The Closed Hull Game and the Closed Interval Game

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Abstract

Given a set S of vertices in a graph G, its geodesic interval is the set $\mathrm{I}(S)$ containing S and all vertices on a shortest path between vertices of S. A set S is convex if $\mathrm{I}(S) = S$. Moreover, the convex hull $\mathcal{H}(S)$ of S is the smallest convex set containing S. In 1984, Harary introduced convexity games where two players, Alice and Bob, alternately select vertices of a graph $G = (V, E)$ such that, if the set of already selected vertices is S, the next player can only select a vertex in $V \setminus \mathrm{I}(S)$ (closed interval game) or in $V \setminus \mathcal{H}(S)$ (closed hull game). Normal and misère versions of these games have been studied and here, we introduced the optimization variants of them. Formally, given a graph G and $k \in \mathbb{N}$, Alice wins if the game ends after at most k vertices have been selected and Bob wins otherwise. The corresponding problem consists of determining which player has a winning strategy.

We prove that the closed interval optimization game is PSPACE-complete in graphs with diameter 4 and that the closed hull optimization game is NP-hard in bipartite graphs and in split graphs. On the positive side, we prove that both games can be solved in polynomial time in trees and that the closed hull optimization game can be solved in polynomial time in cobipartite graphs. We conjecture that the closed interval optimization game is NP-hard in cobipartite graphs and that the closed hull optimization game is PSPACE-complete in general graphs.

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1 Introduction

Convexity has long been a topic of significant interest in mathematics. Its application to graph theory, however, is relatively recent, dating back to approximately 50 years ago. One of the earliest contributions in this direction is the 1972 paper by Erdős et al. [22], which studied convexity in the context of tournaments. According to Duchet [21], the first English-language publication on convexity in graphs was the 1981 paper "*Convexity in Graphs*" by Harary and Nieminen [25].

The main investigated convexity is the *geodesic convexity*. Let G be a graph and $S \subseteq V(G)$ a subset of vertices of G. The *interval* $\mathrm{I}(S)$ of S is the set S together with every vertex outside S that belongs to a shortest path between two vertices of S. We say that S is *convex* if $\mathrm{I}(S) = S$. The *convex hull*, $\mathcal{H}(S)$, of S is the minimum convex set that contains S. We say that S is an *interval set* if $\mathrm{I}(S) = V(G)$ and that S is a *hull set* if $\mathcal{H}(S) = V(G)$. Typical questions are to compute a smallest hull set [23] or a smallest interval set [26] or a largest proper convex set [15] of a given graph.

In 1984, Frank Harary introduced the first graph convexity games in his abstract "*Convexity in Graphs: Achievement and Avoidance Games*" [24]. This line of research on graph convexity games continued until 2003, with the articles [12, 13, 27, 28] investigating Harary's games in simple graph classes. Such games are two-player (Alice and Bob) impartial combinatorial games. The decision problem associated with them is to determine if Alice has a winning strategy.

Among Harary's 1984 games, the following stand out: the Closed Hull Game (CHG) and the Closed Interval Game (CIG). In both, it is given a graph G and, during the game, some vertices are *selected* by the players (forming the set S of selected vertices) and other vertices are considered *covered* (according to specific rules of the game). Every selected vertex is a covered vertex, but not every covered vertex is a selected vertex. Alice and Bob play by alternately selecting uncovered vertices. In the game CHG, a vertex v is covered if $v \in \mathcal{H}(S)$. In the game CIG, a vertex v is covered if $v \in \mathrm{I}(S)$. Each game *ends* when all vertices are covered. That is, CHG ends when S is a hull set and CIG ends when S is an interval set. In the normal (resp. misère) variant, the last person to select a vertex wins (resp. loses).

In 2024, Araújo et al. [4] revived the research on graph convexity games by using the Sprague-Grundy Theory to obtain a polynomial time algorithm for the games CHG-normal and CIG-normal in trees. They also proved that the games CHG-normal and CHG-misère are PSPACE-complete. Other results on similar topics have been obtained in [6, 17]. The PSPACE-hardness of CIG-normal and CIG-misère are still open problems. For the investigation of partizan convexity games, we refer the reader to [5].

Another variant that has been extensively researched in the literature of games on graphs, in addition to the normal and misère variants, is the optimization variant. Here, roughly speaking, Alice wants to minimize some parameter of the game while Bob wants to maximize the same parameter or vice-versa. As an example, the game Kayles [7, 8] of obtaining an independent set of a graph was proved PSPACE-hard in the normal variant [29] in 1978, in the misère variant [14] in 2024 and in the optimization variant in [11] in 2025. Also the Domination game of obtaining a dominating set was proved PSPACE-hard in the optimization variant [9] in 2016 and in the normal and misère variants [10] in 2025.

In this paper, we focus on the optimization variant of CHG and CIG, denoted by CHG-opt and CIG-opt, in which it is also given an integer k and Alice wins if the size of the set S of selected vertices is at most k, no matter who plays last. We often think of the total number of selected vertices as the *time* (or duration) of the game, measured in the total number of *turns* that have been played.

We prove that CHG-opt and CIG-opt are polynomial time solvable in trees and that CHG-opt is polynomial time solvable in cobipartite graphs. We also prove that CIG-opt is PSPACE-complete in graphs with diameter 4 and that CHG-opt is NP-hard in bipartite graphs and in split graphs. Unfortunately, we were unable to prove the PSPACE-hardness of CHG-opt, which is still an open problem.

For simplicity, in the remainder of this paper, CIG and CHG refer to the optimization variants, CIG-opt and CHG-opt.

2 Preliminaries

We define variants of the games CHG and CIG, denoted by CHG* and CIG*, whose inputs are a triple (G, k, C), where G is a graph, k is a positive integer, C a *convex* subset of vertices of G, and where the vertices of C are considered covered (without being selected) at the beginning of the game. Therefore, the set of selected vertices is a set $S \subseteq V - C$. In CHG$^*(G, k, C)$, after S is selected the covered vertices are $\mathcal{H}(S \cup C)$ and in CIG$^*(G, k, C)$ the covered vertices are $\mathrm{I}(S \cup C)$. Alice wins if the total number of selected vertices is at most k. Note that CHG(G, k) is equivalent to CHG$^*(G, k, \emptyset)$ (and similarly for CIG).

In both games, a *strategy* for a player on a graph $G = (V, E)$ where C is initially covered, is a function $f : 2^{V-C} \to V$ such that for every $S \subseteq V - C$, we have that $f(S)$ is the next vertex to be selected by the player when the set of selected vertices is S. Note that this implies that $f(S) \notin \mathcal{H}(C \cup S)$.

A *k-strategy* for Alice is a strategy that ensures that at the end of the game at most k vertices are selected (in total) regardless of how Bob plays (recall that we do not count the vertices in C). A *k-strategy* for Bob is one that ensures that more than k vertices are selected, regardless of how Alice plays. A strategy for Alice (resp. Bob) in a graph G is *optimal* if it is a k-strategy with minimum (resp. maximum) k.

Let the *closed game hull number*, $\mathrm{cghn}(G, C)$ (resp., $\mathrm{cghn}^B(G, C)$), be the smallest integer k such that Alice has a *k-strategy* for CHG$^*(G, k, C)$ when Alice starts (resp., Bob starts). And let the *closed game interval numbers* $\mathrm{cgin}(G, C)$ and $\mathrm{cgin}^B(G, C)$ be defined analogously for the game CIG*. Let $\mathrm{cghn}(G) = \mathrm{cghn}(G, \emptyset)$ and $\mathrm{cgin}(G) = \mathrm{cgin}(G, \emptyset)$. The parameters *closed game hull number* $\mathrm{cghn}(G)$ and *closed game interval number* $\mathrm{cgin}(G)$ were introduced in [4].

In the next lemma, we establish important properties of the parameters cghn and cghn^B. The main idea of the proof for the first part of this lemma is to build a strategy where one "plays ignoring a given set of vertices".

▶ **Lemma 1.** *Let $G = (V, E)$ be a graph and $C \subseteq C' \subseteq V$ where C and C' are convex sets. Then,*
(a) $\mathrm{cghn}(G, C') \leq \mathrm{cghn}(G, C)$ *and* $\mathrm{cghn}^B(G, C') \leq \mathrm{cghn}^B(G, C)$,
(b) $\mathrm{cghn}(G, C) - 1 \leq \mathrm{cghn}^B(G, C) \leq \mathrm{cghn}(G, C) + 1$ *and both bounds are tight.*

▶ **Lemma 2.** *Let G be a connected graph. If there exists a k-strategy for Bob in the game CHG, then there exists one such strategy f such that $f(C)$ is a neighbor of C in G, for every convex set $C \subseteq V(G)$.*

Proof. Let f_0 be a k-strategy for Bob in the game CHG. Suppose that $C \subseteq V(G)$ is a convex set with $f_0(C) = v \notin N_G(C)$ and define $\mathcal{H}(C \cup \{v\}) = C'$. Since $v \notin N_G(C)$ and G is connected, there exists $w \in C' \cap N_G(C)$ that lies on a shortest path from v to some vertex in C. By Lemma 1, since $\mathcal{H}(C \cup \{w\}) \subseteq C'$, we have $\mathrm{cghn}(G, C') \leq \mathrm{cghn}(G, \mathcal{H}(C \cup \{w\}))$. This inequality guarantees that if Bob selects w instead of v, Alice (in her turn) will not be able to make the game finish in a shorter time (if Bob plays optimally). That is, there is a strategy f_1 for Bob in which $f_1(C) = w$ that is not worse than f_0. Such f_1 is a k-strategy for Bob. Repeating this argument, over the course of any game, one can see that Bob can always decide to play at a neighbor of the set of currently covered vertices. Therefore, there exists a strategy f as desired. ◀

Note that, if the graph is not connected, we can prove similarly that, in the game CHG, either Bob plays a neighbor of a covered vertex or the first selected vertex of a component.

3 Hardness results

In this section, first we prove that the Closed Interval Game (optimization variant) is PSPACE-complete; later we show that Closed Hull Game (optimization variant) is NP-hard in bipartite graphs and in split graphs.

3.1 CIG is PSPACE-complete

We start with a (a priori weaker) statement: the extension CIG* (defined in section 2) is PSPACE-complete. Later we use this result to prove that CIG is also PSPACE-complete.

Our reduction is from a variant of the game KAYLES-MAX, which is the optimization version of the classical KAYLES game. In KAYLES-MAX, the instance is a graph G and an integer k and two players, Max and Min (starting with Max), alternate turns selecting vertices of G in such a way that the set of selected vertices must always induce an independent set. Max wins if the number of selected vertices is at least k at the end of the game. Otherwise, Min wins.

KAYLES-MAX was proved PSPACE-complete by Brosse et al. [11] in 2025. One can easily check that the instances constructed in the reduction of [11] have the important property of *pass-stability*. We say that an instance of a game is *pass-stable* if the player with a winning strategy also wins if the players are allowed to pass moves. Note that the notion of pass-stability is defined according to a player that has a winning strategy. Hence, the winning player will never pass and so the game is still finite and well defined even if the opponent is allowed to pass. In some games, every instance is pass-stable, such as the POSCNF game. Other games have instances that are not pass-stable, such as the Graph Coloring Game, which was also proved PSPACE-complete even for pass-stable instances [16].

We first prove in the theorem below that the variant KAYLES-MIN, in which Min is the first to play, is also PSPACE-complete even for pass-stable instances.

▶ **Theorem 3.** *KAYLES-MIN is PSPACE-complete even restricted to pass-stable instances with graphs of even order.*

In the following, we use KAYLES-MIN to prove that CIG* is PSPACE-complete.

▶ **Theorem 4.** *CIG* is PSPACE-complete even restricted to pass-stable instances (G, k, C) such that $n = |V(G)|$ is a multiple of 4, G has diameter 3 and C is a clique of $n/2$ vertices satisfying that every vertex outside C has a neighbor in C. The same is valid for the variant of CIG* in which Bob plays first. In other words,* $\text{cgin}(G, C)$ *and* $\text{cgin}^B(G, C)$ *are PSPACE-hard even when G has diameter 3 and C is a clique with $n/2$ vertices of G, where $n = |V(G)|$.*

Now we prove that the game CIG is PSPACE-complete. We obtain a reduction from CIG*, which we proved PSPACE-complete in Theorem 4 even for pass-stable instances.

▶ **Theorem 5.** *The Closed Interval Game CIG (optimization variant) is PSPACE-complete even for pass-stable instances with diameter at most 4. The same is valid also for the variant in which Bob plays first. In other words,* $\text{cgin}(G)$ *and* $\text{cgin}^B(G)$ *are PSPACE-hard even in graphs with diameter 4.*

Proof. Due to Theorem 4, we can start considering a pass-stable instance (H, k, C) of CIG* such that H is a graph with $4n$ vertices, C is a clique of H with $2n$ vertices (of pre-selected vertices) such that every vertex outside C has a neighbor in C and k is an integer. Let $C = \{c_1, \ldots, c_{2n}\}$.

Let us construct an instance (G, ℓ) for CIG, which we will prove that is pass-stable and will be used to solve CIG* for (H, k, C). The graph G is obtained from H by including, for every vertex $c_i \in C$, the gadget F_i of Figure 1, with $V(F_i) = A_i \cup B_i$, where $B_i = \{b_{i,1}, \cdots, b_{i,8n+1}\}$ is an independent set with $8n+1$ vertices and $A_i = \{a_{i,1}, a_{i,2}\}$ is such that both $a_{i,1}$ and $a_{i,2}$ dominate $B_i \cup \{c_i\}$. Notice that F_i does not contain c_i and that c_i is a cut vertex in G, that is, the removal of c_i disconnects the gadget F_i from the rest of the graph G. Also set $\ell = 8n^2 + 4n + k$.

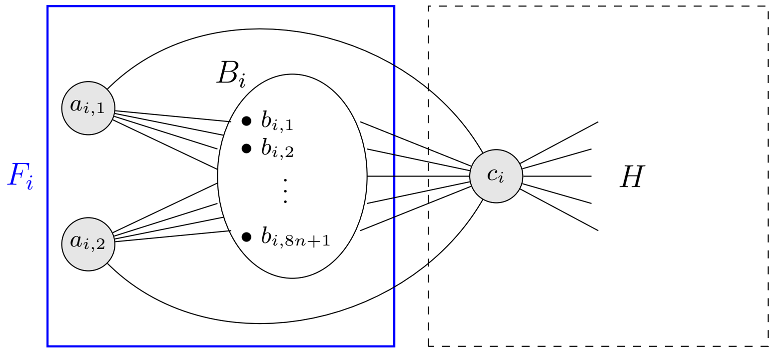

Figure 1 Gadget F_i for every vertex c_i of the clique C, where B_i is an independent set with $8n+1$ vertices and $A_i = \{a_{i,1}, a_{i,2}\}$. Recall that $|V(H)| = 4n$ and $|C| = 2n$.

▷ Claim 6. If Alice is the first to play in F_i (i.e., to select a vertex from F_i), then she can ensure the selection of at most 3 vertices of F_i. If Bob is the first to play in F_i, then he can ensure the selection of at least $8n+1$ vertices of F_i.

Proof. First suppose that Alice is the first to play in F_i. Then she can select $a_{i,1}$. If Bob ever plays in B_i, she plays in $a_{i,2}$, ensuring that no other vertex from B_i can be selected during the rest of the game, summing at most 3 selected vertices in F_i. If Bob plays in $a_{i,2}$, no other vertex of B_i can be selected.

Now suppose that Bob is the first to play in F_i. Then he can select $b_{i,1}$. If Alice plays in F_i, Bob plays (without loss of generality) in $b_{i,2}$, ensuring that at least one of $a_{i,1}$ and $a_{i,2}$ cannot be selected. Since every vertex of B_i belongs to only one shortest path of G and this path is between $a_{i,1}$ and $a_{i,2}$, then every vertex of B_i will eventually have to be selected during the game, ensuring at least $8n+1$ selected vertices in F_i. ◁

Therefore, if Alice is the first to play in more than half of the gadgets F_i, she ensures the selection of at most

$$3 \cdot \left(\frac{|C|}{2} + 1\right) + (8n+1) \cdot \left(\frac{|C|}{2} - 1\right) + 4n = 8n^2 + 2 \text{ vertices.}$$

Moreover, if Bob is the first to play in more than half of the gadgets F_i, he ensures the selection of at least

$$2 \cdot \left(\frac{|C|}{2} - 1\right) + (8n+1) \cdot \left(\frac{|C|}{2} + 1\right) = 8n^2 + 11n - 1 \text{ vertices.}$$

We prove below that both Alice and Bob, in order not to lose the game, have to be the first to play in exactly half of the gadgets F_i, and each gadget controlled by Alice has to have exactly 3 selected vertices and each gadget controlled by Bob has to have exactly $8n+1$ selected vertices, summing exactly

$$3 \cdot \binom{|C|}{2} \;+\; (8n+1) \cdot \binom{|C|}{2} \;=\; 8n^2 + 4n \text{ vertices.}$$

Since $1 \leq k \leq 4n$, we have that $8n^2 + 2 < \ell < 8n^2 + 11n - 1$. Thus, if a player is the first to play in more than half of the gadgets F_i, they win.

So, suppose that a player $P \in \{\text{Alice}, \text{Bob}\}$ wins CIG* on (H, k, C). Let Q be the opponent of P. We say that a vertex v of a gadget F_i is *good* if either it belongs to A_i and P is Alice or it belongs to B_i and P is Bob. Otherwise, we say that v is *bad*.

Consider the following strategy for P on CIG on the graph G with integer ℓ. We divide the strategy in two Phases, (i) and (ii), described below. During the game, we say that a gadget F_i is an Alice's gadget if the number of selected vertices in A_i is greater than in B_i. Analogously for a Bob's gadget. If the number of selected vertices in A_i is equal to in B_i, we say that the gadget has no owner, which can occur when both A_i and B_i have 0 or 1 selected vertex. We say that the game is in Phase (i) until every gadget has an owner; after that, we say that the game is in Phase (ii).

If P is Alice, she first selects $a_{1,1}$ at the beginning of the game, in Phase (i).

(i$_1$) If the player Q selects a bad vertex of a P's gadget F_i (no bad vertex of F_i was selected before and there is exactly one good vertex selected in F_i), then P selects other good vertex of F_i, making F_i a P's gadget again;

(i$_2$) If the player Q selects a good vertex of a gadget F_i in which no other good vertex was selected, then P selects other good vertex of F_i;

(i$_3$) If P is Bob and Q (Alice) selects a bad vertex (A_i) of a gadget F_i in which no vertex was selected before, then P (Bob) selects a good vertex (B_i) of F_i;

(i$_4$) Otherwise, if there is a gadget F_i without selected vertices, then P selects a good vertex of F_i.

From the discussed before, we may assume that the number of Alice's gadgets will be half of the gadgets. The same for Bob's gadgets. To achieve this, at most one vertex of H is selected during Phase (i) and, if a vertex of H was selected, it was by Q=Alice (from P's strategy above). Moreover, we may assume without loss of generality that $a_{i,1}$ (resp. $b_{i,1}$) is selected for any Alice's (resp. Bob's) gadget F_i. Consider now that we are in Phase (ii): every gadget F_i has an owner. Therefore, every vertex c_i is forbidden, because it is in a shortest path between two selected vertices of different gadgets.

If P is Alice, she first selects a vertex of H following her winning strategy in CIG* at the beginning of Phase (ii). After that:

(ii$_1$) If the player Q selects either a vertex of H, or a good vertex of a P's gadget F_i, or a bad vertex of a Q's gadget F_i, then P selects a vertex of H following the winning strategy in CIG* (recall that the graph H is pass-stable in the game CIG*);

(ii$_2$) If the player Q selects a bad vertex of a P's gadget F_i (with exactly one good vertex selected), then P selects other good vertex of F_i, making F_i a P's gadget again;

(ii$_3$) If the player Q selects a good vertex of a Q's gadget F_i, then P selects other good vertex of F_i, making it a P's gadget and winning the game.

With this, notice that, since every shortest path between a vertex of F_i and a vertex outside F_i passes through c_i, the game CIG on G works similarly to the game CIG* on H with C already selected. Moreover, the player P plays in H according to the winning strategy on CIG*, ensuring at most (resp. more than) k selected vertices in H if P is Alice (resp. Bob). Furthermore, if P is Alice, she ensures at most 3 selected vertices in each Alice's gadget and at most $8n+1$ selected vertices in each Bob's gadget, summing at most ℓ selected vertices in G. Finally, if P is Bob, he ensures at least 3 selected vertices in each Alice's gadget and at least $8n+1$ selected vertices in each Bob's gadget, summing more than ℓ selected vertices in G. ◀

3.2 CHG is NP-hard

The Hull Number problem takes a graph G and $k \in \mathbb{N}$ as inputs and aims at deciding whether $\mathrm{hn}(G) \leq k$, where $\mathrm{hn}(G)$ is the minimum size of a hull set of G.

▶ **Theorem 7** ([1]). *The Hull Number problem is NP-complete in the class of bipartite graphs.*

Here we prove the following theorem about the closed *game* hull number.

▶ **Theorem 8.** *Deciding whether* $\mathrm{cghn}(G) \leq k$ *holds is NP-hard in the class of bipartite graphs.*

Sketch of proof. Let G be a bipartite instance of the Hull Number problem and let H be the bipartite graph obtained by subdividing all edges of G twice. That is, every edge $\{x, y\} \in E(G)$ is replaced by a path (x, v_x, v_y, y). Note that, for every $x, y \in V(G)$ (called original vertices), $\mathrm{dist}_H(x, y) = 3 \cdot \mathrm{dist}_G(x, y)$. Therefore, for every $x, y \in V(G)$, every shortest (x, y)-path in G corresponds to one shortest (x, y)-path in H. Moreover, for every $v_x \in V(H) \setminus V(G)$ (where v_x is the neighbor of x in the path resulting from the subdivision of $\{x, y\} \in E(G)$) and every $v \in V(H)$ and every shortest (v_x, v)-path P in H: either $x \in V(P)$ and $P \setminus \{v_x\}$ is a shortest (x, v)-path in H, or $P' = x \cdot P$ (adding x to P) is a shortest (x, v)-path in H. Therefore, for every hull set S of H, there exists a hull set $S' \subseteq V(G)$ of H with $|S'| \leq |S|$ (indeed, for every $v_x \in S \setminus V(G)$, remove v_x and add x if $x \notin S$). Reciprocally, for every hull set $S \subseteq V(G)$ of H, S is also a hull set of G. Hence, $\mathrm{hn}(G) = \mathrm{hn}(H)$ and there exists a minimum hull set S of H using only original vertices (i.e., $S \subseteq V(G)$).

Next, we shall prove that $\mathrm{cghn}(H) = 2\,\mathrm{hn}(G) - 1$ and so prove the NP-hardness. ◀

▶ **Theorem 9.** *Deciding whether* $\mathrm{cghn}(G) \leq k$ *(resp.* $\mathrm{cgin}(G) \leq k$*) is NP-hard in the class of split graphs.*

Sketch of proof. Let (G, k) be an instance of the classical problem of Minimum Dominating Set, which is NP-complete and not approximable up to $\log n$ factor [18]. Let $V = V(G) = \{v_1, \cdots, v_n\}$.

Let G' be obtained from G as follows. Let $V_S = \{v_1^S, \cdots, v_n^S\}$ be a stable set of order n. Let C be a clique on n^2 vertices: $\{v_1^1, \cdots, v_n^1, v_1^2, \cdots, v_n^2, \cdots, v_1^n, \cdots, v_n^n\}$. For any $1 \leq i, j \leq n$, make v_i^S adjacent to $v_1^j, \cdots, v_n^j$ if and only if $i = j$ or $\{v_i, v_j\} \in E(G)$. Clearly G' is a split graph. Note that, for any $1 \leq j \leq n$, $v_1^j, \cdots, v_n^j$ are true twins in G' (vertices with same closed neighborhood). Also note that the vertices of V_S are simplicial in G' (and so belong to any interval/hull set) and that the (unique) minimum hull set (resp., interval set) of G' is V_S.

It is well known that, in a split graph, the interval of any set of vertices equals its convex hull and so both interval and hull games behave the same [19, 20]. Now, $\mathrm{cghn}(G) \leq n + k - 1$ if and only if $\gamma(G) \leq k$, where $\gamma(G)$ denotes the minimum size of a dominating set in G. ◀

4 Polynomial algorithms in trees and cobipartite graphs

4.1 CHG and CIG are polynomial in Trees

In this section, we show that the games CHG and CIG are polynomial time solvable in trees.

Let T be a tree with at least 3 vertices. We will denote by $L(T)$ its set of leaves. Note that for every $\ell \in L(T)$, there is no path in T for which ℓ is an internal vertex. So all leaves of T will have to be selected by either Alice or Bob before the game ends. A vertex is *internal* (of the tree) if it is not a leaf. We first prove the following two technical lemmas.

▶ **Lemma 10.** *Let T be a tree. If there exists a k-strategy for Alice, then there exists a k-strategy for Alice such that she only selects leaves.*

▶ **Lemma 11.** *Let $T = (V, E)$ be a tree, $C \subseteq V$ be a non-empty convex set such that $V \setminus C$ contains at least one internal vertex. If Alice has a k-strategy in (T, C), then she has a k-strategy first selecting a leaf whose only neighbor is not covered (not in C).*

The *eccentricity* of a vertex v is the maximum distance from v to the other vertices. The *radius* $r(G)$ of a graph G is the minimum eccentricity among all vertices of G. The *center* of a graph consists of the vertices with minimum eccentricity. It is known that the center of a tree is either one vertex or two adjacent vertices.

▶ **Theorem 12.** *Let T be a tree. If $|V(T)| \in \{1, 2\}$ or T has radius 1, then $\mathrm{cghn}(T) = \mathrm{cghn}^B(T) = |V(T)|$. Otherwise,*

$$\mathrm{cghn}(T) = \left\lfloor \frac{\mathrm{cghn}(T')}{2} \right\rfloor + |L(T)| \qquad \text{and} \qquad \mathrm{cghn}^B(T) = \left\lceil \frac{\mathrm{cghn}^B(T')}{2} \right\rceil + |L(T)|,$$

where $T' = T - L(T)$ is the tree obtained from T by removing its leaves. Thus, $\mathrm{cghn}(T)$ and $\mathrm{cghn}^B(T)$ can be computed in polynomial time. Moreover, if Bob is allowed to pass turns, it cannot increase the length (the number of vertices actually selected by the players) of the game.

Proof. The proof is by induction on the radius of T. If $|V(T)| \in \{1, 2\}$, then clearly $\mathrm{cghn}(T) = \mathrm{cghn}^B(T) = |V(T)|$. If T has radius 1 and at least 3 vertices, then T is a star and Bob can always select the only internal vertex of T and then $\mathrm{cghn}(T) = \mathrm{cghn}^B(T) = |L(T)| + 1 = |V(T)|$. Moreover, in these cases, if Bob can pass turns, this cannot increase the length of the game.

So, assume that T has at least 4 vertices and radius $r \geq 2$ and let $T' = T - L(T)$. Note that the radius $r(T') = r(T) - 1$ and, by induction, let f' be an optimal (i.e., finishing in $\mathrm{cghn}(T')$ turns) strategy for Alice, playing only in the leaves of T'. Also, assume by induction that, following f', the number of turns cannot increase if Bob can pass turns.

Alice's strategy. We first describe an Alice's strategy f when she is first to play in T. Let ℓ'_1 be the leaf of T' first selected by Alice when playing f' in T'. Then Alice selects one leaf ℓ_1 of T adjacent to ℓ'_1. From now on, let v_i be the vertex selected by Bob just after Alice has selected her i-th vertex in the game on T (unless if Bob passes his turn). From this, Alice will assume that Bob played on a certain vertex v'_i on the game in T' described below or passed his turn.

If v_i is a leaf of T and its neighbor v'_i in T' is already covered but there is a vertex not covered on T', or if Bob passes his turn, Alice assumes that Bob passed his turn in the game on T'. That is, let ℓ'_{i+1} be the leaf of T' selected by Alice following f' assuming that Bob has passed his last turn. Then, in T, Alice selects any leaf ℓ_{i+1} of T adjacent to ℓ'_{i+1}.

If v_i is a leaf of T and its neighbor u in T' is not already covered, let $v'_i = u$. Moreover, if v_i is not a leaf of T, let $v'_i = v_i$. Then, let ℓ'_{i+1} be the leaf of T' selected by Alice in her optimal strategy f' in T', considering that Bob played on v'_i at its i-th turn. Then Alice plays on a leaf ℓ_{i+1} of T adjacent to ℓ'_{i+1} in the game on T. Finally, if v_i is a leaf of T and all vertices of T' are already covered, then Alice will select any leaf of T.

Note that, while all vertices of T' are not covered, each move of Alice following f covers exactly the same vertices of T' as when she follows f' in T'. Hence, all vertices of T' are covered after at most $\text{cghn}(T')$ steps, i.e., after Alice has selected $\left\lceil \frac{\text{cghn}(T')-x}{2} \right\rceil + x$ leaves of T and Bob has selected $\left\lfloor \frac{\text{cghn}(T')-x}{2} \right\rfloor$ vertices (where x is the number of turns Bob passed). Once all vertices of T' have been covered, it remains at most $|L(T)| - \left(\left\lceil \frac{\text{cghn}(T')-x}{2} \right\rceil + x \right)$ leaves of T that are not covered (since Alice has selected $\left\lceil \frac{\text{cghn}(T')-x}{2} \right\rceil + x$ leaves and Bob might have selected ones too). Hence, if Alice follows f, the game ends in at most $|L(T)| - \left(\left\lceil \frac{\text{cghn}(T')-x}{2} \right\rceil + x \right) + \text{cghn}(T') = L(T) + \left\lfloor \frac{\text{cghn}(T')-x}{2} \right\rfloor \leq |L(T)| + \left\lfloor \frac{\text{cghn}(T')}{2} \right\rfloor +$ steps.

Bob's strategy. Now we show a Bob's strategy with at least $\left\lfloor \frac{\text{cghn}(T')}{2} \right\rfloor + |L(T)|$ vertices in the game in which Alice is first to play, given Bob's best strategy in T'. Recall from Lemma 10 that we may assume that Alice only plays on leaves and that, by Lemma 11 that, unless all internal vertices (i.e., of T') are covered, she selects a leaf whose neighbor is not covered. Let ℓ_i be the leaf of T selected by Alice in her i-th turn of the game on T and let ℓ'_i be the leaf of T' adjacent to ℓ_i. If all vertices of T' are already covered (in the game in T), Bob can only select leaves of T. Otherwise, we may assume that ℓ'_i was not already covered by remark above (by Lemma 11). Since ℓ'_i was not selected before (considering the game on T'), Bob assumes that Alice played on ℓ'_i in the game on T'. From this, let v'_{i+1} be the vertex of T' in which Bob would play in his best strategy on T' after Alice selects ℓ'_i. Then Bob plays on $v_{i+1} = v'_{i+1}$ in the game on T. Notice that, at the end of the game, Bob selected at least $\lfloor \text{cghn}(T')/2 \rfloor$ vertices of T' in the game on T, and we are done (since the number of steps of the game equals the number of selected internal vertices plus the number of leaves of T). Analogously for $\text{cghn}^B(T)$. ◀

4.2 CHG is polynomial in Cobipartite graphs

A graph $G = (V, E)$ is a cobipartite graph (complement of a bipartite graph) if there is a partition $V = A \cup B$ such that A and B are cliques. Denote by S the set of simplicial vertices of G. Let $S_A = S \cap A$ and $S_B = S \cap B$. Let U be the set of universal vertices of G. Note that if G is not a complete graph, then $U \cap S = \emptyset$. Let F be the graph obtained from G by removing the vertices in S and U, and removing the edges intra-cliques, i.e., $V(F) = V \setminus (U \cup S)$ and $E(F) = \{uv \in E : u \in A \cap V(F) \text{ and } v \in B \cap V(F)\}$. Denote by $\mathcal{C} = \{C_1, C_2, \ldots, C_r\}$ with $r \geq 1$ the set of connected components C_i of F. Note that if G is neither one complete graph nor the disjoint union of complete graphs, F is not empty, and every component C_i has at least two vertices, for every $i \in \{1, \ldots, r\}$. Indeed, any vertex in $A \setminus S_A$ (resp., in $B \setminus S_B$) has a neighbor in $B \cap V(F)$ (resp., in $A \cap V(F)$).

In the following, we will use the results presented by Araujo et al. [1] for the hull number in cobipartite graphs. In particular, to show that computing $\text{hn}(G)$ can be done in polynomial time in the class of cobipartite graphs G, they precisely characterized minimum hull sets of such graphs. We recall their result:

▶ **Theorem 13** ([1]). *Let $G = (A \cup B, E)$ be a cobipartite graph (with the notations defined above). Recall that any hull set must contain S. Furthermore:*

- *If $U = \emptyset$ and $S_A \neq \emptyset$ and $S_B \neq \emptyset$ then S is a minimum hull set of G.*
- *Otherwise (if $U \neq \emptyset$ or $S_A = \emptyset$ or $S_B = \emptyset$):*
  - *If $r \geq 2$, let $v_i \in C_i \cap A$ for $i < r$ and $v_r \in C_r \cap B$, then $S \cup \{v_1, \cdots, v_r\}$ is a minimum hull set of G.*
  - *Otherwise (if $r = 1$), let C be the unique component of F.*
    - *If $S_A \neq \emptyset$ and $S_B \neq \emptyset$ (and so $U \neq \emptyset$), then $S \cup \{v\}$ is a minimum hull set for every $v \in C$.*
    - *If $S_A \neq \emptyset$ and $S_B = \emptyset$, then $h(G) \leq |S| + 2$. Precisely, let $v_A \in A \cap C$ be such that $|N(v_A) \cap B \cap C|$ is maximum and let $x \in (B \cap C) \setminus N(v_A) \neq \emptyset$, then $S \cup \{v_A, x\}$ is a hull set of G.*
    - *If $S_A = \emptyset$ and $S_B = \emptyset$, then $h(G) \leq 4$.*

We first specify one of the cases in the above theorem.

▶ **Lemma 14.** *Let G be a cobipartite graph that is not a complete graph. If $S_A \neq \emptyset$ and $S_B = \emptyset$ and $r = 1$, then* $\text{hn}(G) = |S| + 2 = |S_A| + 2$.

We are now ready to prove the main result of this section.

▶ **Theorem 15.** *The* $\text{cghn}(G)$ *(resp.,* $\text{cghn}^B(G)$*) can be computed in polynomial time in the class of cobipartite graphs G.*

Sketch of proof. We prove the theorem in the case of $\text{cghn}(G)$, the case $\text{cghn}^B(G)$ is similar. The proof follows the characterization of Theorem 13 (each of the following 5 claims corresponds to one case of Theorem 13).

Note that if G is a complete graph or a disjoint union of complete graphs, then $\text{cghn}(G) = |V(G)|$. From now on, let G be a cobipartite graph that is neither a complete graph nor the disjoint union of complete graphs. We prove the following.

- Let G be a cobipartite graph other than a complete graph and the disjoint union of complete graphs and $U = \emptyset$, $S_A \neq \emptyset$ and $S_B \neq \emptyset$. Then, $\text{cghn}(G) = |S| + 1$.
- Let G be a cobipartite graph other than a complete graph. If $U \neq \emptyset$ or $S_B = \emptyset$, and $r \geq 2$, then $\text{cghn}(G) = |S| + r + 1$.
- Let G be a cobipartite graph, not a complete graph or the disjoint union of complete graphs. If $U \neq \emptyset$, $S_A \neq \emptyset$ and $S_B \neq \emptyset$ and $r = 1$, then $\text{cghn}(G) = |S| + 2$.
- Let G be a cobipartite graph, not a complete graph or the disjoint union of complete graphs. If $r = 1$ and $S_A \neq \emptyset$ and $S_B = \emptyset$, then $\text{cghn}(G) = |S| + 3$.
- Let G be a cobipartite graph with $|V(G)| \geq 3$. If $r = 1$ and $S_A = S_B = \emptyset$, then $\text{cghn}(G) \leq 7$ can be computed in polynomial time. In particular, if $\text{hn}(G) = 2$, $\text{cghn}(G) = 3$. ◀

5 Conclusion

The main remaining open question is whether the closed hull game, CHG, (optimization variant) is PSPACE-complete, which we conjecture to be true. It is also interesting to find other graph classes in which the decision problems of the games CHG and CIG (optimization variants) are polynomial time solvable. Unfortunately, it seems that the closed interval game CIG is hard on cobipartite graphs as the following argument suggests. First we prove that deciding if a cobipartite graph admits an interval set of size k is NP-complete (to the best of our knowledge, this was only proved for oriented graphs [3]).

▶ **Theorem 16.** *The problem of deciding if a cobipartite graph has an interval set (in the geodesic or P_3 convexities) with at most k vertices is NP-complete.*

The above reduction implies that, in the closed interval game CIG, Alice must ensure the selection of a dominating set of H, which also must be an independent set (since the game is closed). In other words, she aims at obtaining an independent set of H as small as possible. That is, Alice and Bob are playing the optimization variant of the Kayles game on H where the first player aims at minimizing the obtained independent set. This problem is PSPACE-complete by Theorem 3. However, since H is bipartite, to prove that the closed interval game is PSPACE-complete in cobipartite graphs, we would need to show that the optimization variant of the Kayles game starting with Bob is PSPACE-complete in bipartite graphs, which is still an open question.

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