$. The total stain Q is the union of all these regions.

First, suppose there exists a valid 3-coloring extension $c : V \to \{1, 2, 3\}$ of φ. For each vertex $v \in V$, place a copy of P with orientation $P_{c(v)}$ whose center is at $m(v)$. If $v \notin S$, the sub-region at B_v is Q_0, and by Property 1 this single sticker covers it. If $v \in S$, then $c(v) = \varphi(v) = i$, the sub-region is the copy of P_i centered at $m(v)$, and by Property 2 this single sticker covers it. Thus every sub-region is covered, so all of Q is covered. Moreover, for each edge $\{u, v\} \in E$, proper coloring gives $c(u) \neq c(v)$, so by Property 3 the two stickers centered at $m(u)$ and $m(v)$ do not overlap. Therefore these placements form a valid cover of Q by P.

Conversely, let C be any valid cover of Q by non-overlapping copies of P. We will show that C corresponds to a valid 3-coloring extension of φ. For each vertex $v \in V$, the stain Q contains a sub-region associated with v, namely either Q_0 on B_v (if $v \notin S$) or a copy of P_i centered at $m(v)$ (if $v \in S$). Let $C_v \subseteq C$ be the set of stickers in C that intersect this sub-region. Since C covers Q, C_v covers the sub-region. Moreover, every sticker in C_v intersects the sub-region, so C_v is a minimal cover of the sub-region in the above sense. By Property 1 (for Q_0) and Property 2 (for P_i), any such minimal cover consists of exactly one sticker, centered exactly at $m(v)$, with an orientation corresponding to some color $c(v) \in \{1, 2, 3\}$ (and for $v \in S$, $c(v) = \varphi(v)$). This ensures that any cover is strictly aligned with the 8×8 anchor grid, precluding any larger coverings that emerge for other reasons. Since the stickers in the global cover C are mutually non-overlapping, the stickers centered at $m(u)$ and $m(v)$ for any edge $\{u, v\} \in E$ cannot overlap. By Property 3, this implies $c(u) \neq c(v)$. Thus, c is a valid proper 3-coloring extension of φ.

Since G is connected, and every placed stain piece (either Q_0 or some P_i) contains the same copy of Q_0 at each vertex position, the above connected-adjacency property of Q_0 implies that for every edge $\{u, v\} \in E$, the stain pieces at u and v are connected within Q. Hence Q is connected. Thus, both P and Q are polyominoes, concluding the proof. ◀

5 NP-completeness in One Dimension

In the previous section, we showed that FLAT COVER is NP-complete for 2D polyominoes. While the problem is computationally trivial for connected 1D segments (as a single segment can trivially cover the line), generalizing this base case in two distinct directions – either by increasing the dimension to 2D (as shown previously) or by removing the connectivity constraint in 1D (as shown in this section) – leads to NP-completeness.

▶ **Theorem 15.** *Let $\mathcal{P}$ be the set of all finite subsets of $\mathbb{Z}$, and let $\mathcal{Q} = \{[0, L) \cap \mathbb{Z} \mid L \in \mathbb{Z}^+\}$ be the set of solid segments starting at 0. Let $\sim_T$ be the equivalence relation under translation. The problem* FLAT COVER$(\mathcal{P}, \sim_T, \mathcal{Q})$*, which asks if a segment of length L can be covered by non-overlapping translations of a single template $T \in \mathcal{P}$, is* NP*-complete.*

Membership in NP is trivial. We prove NP-hardness of this problem by a reduction from X3C, which is an NP-complete problem [9] defined as follows: Given a universe U of size $3q$ and a collection $\mathcal{S} = \{S_1, \dots, S_r\}$ of 3-element subsets of U, does there exist a subcollection $\mathcal{S}' \subset \mathcal{S}$ such that every element of U belongs to exactly one subset in $\mathcal{S}'$?

The idea of the reduction is as follows. We introduce two types of gadgets: one "frame" gadget G_0 and r "set" gadgets $G_1, \dots, G_r$, and place them in a template T with strategic distances based on a *Golomb ruler* [18, 1] (a set of integers with distinct pairwise differences). The construction enforces that if there exists a flat cover of the target segment by T, it must use $q + 1$ copies of T, where the segment cover is contributed by exactly one frame gadget and q set gadgets, which represents a solution to the X3C instance. The Golomb ruler property ensures that the templates do not overlap with each other, and the shape of the gadgets which we detail below ensures different types of solutions are impossible.

Given an instance $(U = \{0, \dots, 3q - 1\}, \mathcal{S} = \{S_1, \dots, S_r\})$ of X3C, we construct a target interval $[0, L)$ and a single template $T \subset \mathbb{Z}$.

We let $N = 10(3q + r + 1)$, which we call *element size*, and $L = 2N^2 + 3qN$ be the size of the target interval $Q = \{0, \dots, L - 1\}$. We also define $W = L + 10(r + 1)$, which we call *gadget size*, because each gadget is represented as a binary string of length W.

The template T is constructed as a union of the gadgets $G_0, G_1, \dots, G_r$, placed at relative positions $0, 2a_1W, 2a_2W, \dots, 2a_rW$, where $\{a_0 = 0, a_1, \dots, a_r\}$ forms a Golomb ruler, i.e., all differences $a_i - a_j$ $(i \neq j)$ are non-zero and distinct. While finding an optimal Golomb ruler is hard, a ruler with $a_r \in O(r^2)$ can be explicitly constructed in polynomial time [6, 5].

Each gadget consists of a *left stopper*, a *main body*, and a *right stopper*, whose lengths are $5(r + 1)$, L, and $5(r + 1)$ respectively. Each right stopper is a reversed binary string of the left stopper. We denote a run of k ones as 1^k and k zeros as 0^k.

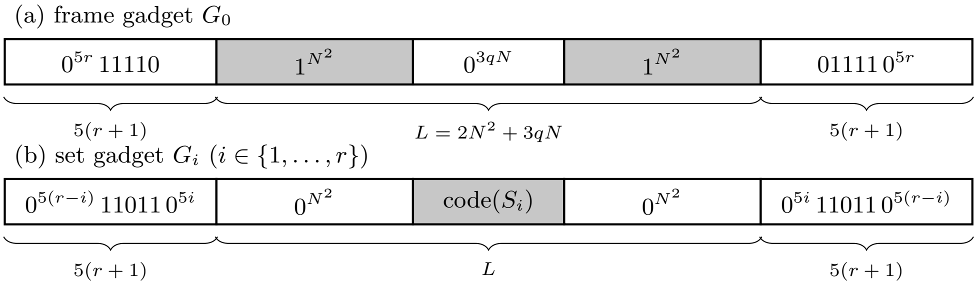

Figure 14 Schematic of the frame gadget G_0 and a set gadget G_i in the 1D reduction (not to scale). These gadgets are parts of one template T at fixed offsets, not separate stickers. Each gadget consists of a left stopper, a main body of length L, and a right stopper (the reverse of the left stopper).

The frame gadget G_0 is defined as follows.

$$G_0 = \underbrace{0^{5r}11110}_{\text{left stopper}} \circ \underbrace{1^{N^2}\,0^{3qN}\,1^{N^2}}_{\text{main body}} \circ \underbrace{011110^{5r}}_{\text{right stopper}}$$

For each $i \in \{1, \ldots, r\}$, G_i (set gadget) is defined as follows.

$$G_i = 0^{5(r-i)}110110^{5i} \circ 0^{N^2}\text{code}(S_i)0^{N^2} \circ 0^{5i}110110^{5(r-i)}$$

where $\text{code}(S_i) = \bigcup_{j \in S_i} 0^{jN}1^N0^{(3q-1-j)N}$ is a binary string of length $3qN$ encoding the set S_i.

The template is defined as $T = \bigcup_{i \in \{0,\ldots,r\}} 0^{2a_iW}G_i$. The total length of T is polynomial in the input size since $a_r \in O(r^2)$. By construction, the distance between different gadgets G_i and G_j is $2(a_i - a_j)W$, and placing two copies of T with relative shift of $2(a_i - a_j)W$ results in a collision if and only if $i, j > 0$ and $S_i \cap S_j \neq \emptyset$.

Note the following properties:

- Due to the block of 1's in the main body of the frame gadget, placing two copies of T with relative shift of less than L results in a collision.
- In each gadget, every 1 has another 1 to its immediate left or right, meaning filling a single 0 between 1's without collision is impossible.

Proof.

Completeness: Suppose $S_{k_1}, \ldots, S_{k_q} \subset \mathcal{S}$ is an exact cover. We place one copy of T so that the main body of its G_0 aligns with Q. For each $i \in \{1, \ldots, q\}$, we place one copy of T so that the main body of its G_{k_i} aligns with Q. This forms a valid cover of Q by T, because the solution is an exact cover, so the selected sets are disjoint, ensuring that the placements do not collide, and Q is clearly fully covered.

Soundness: Suppose Q is covered by a set of non-overlapping $m+1$ copies of T, and let $G_{k_0}, \ldots, G_{k_m}$ be all the gadgets contributing to the cover of Q. We can assume $m+1$ gadgets here because due to the distances between gadgets, at least one gadget per a copy of T can intersect with Q.

We first prove that $\{k_i\}$ must contain 0 (frame gadget), assuming otherwise, i.e., Q is fully covered by set gadgets. Since each set gadget has exactly $8 + 3N < 4N$ 1's, and Q has more than $2N^2$ 1's, the number of used gadgets must be at least $N/2$, which is larger than $5r$, indicating that there exists an index (say j) contained in $\{k_i\}$ more than 4 times.

Because of the length of Q (which is L) and G_j (which is $W < 2L$), simple arithmetic shows that there exist two copies of T with relative shift less than L. However, this is impossible because it produces a collision on the frame gadgets.

Let's consider how G_0 covers Q. Since its main body and left/right stoppers are separated by a single 0 between 1's, the intersection of G_0 and Q must either entirely lie on the main body, or its left or right stopper. If we suppose Q never intersects with G_0's main body, since the stoppers have only four 1's, we can easily lead to a contradiction with the same argument as we used in the previous paragraph. Therefore, there exists a copy of T covering Q precisely with G_0's main body. It is easy to see no other copy of T can cover Q with G_0 without collision, so we can set $k_0 = 0$ and $k_i > 0$ for $i \in \{1, \ldots, m\}$.

Now that G_0's main body is placed at Q, the remaining central blank of length $3qN$ should be filled by m set gadgets. For each of them, it is never possible for its left or right stopper to intersect with the blank, because of the pattern 11011 having a 0 impossible to fill without collision. That means the blank is filled by 1's in the main body of each set gadget, having the consecutive length that is a multiple of N, meaning the alignment of multiple of N must hold. That is, a possible placement of a set gadget is placing its main body to precisely cover Q or with a shift of sN with an integer s with $|s| < 3q$. However, choosing $s \neq 0$ is impossible as it causes its left or right stopper to collide with the 1's in the already-placed G_0's main body. Thus, $m = q$ set gadgets' main bodies must align with Q, completing the proof. ◀

6 Open Problems

We have shown that Troublesome Sticker is decidable for polyominoes, but what if the domain is not polyominoes? The following problem is, to the best of our knowledge, unsolved even for $d = 1$.

▶ **Question 1.** *Let d be a positive integer and $\mathcal{P}$ be the set of all finite subsets of $\mathbb{Z}^d$. Define $\sim_T$ on $\mathcal{P}$ so that for $A, B \in \mathcal{P}$, $A \sim_T B \iff \exists x \in \mathbb{Z}^d, A = B + x$. Is Troublesome Sticker$(\mathcal{P}, \sim_T, \mathcal{P})$ decidable?*

As we have proven, Flat Cover is NP-complete for general polyominoes. What would happen if we limited the height of the stain?

▶ **Question 2.** *Let $\mathcal{P}$ be the set of polyominoes and $\sim_E$ be the equivalence relation under translation, rotation, and reflection. Let $\mathcal{Q}$ be the set of all polyominoes with a height of 1. Is Flat Cover$(\mathcal{P}, \sim_E, \mathcal{Q})$ NP-complete?*

In Troublesome Sticker, the stain was given as input, but what would be the computational complexity of the problem of, conversely, finding a stain that cannot be covered by a given sticker?

▶ **Question 3.** *Is the problem of finding a polyomino with the minimum number of cells that a given polyomino cannot cover NP-hard?*

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A Complete Proof of Lemma 9

Proof. From Lemma 7, 8 and the following Lemma 16, 17, 18, 19, every polyomino in $\mathcal{J}$ is always-coverable and thus Lemma 9 holds. ◀

▶ **Lemma 16.** *Pentomino F (Figure 15) is always-coverable.*

Proof. Let Q be a pentomino F and assume that Q cannot be covered by a sufficiently large polyomino P. Let L_0 (resp. R_0) be the polyomino obtained by reflecting P about its top side and shifting it to the left (resp. right) by one square, and let L_i, R_i be polyominoes obtained by shifting L_0, R_0 down by i squares, respectively. Take the smallest j such that

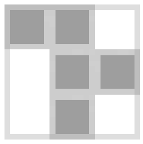


Figure 15 Pentomino "F".

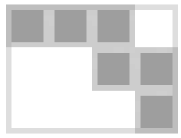


Figure 16 Hexomino "3".

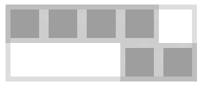


Figure 17 Hexomino "N".

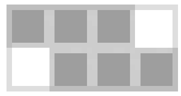
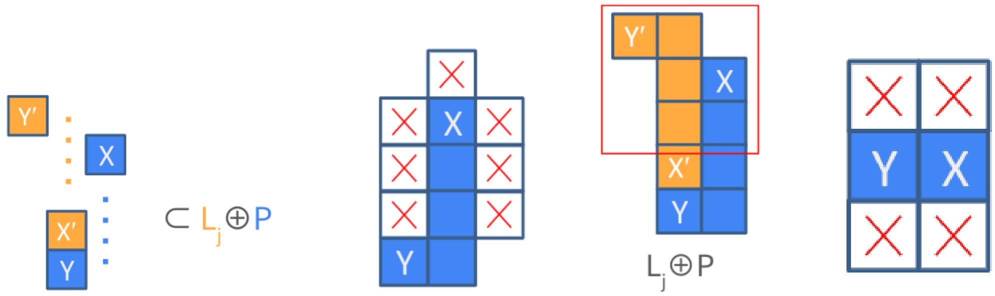


Figure 18 Hexomino "S".



(a) **(b)** **(c)** **(d)**

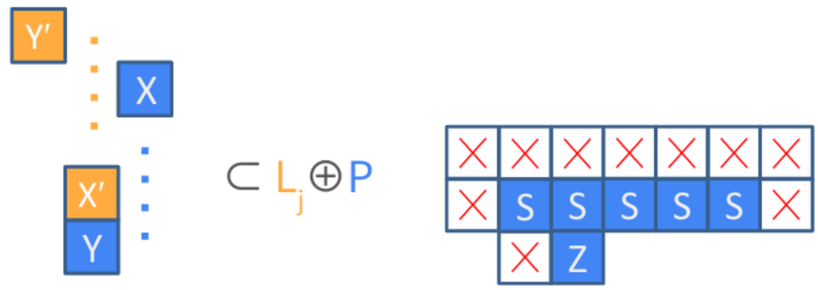
Figure 19 Case analysis for Lemma 16.

$L_{j+1} \cap P$ is non-empty and the smallest k such that $R_{k+1} \cap P$ is non-empty. From symmetry, we may assume that $j \leq k$. That is, none of $L_0, \cdots L_j, R_0, \cdots, R_j$ overlaps P. From the fact that L_{j+1} overlaps P, we can take $X' \in L_j$ and $Y \in P$ such that Y is one square below X'. Let X be the cell in P corresponding to X' (Figure 19a). In the following, we divide cases according to the difference between the row index of X and the row index of Y in P.

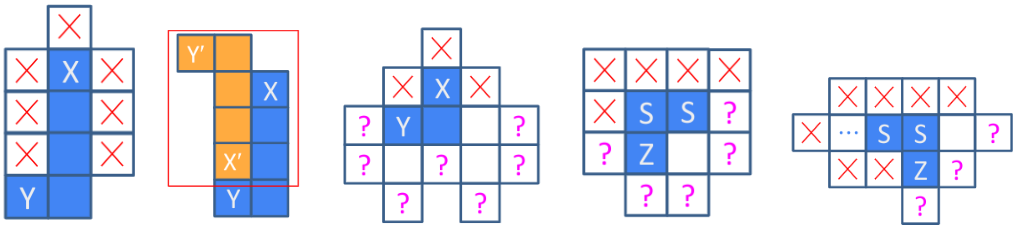
- If the row index of X is smaller than the row index of Y, P does not contain the squares marked by $\times$ in Figure 19b, since L_j does not contain the square below X', and $L_0, \cdots, L_j, R_0, \cdots, R_j$ do not overlap P. Therefore, by connectedness, P contains all the squares below X up to the square to the right of Y. Then $L_j \oplus P$ contains Q, which is a contradiction (Figure 19c).
- If the row index of X is greater than the row index of Y, by reversing up and down and reversing the roles of L_j and P, a contradiction is derived in the same way.
- If the row index of X equals the row index of Y, P does not contain any squares above or below X and Y (Figure 19d), since L_j and R_j do not overlap with P, and $L_j \oplus P$ and $R_j \oplus P$ do not include Q. If the number of consecutive cells to the left and right of Y is n, the same argument applies sequentially to all cells to the left of Y and to the right of X, establishing that there are no cells above or below them, so P is a polyomino of height 1 and width n, which contradicts the assumption that P is sufficiently large. ◀

▶ **Lemma 17.** *Hexomino 3 (Figure 16) is always-coverable.*

Proof. Let Q be a hexomino 3 and assume that Q cannot be covered by a sufficiently large polyomino P. Let L_0 (resp. R_0) be the polyomino obtained by reflecting P about its top side and shifting it to the left (resp. right) by one square, and let L_i, R_i be polyominoes obtained by shifting L_0, R_0 down by i squares, respectively. Take the smallest j such that $L_{j+1} \cap P$ is non-empty and the smallest k such that $R_{k+1} \cap P$ is non-empty. From symmetry, we may assume that $j \leq k$. That is, none of $L_0, \cdots L_j, R_0, \cdots, R_j$ overlaps P. From the fact that L_{j+1} overlaps P, we can take $X' \in L_j$ and $Y \in P$ such that Y is one square below X'. Let X be the cell in P corresponding to X' (Figure 20a). In the following, we divide cases according to the difference between the row index of X and the row index of Y in P.



(a) (b)



(a) (b) (c) (d) (e)

Figure 21 Case analysis for Lemma 17.

- If the row index of X is smaller than the row index of Y, P does not contain the squares marked by $\times$ in Figure 21a, since L_j does not contain the square under X', and $L_0, \cdots, L_j, R_0, \cdots, R_j$ do not overlap P. Therefore, by connectedness, P contains all squares below X up to the right of Y. If the row index of X is smaller than the row index of $Y-1$, $L_j \oplus P$ includes Q (Figure 21b), thus it must hold that the row index of X equals the row index of $Y-1$. In this case, there is a cell at one of the positions marked by ? in Figure 21c that connects to X without passing through any other ? , but in any case, $L_j \oplus P$ or $R_j \oplus P$ includes Q, which is a contradiction.
- If the row index of X is greater than the row index of Y, by reversing up and down and reversing the roles of L_j and P, a contradiction can be derived in the same way.
- If the row index of X equals the row index of Y, let S be the set of cells that lie consecutively to the left and right of Y. Since L_j and R_j do not overlap P, P does not contain any square that is the upper left or upper right of a cell in S. Since P is not of height 1, it contains at least one square below S. Let Z be the leftmost of them (Figure 20b). If Z is adjacent to the leftmost cell in S, then there is a cell in one of the positions marked by ? in Figure 21d and connects to Z without passing through any other ? , but in any case, $L_j \oplus P$ or $R_j \oplus P$ includes Q, which is a contradiction. Thus, Z is adjacent to a non-leftmost cell of S. In this case, there is a cell in one of the positions marked by ? in Figure 21e that connects to Z without passing through any other ? , but in any case, $L_j \oplus P$ or $R_j \oplus P$ includes Q, which is a contradiction. ◀

▶ **Lemma 18.** *Hexomino N (Figure 17) is always-coverable.*

Proof. Let Q be a hexomino N and assume that Q cannot be covered by a sufficiently large polyomino P. Let $L \subset \mathbb{R}^2$ be the uppermost 45 degree line of the shape of / that has a non-empty intersection with P (Figure 22a). All the following polyominoes do not overlap P.

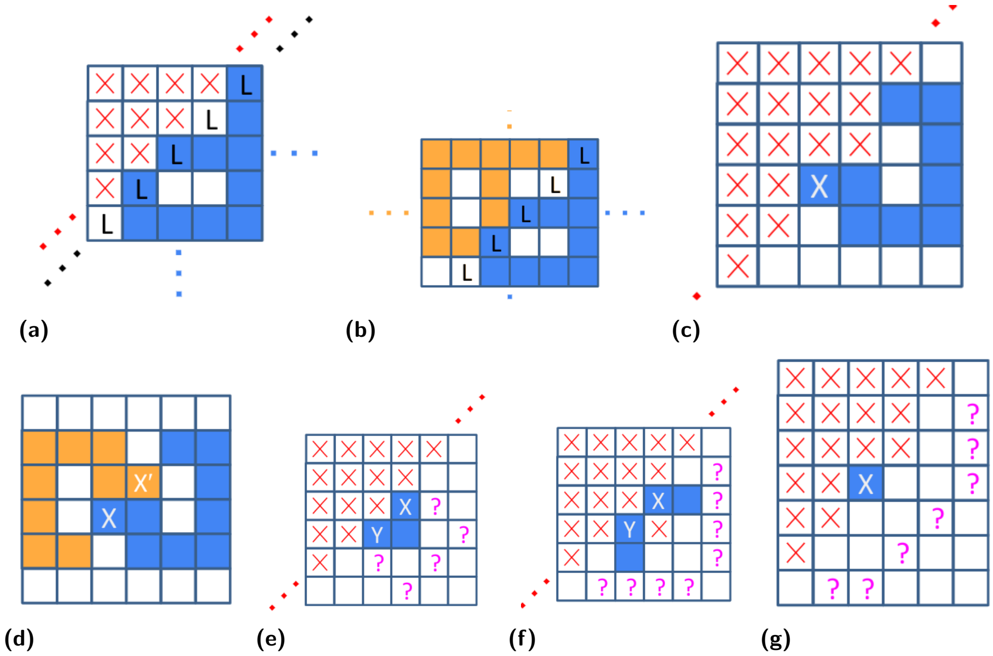


Figure 22 Case analysis for Lemma 18.

- Polyomino P_{L}, one which is obtained by reflecting P about L and shifting it by one square left (Figure 22b).
- Polyomino P_{U}, one which is obtained by reflecting P about L and shifting it by one square up.
- Polyomino P_{L2}, one which is obtained by shifting P_{L} by one square left and down.
- Polyomino P_{U2}, one which is obtained by shifting P_{U} by one square right and up.

We show the following property first.

$\star$ There are no two cells in $P \cap L$ that are diagonally consecutive (with Euclidean distance $\sqrt{2}$).

Assume that there are diagonally consecutive two cells, with X at the upper right and Y at the lower left. We can say that there is no cell below X (Proof: Assume that there is a cell below X. There is a cell at one of the positions marked by ? in Figure 22e and connects to X without passing through any other ? , but in any case, one of $P_{\mathrm{L}} \oplus P, P_{\mathrm{U}} \oplus P, P_{\mathrm{L2}} \oplus P, P_{\mathrm{U2}} \oplus P$ includes Q). Therefore, there are cells to the right of X and below Y. There is a cell at one of the positions marked by ? in Figure 22f and connects to X without passing through any other ? , but in any case, one of $P_{\mathrm{L}} \oplus P, P_{\mathrm{U}} \oplus P, P_{\mathrm{L2}} \oplus P, P_{\mathrm{U2}} \oplus P$ includes Q, which is a contradiction. Therefore, $\star$ has been proven.

From $\star$, the following polyominoes also do not overlap with P.

- Polyomino P_{LD}, one which is obtained by shifting P_{L} by one square down.
- Polyomino P_{UR}, one which is obtained by shifting P_{U} by one square right.

Let X be the bottom leftmost cell in $L \cap P$ (Figure 22c). The following polyominoes also do not overlap with P.

- Polyomino L_0, one which is obtained by rotating P 180 degrees around the lower left point of the square X.
- Polyomino L_1, one which is obtained by shifting L_0 by one square up.

- (From $\star$) Polyomino R_0, one which is obtained by rotating P 180 degrees around the upper right point of the square X (Figure 22d).
- Polyomino R_1, one which is obtained by shifting R_0 by one square right.

There is a cell in one of the positions marked by ? in Figure 22g and connects to X without passing through any other ? , but in any case, the union of P and one of the polyominoes mentioned above as a polyomino that does not overlap with P includes Q, which is a contradiction. ◀

▶ **Lemma 19.** *Hexomino S (Figure 18) is always-coverable.*

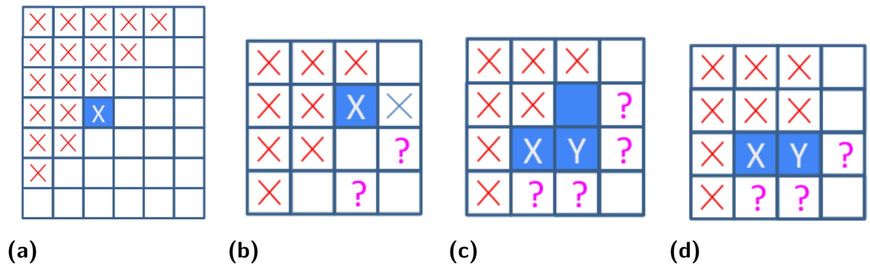


Figure 23 Case analysis for Lemma 19.

Proof. Let Q be a hexomino S, and assume that Q cannot be covered by a sufficiently large polyomino P. Let $L \subset \mathbb{R}^2$ be the uppermost 45 degree line of the shape of / that has a non-empty intersection with P, and let X be the leftmost cell of $L \cap P$ (Figure 23a). The following polyominoes do not overlap P.

- Polyomino L_0, one which is obtained by rotating P 180 degrees around the lower left point of the square X.
- Polyomino P_{L}, one which is obtained by reflecting P about L and shifting it by one square left.
- Polyomino P_{U}, one which is obtained by reflecting P about L and shifting it by one square up.

There is a cell to the right of X (Proof: otherwise, there would be a cell in one of the positions marked by ? in Figure 23b and connects to X without passing through any other ?, but in any case $L_0 \oplus P$ includes Q, which is a contradiction). Let Y denote the cell. There is no cell above Y (Proof: assuming there is a cell above Y, there is a cell in one of the positions marked by ? in Figure 23c, but in any case $P_{\mathrm{L}} \oplus P$ or $P_{\mathrm{U}} \oplus P$ includes Q, which is a contradiction). Therefore, the following polyomino also does not overlap with P.

- Polyomino R_0, one which is obtained by rotating P 180 degrees around the upper right point of the square X.

There is a cell in one of the positions marked by ? in Figure 23d, but in any case $R_0 \oplus P$ includes Q, which is a contradiction. ◀

B Polyominoes for Proof of Lemma 5

The table below shows each polyomino in $\mathcal{I}$ on the left, and a polyomino that cannot cover it (counterexample) on the right: the two numbers in the first row indicate the height and width of the polyomino, and # indicates the cells of the polyomino.

The counterexamples in this appendix are representative outputs of our search procedure. We do not claim minimality, and especially for larger cases the displayed shapes may contain avoidable detours or extra cells. Additional post-processing to further reduce the number of cells may be possible, but we did not pursue it in this work.

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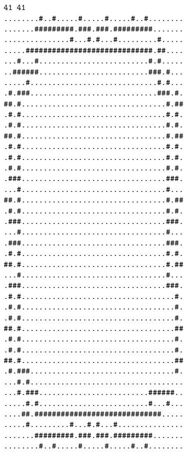
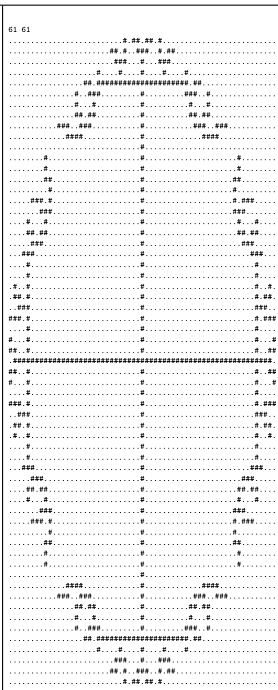
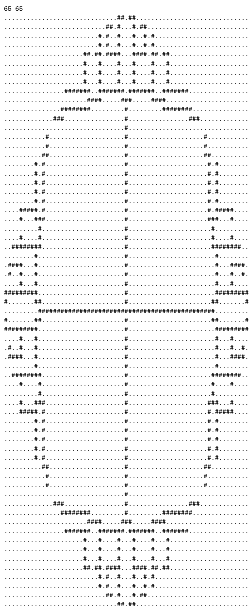
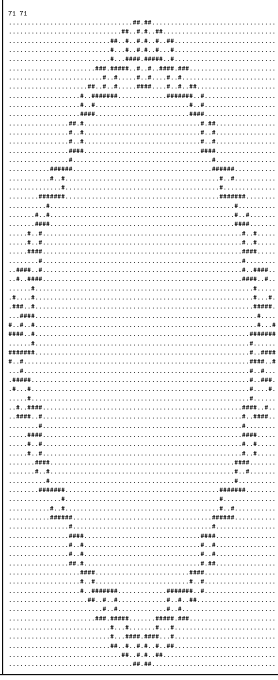
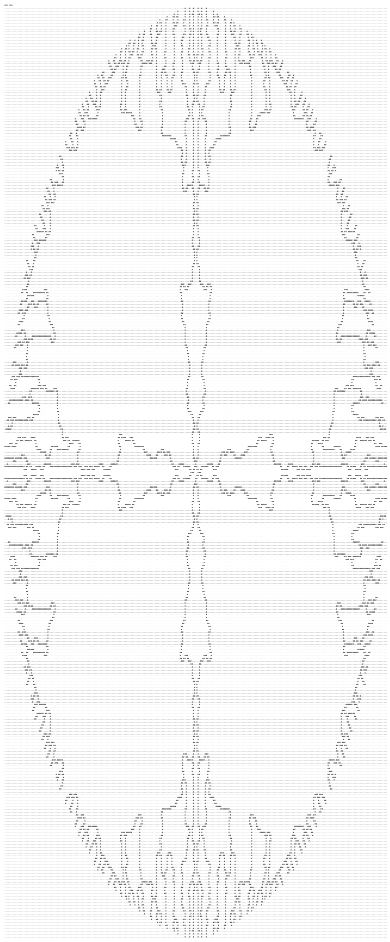
4 3 .## ##. .#. .#.		4 2 ## ## #. #.	
3 3 ##. .#. .##		2 3 ### #.#	

3 3 ### #.. #..		5 2 #. #. ## .# .#