Solving Small Rubik's Cubes as Slowly as Possible

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Abstract

We study, for different Rubik's puzzles, whether from any starting state one can solve the puzzle *as slowly as possible*, visiting every reachable state exactly once before reaching the solved configuration. This question corresponds to the existence of Hamiltonian paths (ending in the solved state) in the Cayley graphs associated with these puzzles. A major conjecture attributed to Lovász is that every Cayley graph has a Hamiltonian path. An even stronger version of the conjecture, considered by Dupuis and Wagon (2015) and Gregor et al. (2024), is that every Cayley graph of degree at least 3 is either bipartite and has Hamiltonian paths between any pair of vertices on opposite parts, or is non-bipartite and has Hamiltonian paths between any pair of vertices. Our study of slowly solving Rubik's puzzles amounts to studying this *Strong Lovász Conjecture* in their respective Cayley graphs. We first verify the Strong Lovász Conjecture computationally for small Rubik's puzzles like the $1\times2\times3$ or $1\times3\times3$ cuboids, which have under 200 states. This approach, however, becomes infeasible for the $2\times2\times2$, which has over 3.6 million states. Our main result is then showing that the Strong Lovász Conjecture holds for the $2\times2\times2$ cube, using a careful graph-theoretic construction based on the subgroup induced by the R and U turns.

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Supplementary Material *Software*: `https://github.com/bsubercaseaux/rubik-hamiltonian` archived at `swh:1:dir:0f9311f69998251f6e55d18e982d2bbef4e0124f`

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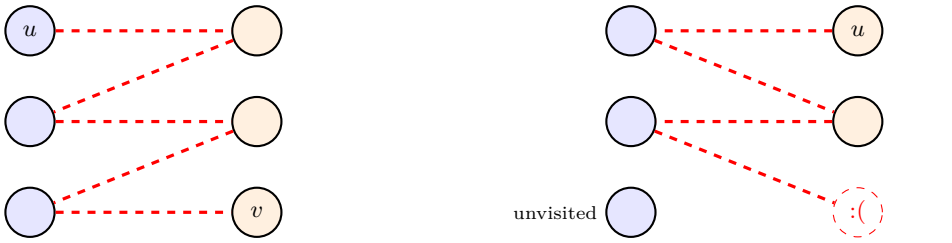
1 Introduction

Little Ernie, a clever Hungarian boy, often gets reprimanded at the dinner table by his mom for playing with the Rubik's cube. His mom, fed up one day with the constant noise of the cube, tells Ernie *"It's over; this will be your last solve"*. Clever Ernie, in turn, decides to maximize his fun by taking as many steps as possible in what will be his final solve. But how many steps can he prolong the solve for? While he could obviously keep turning a single face forever, without making any progress, this wouldn't be any fun for Ernie. Moreover, Ernie's mom is no fool, and would realize that Ernie has had the cube in the exact same state before, which could end up with tragically scattered cube pieces all over the floor.

Being Hungarian, little Ernie has a good command of combinatorics and quickly realizes that his problem is equivalent to finding the longest path from the starting scrambled state of the cube to the solved state, and thus ideally, a Hamiltonian path. In this paper, we address the mathematical question at the core of Ernie's dilemma: *is there a Hamiltonian path from any starting state that ends in the solved state?*



(a) Even n. Endpoints of Hamiltonian paths must be on opposite sides.

(b) Odd n. Hamiltonian paths cannot start on the smaller side.

Figure 1 Illustration of the impossibility of Hamilton-connectivity for bipartite graphs.

While any $2\times2\times2$ cube can be solved in at most 14 moves (or 11 if we count 180° moves as one) [27], we will show in this paper that Ernie can succeed in solving such a cube, without repeating any state, for at least 3,674,158 steps.

This seemingly innocent question concerning Rubik's puzzles is a concrete instance of a much more general problem in graph theory: understanding the relationship between *symmetry* and *connectivity*. Concretely, a famous open question of Lovász [16] is whether every *vertex-transitive* graph has a Hamiltonian path. A slightly weaker variant, also referred to as Lovász conjecture, is whether every Cayley graph has a Hamiltonian path. Curran and Gallian have written a nice, albeit somewhat old, survey on Hamiltonicity properties of Cayley graphs [6].

In this article, we are interested in the much stronger notion of *Hamilton-connectivity*: a graph G is said to be Hamilton-connected if for every pair of vertices $u, v \in V(G)$ there is a Hamiltonian path from u to v. Unfortunately, bipartite graphs cannot be Hamilton-connected for a rather trivial parity reason (illustrated in Figure 1): all Hamiltonian paths of a graph on n vertices have $n-1$ edges, and thus if $n-1$ is odd, then all Hamiltonian paths end in the opposite part of the graph they started from. If $n-1$ is even, then n is odd, and thus the two parts of the graph must have different sizes. But then it is not possible to have a Hamiltonian path that starts in the smaller part.

This motivates the notion of *Hamilton-laceability*: a bipartite graph $(A \sqcup B, E)$ is said to be *Hamilton-laceable* if for every pair of vertices $a \in A$ and $b \in B$ there is a Hamiltonian path between a and b. To avoid having to case on whether a graph is bipartite or not, we consider the following notion.

▶ **Definition 1** ($H^\star$-connected). *A graph G is said to be $H^\star$-connected if it is either Hamilton-connected, or bipartite and Hamilton-laceable.*

We can now state the conjecture at the core of our work, which is a slight variant of the "Strong Lovász Conjecture" [11], considered earlier by Dupuis and Wagon [9].

▶ **Conjecture 2.** *Every Cayley graph of degree at least 3 is $H^\star$-connected.*

Naturally, in the statement above, the degree of the graph refers to the degree of any of its vertices, since Cayley graphs are regular (see Definition 4). The purpose of the mild degree condition is simply to avoid cycles as a trivial exception.

Conjecture 2 is known to hold for Cayley graphs arising from some families of groups, such as Abelian groups [5], Hamiltonian groups [1], groups of order less than 48 [29], or groups generated by transpositions [2]. Unfortunately, none of these classes encompasses the graphs arising from Rubik's cubes (see Section 2).

Our main contribution is thus to study Conjecture 2 in the context of Rubik's cubes, where it corresponds to the possibility of solving cubes *as slowly as possible*. Our main result is the following (stated informally since a full statement requires defining the exact groups and generators of the Rubik's cubes, as we do in Section 2).

▶ **Theorem 3** (Informal). *The Cayley graphs of the* $1\times2\times3$, $1\times3\times3$, $1\times2\times4$, $1\times2\times5$ *and* $2\times2\times2$ *Rubik's cubes are all* $H^\star$*-connected.*

Related Work. For a general reference on the mathematical treatment of Rubik's cubes (and other puzzles) as groups, we direct the reader to the nice book of Mulholland [18], or his interactive notes [19]. The most studied question regarding Rubik's cube puzzles is that of *"God's number"*, corresponding to the diameter of the associated Cayley graph [17, 14, 13, 25, 24], with the most prominent result being the fact that in the *Half Turn Metric*, where 180° face rotations count as one move, the diameter of the $3\times3\times3$ cube is 20 [25]. The asymptotic diameter of an $n\times n\times n$ cube is known to be $\Theta(n^2 \lg n)$ [7], and optimal solutions from a given position are known to be NP-hard to compute [8]. Norskog has found Hamiltonian cycles on both the $2\times2\times2$ and $3\times3\times3$ puzzles [21, 20]. It is worth clarifying that finding a single Hamiltonian cycle is much weaker than Hamilton-connectivity or Hamilton-laceability. For instance, while a cycle graph C_n trivially has a Hamiltonian cycle, it does not have Hamiltonian paths between vertices that are not adjacent.

2 Preliminaries

We will treat groups $G = (A, \cdot)$ in multiplicative notation, using ab to denote $a \cdot b$. Recall that, given a subset $S \subseteq A$ of the elements of group G, the subgroup *generated* by S, denoted $\langle S\rangle$, is the subgroup of all elements that can be expressed as a combinations of elements of S and their inverses, or more formally, as finite products of the form $s_1^{\sigma_1} s_2^{\sigma_2} \dots s_k^{\sigma_k}$ with $\sigma_i \in \{-1, 1\}$. When $\langle S\rangle = G$, we say that S generates G, or that S is a set of generators for G.

The group of the $2\times2\times2$ Rubik's cube, which we will denote $\mathcal{R}$, can be generated by just three moves: R (rotating the right face clockwise by 90°), U (rotating the top face clockwise by 90°) and F (rotating the front face clockwise by 90°). By labeling the 24 stickers as shown in Figure 2, we think of this group as a subgroup of S_{24}, with the following generators:

$$\mathsf{R} = (2\ 19\ 22\ 10)\,(4\ 17\ 24\ 12)\,(13\ 14\ 16\ 15)$$
$$\mathsf{U} = (1\ 2\ 4\ 3)\,(5\ 17\ 13\ 9)\,(6\ 18\ 14\ 10)$$
$$\mathsf{F} = (3\ 13\ 22\ 8)\,(4\ 15\ 21\ 6)\,(9\ 10\ 12\ 11).$$

In other words, the formal definition of $\mathcal{R}$ is simply $\langle \mathsf{R}, \mathsf{U}, \mathsf{F}\rangle$. It is worth clarifying right away that, while in certain contexts it is convenient to define 180° turns as a single move (this is known as the *Half Turn Metric*), for our purposes of visiting each state exactly once, we believe it makes more sense to consider 90° turns only. The reason is that, when performing a 180° turn, the cube must still go momentarily through the intermediate state corresponding to a 90° turn, which we would want to count as a visited state.

Our treatment of the cube is mostly graph theoretic. Intuitively, the graph of a Rubik's puzzle has all the possible configurations of the puzzle as vertices, and there are edges between configurations that are one move away.

▶ **Definition 4** (Cayley Graph). *Given a group* Γ *and a set of generators* $S \subseteq \Gamma$*, the Cayley graph* $\mathsf{Cay}(\Gamma, S)$ *is the undirected graph whose vertex set is* Γ *and whose edges are the sets* $\{a, b\}$ *such that* $a^{-1}b \in (S \cup S^{-1})$*.*

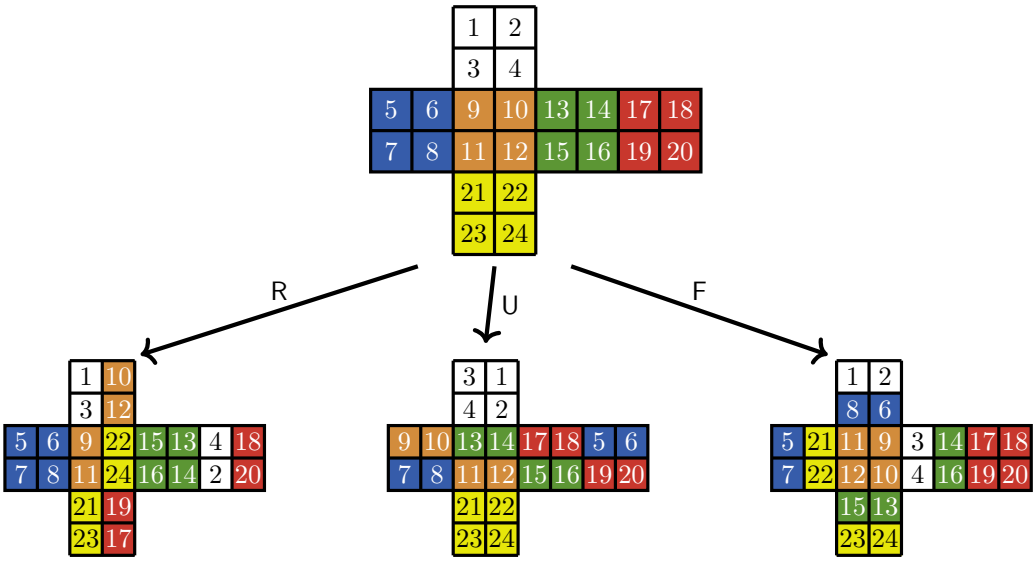


Figure 2 Illustration of the $2\times2\times2$ Rubik's cube with a labeling of its stickers, and the three basic moves $\mathsf{R}, \mathsf{U}, \mathsf{F}$.

Let $G_{\mathcal{R}}$ denote $\mathsf{Cay}(\mathcal{R}, \{\mathsf{R}, \mathsf{U}, \mathsf{F}\})$, and note that Definition 4 implies that $G_{\mathcal{R}}$ is a 6-regular graph, since for each vertex (i.e., puzzle states) there are 6 possible edges; 3 stemming from $\{\mathsf{R}, \mathsf{U}, \mathsf{F}\}$, and 3 from their inverses $\{\mathsf{R}^{-1}, \mathsf{U}^{-1}, \mathsf{F}^{-1}\}$. As it is common in the Rubik's cube literature, we will use notation R' to denote R^{-1}, and analogously for U' and F'.

Given a graph $G = (V, E)$, a *graph automorphism* is a function $\varphi\colon V \to V$ such that for every pair of vertices $u, v \in V$, it holds that $\{u, v\} \in E$ if and only if $\{\varphi(u), \varphi(v)\} \in E$.

▶ **Definition 5** (Vertex transitivity). *A graph $G = (V, E)$ is said to be* vertex transitive *if for every pair of vertices $u, v \in V$, there is an automorphism φ of G such that $\varphi(u) = v$.*

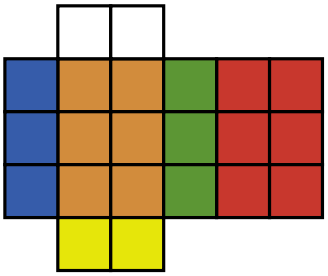
We will use throughout a standard fact: Cayley graphs are always vertex transitive.

▶ **Lemma 6.** *For any group Γ, and generating set $S \subseteq \Gamma$, the Cayley graph $\mathsf{Cay}(\Gamma, S)$ is vertex transitive.*

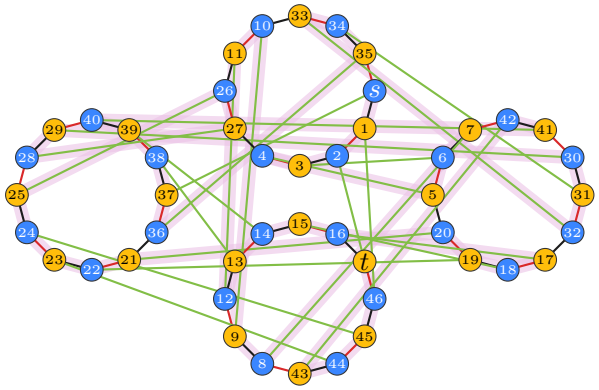
To prove Lemma 6 it suffices to consider, for every pair of vertices u, v, to consider the automorphism $x \mapsto vu^{-1}x$ (see [17, Lemma 3] for a proof that matches our notation).

We have therefore that $G_{\mathcal{R}}$ is vertex transitive. We will next see that $G_{\mathcal{R}}$ is bipartite, which requires introducing some notation regarding permutations.

Recall that the *parity* of a permutation $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ corresponds to the parity of the number of *inversions* (i.e., pairs i, j such that $i < j$ but $\sigma(i) > \sigma(j)$). For example, $(2, 1, 4, 3)$ is an even permutation since it has 2 inversions, whereas $(2, 3, 4, 1)$ is odd. A folklore fact about the parity of permutations is that a permutation is odd if and only if it consists of an odd number of cycles of even length [26, §1]. Therefore, the permutations $\mathsf{R}, \mathsf{U}, \mathsf{F}$ and their inverses, which consist of 3 cycles of length 4, are odd permutations. It follows that $G_{\mathcal{R}}$ is bipartite, with the two parts being $O := \{v \in V(G_{\mathcal{R}})\colon v \text{ is odd}\}$, and $E := \{v \in V(G_{\mathcal{R}})\colon v \text{ is even}\}$, as applying any move (an odd permutation) to an element of O yields an element of E, and viceversa.



(a) Net of the 1×2×3 cuboid. Moves are defined as for the 2×2×2 cube, and thus we omit enumerating stickers.



(b) The Cayley graph of the 1×2×3 cuboid. Edges corresponding to $\mathsf{R}_\star$ are colored red, while those corresponding to $\mathsf{U}_\star$ and $\mathsf{D}_\star$ are colored black and green, respectively. A Hamiltonian path $s \to 1 \to 2 \to \cdots \to 46 \to t$ is highlighted.

Small Puzzles

We start by exploring the possibilities and limits of computational methods to directly confirm Conjecture 2 in Rubik's puzzles with relatively few states. First, we consider the 1×2×3 cuboid (illustrated in Figure 3a), which has only 48 states. Let us denote its group $\mathcal{R}_{1\times2\times3}$. We consider its associated graph $\mathsf{Cay}(\mathcal{R}_{1\times2\times3}, \{\mathsf{R}_\star, \mathsf{U}_\star, \mathsf{D}_\star\})$, illustrated in Figure 3b, where the moves $\mathsf{R}_\star, \mathsf{U}_\star, \mathsf{D}_\star$ are defined in the natural way by 180° twists of the right, top, and bottom face, respectively. Similarly, we consider the 1×3×3 cuboid, which has 192 states under its natural 4 generators, and the 1×2×4 cuboid, with 144 states under its natural 5 generators (left, right, top, bottom, and one of the two intermediate slices), and finally, we consider as well the 1×2×5, with 1152 states, as a stress test.

Since the Hamiltonian path problem is NP-complete, we opted for SAT solving (for a general reference, see [4]). There is a body of research on different SAT encodings and techniques to deal with Hamiltonian paths or cycles [22, 28, 31, 30, 12]. For simplicity, we used the incremental approach of [28]. A summary of results is presented in Table 1. We used the CaDiCaL solver [3] due to its incremental interface, running experiments on a personal computer (MacBook Pro M1 2020). As these graphs are bipartite and vertex-transitive, we considered an arbitrary vertex s and found Hamiltonian paths ending in each state t on the opposite part of s; Table 1 displays the maximum, minimum, and average runtime over the choices of t. In a nutshell, satisfiability solving allows us to efficiently confirm the first part of Theorem 3, but it already reaches its limit around the 1×2×5 cuboid, implying the 2×2×2 puzzle requires different techniques. All our code, using the `PySat` library [15], is publicly available at `https://github.com/bsubercaseaux/rubik-hamiltonian/`.

Table 1 Summary of computational results.

Cuboid dims.	Bipartite	$\lvert V\rvert$	$\lvert E\rvert$	Avg ± stddev [s]	Max [s]	Min [s]	Total [s]
1×2×3	Yes	48	72	0.005 ± 0.007	0.028	0.001	0.12
1×2×4	Yes	144	288	0.113 ± 0.587	4.365	0.008	8.13
1×3×3	Yes	192	384	0.024 ± 0.023	0.209	0.011	2.33
1×2×5	Yes	1152	2880	1.438 ± 3.529	57.143	0.319	828.11

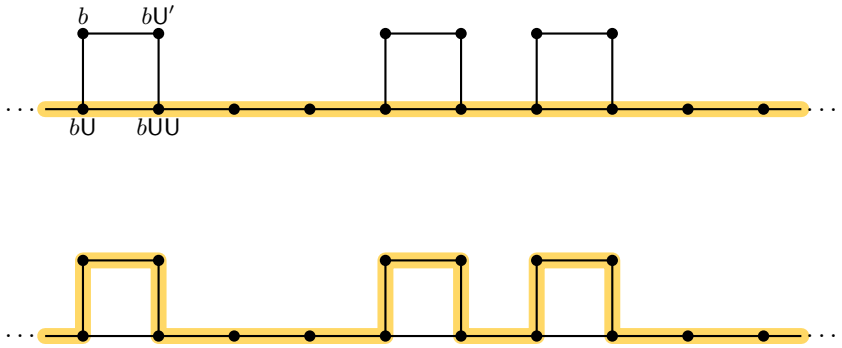


Figure 4 Intuitively, a path π_2 for $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ whose properties are as described Section 3 can be modified to visit every vertex, giving us a Hamiltonian path for $G_{\langle \mathsf{R},\mathsf{U}\rangle}$.

3 Overview of the proof for the $2\times2\times2$

Recall that $G_{\mathcal{R}}$ denotes the Cayley graph for the 2×2×2 Rubik's cube. As this graph contains over 3.6 million nodes, the previous SAT-solving technique for proving Hamilton laceability becomes infeasible. We will instead construct Hamiltonian paths using a group-theoretic approach. Our construction is based on considering the subgroup $\langle \mathsf{R}, \mathsf{U}\rangle$, which has 29,160 elements, and proving:

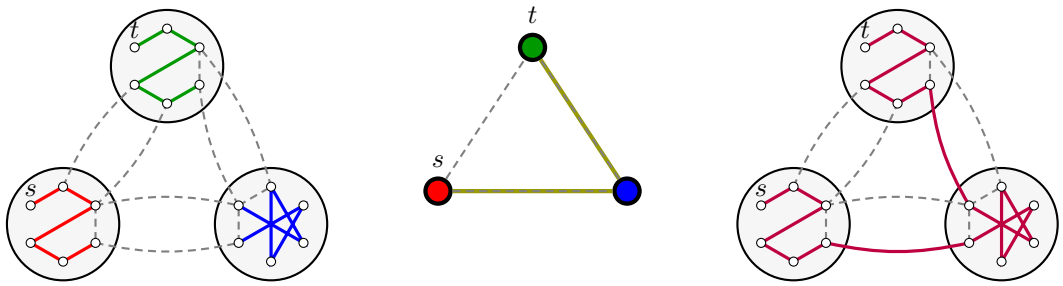
1. **(Subgroup Laceability)** that its associated Cayley graph $G_{\langle \mathsf{R},\mathsf{U}\rangle} := \mathsf{Cay}(\langle \mathsf{R}, \mathsf{U}\rangle, \{\mathsf{R}, \mathsf{U}\})$ is Hamilton laceable, and
2. **(Cycle stitching)** that we can "stitch together" parts of the Hamiltonian paths from the different cosets of $\langle \mathsf{R}, \mathsf{U}\rangle$ to form Hamiltonian paths for $G_{\mathcal{R}}$. Figure 5 illustrates this part.

Let us first elaborate on the subgroup laceability step. Since $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ is a Cayley graph and thus vertex transitive, it will suffice to show the existence of a path from an arbitrary odd $a \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$ to $1_{\mathcal{R}}$ (the solved state, which is also the identity element). In order to do this, we start by showing that there exists an *"almost Hamiltonian"* path π_2 between a and $1_{\mathcal{R}}$, which does not repeat vertices but might not visit certain structured subsets of the vertices. Our strategy will then be to extend these almost Hamiltonian paths into true Hamiltonian paths, in the way illustrated by Figure 4.

To state the sufficient condition for the extendability of these π_2 paths, let us say that a set of two vertices forms a *couple* if it is of the form $\{b, b\mathsf{U}'\}$, where b is even. Then, the condition for a path π_2 to be extendable into a Hamiltonian path is that for every such couple of vertices in $G_{\langle \mathsf{R},\mathsf{U}\rangle}$,

1. Either b and $b\mathsf{U}'$ are both in π_2, or they are both not in π_2.
2. If the vertices of the couple $\{b, b\mathsf{U}'\}$ are not in π_2, then the vertices $b\mathsf{U}$ and $b\mathsf{UU}$ are both in π_2, and $\{b\mathsf{U}, b\mathsf{UU}\}$ is an edge in π_2.

Figure 4 shows how, given a path π_2 with these properties, we can intuitively extend π_2 to form a Hamiltonian path on $G_{\langle \mathsf{R},\mathsf{U}\rangle}$. However, it is not immediately obvious that such a π_2 exists, so we begin by constructing an "almost-almost Hamiltonian" path π_1 whose properties are a relaxed version of those of π_2, and from which we will obtain π_2 later on. We partition $V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$ into cycles, and given a vertex $a \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, let $[a]$ denote the cycle-part to which a belongs; a similar cycle decomposition for the $1\times2\times3$ cuboid is displayed in Figure 3b. We want π_1 to have the following properties: for any couple of vertices $\{b, b\mathsf{U}'\}$,



(a) Hamiltonian paths within cosets. **(b)** Hamiltonian path in the quotient graph. **(c)** Stitched Hamiltonian path.

Figure 5 Schematic "stitching" of Hamiltonian paths across subgroup cosets.

1. Either b and $b\mathsf{U}'$ are both in π_1, or they are both not in π_1.
2. If the vertices of the couple $\{b, b\mathsf{U}'\}$ are not in π_1, then at least one of the following is true.
   a. The vertices $b\mathsf{U}$ and $b\mathsf{UU}$ are both in π_1, and $\{b\mathsf{U}, b\mathsf{UU}\}$ is an edge in π_1.
   b. None of the vertices in $[b]$ are visited by π_1.
   c. None of the vertices in $[b\mathsf{U}]$ are visited by π_1.
3. Either π_1 visits all cycle-parts at least once, or it is possible to augment π_1 to form a new path, π_1', such that the set of cycle-parts visited by π_1' is a strict superset of the set of cycle-parts visited by π_1.

Once we have a π_1 that satisfies the above properties, we will argue that we can create a path π_2 that satisfies the properties for π_1 and also visits every cycle, which will be enough to ensure π_2 is indeed almost Hamiltonian.

Once proved that $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ is Hamilton laceable, we will consider the *quotient* graph, which describes how the different cosets of the subgroup $\langle \mathsf{R}, \mathsf{U}\rangle$ are related. It turns out that this quotient graph is very dense, which allows us to use a classic result of Ore [23] to argue it must be Hamilton connected. We finish the proof by using the Hamilton connectedness of the quotient graph will allow us to *thread* the Hamiltonian paths from the different cosets into one large Hamiltonian path, as illustrated in Figure 5.

4 Hamilton Laceability of $G_{\langle \mathsf{R},\mathsf{U}\rangle}$

4.1 Partitioning $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ into cycles

We begin by formally defining the cycle-partition. Let $a, b \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$. We say that $a \sim b$ if either:

1. a is even and $b = a(\mathsf{RU})^k\mathsf{R}^\sigma$ for some $k \in \mathbb{N}, \sigma \in \{0, 1\}$
2. a is odd and $b = a\mathsf{U}(\mathsf{RU})^k\mathsf{R}^\sigma$ for some $k \in \mathbb{N}, \sigma \in \{0, 1\}$

We will next show that $\sim$ is an equivalence relation, and intuitively, its equivalence classes will correspond to the cycle of vertices visited by repeating the sequence RU an arbitrary number of times.

▶ **Lemma 7.** *The relation $\sim$ is an equivalence relation.*

Proof. We make use of the fact that $a(\mathsf{RU})^{15} = a$. For even a, we observe that $a = a(\mathsf{RU})^{15}\mathsf{R}^0$ and for odd a, $a = a\mathsf{U}(\mathsf{RU})^{14}\mathsf{R} = a(\mathsf{RU})^{15}$. This suffices to show reflexivity.

For symmetry, suppose $a \sim b$. If a is even, then $b = a(\mathsf{RU})^k\mathsf{R}^\sigma$ implies

$$b\mathsf{U}^\sigma(\mathsf{RU})^{15-\sigma-k} = a(\mathsf{RU})^k\mathsf{R}^\sigma\mathsf{U}^\sigma(\mathsf{RU})^{15-\sigma-k} = a.$$

Similarly, if a is odd, then one can check that $b(\mathsf{R})^{1-\sigma}(\mathsf{RU})^{14-k} = a$. Thus, $b \sim a$.

Now assume $a \sim b$ and $b \sim c$. We observe that the parity of b is dependent on the parity of a and σ. In particular, b is even if and only if $\sigma = 0$ with a even or if $\sigma = 1$ with odd a. We now prove that $a \sim c$ by cases depending on the parities of a and b:

- (a and b are both even) Since $b \sim c$, then

$$c = b(\mathsf{RU})^\ell\mathsf{R}^\tau = a(\mathsf{RU})^k(\mathsf{RU})^\ell\mathsf{R}^\tau.$$

- (a is even and b is odd)

$$c = b\mathsf{U}(\mathsf{RU})^\ell\mathsf{R}^\tau = a(\mathsf{RU})^k\mathsf{RU}(\mathsf{RU})^\ell\mathsf{R}^\tau = a(\mathsf{RU})^{k+\ell+1}\mathsf{R}^\tau.$$

- (a and b are both odd)

$$c = b\mathsf{U}(\mathsf{RU})^\ell\mathsf{R}^\tau = (a\mathsf{U}(\mathsf{RU})^k\mathsf{R})\mathsf{U}(\mathsf{RU})^\ell\mathsf{R}^\tau = a\mathsf{U}(\mathsf{RU})^{k+\ell+1}\mathsf{R}^\tau$$

- (a is odd and b is even)

$$c = b(\mathsf{RU})^\ell\mathsf{R}^\tau = (a\mathsf{U}(\mathsf{RU})^k)(\mathsf{RU})^\ell\mathsf{R}^\tau = a\mathsf{U}(\mathsf{RU})^{k+\ell}\mathsf{R}^\tau.$$

In all cases, $a \sim c$, so $\sim$ is transitive. Since $\sim$ is reflexive, symmetric, and transitive, we conclude the proof. ◀

▶ **Definition 8.** *For $a \in G_{\langle \mathsf{R},\mathsf{U}\rangle}$, let $[a] = \{b : b \sim a\}$ denote the equivalence class of a given by the equivalence relation $\sim$.*

Intuitively, we can think of $[a]$ as the set of positions reached by repeating the moves R and U in an alternating manner. If a is even, the first move in the alternating sequence is R, and if a is odd, the alternating moves begin with U.

▶ **Lemma 9.** *For all $a \in G_{\langle \mathsf{R},\mathsf{U}\rangle}$, $|[a]| = 30$*

Proof. First suppose a is even. It can be verified computationally that RU has order 15 (see Appendix A). So $[a]_{even} = \{a(\mathsf{RU})^k : k \in \mathbb{N}\}$ has size 15. Then $[a]_{odd} = [a]_{even}\mathsf{R}$ also has size 15, so

$$|[a]| = |[a]_{even}| + |[a]_{odd}| = 15 + 15 = 30.$$ ◀

Thus we can equivalently define $[a]$ by

$$[a] = \begin{cases} \{a(\mathsf{RU})^k\mathsf{R}^\sigma : 0 \leqslant k < 15, \sigma \in \{0,1\}\} & \text{for } a \text{ even,} \\ \{a\mathsf{U}(\mathsf{RU})^k\mathsf{R}^\sigma : 0 \leqslant k < 15, \sigma \in \{0,1\}\} & \text{for } a \text{ odd.} \end{cases}$$

We can visualize the equivalence classes with respect to the graph $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ as 30-cycles, as illustrated in Figure 6.

Let Q be the graph whose vertex set is $\{[a] : a \in \langle \mathsf{R}, \mathsf{U}\rangle\}$ and where two vertices $A, B \in V(Q)$ are adjacent if and only if there exists an $a \in A$ and $b \in B$ such that either $a = b\mathsf{U}$ or $a = b\mathsf{U}'$. Given two cycle-classes $A = [a]$ and $B = [b]$ that are adjacent in this sense, we are interested in the appearance of $G_{\langle \mathsf{R},\mathsf{U}\rangle}[A \cup B]$. The following few lemmas explore how a cycle-class interacts with its neighbors.

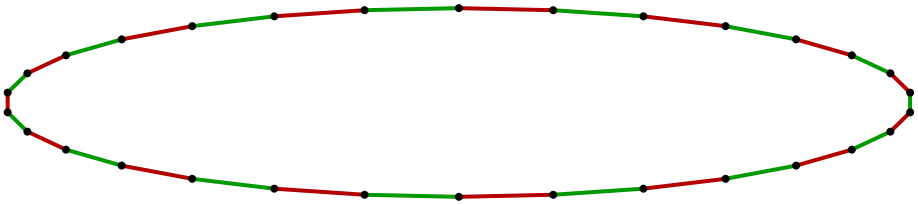


Figure 6 A subgraph of $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ induced by an equivalence class. Red edges represent the moves R and R', while green edges represent the moves U and U'.

▶ **Lemma 10.** *For any two even $a, b \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, the function $f\colon x \mapsto ba^{-1}x$ is a graph automorphism that satisfies:*

1. *$f(a) = b$,*
2. *for any $c, d \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, $f(cd) = f(c)d$,*
3. *for all $c \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, such that $[f(c)] = f([c])$.*

Proof. To check that f is indeed a graph automorphism, let $c, d \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, and note that $f(c)^{-1}f(d) = (ba^{-1}c)^{-1}(ba^{-1}d) = c^{-1}d$, from where

$$\begin{aligned}\{c,d\} \in E(G_{\langle \mathsf{R},\mathsf{U}\rangle}) &\iff c^{-1}d \in \{\mathsf{R},\mathsf{U}\}\\ &\iff f(c)^{-1}f(d) \in \{\mathsf{R},\mathsf{U}\}\\ &\iff \{f(c),f(d)\} \in E(G_{\langle \mathsf{R},\mathsf{U}\rangle}).\end{aligned}$$

To verify the first condition, we have $f(a) = ba^{-1}a = b$. To verify the second condition, let $c, d \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, and note that $f(cd) = ba^{-1}cd = f(c)d$.

For the third condition, suppose without loss of generality that $c \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$ is even (the odd case is almost identical) and observe that

$$\begin{aligned}[f(c)] =& \{f(c)(\mathsf{RU})^k\mathsf{R}^\sigma : 0 \leqslant k < 15, \sigma \in \{0,1\}\}\\ =& \{f(c(\mathsf{RU})^k\mathsf{R}^\sigma) : 0 \leqslant k < 15, \sigma \in \{0,1\}\}\\ =& f([c]).\end{aligned}$$

Thus f satisfies the conditions required by the lemma. ◀

▶ **Lemma 11.** *The graph Q is vertex-transitive.*

Proof. It suffices to show that for all $A, B \in V(Q)$, there exists an automorphism f such that $f(A) = B$. Let $A, B \in V(Q)$, and choose $a \in A$ and $b \in B$ both even. Apply Lemma 10 on a and b to obtain f. By Lemma 10, f maps Q to itself, and since $f(a) = b$, we have that $f(A) = B$. ◀

▶ **Lemma 12.** *For all even $a \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, we have $[a\mathsf{U}'\mathsf{U}'] = [a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']$ if and only if k is a multiple of 5.*

Proof. This can be verified computationally for $a = 1_{\mathcal{R}}$ (see Appendix A). Thus, we have that k is a multiple of 5 if and only if $[1_{\mathcal{R}}\mathsf{U}'\mathsf{U}'] = [1_{\mathcal{R}}(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']$, and since $1_{\mathcal{R}}$ is the group identity, we have

$$k \text{ is a multiple of } 5 \iff [\mathsf{U}'\mathsf{U}'] = [(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']. \tag{1}$$

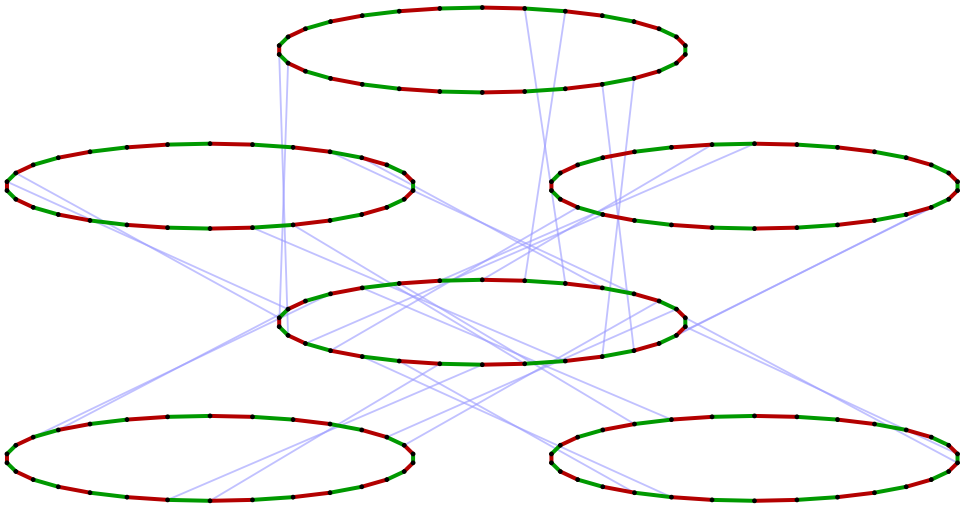


Figure 7 A subgraph of $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ induced by an equivalence class and its neighbors with respect to Q. Blue edges represent the moves U and U' that connect positions of different equivalence classes.

For an even vertex a where $a \neq 1_{\mathcal{R}}$, let f be the automorphism given by Lemma 10 with $f(1_{\mathcal{R}}) = a$. Then, combining Equation (1) with the properties of f given by Lemma 10, we obtain

$$\begin{aligned} k \text{ is a multiple of } 5 &\iff [\mathsf{U}'\mathsf{U}'] = [(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}'] \\ &\iff f([\mathsf{U}'\mathsf{U}']) = f([(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']) \\ &\iff [f(\mathsf{U}'\mathsf{U}')] = [f((\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}')] \\ &\iff [f(1_{\mathcal{R}})\mathsf{U}'\mathsf{U}'] = [f(1_{\mathcal{R}})(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}'] \\ &\iff [a\mathsf{U}'\mathsf{U}'] = [a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']. \end{aligned}$$

◀

▶ **Lemma 13.** *The graph Q is 5-regular, and for any $A \in V(Q)$ and even $a \in A$, the neighbors of A in Q are exactly the elements of the set*

$$\{[a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']\colon 0 \leqslant k < 5\}.$$

Proof. It can be checked that the vertex $[1_{\mathcal{R}}]$ has degree 5, so by vertex-transitivity (Lemma 11), Q is 5-regular. Let $A \in V(Q)$ and pick any even $a \in A$. By Lemma 12, no two elements in the set $\{a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}'\colon 0 \leqslant k < 5\}$ are in the same equivalence class, so $\{[a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']\colon 0 \leqslant k < 5\}$ has size 5. It remains to show that each of these elements is adjacent to A in Q. Indeed, for any $0 \leqslant k < 5$, $b = a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'$ is odd and, solving for a we have $a = b\mathsf{U}(\mathsf{R}\mathsf{U})^k$, so $[b] = [a] = A$. As $a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}' = b\mathsf{U}'$ is adjacent to $b \in A$, we have that $[a(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']$ is adjacent to A. ◀

A visualization of the results of Lemma 12 and Lemma 13 (In particular, the 5-regularity of Q) is shown by Figure 7.

4.2 Constructing Hamiltonian paths

We turn towards the construction of Hamiltonian paths in $G_{\langle \mathsf{R},\mathsf{U}\rangle}$. Given an even $a \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, we will construct a path π_1 from $1_{\mathcal{R}}$ to a with certain properties that allow us to extend π_1 into π_2, where π_2 visits every cycle-class at least once and also has properties that allow us to extend it into a Hamiltonian path.

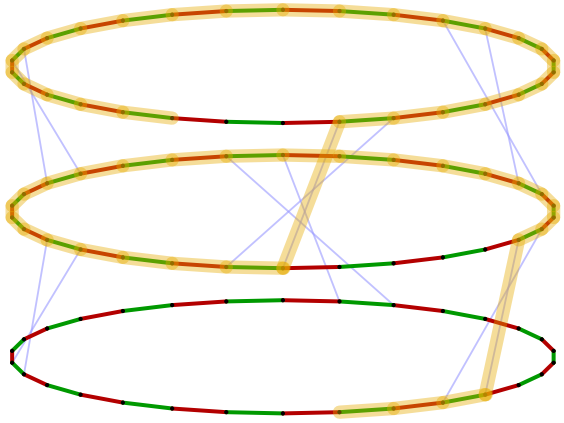


Figure 8 An example of π_1 from where $m = 3$, $k_1 = 13$, $k_2 = 12$, and $k_3 = 1$.

We begin by rigorously defining the properties of π_1, and by arguing that paths with such properties exist. Examples of paths with these properties are shown in Figure 8 and Figure 9a.

▶ **Lemma 14.** *Let $a \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$ be odd. Then there exists a path π_1 from $1_{\mathcal{R}}$ to a with the following properties:*

1. ***(Pair inclusion)*** *For all even $b \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, $b\mathsf{U}' \in \pi_1$ if and only if $b \in \pi_1$*
2. ***(Cycle options)*** *Let $C = \{[c] : c \in \pi_1\}$. For all even $b \in \bigcup_{[c]\in C}[c]$ such that $b \notin \pi_1$, either $b\mathsf{U} \in \pi_1$ or $[b\mathsf{U}] \notin C$.*
3. ***(Neighbor-connectivity)*** *For all $A \in C\backslash\{[a]\}$, if B is a neighbor of A with respect to Q, then there exists an even $d \in A$ such that $d \in \pi_1$ and $d\mathsf{U} \in B$.*
4. ***(Path construction invariant)*** *For all even $b \in \pi_1$, the vertex immediately following b in π_1 is either $b\mathsf{U}$ or $b\mathsf{U}'$.*

Proof. Let $\pi_Q = S_1 S_2 \dots S_m$ be a shortest path in Q from $[1_{\mathcal{R}}]$ to $[a]$.

By Lemma 13, for all i such that $1 \leqslant i < m$, given an even vertex $a_i \in S_i$, there exists a unique k_i (based on a_i and S_{i+1}) such that $10 \leqslant k_i \leqslant 14$ and $a_i(\mathsf{U}'\mathsf{R}')^{k_i}\mathsf{U}'\mathsf{U}' \in S_{i+1}$.

Consider the sequence $a_1, \dots, a_m$ defined by $a_1 := 1_{\mathcal{R}}$ and $a_{i+1} := a_i(\mathsf{U}'\mathsf{R}')^{k_i}\mathsf{U}'\mathsf{U}'$ for $1 \leqslant i \leqslant m-1$. Intuitively, this is the same as constructing our path greedily; we construct our π_1 such that π_1 covers as much of each S_i as possible whilst traversing $S_1 S_2 \dots S_m$.

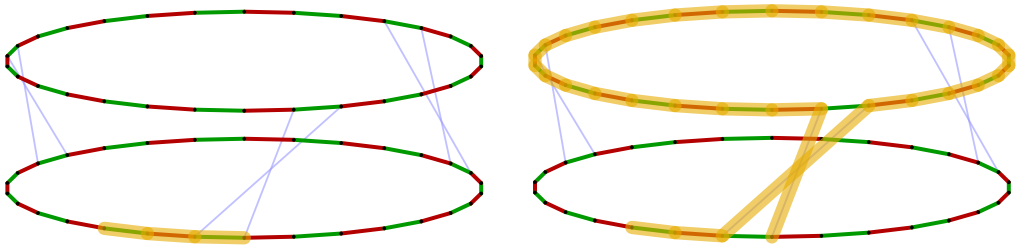
For $1 \leqslant i < m$, let π_{1_i} be the path that starts at a_i and records the positions visited by the sequence of moves $(\mathsf{U}'\mathsf{R}')^{k_i}\mathsf{U}'$, ending on $a_i(\mathsf{U}'\mathsf{R}')^{k_i}\mathsf{U}'$. Note a_m is even and $a_m \in S_m = [a]$. As a is odd, $a\mathsf{U}(\mathsf{R}\mathsf{U})^{k_m} = a_m$ for some k_m such that $0 \leqslant k_m \leqslant 14$. Then $a_m(\mathsf{U}'\mathsf{R}')^{k_m}\mathsf{U}' = a$. Let π_{1_m} be the path that starts at a_m and records the positions visited by performing the sequence of moves $(\mathsf{U}'\mathsf{R}')^{k_m}\mathsf{U}'$, ending on a.

Then $\pi_1 = \pi_{1_1}\pi_{1_2}\dots\pi_{1_m}$ is a path in $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ that starts on $1_{\mathcal{R}}$ and ends on a. It remains to show that π_1 satisfies the desired properties.

Note that any even position a in π_1 is followed immediately by $a\mathsf{U}'$, so condition 4 is satisfied. As π_1 visits an equal number of even and odd positions, condition 1 is satisfied as well.

To prove condition 2, suppose for the sake of contradiction that there is an even $b \in \bigcup_{[c]\in C}[c]$ such that $b \notin \pi_1$, $b\mathsf{U} \notin \pi_1$, and $[b\mathsf{U}] \in C$. Since $[b] \in C$ and π_1 is contained in $\bigcup_{j=1}^{m} S_j$, we have that $[b] \in \pi_Q$ and $[b\mathsf{U}] \in \pi_Q$. Suppose $[b] = S_i$.

Since $b \notin \pi_1$, it follows that $b = a_i(\mathsf{U}'\mathsf{R}')^k$ for some $k_i k < 15$. Note $a_i\mathsf{U}'\mathsf{U}' \in S_{i-1}$ (if it exists) and $a_i(\mathsf{U}'\mathsf{R}')^{k_i}\mathsf{U}'\mathsf{U}' \in S_{i+1}$ (if it exists).



(a) An example of π_1 where $|C| = 1$.

(b) The resulting π_1' using π_1 as shown in Figure 9a.

Figure 9 Illustration of the modification used in the π_1' construction.

Since $[b\mathsf{U}]$ is odd, $[b\mathsf{U}] = [b\mathsf{UU}]$, and by Lemma 12, $[b\mathsf{U}'\mathsf{U}'] = [a_i(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']$. Because $\mathsf{UU} = \mathsf{U}'\mathsf{U}'$, we have that $[b\mathsf{U}] = [a_i(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']$. Now we note that since $10 \leqslant k_i < k < 15$, k is neither a multiple of 5 nor does it differ from k_i by a multiple of 5, so $[a_i(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']$ is neither S_{i-1} nor S_{i+1}. Thus, while $[b\mathsf{U}]$ and $[b] = S_i$ are both in π_Q, π_Q does not visit these two vertices consecutively. Since $[b\mathsf{U}]$ and $[b]$ are adjacent in Q, this contradicts the fact that π_Q is a shortest path.

To prove condition 3, let $A = S_i$ where $i < m$. By Lemma 13, $B = [a_i(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}']$ for some $0 \leqslant k \leqslant 4$. Then $d = a_i(\mathsf{U}'\mathsf{R}')^k$ is in both π_1 and A, and

$$d\mathsf{U} = a_i(\mathsf{U}'\mathsf{R}')^k\mathsf{U} = a_i(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}'\mathsf{U}' \in [a_i(\mathsf{U}'\mathsf{R}')^k\mathsf{U}'\mathsf{U}'] = B.$$

Thus π_1 satisfies all four conditions, as desired. ◀

▶ **Theorem 15** (Godsil and Royle [10]). *A vertex-transitive graph with valency k has vertex connectivity at least $\frac{2}{3}(k+1)$*

▶ **Lemma 16.** *The graph Q is 2-connected.*

Proof. By Lemma 11 and Lemma 13, Q is a vertex-transitive graph with valency 5. Then by Theorem 15, Q has connectivity at least $\frac{2}{3}(5+1) = 4$. Thus Q is 4-connected, meaning it is also 2-connected. ◀

We will now argue that we can extend π_1 to visit more cycles while preserving its properties. Intuitively, we take our original path, find a neighboring cycle, deviate from our original path for a moment and fully fill out an untouched neighboring cycle, and finally pick up where we left off with the original path. An example of such a modification can be seen in the difference between Figure 9a and Figure 9b.

▶ **Lemma 17.** *Let π_1 be a path and C be a set as described in Lemma 14. Suppose $C \subsetneq V(Q)$. Then there exists a path π_1' and $C' = \{[c] : c \in \pi_1'\}$ such that $C \subsetneq C'$ and π_1' satisfies the conditions for π_1 given by Lemma 14.*

Proof. We will case on whether $|C| > 1$.

- **Case 1.** If $|C| = 1$, then as $1_{\mathcal{R}} \in \pi_1$, it follows that $C = \{[1_{\mathcal{R}}]\}$, so all vertices in π_1 are in $[1_{\mathcal{R}}]$. Then as $\mathsf{U} \notin [1_{\mathcal{R}}]$, it follows that $\pi_1 = v_1v_2\pi_a$ for some path π_a where $v_1 = 1_{\mathcal{R}}$ and $v_2 = \mathsf{U}'$.
  Let π_b be the path that begins on $1_{\mathcal{R}}$ and records the positions visited by performing the sequence of moves $\mathsf{UR}'(\mathsf{U}'\mathsf{R}')^{14}\mathsf{U}$. Then π_b ends on the position $\mathsf{UR}'(\mathsf{U}'\mathsf{R}')^{14}\mathsf{U} = \mathsf{UUU} = \mathsf{U}'$. A visualization thereof is shown with Figure 9a and Figure 9b. Let $\pi_1' = \pi_b\pi_a$, so $C' = \{[1_{\mathcal{R}}], [\mathsf{U}]\}$, so $C \subsetneq C'$. Since the set of vertices visited by π_1' is exactly the union of the set of vertices visited by π_1 and $[\mathsf{U}]$, condition 1 of Lemma 14 is satisfied.

For condition 2, suppose $b \in \bigcup C'$ and $b \notin \pi_1'$. Then since all vertices in $[\mathsf{U}]$ are in π_1', it follows that $b \in \bigcup C$. Then either $b\mathsf{U} \in \pi_1$, in which case $b\mathsf{U} \in \pi_1'$, or $[b\mathsf{U}] \notin C$. In the latter case, either $[b\mathsf{U}] = [\mathsf{U}]$, which implies $b\mathsf{U} \in \pi_1'$, or $[b\mathsf{U}] \neq [\mathsf{U}]$, which implies $[b\mathsf{U}] \notin C'$.

For condition 3, we only need to check $A = [\mathsf{U}]$. Note that all vertices in A are in π_1'. By Lemma 13, for any neighboring B, there exists an even $d \in A$ such that $B = [d\mathsf{U}'\mathsf{U}']$. Since $d\mathsf{U}'\mathsf{U}' \in B$ is even, $d\mathsf{U}'\mathsf{U}'\mathsf{U}' = d\mathsf{U} \in B$. As all vertices in $A = [\mathsf{U}]$ are also in π_1', we have that $d \in \pi_1'$, and we are done.

Condition 4 is satisfied because $\pi_1' = \pi_b\pi_a$, where vertices π_a satisfy the condition because π_a is a subpath of π_1 and vertices in π_b satisfy the condition because π_b begins on $1_{\mathcal{R}}$, an even vertex, and is constructed by the move sequence $\mathsf{UR}'(\mathsf{U}'\mathsf{R}')^{14}\mathsf{U}$.

Case 2. If $|C| > 1$, then $C\backslash\{[a]\}$ is nonempty. By Lemma 16, the graph of Q with the vertex $[a]$ removed is connected, so there exists a $B \notin C_i$ that is adjacent to some $A \in C_i\backslash\{[a]\}$. By the third clause of Lemma 14, there exists an even $d \in A$ such that $d \in \pi_1$ and $d\mathsf{U} \in B$. Since $B \notin C$, $d\mathsf{U} \notin \pi_1$, so the vertex immediately following d in π_1 must be $d\mathsf{U}'$. Note that

$$d\mathsf{U}' = d\mathsf{UUU} = d\mathsf{UR}'(\mathsf{U}'\mathsf{R}')^{14}\mathsf{U}.$$

Let π_a be the prefix of π_1 that ends at d, exclusive. Let π_b be the path that starts at d and records the positions visited by performing the move sequence $\mathsf{UR}'(\mathsf{U}'\mathsf{R}')^{14}\mathsf{U}$, ending on the position $d\mathsf{U}'$. Let π_c be a suffix of π_1 that begins immediately after $d\mathsf{U}'$. Let $\pi_1' = \pi_a\pi_b\pi_c$. Then $C' = C \cup \{B\}$, so $C \subsetneq C'$. It remains to prove that π_1' satisfies the properties given by Lemma 14. Since π_1' traverses exactly the union of B and the set of points traversed by π_1, condition 1 remains true.

For condition 2, we have that $C' = \{[c] \colon c \in \pi_1'\}$. Suppose $b \in \bigcup C'$ and $b \notin \pi_1'$. Then $b \notin \pi_1$. Since all points in B are also in π_1', we also have that $b \in \bigcup C$. As π_1 satisfies property 2 from Lemma 14, it follows that either $b\mathsf{U} \in \pi_1$ or $[b\mathsf{U}] \notin C$. If $b\mathsf{U} \in \pi_1$, then $b\mathsf{U} \in \pi_1'$. Otherwise, $[b\mathsf{U}] \notin C$, so either $[b\mathsf{U}] = B$, in which case $b\mathsf{U} \in \pi_1'$, or $[b\mathsf{U}] \notin C'$.

For condition 3, suppose $A \in C'\backslash\{[a]\}$ and B' is a neighbor of A with respect to Q. We will case on whether $A \in C\backslash\{[a]\}$. If $A \in C\backslash\{[a]\}$, then because π_1 satisfies condition 3, there exists an even $d \in A$ such that $d\mathsf{U} \in B'$ and $d \in \pi_1$, which means $d \in \pi_1'$ as well. If $A \notin C\backslash\{[a]\}$, then $A = B$. Then there exists a $d \in B$ such that either $d\mathsf{U} \in B'$ or $d\mathsf{U}' \in B'$. If $d\mathsf{U} \in B'$, then d must be even, since if d is odd, then $d\mathsf{U} \in B \neq B$, so we are done. Otherwise, if $d\mathsf{U}' = d(\mathsf{RU})^k\mathsf{R} \in B'$, then d must be odd, so $d\mathsf{U} \in A$ is even and $d\mathsf{U}' \in B'$ is even. Then $d\mathsf{UU} = d\mathsf{U}'\mathsf{U}' \in B$.

To show condition 4 is true for $\pi_1' = \pi_a\pi_b\pi_c$, since π_a and π_c are both subpaths of π_1 that start on even positions (and π_1 satisfies condition 4), it suffices to show that π_b satisfies condition 4. This can be verified by observing that π_b starts on d – which is an even position – and is formed by the move sequence $\mathsf{UR}'(\mathsf{U}'\mathsf{R}')^{14}\mathsf{U}$. ◀

By applying Lemma 17 iteratively, we obtain a path with the properties of π_1 that also visits every cycle, upon which new properties emerge.

▶ **Lemma 18.** *Let $a \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$ be odd. Then there exists a path π_2 from $1_{\mathcal{R}}$ to a with the following properties:*

1. *For all even $b \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, $b\mathsf{U}' \in \pi_2$ if and only if $b \in \pi_2$*
2. *For all even $b \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, either $b \in \pi_2$ or $b\mathsf{U} \in \pi_2$*
3. *For all even $b \in \pi_2$, the vertex immediately following b in π_2 is either $b\mathsf{U}$ or $b\mathsf{U}'$.*

Proof. Let $\pi_{1,1}$ be a path given by Lemma 14, and define $C_i = \{[c] \colon c \in \pi_{1,i}\}$. Given $\pi_{1,i}$ satisfying the properties given by Lemma 14, suppose $C_i \subsetneq Q$. Then by Lemma 17, there exists a $\pi_{1,i+1}$ such that $C_i \subsetneq C_{i+1}$. By repeating this argument (no more than $|Q|$ times), there is a $\pi_{1,n}$ such that $C_n = Q$ and $\pi_{1,n}$ satisfies the properties of Lemma 14. Let $\pi_2 = \pi_{1,n}$. Note π_2 satisfies properties 1 and 3 by properties 1 and 4 from Lemma 14. Furthermore, $\{[c] \colon c \in \pi_2\} = C_n = Q$ and $\bigcup_{[c]\in C_n} C_n = G_{\langle \mathsf{R},\mathsf{U}\rangle}$, so property 2 follows from the property 2 from Lemma 14. ◀

We will now argue that a path π_2 as described in Lemma 18 can be extended into a Hamiltonian path. We make use of the fact that $\mathsf{U}' = \mathsf{UUU}$. If we have $b\mathsf{UU}$ and $b\mathsf{U}$ as consecutive vertices in π_2, and b and $b\mathsf{U}'$ are not in π_2, then we can extend π_2 to include b and $b\mathsf{U}'$ by performing UUU in place of U' on $b\mathsf{UU}$.

▶ **Theorem 19.** *The graph $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ is Hamilton laceable.*

Proof. Since $G_{\langle \mathsf{R},\mathsf{U}\rangle}$ is vertex-transitive, it suffices to show that for an arbitrary odd $a \in V(G_{\langle \mathsf{R},\mathsf{U}\rangle})$, there exists a path from $1_{\mathcal{R}}$ to a that is Hamiltonian for $G_{\langle \mathsf{R},\mathsf{U}\rangle}$. Fix a. Let $\pi_2^{(1)} = \pi_2$ be a path from $1_{\mathcal{R}}$ to a as given by Lemma 18. If $\pi_2^{(1)}$ is Hamiltonian, we are done. Otherwise, we claim that there exists a path $\pi_2^{(2)}$ from $1_{\mathcal{R}}$ to a that satisfies the conditions for π_2 from Lemma 18 and is strictly longer than $\pi_2^{(1)}$. If the claim holds, then by repeating this argument (no more than $|G_{\langle \mathsf{R},\mathsf{U}\rangle}| - 1$ times) we must end with a Hamiltonian path.

We now prove the claim. Suppose the path $\pi_2^{(i)}$, for some $i \geqslant 1$, is not Hamiltonian, and thus there is some vertex $b \notin \pi_2^{(i)}$. Then, by the first clause of Lemma 18, we can assume without loss of generality that b is even and $b\mathsf{U}' \notin \pi_2^{(i)}$. Then, by the second clause of Lemma 18, $b\mathsf{U} \in \pi_2^{(i)}$. As $b \notin \pi_2^{(i)}$, it follows from the third clause of Lemma 18 that the vertex before $b\mathsf{U}$ in $\pi_2^{(i)}$ is $c := b\mathsf{UU}$. Thus, c is followed immediately by $c\mathsf{U}' = c\mathsf{UUU} = b\mathsf{U}$ in $\pi_2^{(i)}$, while $b = c\mathsf{UU}$ and $b\mathsf{U}' = c\mathsf{U}$ are both not in $\pi_2^{(i)}$. Let π_a be the prefix of $\pi_2^{(i)}$ that ends with c and let π_b be the suffix of $\pi_2^{(i)}$ that begins on $c\mathsf{U}'$. Then, $\pi_2^{(i+1)} := \pi_a(b)(b\mathsf{U})\pi_b$ is a path from $1_{\mathcal{R}}$ to a, strictly longer than $\pi_2^{(i)}$, and also holds the conditions from Lemma 18. This establishes the claim, and thus concludes the proof. ◀

5 Extending Laceability to $G_{\mathcal{R}}$

We turn to prove Hamilton laceability for $G_{\mathcal{R}}$. First, observe that for arbitrary $a \in V(G_{\mathcal{R}})$, we have that $G_{\mathcal{R}}[a\langle \mathsf{R},\mathsf{U}\rangle]$ is isomorphic to $G_{\langle \mathsf{R},\mathsf{U}\rangle}$. This gives us the following generalization of Theorem 19.

▶ **Corollary 20.** *For any $a \in \mathcal{R}$, $G_{\mathcal{R}}[a\langle \mathsf{R},\mathsf{U}\rangle]$ is Hamilton laceable.*

Let $Q_{\mathcal{R}}$ be the graph whose vertex set is $\mathcal{R}/\langle \mathsf{R},\mathsf{U}\rangle$ and where two vertices A, B are adjacent if there exist $a_0 \in A$, $b_0 \in B$ even and $a_1 \in A$, $b_1 \in B$ odd such that $\{a_0, b_1\}$ and $\{a_1, b_0\}$ are both edges in $G_{\mathcal{R}}$. It can be checked computationally that $|\mathcal{R}| = 3{,}674{,}160$ and $|\langle \mathsf{R},\mathsf{U}\rangle| = 29{,}160$, so $|Q_{\mathcal{R}}| = 3{,}674{,}160/29{,}160 = 126$ (see Appendix A).

▶ **Lemma 21.** *The graph $Q_{\mathcal{R}}$ is vertex-transitive.*

Proof. Let $a\langle \mathsf{R},\mathsf{U}\rangle, b\langle \mathsf{R},\mathsf{U}\rangle$ be two arbitrary vertices in $Q_{\mathcal{R}}$. We want to show that there exists a graph automorphism that maps $a\langle \mathsf{R},\mathsf{U}\rangle$ to $b\langle \mathsf{R},\mathsf{U}\rangle$ Define a function f on $\mathcal{R}/\langle \mathsf{R},\mathsf{U}\rangle$ by

$$x\langle \mathsf{R},\mathsf{U}\rangle \mapsto ba^{-1}x\langle \mathsf{R},\mathsf{U}\rangle.$$

Note f is well-defined because if $x'\langle \mathsf{R}, \mathsf{U}\rangle = x\langle \mathsf{R}, \mathsf{U}\rangle$, then $x' = xh$ for some $h \in \langle \mathsf{R}, \mathsf{U}\rangle$. Then

$$f(x'\langle \mathsf{R}, \mathsf{U}\rangle) = f(xh\langle \mathsf{R}, \mathsf{U}\rangle) = ba^{-1}xh\langle \mathsf{R}, \mathsf{U}\rangle = ba^{-1}x\langle \mathsf{R}, \mathsf{U}\rangle = f(x\langle \mathsf{R}, \mathsf{U}\rangle).$$

Since $f^{-1}\colon x\langle \mathsf{R}, \mathsf{U}\rangle \mapsto ab^{-1}x\langle \mathsf{R}, \mathsf{U}\rangle$ is a two-sided inverse, f is bijective.

To show that f preserves edges, suppose C and D are adjacent by edges $\{c_0, d_1\}$ and c_1, d_0. Then $ba^{-1}c_0, ba^{-1}c_1 \in f(C)$ and $ba^{-1}d_0, ba^{-1}d_1 \in f(D)$, where $\{c_0, d_1\}$ and $\{c_1, d_0\}$ are both edges in $G_{\mathcal{R}}$, so $f(C)$ is adjacent to $f(D)$.

Thus f is a graph automorphism, and it can be easily checked that f maps $a\langle \mathsf{R}, \mathsf{U}\rangle$ to $b\langle \mathsf{R}, \mathsf{U}\rangle$. ◀

▶ **Lemma 22.** *The graph $Q_{\mathcal{R}}$ is 90-regular*

Proof. It follows from Lemma 21 that $Q_{\mathcal{R}}$ is regular. Additionally, it can be checked computationally that the vertex $\langle \mathsf{R}, \mathsf{U}\rangle$ has degree 90 (see Appendix Appendix A), so $Q_{\mathcal{R}}$ is 90-regular. ◀

▶ **Lemma 23.** *The graph $Q_{\mathcal{R}}$ is 3-connected*

Proof. By Lemma 21 and Lemma 22, $Q_{\mathcal{R}}$ is a vertex-transitive graph with a valency of 90. It follows from Theorem 15 that $Q_{\mathcal{R}}$ has vertex-connectivity at least $\frac{2}{3}(90+1) > 3$. Thus $Q_{\mathcal{R}}$ is 3-connected. ◀

▶ **Theorem 24** (Ore [23])**.** *Let G be a 2-connected graph on n vertices satisfying $d(x)+d(y) \geqslant n+1$ for every pair of non-adjacent vertices $x, y \in G$. Then, G is Hamilton connected.*

▶ **Lemma 25.** *The graph $Q_{\mathcal{R}}$ is Hamilton connected, and so is the induced subgraph $Q'_{\mathcal{R}} := Q_{\mathcal{R}} \backslash \{1_{\mathcal{R}}\langle \mathsf{R}, \mathsf{U}\rangle\}$.*

Proof. By Lemma 23, both $Q_{\mathcal{R}}$ and $Q'_{\mathcal{R}}$ are 2-connected. Recall that $Q_{\mathcal{R}}$ is a 90-regular graph on 126 vertices, and as $Q'_{\mathcal{R}}$ is the same graph but with one vertex removed, every remaining vertex has degree at least 89. Therefore, both $Q_{\mathcal{R}}$ and $Q'_{\mathcal{R}}$ satisfy the hypotheses of Theorem 24 ($d(x) + d(y) = 180 \geqslant 127$ for $Q_{\mathcal{R}}$, and $d(x) + d(y) \geqslant 178 \geqslant 126$ for $Q'_{\mathcal{R}}$) and are thus Hamilton connected. ◀

Finally, we prove our main result:

▶ **Theorem 26.** *The graph $G_{\mathcal{R}}$ is Hamilton laceable.*

Proof. First, observe that by vertex transitivity it suffices to consider an arbitrary odd position b and show that there is a Hamiltonian path whose endpoints are b and $1_{\mathcal{R}}$. Now the proof is by cases depending on whether $b \in \langle \mathsf{R}, \mathsf{U}\rangle$ or not.

- **Case 1.** Suppose $b \notin \langle \mathsf{R}, \mathsf{U}\rangle$. By Lemma 25, we can find a Hamiltonian path $\pi_Q = S_1 S_2 \dots S_{126}$ on $Q_{\mathcal{R}}$ such that $1_{\mathcal{R}} \in S_1$ and $b \in S_{126}$. Then we can find distinct $b_1, b_2, \dots, b_{126}$, all odd, such that $b_{126} = b$ and for all $1 \leqslant i < 126$, $b_i \in S_i$ and b_i is adjacent to some $a_{i+1} \in S_{i+1}$. Furthermore, let $a_1 = 1_{\mathcal{R}}$. For all $1 \leqslant i \leqslant 126$, a_i is even and b_i is odd, so by Corollary 20 we can find a path π_i that begins at a_i, ends at b_i, and is Hamiltonian for $G_{\mathcal{R}}[S_i]$. Then $\pi_1\pi_2 \dots \pi_{126}$ is a path that starts at $a_1 = 1_{\mathcal{R}}$, ends at $b_{126} = b$, and is Hamiltonian for $G_{\mathcal{R}}$.
- **Case 2.** Now suppose $b \in \langle \mathsf{R}, \mathsf{U}\rangle$. We can find a path $\pi_0 = b \dots x1_{\mathcal{R}}$ that is Hamiltonian for $G_{\langle \mathsf{R},\mathsf{U}\rangle}$. Then $x \in \{\mathsf{R}, \mathsf{U}, \mathsf{R}', \mathsf{U}'\}$. It can be checked that for any $x \in \{\mathsf{R}, \mathsf{U}, \mathsf{R}', \mathsf{U}'\}$, $x\mathsf{F} \notin \mathsf{F}\langle \mathsf{R}, \mathsf{U}\rangle$, so $x\mathsf{F}\langle \mathsf{R}, \mathsf{U}\rangle \neq \mathsf{F}\langle \mathsf{R}, \mathsf{U}\rangle$. Then by Lemma 25, we can find a Hamiltonian path $\pi_{Q'} = S_1 S_2 ... S_{125}$ on $Q'_{\mathcal{R}}$ such that $x\mathsf{F} \in S_1$ and $\mathsf{F} \in S_{125}$. We can find $b_1, b_2, \dots, b_{126}$,

all odd, such that $b_{125} = \mathsf{F}$ and for all $1 \leqslant i < 125$, $b_i \in S_i$ and b_i is adjacent to some $a_{i+1} \in S_{i+1}$. Let $a_1 = x\mathsf{F}$. By Corollary 20 for all $1 \leqslant i \leqslant 125$, we can find a path π_i that begins at a_i, ends at b_i, and is Hamiltonian for $G_{\mathcal{R}}[S_i]$. Let π'_0 be such that $\pi_0 = \pi'_0 1_{\mathcal{R}}$. Then $\pi'_0\pi_1\pi_2 \dots \pi_{125}1_{\mathcal{R}}$ is a path that starts on b, ends on $1_{\mathcal{R}}$, and is Hamiltonian for $G_{\mathcal{R}}$. ◀

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A GAP Code for Computational Checks

Listing 1 GAP code for computational checks. Also publicly available at `https://github.com/bsubercaseaux/rubik-hamiltonian/`.

```
# 2x2x2 Rubik's Cube group tests
# --- Generators on 24 "stickers" (clockwise when looking at that
    face) ---
R := (2,19,22,10)(4,17,24,12)(13,14,16,15);;
U := (1,2,4,3)(5,17,13,9)(6,18,14,10);;
F := (3, 13,22,8)(4,15,21,6)(9,10,12,11);;

cube_2x2x2 := Group(R, U, F);;
Print("Order(cube_2x2x2) = ", Order(cube_2x2x2), "\n");; # Should be
    3674160


# Check for Lemma 9
RU := R * U;;
subRU := Group(RU);;
Print("Order(subRU) = ", Order(subRU), "\n");; # Should be 15


# Check for Lemma 12
# Helper: check evenness of a permutation
IsEven := g -> SignPerm(g) = 1;

# Cycle class [g] for EVEN g, per Definition 8 / Lemma 9 discussion:
#   [g] = { g*(RU)^k*R^sigma : 0<=k<15, sigma in {0,1} }
CycleClassEven := function(g)
  local k, L;
  if not IsEven(g) then
    Error("CycleClassEven expects an even element");
  fi;
  L := [];
  for k in [0..14] do
    Add(L, g * (RU^k));          # sigma = 0
    Add(L, g * (RU^k) * R);      # sigma = 1
  od;
  return Set(L);                # make order/copies irrelevant for
      equality tests
end;

UU := U^-1 * U^-1;                    # U^{-1}U^{-1} = U^{-2}, this is
    even
UpRp := U^-1 * R^-1;

base := CycleClassEven(UU);

Print("k values where [U^{-2}] = [(U^{-1}R^{-1})^k U^{-2}]:\n");
for k in [0..14] do
  if base = CycleClassEven((UpRp^k) * UU) then
    Print(k, " ");
  fi;
od;
Print("\nExpected: 0 5 10\n");
```

```
# Section 5
R_U_subgroup := Group(R, U);;
Print("Order(R_U_subgroup) = ", Order(R_U_subgroup), "\n"); # Should
    be 29160

# --- Lemma 22 check: deg_{Q_R}(<R,U>) = 90 ---
Print("Index(<R, U, F>,<R, U>) = ", Index(cube_2x2x2,R_U_subgroup), "
    \n");;   # should be 126

# Vertices of Q_R are LEFT cosets gH
cosets := LeftCosets(cube_2x2x2, R_U_subgroup);;
Print("|Q_R| = ", Length(cosets), "\n");;    # should be 126

# Precompute representatives so we can locate cosets quickly
reps := List(cosets, Representative);;

# Map x in cube_2x2x2 to the index i such that x is in reps[i]*
    R_U_subgroup (i.e., xR_U_subgroup is that coset)
CosetIndex := function(x)
  local i;
  for i in [1..Length(reps)] do
    # x in reps[i]*R_U_subgroup  <=>  reps[i]^-1 * x in R_U_subgroup
    if (reps[i]^-1 * x) in R_U_subgroup then
      return i;
    fi;
  od;
  Error("CosetIndex: could not locate coset for given element");
end;

# Identify the vertex R_U_subgroup itself (the coset containing the
    identity)
id := Identity(cube_2x2x2);;
idIdx := CosetIndex(id);;

Finv := F^-1;;

# --- Definition of adjacency in Q_R specialized to A = H ---
# B is adjacent to R_U_subgroup iff:
#  - there is some EVEN a in R_U_subgroup with a*F^{+-1} landing in B
    (odd vertex in B),
#  - and some ODD  a in R_U_subgroup with a*F^{+-1} landing in B (
    even vertex in B).
#
# So compute:
#   evenHit = { (a*F^{+-1})R_U_subgroup : a in R_U_subgroup even }
#   oddHit  = { (a*F^{+-1})R_U_subgroup : a in R_U_subgroup odd  }
# Neighbors are Intersection(evenHit, oddHit), excluding R_U_subgroup
    itself.

evenHit := [];;
oddHit  := [];;

for a in Elements(R_U_subgroup) do
  if IsEven(a) then
```

```
    Add(evenHit, CosetIndex(a * F));
    Add(evenHit, CosetIndex(a * Finv));
  else
    Add(oddHit,  CosetIndex(a * F));
    Add(oddHit,  CosetIndex(a * Finv));
  fi;
od;

evenHit := Set(evenHit);;
oddHit  := Set(oddHit);;

neighbors := Intersection(evenHit, oddHit);;

# Remove self if it appears (it *shouldn't* be adjacent to itself)
if idIdx in neighbors then
  RemoveSet(neighbors, idIdx);
fi;

Print("deg_{Q_R}(<R,U>) = ", Length(neighbors), "\n");;
Print("Expected: 90\n");
```