

The Squishy Grid Problem

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Abstract

In this paper we consider the problem of approximating Euclidean distances by the infinite integer grid graph. Although the topology of the graph is fixed, we have control over the edge-weight assignment $w : E \rightarrow \mathbb{R}_{\geq 0}$, and hope to have grid distances be asymptotically isometric to Euclidean distances, that is:

$$\text{For all grid points } u, v, \text{ dist}_w(u, v) = (1 \pm o(1))\|u - v\|_2.$$

We give three methods for solving this problem, each attractive in its own way.

- Our first construction is based on an embedding of the recursive, non-periodic *pinwheel tiling* of Radin and Conway [15, 16, 9] into the integer grid. Distances in the pinwheel graph are asymptotically isometric to Euclidean distances, but no explicit bound on the rate of convergence was known. We prove that the multiplicative distortion of the pinwheel graph is $(1 + 1/\Theta(\log^\xi \log D))$, where D is the Euclidean distance and $\xi = \Theta(1)$. The pinwheel tiling approach is conceptually simple, but can be improved quantitatively.
- Our second construction is based on a hierarchical arrangement of *highways*. It is simple, achieving stretch $(1 + 1/\Theta(D^{1/9}))$, which converges doubly exponentially faster than the pinwheel tiling approach.
- The first two methods are deterministic, with rigorous guarantees. An even simpler approach is to sample the edge weights independently and randomly from a common distribution $\mathcal{D}$. Whether there exists a distribution $\mathcal{D}^*$ that makes grid distances Euclidean, asymptotically and in expectation, is major open problem in the theory of *first passage percolation*. Previous experiments show that when $\mathcal{D}$ is a Fisher distribution (which is continuous), grid distances are within 1% of Euclidean distances. We demonstrate experimentally that this level of accuracy can be achieved by a simple 2-point distribution that assigns weights 0.41 or 4.75 with probability 44% and 56%, respectively.

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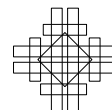
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1 Introduction

In this paper we consider a natural geometric problem tangentially related to metric embeddings, spanners, and, in its randomized form, percolation theory. Suppose we wish to approximate Euclidean distances between points on the plane, but with a simple *discrete* structure: the integer grid graph $\text{Grid} = (\mathbb{Z} \times \mathbb{Z}, \{\{u, v\} \mid \|u - v\|_1 = 1\})$. If we consider all edges of $E(\text{Grid})$ to have unit length, then Grid can be regarded as a $\sqrt{2}$ -spanner since for any $(u, v) \in (\mathbb{Z}^2)^2$,

$$\|u - v\|_2 \leq \text{dist}_{\text{Grid}}(u, v) \leq \sqrt{2} \cdot \|u - v\|_2.$$

Now define $\text{Grid}[w]$ to be Grid endowed with a non-negative edge-weight assignment $w : E(\text{Grid}) \rightarrow \mathbb{R}_{\geq 0}$, and let dist_w be the distance function with respect to w . We consider the natural question: does there exist a $\text{Grid}[w^*]$ that is an asymptotic 1-spanner of the Euclidean plane?

► **Question 1 (The Squishy Grid Problem).** Does there exist a weight function w^* such that for all $u, v \in V(\text{Grid})$ dist_{w^*} is asymptotically Euclidean? That is,

$$\text{dist}_{w^*}(u, v) = (1 \pm o(1))\|u - v\|_2.$$

If so, we may distinguish various types of convergence:

Polynomial. $\text{dist}_{w^*}(u, v) = \|u - v\|_2 \pm O(\|u - v\|_2^{1-\Omega(1)})$.

Subpolynomial. $\text{dist}_{w^*}(u, v) = \|u - v\|_2 \pm (\|u - v\|_2)^{o(1)}$.

Constant. $\text{dist}_{w^*}(u, v) = \|u - v\|_2 \pm O(1)$.

Before discussing our approach to answering Question 1 we review the history of Question 1 and its connections to percolation theory.

1.1 History of the Problem and Related Results

G. Tardos (personal communication) made us aware of a 1990 book chapter of Pach, Pollack, and Spencer [14] who attributed some version of Question 1 to Paul Erdős. Pach et al. [14] proved that for any fixed $\epsilon > 0$ there exists a graph $G[\epsilon]$ on the vertex set $\mathbb{Z} \times \mathbb{Z}$ such that for all u, v ,

$$\|u - v\|_2 \leq \text{dist}_{G[\epsilon]}(u, v) \leq (1 + \epsilon)\|u - v\|_2 + O(5^{1/\epsilon}).$$

Unfortunately, $G[\epsilon]$ is not planar and obviously depends on ϵ , so it does not lead to a resolution of Question 1. Burago and Ivanov [6] exhibited a regular, weighted graph G on the vertex set $\mathbb{Z} \times \mathbb{Z}$ for which

$$\|u - v\|_2 \leq \text{dist}_G(u, v) \leq \|u - v\|_2 + C,$$

for some absolute constant C . However, G is also not planar.

Borradaile and Eppstein [4] considered a more general problem: given a point set $P \in \mathbb{R}^2$, compute a weighted planar graph $G = (P \cup S, E)$ with *Steiner points* S such that $\text{dist}_G(u, v)$ $(1 + \epsilon)$ -approximates the Euclidean distance $\|u - v\|_2$. They proved that $|S| = O_{\epsilon, \alpha}(|P|)$ suffices, where α is the sharpest angle in the Delaunay triangulation of P . A result of Chang, Krauthgamer, and Tan [8] implies an upper bound of $O_{\epsilon}(|P| \text{polylog } |P|)$, which is slightly superlinear but independent of α .

The problem was first posed to us by G. Bodwin, not as a deterministic design problem (Question 1) but as a *randomized* one. Whenever $\mathcal{D}$ is a distribution over $\mathbb{R}_{\geq 0}$, let $\text{Grid}[\mathcal{D}]$ be the distribution of weighted graphs such that for each $e \in E(\text{Grid})$, $w(e) \sim \mathcal{D}$ is sampled independently from the distribution. Is it possible to find a distribution $\mathcal{D}^*$ such that distances in $\text{Grid}[\mathcal{D}^*]$ are Euclidean in expectation? In more detail:

► **Question 2 (Randomized Squishy Grid Problem).** Does there exist a distribution $\mathcal{D}^*$ over $\mathbb{R}_{\geq 0}$ such that if $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}^*]$ is a randomly weighted graph, for all $u, v \in V(\text{Grid})$,

$$\mathbb{E}(\text{dist}_w(u, v)) = (1 \pm o(1))\|u - v\|_2.$$

The randomized process implicit in Question 2 is actually not new, but dates back to at least a 1965 paper of Hammersley and Welsh [11], who called it *first passage percolation*. They imagined an *orchard* in which trees were planted on the integer lattice. One tree is initially infected, and the time taken for an infected tree to infect a cardinal neighbor is governed by a distribution $\mathcal{D}$ on $\mathbb{R}_{\geq 0}$. One can then ask: how far does the infection spread by time t ? and what does the set of infected trees look like?

Many basic questions in first passage percolation theory remain open, and we can quickly summarize the known facts related to Question 2. Let $\mathbf{0} = (0, 0)$ be the origin and e_θ be the unit vector with angle θ degrees. We interpret ne_θ to mean the integer point in $V(\text{Grid})$ nearest to ne_θ . The *time constant* $\mu_0(\mathcal{D})$ is such that $\lim_{n \rightarrow \infty} \text{dist}_w(\mathbf{0}, ne_0)/n = \mu_0$ almost surely, which exists if, whenever $w_1, \dots, w_4 \sim \mathcal{D}$ are independently sampled, $\mathbb{E}(\min\{w_1, w_2, w_3, w_4\}) < \infty$ [12]. It follows that $0 \leq \mu_0 \leq \mathbb{E}(w_1 \sim \mathcal{D})$, with the latter inequality holding with equality only if $w_1 \sim \mathcal{D}$ is constant almost surely [11]. Similarly, the time constants for other angles $\mu_\theta(\mathcal{D}) = \lim_{n \rightarrow \infty} \text{dist}_w(\mathbf{0}, ne_\theta)/n$ exist, and collectively define the limiting *shape* of the balls under distribution $\mathcal{D}$. Let $B(t) = \{u \in \mathbb{Z}^2 \mid \text{dist}_w(\mathbf{0}, u) \leq t\}$ be the ball of radius t around the origin. The Cox-Durrett shape theorem [10] shows that with probability 1, as $t \rightarrow \infty$, $B(t)/t$ tends to a fixed *limit shape* $\mathcal{B}(\mathcal{D}) \subset \mathbb{R}^2$. When $\mu_0(\mathcal{D}) > 0$, $\mathcal{B}(\mathcal{D})$ is bounded, convex, and has the same symmetries as $\mathbb{Z}^2$, and when $\mu_0(\mathcal{D}) = 0$, $\mathcal{B}(\mathcal{D})$ is $\mathbb{R}^2$ itself. See [2] for an extensive survey of first passage percolation theory.

In the context of answering Question 2 we can rescale any non-trivial distribution $\mathcal{D}$ so that its time constant $\mu_0(\mathcal{D}) = 1$, i.e., distances from the origin to points on the x - and y -axes are asymptotically isometric. In light of the Cox-Durrett theorem, Question 2 asks whether there exists a $\mathcal{D}$ for which $\mathcal{B}(\mathcal{D})$ is the unit L_2 ball $\{x \mid \|x\|_2 \leq 1\}$.

Unfortunately, there are no results characterizing $\mathcal{B}(\mathcal{D})$ for *any* non-trivial distribution $\mathcal{D}$. It is not even known whether there exists $\mathcal{D}$ such that

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\text{dist}_w(\mathbf{0}, ne_0))}{n} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\text{dist}_w(\mathbf{0}, ne_{45}))}{n} = 1, \quad (1)$$

i.e., $\mathcal{B}(\mathcal{D})$ coincides with the unit L_2 -ball on the eight (inter)cardinal directions. On the other hand, we have solid experimental evidence that $\mathcal{B}(\mathcal{D})$ can get within 1% of the unit L_2 -ball, for certain distributions $\mathcal{D}$. An experimental study of Alm and Deijfen [1] looked at various continuous distributions $\mathcal{D}$. When $\mathcal{D}$ is the uniform distribution, the limit shape $\mathcal{B}(\mathcal{D})$ approximates the L_2 -ball to within 4%, whereas when $\mathcal{D}$ is exponential the limit shape is about 1.5% away from the L_2 -ball. The best empirical approximation to the L_2 -ball came from a Fisher distribution, with error less than 1%.

1.2 Results and Findings

We provide two approaches to answering Question 1 and present experimental evidence that Question 2 can be answered in the affirmative, using simple discrete distributions.

Our first construction is based on Radin and Conway’s *pinwheel tiling* [15, 16, 9], a conceptually simple tiling that emerges from the observation that a right triangle with proportions $1 : 2 : \sqrt{5}$ can be partitioned into five right triangles with the same proportions. It is known [16] that when regarded as a plane graph G_{PW} with edges weighted according to Euclidean distance, distances in the pinwheel tiling are Euclidean *in the limit*, that is,

$$\lim_{d \rightarrow \infty} \max_{u, v: \|u-v\| > d} \frac{\text{dist}_{G_{\text{PW}}}(u, v)}{\|u - v\|_2} = 1.$$

However the rate of convergence is unknown. We embed the pinwheel tiling into the grid graph, and prove a bound on its convergence, namely that for a constant $\xi = \Theta(1)$,

$$\text{dist}_{G_{\text{PW}}}(u, v) = \left(1 + O\left(\frac{1}{\log^\xi \log \|u - v\|_2}\right)\right) \|u - v\|_2.$$

A natural problem is to optimize the *convergence* rate of the construction. We give a new, simple construction of a weight function w of the grid that is asymptotically Euclidean, with a **polynomial** convergence rate, namely: $\text{dist}_w(u, v) = \|u - v\|_2 + O(\|u - v\|_2^{8/9})$.

The construction is based on laying out “highways” in the plane, which are paths cleaving closely to a line with a certain slope a , whose edge weights are equal and chosen to approximate Euclidean distances along the highway. For example, when $a \in [0, 1]$, the weights are $\frac{\sqrt{a^2+1}}{a+1}$. In order to get a $(1 + o(1))$ -distance approximation, it is necessary that the set of slopes of all highways be dense in $[0, \pi)$. Thus, there are infinitely many slopes, and infinitely many parallel highways of each slope, whose intersection pattern is quite complicated. The tricky part in the design stage is to decide what to do with intersecting highways. We give a simple method that eliminates intersections while guaranteeing polynomial convergence.

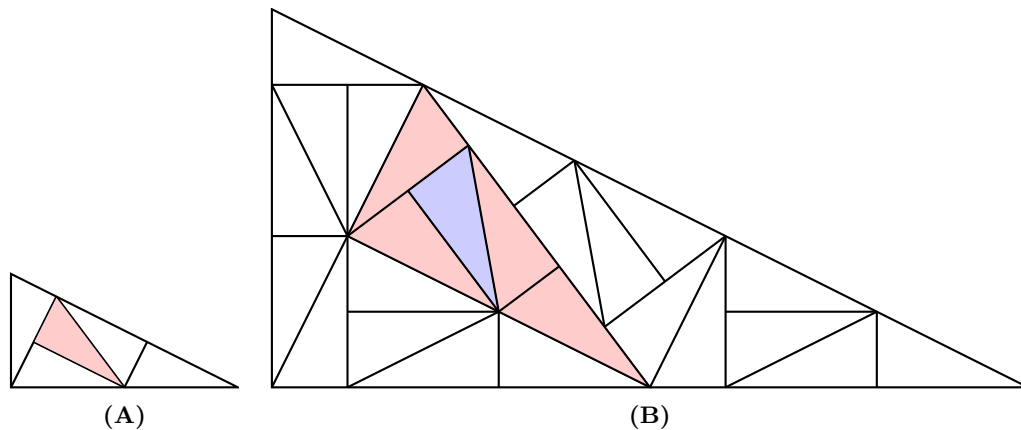
Alm and Deijfen’s [1] experimental study of first passage percolation selected $\mathcal{D}$ from various continuous distributions such as uniform, exponential, Gamma, and Fisher distributions. For several of these distributions the limit shape $\mathcal{B}(\mathcal{D})$ approximated the L_2 -ball within a few percent, with a Fisher distribution being the best. Our experiments show that very simple distributions with support size 2 or 3 can replicate the accuracy of the continuous distributions [1]. For example, the improbable distribution $\mathcal{D}_2$:

$$\Pr_{w_0 \sim \mathcal{D}_2} \left(w_0 = \begin{Bmatrix} 0.41401 \\ 4.75309 \end{Bmatrix} \right) = \begin{Bmatrix} 0.44273 \\ 0.55727 \end{Bmatrix}$$

empirically approximates the Euclidean metric to within about 0.75%, and a certain 3-point distribution $\mathcal{D}_3$ approximates it to within 0.622%. A very natural question is whether other L_p metrics can be approximated, in expectation, by various distributions. We illustrate that the uniform distribution and some Gamma distributions approximate L_p metrics with $p < 2$. None of our experiments support the possibility that L_p metrics with $p > 2$ can be approximated.

1.3 Organization

We present the construction based on pinwheel tilings in Section 2, as well as new bounds on the convergence of the stretch of the pinwheel graph. The highway construction is presented in Section 3, having polynomial convergence. We present the experimental findings in Section 4 and conclude with a discussion of several open problems related to Questions 1 and 2 in Section 5. Sections 2, 3, and 4 are written to be independent.



■ **Figure 1** (A) $\mathcal{T}_1$, containing $\mathcal{T}_0$ in red. (B) $\mathcal{T}_2$, containing $\mathcal{T}_1$ in red, and $\mathcal{T}_0$ in blue.

2 A Deterministic Construction Based on Pinwheel Tilings

The “pinwheel” tiling of Radin [15] is an example of a non-periodic tiling using a single tile type (and its reflection). Let $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \dots$ be a series of tilings of ever larger triangular swatches of the plane, and let $\mathcal{T}_\omega$ be the tessellation of the plane achieved in the limit. $\mathcal{T}_0$ consists of a single right triangle with side lengths $1, 2, \sqrt{5}$. In general $\mathcal{T}_{i+1}$ is formed from $\mathcal{T}_i$ by taking four additional copies of $\mathcal{T}_i$, suitably reflected, rotated, and translated, so that they form a larger triangle with the same $1 : 2 : \sqrt{5}$ proportions. Figure 1 illustrates the construction of $\mathcal{T}_2$ from $\mathcal{T}_1$ and $\mathcal{T}_0$.

2.1 Pinwheel Tilings

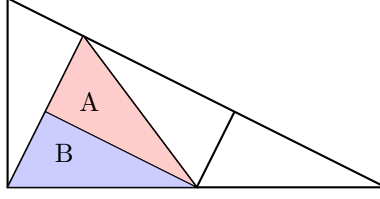
By construction $\mathcal{T}_\omega$ is a tiling of the plane using *atomic* triangles with side lengths $1, 2, \sqrt{5}$. Due to the recursive nature of the construction, we can also regard $\mathcal{T}_\omega$ as a tiling using $\sqrt{5}^i, 2\sqrt{5}^i, \sqrt{5}^{i+1}$ triangles, for any integer $i \geq 0$. Observe from Figure 1 that the boundary of $\mathcal{T}_{i+1}$ is obtained from the boundary $\mathcal{T}_i$ by scaling by $\sqrt{5}$, translation, and rotation by $\arctan(1/2)$. We will henceforth define $\gamma = \arctan(1/2)$. As $\gamma/(2\pi)$ is irrational, the orientation of tiles in $\mathcal{T}_\omega$ is uniformly distributed in $[0, 2\pi)$. Radin and Sadun [16] used this fact to prove an isoperimetric property of pinwheel tilings, namely that there are finite subsets of tiles from $\mathcal{T}_\omega$ whose area/perimeter² is arbitrarily close to that of the circle. Suppose we regard $\mathcal{T}_\omega$ as a plane graph G_ω , whose vertices and edges are the union of the vertices and edges of all atomic triangles. Radin and Sadun [16] proved that for $u, v \in V(G_\omega)$, $\text{dist}_{G_\omega}(u, v) = (1 + o(1))\|u - v\|_2$. Although the multiplicative stretch is 1 *in the limit*, their proof implies no particular rate of convergence. We prove the following.

► **Theorem 1.** *Let G_ω be the plane graph of the pinwheel tiling $\mathcal{T}_\omega$, whose edges are weighted according to the Euclidean distance between their endpoints. Then for any $u, v \in V(G_\omega)$,*

$$\|u - v\|_2 \leq \text{dist}_{G_\omega}(u, v) \leq (1 + O(1/(\log \log \|u - v\|_2)^\xi)) \cdot \|u - v\|_2, \quad \text{for some } \xi > 0.$$

2.2 Distribution of Tile Orientations

If T is a triangle in the recursive tiling with dimensions $\sqrt{5}^i, 2\sqrt{5}^i, \sqrt{5}^{i+1}$, we call T a *level- i* triangle. As a first step toward proving Theorem 1, we analyze the orientations of the triangles contained within a single large triangle. Given a triangle T of level x , let $\text{Ang}(T, k)$ denote the set of angles attained by the hypotenuses of all level- $(x - k)$ triangles contained within T . We observe that the elements of this set are characterized by an arithmetic recurrence.



■ **Figure 2** Two triangles within T . The hypotenuses of T, A, B have angles $-\gamma, -2\gamma, 0$, respectively.

► **Observation 2.** Suppose a triangle T of level x has its hypotenuse at angle θ . For $0 \leq k \leq x$, $\text{Ang}(T, k) \supseteq \{\theta + (2t - k)\gamma \mid t \in \{0, \dots, k\}\}$.

Proof. We proceed by induction on k , with the base case $k = 0$ being trivial. Consult Figure 2, where two triangles A and B of level $(x - 1)$ are depicted. Observe that the hypotenuse of A (resp. B) is rotated by an angle of $-\gamma$ (resp. $+\gamma$) relative to that of the exterior triangle. By the inductive hypothesis,

$$\text{Ang}(A, k - 1) \supseteq \{\theta - \gamma + (2t - k + 1)\gamma \mid t \in [0, k - 1]\} = \{\theta + (2t - k)\gamma \mid t \in [0, k - 1]\}$$

$$\text{Ang}(B, k - 1) \supseteq \{\theta + \gamma + (2t - k + 1)\gamma \mid t \in [0, k - 1]\} = \{\theta + (2t - k)\gamma \mid t \in [1, k]\},$$

and the union of these sets is exactly $\{\theta + (2t - k)\gamma \mid t \in \{0, \dots, k\}\}$. ◀

Consider some $\lambda \in \mathbb{R}$, and denote by $\{\lambda\} = \lambda - \lfloor \lambda \rfloor$ the fractional part of λ . It is well known due to Weyl's criterion that if λ is irrational, the set $S(\lambda, N) = \{\{kx\} : 1 \leq k \leq N\}$ becomes uniformly distributed in $[0, 1)$ as $N \rightarrow \infty$ [5]. Moreover, the rate of convergence is controlled by the *irrationality exponent* $\mu(\lambda)$ which measures the asymptotic quality of rational approximations to λ . For a real number λ , $\mu(\lambda)$ is defined to be the supremum of the set of real numbers μ such that $0 < \left| \lambda - \frac{p}{q} \right| < \frac{1}{q^\mu}$ has infinitely many solutions for integers p, q with $q > 0$. A finite irrationality exponent implies that a number does not have a sequence of rational approximations that are “too good.” For sets $S \subseteq U \subseteq \mathbb{R}$, we say S is an ε -cover of U if for every $x_1 \in U$ there exists $x_2 \in S$ satisfying $|x_1 - x_2| < \varepsilon$. When $\mu(\lambda)$ is finite, $S(\lambda, N)$ is an ε -cover of $[0, 1)$ for $N = \varepsilon^{1-\mu(\lambda)-o(1)}$ [13]. The following corollary follows by applying this fact to Observation 2.

► **Corollary 3.** Given a triangle T of level- x , $\text{Ang}(T, k)$ is a θ -cover of $[0, 2\pi)$ for $k = \theta^{1-\mu(\gamma/\pi)-o(1)}$.

Before bounding the irrationality exponent $\mu(\gamma/\pi)$, we quickly review some terminology related to the algebraic numbers $\overline{\mathbb{Q}}$. For $\alpha \in \overline{\mathbb{Q}}$, its *minimal polynomial* is the unique polynomial $P \in \mathbb{Z}[x]$ of lowest degree with relatively prime coefficients such that $P(\alpha) = 0$. We say the *degree* of α is the degree of P , while the *height* of α is the absolute value over coefficients of P .

Both γ and π can be expressed in the form $\beta \ln \alpha$ where $\alpha \in \overline{\mathbb{Q}}$ and $\beta \in \mathbb{Q}(i)$. Since $e^{i\gamma} = \frac{2+i}{\sqrt{5}}$, $\gamma = -i \ln\left(\frac{2+i}{\sqrt{5}}\right)$. Similarly, π can be written as $-i \ln(-1)$. A great deal of work in the mid-20th century yielded various lower bounds on linear forms in logarithms of algebraic numbers. A history of the problem up to 1976 can be found in [3]. For our purposes, we choose a simple bound due to Baker.

► **Theorem 4 (Baker [3]).** Suppose for $n \geq 1$ we have algebraic numbers $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \overline{\mathbb{Q}} \setminus 0$. If the logarithms $\ln \alpha_i$ are linearly independent over the rational numbers, then

$$|\beta_1 \ln \alpha_1 + \dots + \beta_n \ln \alpha_n| > H^{-C},$$

where H is the maximum of the heights of the β_i and C is a function of n , the numbers α_i , and the degrees of the numbers β_i .

Consider arithmetic expressions of the form $\beta_1 \ln \alpha_1 + \beta_2 \ln \alpha_2$ where $\alpha_1, \alpha_2 \in \overline{\mathbb{Q}} \setminus 0$ are fixed and $\beta_1, \beta_2 \in \mathbb{Z} \setminus 0$. Now β_1 and β_2 have degree 1, so the exponent C becomes a constant, and $H = \max\{|\beta_1|, |\beta_2|\}$. Applying Baker's theorem (Theorem 4) and dividing both sides by $|\beta_1 \ln \alpha_2|$ yields $\left| \frac{\ln \alpha_1}{\ln \alpha_2} + \frac{\beta_2}{\beta_1} \right| > \frac{1}{|\beta_1 \ln \alpha_2| H^C} \geq \frac{1}{|\ln \alpha_2| H^{C+1}}$. By choosing $\alpha_1 = \frac{2+i}{\sqrt{5}}$ and $\alpha_2 = -1$, the ratio $\frac{\ln \alpha_1}{\ln \alpha_2}$ is exactly γ/π . Recall that this number is irrational, satisfying the criteria that $\ln \alpha_1$ and $\ln \alpha_2$ are linearly independent over the rationals. Since $\gamma/\pi < 1$, clearly we can replace H by β_1 to obtain $\left| \frac{\ln \alpha_1}{\ln \alpha_2} + \frac{\beta_2}{\beta_1} \right| > \frac{1}{|\ln \alpha_2| \beta_1^{C+1}}$. Comparing this with the definition of the irrationality exponent, this is sufficient to see that $\mu(\gamma/\pi)$ is at most $C+1$ and therefore finite. Henceforth let $\mu = \mu(\gamma/\pi)$. It is worth noting that the constant C given by Theorem 4 is effectively computable, though the order of magnitude is impractical.¹

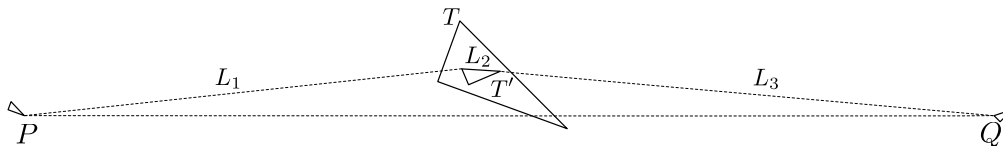
2.3 Convergence of Stretch

We define $f(d)$ to be the maximum stretch guaranteed by G_ω , over all pairs of vertices at Euclidean distance at least d .

$$f(d) = \sup \left\{ \frac{\text{dist}_{G_\omega}(u, v)}{\|u - v\|_2} \mid u, v \in V(G_\omega) \text{ and } \|u - v\|_2 \geq d \right\}.$$

► **Lemma 5.** Fix any distance d and let $f(d) = 1 + \varepsilon$. There exists a function $n = \varepsilon^{(1-\mu)/2 - o(1)}$ such that if $d = \Omega(5^n/\varepsilon)$, then $f(3d) < 1 + \varepsilon - \Omega(5^{-n})$.

Proof. Consider any two vertices $P, Q \in G_\omega$ with $|\overline{PQ}| = D \geq 3d$ when we regard them as points in the plane. Begin by choosing a parameter $\delta = \delta(\varepsilon)$ and identifying a triangle T satisfying the following properties: T intersects $\overline{PQ}$, the projection of T onto $\overline{PQ}$ lies entirely within the middle third of $\overline{PQ}$, and the hypotenuse of T has length exactly δD . Now choose a parameter $\theta = \theta(\varepsilon)$ and let $n = n(\theta)$ be large enough such that $\text{Ang}(T, n)$ is a θ -cover of $[0, \pi)$. Finally, select a triangle T' within T , and exactly n levels below T , whose hypotenuse creates an angle less than θ with $\overline{PQ}$.



We now construct a path as follows. Let p and q be the endpoints of the hypotenuse of T' closer to P and Q respectively. Let $L_1 = \overline{Pp}$, $L_2 = \overline{pq}$, and $L_3 = \overline{Qq}$. $|L_1|$ and $|L_3|$ must each be at least d , so by the definition of the stretch function f , G_ω approximates L_1 and L_3 to within $1 + \varepsilon$ stretch.

Let ℓ_1 , ℓ_2 , and ℓ_3 be the projections of L_1 , L_2 , and L_3 onto $\overline{PQ}$. We claim that $\text{dist}_{G_\omega}(P, p)/|\ell_1| < (1 + \varepsilon)(1 + 5\delta^2)$. Since p lies within T , its projection onto $\overline{PQ}$ is a distance at most δD from p and at least $D/3$ from P . Therefore, $|L_1|/|\ell_1|$ is bounded by $\sqrt{(D/3)^2 + (\delta D)^2}/(D/3) = \sqrt{1 + 9\delta^2} < 1 + 5\delta^2$. By definition of f , $\text{dist}_{G_\omega}(P, p)/|L_1| \leq 1 + \varepsilon$.

¹ One estimate, also due to Baker [3], states that $C = \ln A_1 \ln^2 A_2 (16nd)^{200n}$ suffices when the β_i are rational, where A_i is the height of α_i and d is the degree of the field extension $\mathbb{Q}[\alpha_1, \alpha_2]/\mathbb{Q}$. For our purposes, $A_1 = 4$, $A_2 = 2$, $n = 2$, and $d = 4$, giving $C \approx 2^{2800}$. Baker remarks it is possible to argue C is much lower in reality, but the required analysis is quite technical and beyond the scope of this paper.

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The same is true for $\text{dist}_{G_\omega}(q, Q)/|\ell_3|$. On the other hand, $\text{dist}_{G_\omega}(p, q)/|\ell_2|$ is trivially bounded by $|L_2|/|\ell_2| \leq 1/\cos \theta$ since p and q are joined directly by an edge in G_ω . Combining the bounds on each component, we write

$$\text{dist}_{G_\omega}(P, Q) < (1 + \varepsilon)(1 + 5\delta^2)(|\ell_1| + |\ell_3|) + \frac{|\ell_2|}{\cos \theta}.$$

We normalize by $1/D$ to obtain the stretch, and because this inequality holds for arbitrary P, Q satisfying $|\overline{PQ}| \geq 3d$, it holds for the supremum as well.

$$\begin{aligned} f(3d) &= \sup_{P, Q \in V(G_\omega)} \left\{ \frac{\text{dist}_{G_\omega}(P, Q)}{|\overline{PQ}|} \mid |\overline{PQ}| \geq 3d \right\} \\ &< (1 + \varepsilon)(1 + 5\delta^2) \left(1 - \frac{|\ell_2|}{D} \right) + \frac{|\ell_2|}{D \cos \theta} \\ &\leq (1 + \varepsilon)(1 + 5\delta^2)(1 - 5^{-n/2} \delta \cos \theta) + 5^{-n/2} \delta \end{aligned}$$

Letting $\kappa = 5^{-n/2}$, this is upper bounded by

$$< 1 + \varepsilon + 6\delta^2 + \kappa\delta(1 - (1 + \varepsilon) \cos \theta)$$

At this point we fix $\theta = \sqrt{\varepsilon}$, so $\cos \theta > 1 - \varepsilon/2$.

$$< 1 + \varepsilon + 6\delta^2 - \kappa\delta\varepsilon(1 - \varepsilon)/2 = 1 + \varepsilon + \delta(6\delta - \kappa\varepsilon(1 - \varepsilon)/2)$$

Finally, we pick $\delta < \kappa\varepsilon/24$.

$$\leq 1 + \varepsilon - \Omega(\delta^2) = 1 + \varepsilon - \Omega(5^{-n}).$$

By Corollary 3, we choose $n(\theta) = n(\sqrt{\varepsilon}) = (\sqrt{\varepsilon})^{1-\mu-o(1)}$. The only remaining detail is to address that the triangles T and T' used in this argument do indeed exist. The length of the hypotenuse of T was taken to be δD , and then we chose T' to be n levels beneath that of T . Therefore, we have implicitly assumed $D \geq 5^{n/2}/\delta > \Omega(5^n/\varepsilon)$, consistent with the assumption in the statement of the lemma. ◀

Theorem 1 now follows easily from this lemma; see the full version [7].

2.4 Pinwheel Tilings on the Grid

► **Theorem 6.** *There exists a weight function w such that $\text{Grid}[w]$ has stretch 1, asymptotically. In particular, for any $u, v \in V(\text{Grid})$,*

$$\|u - v\|_2 - O(1) \leq \text{dist}_w(u, v) \leq (1 + O(1/(\log \log \|u - v\|_2)^\xi)) \cdot \|u - v\|_2, \quad \text{for some } \xi > 0.$$

The proof of this theorem can be found in the full version [7].

3 A Deterministic Construction with Faster Convergence

In this section, we give a deterministic construction based on *highways* with faster convergence. Specifically, we establish the following result as Theorem 7:

► **Theorem 7.** *There exists an assignment $W : E(\text{Grid}) \rightarrow \mathbb{R}_{\geq 0}$ such that for any $u, v \in V(\text{Grid})$,*

$$\|u - v\|_2 - 1 \leq \text{dist}_W(u, v) \leq \|u - v\|_2 + O\left(\|u - v\|_2^{\frac{8}{9}}\right).$$

We prove Theorem 7 in two steps. First, we show that the same statement holds for the finite square grid $[n] \times [n]$ (Theorem 8), then we give a black-box reduction from the infinite case to the finite case.

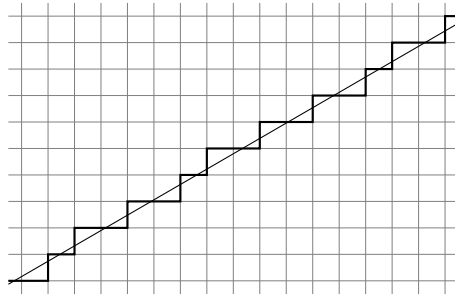
► **Theorem 8.** *There exists a weight assignment $W(n)$ to edges of the finite grid on $[n] \times [n]$ such that for any $u, v \in [n]^2$,*

$$\|u - v\|_2 - 1 \leq \text{dist}_{W(n)}(u, v) \leq \|u - v\|_2 + O\left(\|u - v\|_2^{\frac{8}{9}}\right).$$

The proof of Theorem 8 follows from the constructions presented in Section 3.1 and Section 3.2.

3.1 Highways

Given a line $\ell = ax + b$ in $\mathbb{R}^2$ we define the $\text{Highway}(\ell)$ to be a grid-path that tracks ℓ ; see Figure 3.



■ **Figure 3** A grid-path that tracks ℓ .

Specifically, let $v \in V(\text{Grid})$ be a grid-point, and consider the 1×1 square $[v - 0.5, v + 0.5) \times [v - 0.5, v + 0.5)$. If ℓ intersects this square, then we include v in $V(\ell)$. Whenever $u, v \in V(\ell)$ are adjacent grid points, $\text{Highway}(\ell)$ contains the edge $\{u, v\} \in E(\text{Grid})$. If $a \in [-1, 1]$ we let w_ℓ assign every edge in $\text{Highway}(\ell)$ the weight $\frac{\sqrt{a^2+1}}{|a|+1}$, which is the asymptotic ratio between the Euclidean distance of points on ℓ and the number of edges taken along the grid path $\text{Highway}(\ell)$. Otherwise we write ℓ as $x = a^{-1}(y - b)$ with $a^{-1} \in [-1, 1]$ and use weight $\frac{\sqrt{a^{-2}+1}}{|a^{-1}|+1}$. All off-highway edges have weight ∞ .

This weight assignment guarantees a discrepancy of at most 1 between Euclidean distances and grid distances along $\text{Highway}(\ell)$. The proof of Lemma 9 appears in [7].

► **Lemma 9.** *Let $\text{Highway}(\ell)$ denote the highway that approximates a line ℓ of the form $y = ax + b$. Then for any two points $u, v \in V(\text{Highway}(\ell))$, $|\text{dist}_{w_\ell}(u, v) - \|u - v\|_2| \leq 1$.*

The highway transformation can also be applied to a line segment s . We use the same notation $\text{Highway}(s)$.

3.2 The Hierarchical Highway Construction

We are trying to find a weight assignment for the finite grid $[n] \times [n]$ in order to prove Theorem 8. The construction is parameterized by $(k_i)_{1 \leq i \leq m}$, where $k_1 = \lfloor n^{1/5} \rfloor$, $k_{i+1} = \lfloor k_i^{1/2} \rfloor$, and m is minimum such that $k_m < 100$.

27:10 The Squishy Grid Problem

The construction is based on a hierarchical system of lines in $\mathbb{R}^2$ which will eventually be embedded as highways in the grid. The lines at level i have angles selected from $(\theta_{i,j})_{0 \leq j < k_i}$, where $\theta_{i,j} = \frac{\pi \cdot j}{k_i}$. Fixing i and one such angle $\theta_{i,j}$, there are many lines with angle $\theta_{i,j}$, spaced at distance k_i^4 . For $i \in [1, m], j \in [0, k_i - 1], t \in \mathbb{Z}$, define

$$\ell_{i,j,t} = \{(x, y) \in \mathbb{R}^2 \mid y \cdot \cos \theta = x \cdot \sin \theta + t \cdot k_i^4\}.$$

Define $\text{Lines}[i] = \{\ell_{i,j,t}\}$ to be the set of all lines at level i .

We cannot choose a weight function $W(n)$ that agrees with $w_{\ell_{i,j,t}}$ for *every* line $\ell_{i,j,t}$ due to intersections. Our solution is to avoid this issue by removing all line intersections, which introduces distortions in distances that must be bounded.

Below we define a procedure to remove parts of $\ell_{i,j,t}$, leaving a set of line *segments* $\mathcal{L}_{i,j,t}$. Define $\text{Segments}[i] = \bigcup \mathcal{L}_{i,j,t}$ to be the set of all line segments at level i . If O is an object or collection of objects, define

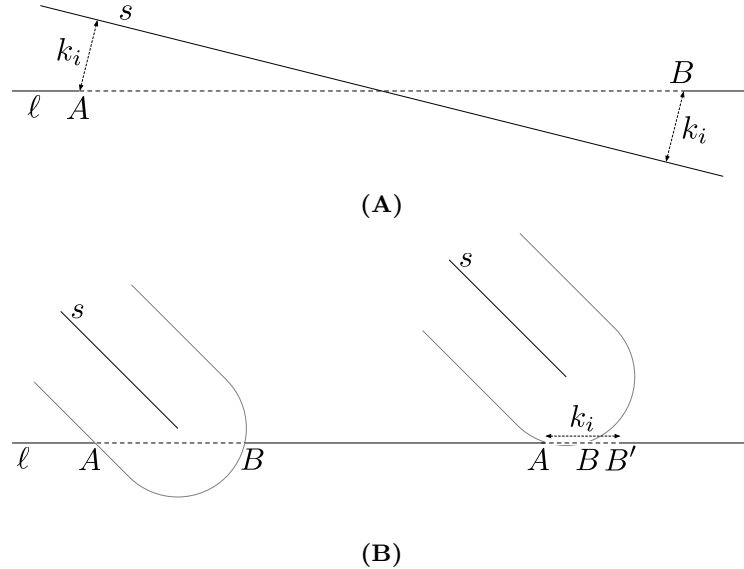
$$\text{Fat}(O, \delta) = \{p \mid \exists q \in O \text{ such that } \|p - q\|_2 \leq \delta\}$$

to be all points within distance δ of O . Specifically, $\text{Fat}(\ell, \delta)$ is a strip if ℓ is a line, and a hippodrome if ℓ is a segment.

Once $\text{Segments}[1], \dots, \text{Segments}[i-1]$ are constructed, we construct $\text{Segments}[i]$ as follows. For each $\ell_{i,j,t} \in \text{Lines}[i]$ initialize $\ell \leftarrow \ell_{i,j,t}$ and proceed to remove parts of ℓ in Steps 1 and 2.

Step 1. Set $\ell \leftarrow \ell - \text{Fat}(\text{Lines}[i] - \{\ell\}, k_i)$, i.e., we remove every part of ℓ within distance k_i of any other line at level i .

Step 2. For each segment $s \in \bigcup_{i' < i} \text{Segments}[i']$ such that $\text{Fat}(s, k_i) \cap \ell \neq \emptyset$, let $\overline{AB} = \text{Fat}(s, k_i) \cap \ell$. If $\|A - B\|_2 \geq k_i$, set $\ell \leftarrow \ell - \overline{AB}$. Otherwise, let $B' \in \ell_{i,j,t}$ be such that $\|A - B'\| = k_i$ and $B \in \overline{AB'}$, and set $\ell \leftarrow \ell - \overline{AB'}$. See Figure 4. Define $\mathcal{L}_{i,j,t} = \ell$, and include all line segments of $\mathcal{L}_{i,j,t}$ in $\text{Segments}[i]$.



■ **Figure 4** Illustrations of various cases in Step 2. **(A)** ℓ and s intersect. The segment $\ell \cap \text{Fat}(s, k_i)$ is removed from ℓ . **(B)** ℓ and s do not intersect. Left: $\overline{AB} \geq k_i$ and $\overline{AB}$ is removed from ℓ . Right: $\overline{AB} < k_i$ and B' is such that $\overline{AB'} = k_i$ is removed from ℓ .

The weight assignment $W(n)$ is now constructed as follows. For each line segment $s \in \bigcup_i \text{Segments}[i]$, let $W(n)$ agree with w_s at all edges in the corresponding highway segment $\text{Highway}(s)$. All edges not appearing in any line segment have weight 2.

► **Lemma 10.** *Fix any $s \in \text{Segments}[i]$, $s' \in \text{Segments}[i']$, where $s \neq s'$ and $i \geq i'$. For any points $u \in s, u' \in s'$, $\|u - v\|_2 \geq k_i$.*

Proof. Steps 1 and 2 ensure that all remaining points on $\mathcal{L}_{i,j,t}$ lie outside

$$\text{Fat} \left(\bigcup_{i' < i} \text{Segments}[i'] \cup (\text{Lines}[i] - \{\ell_{i,j,t}\}), k_i \right),$$

and that any two consecutive segments on $\mathcal{L}_{i,j,t}$ are separated by distance at least k_i . This implies that the Euclidean distance between any $u \in s$ and $u' \in s'$ is at least k_i . ◀

Lemma 11 shows that distances under $W(n)$ are approximately Euclidean, up to a multiplicative stretch of $1 + O(k_i^{-1})$ and additive stretch $O(k_i^4)$, for every index i .

► **Lemma 11.** *For any $u, v \in [n]^2$ and $i \in [m]$.*

$$\|u - v\|_2 - 1 \leq \text{dist}_{W(n)}(u, v) \leq \|u - v\|_2 + O(k_i^{-1}\|u - v\|_2 + k_i^4).$$

Proof. Observe that the highways corresponding to all segments in $\bigcup_i \text{Segments}[i]$ are vertex-disjoint. Thus, every grid path P from u to v can be written as $B_0 A_1 B_1 A_2 B_2 \cdots A_k B_k$, where each $A_j \subset \text{Highway}(s)$ for some $s \in \bigcup_i \text{Segments}[i]$ and each B_j is disjoint from all highway segments, and therefore consists of only weight-2 edges. (One or both of B_0, B_k may be empty.) By Lemma 9, $W(A_j)$ is at least the Euclidean distance between its endpoints minus one, and $W(B_j)$ is at least the Euclidean distance multiplied by 2, which implies that $W(B_j)$ is at least the Euclidean distance plus 1 for $1 \leq j \leq k - 1$, so $W(P)$ is at least $\|u - v\|_2 - 1$.

Turning to the upper bound, We bound the distance $\text{dist}_{W(n)}(u, v)$ by explicitly constructing a path that stays within the vicinity of a single line $\ell_{i,j,t}$. Since level- i lines occur at angular intervals of $\frac{\pi}{k_i}$ and parallel lines are spaced k_i^4 apart, we can always find an $\ell^* = \ell_{i,j,t}$ satisfying the following properties. First, the difference in angle between ℓ^* and $\overline{uv}$ is at most $\frac{\pi}{2k_i}$. Second, the distance from u to ℓ^* is at most $k_i^4/2$. Let A and B be the closest grid points on $\text{Highway}(\ell^*)$ from u and v respectively. It follows that

$$\text{dist}_{W(n)}(u, A) + \text{dist}_{W(n)}(v, B) = O\left(k_i^4 + \|u, v\|_2 \sin\left(\frac{\pi}{2k_i}\right)\right) = O(k_i^4 + k_i^{-1}\|u - v\|_2). \quad (2)$$

It remains to bound $\text{dist}_{W(n)}(A, B)$. A trivial upper bound is $\text{dist}_{W(n)}(A, B) \leq 2 \cdot \sqrt{2} \|A - B\|_2$, so if $\|A - B\|_2 < 100k_i^4$ we are done. Henceforth we shall assume that $\|A - B\|_2 \geq 100k_i^4$. We would prefer to follow the A - B path along $\text{Highway}(\ell^*)$, but sections of this highway have effectively been removed by Steps 1 and 2 of the construction. We bound the stretch induced by the gaps in the highway introduced in Steps 1 and 2 separately.

Step 1 Stretch. Fix a direction $\theta_{i,j'}$ different from ℓ^* 's direction $\theta_{i,j}$. Whenever ℓ^* intersects a line with angle $\theta_{i,j'}$, Step 1 causes $\text{Highway}(\ell^*)$ to lose $\frac{k_i}{\sin|\theta_{i,j} - \theta_{i,j'}|} = O(k_i^2)$ edges. There are k_i angles, and parallel lines with angle $\theta_{i,j'}$ are spaced k_i^4 apart, so the total number of edges removed from the A - B path in $\text{Highway}(\ell^*)$ in Step 1 is

$$O\left(k_i^2 \cdot k_i \cdot \frac{\|A - B\|_2}{k_i^4}\right) = O(k_i^{-1}\|A - B\|_2). \quad (3)$$

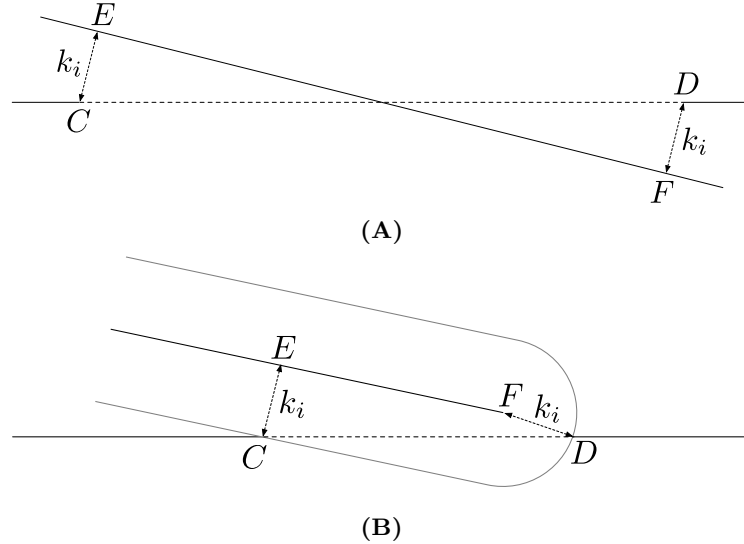
The additive stretch induced by walking across the gaps induced by Step 1 is also $O(k_i^{-1}\|A - B\|_2)$ as all these edges have weight 2.

Step 2 Stretch. Whenever part of $\text{Highway}(\ell^*)$ is removed by Step 2 we do *not* walk precisely in the direction of ℓ^* but take a *detour* to a lower level highway. Suppose that in Step 2, a segment $s \in \bigcup_{i' < i} \text{Segments}[i']$ causes an interval $\overline{CD}$ of ℓ^* to be removed. Define E, F to be the points on s closest to C, D , respectively. When our path reaches C , we walk from C to E , then to F along $\text{Highway}(s)$, then to D . See Figure 5. By construction $\|C - E\|_2, \|D - F\|_2 = O(k_i)$. Since $\|E - F\|_2 \leq \|C - D\|_2$ and by Lemma 9 $\text{dist}_{W(n)}(E, F) = \|E - F\|_2 \pm 1$, the additive stretch due to the conflict with s is at most

$$\text{dist}_{W(n)}(C, E) + \text{dist}_{W(n)}(E, F) + \text{dist}_{W(n)}(F, D) - \text{dist}_{w_{\ell^*}}(C, D) = O(k_i).$$

The last task is to bound the number of such segments interfering with the A - B path. Observe that s is a segment of a line at level $i - 1$ or lower. Thus, by Lemma 10 any two such segments s, s' are at distance at least $k_{i-1} \geq k_i^2$, and the total additive stretch caused by Step 2 detours is

$$O\left(k_i \cdot \frac{\|A - B\|_2}{k_i^2}\right) = O(k_i^{-1} \|A - B\|_2). \quad (4)$$



■ **Figure 5** Step 2 detours from the proof of Lemma 11. (A) The case when segment s intersects ℓ^* . (B) When segment s does not intersect ℓ^* , but $\text{Fat}(s, k_i)$ does.

Combining Equations (2)–(4), we conclude that

$$\begin{aligned} \text{dist}_{W(n)}(u, v) &\leq \text{dist}_{W(n)}(u, A) + \text{dist}_{W(n)}(A, B) + \text{dist}_{W(n)}(B, v) \\ &\leq \|u - v\| + O(k^{-1} \|A - B\|_2 + k_i^{-1} \|u - v\|_2 + k_i^4) \\ &= \|u - v\|_2 + O(k^{-1} \|u - v\|_2 + k_k^4). \end{aligned} \quad \blacktriangleleft$$

Proof of Theorem 8. Recall that by the definition of the sequence (k_i) , for any pair of points u, v with Euclidean distance $d = \|u - v\|_2 > 100^9$, there exists some index $1 \leq i \leq m$ such that $k_i \in [d^{1/9} - 1, d^{2/9}]$. Applying Lemma 11, we have

$$\text{dist}_W(u, v) \leq \|u - v\|_2 + O\left(\frac{1}{k_i} \cdot \|u - v\|_2 + k_i^4\right) \leq \|u - v\|_2 + O\left(\|u - v\|_2^{8/9}\right).$$

The theorem holds trivially when $d \leq 100^9$, which completes the proof of Theorem 8. $\blacktriangleleft$

In the full version [7] we give a black-box reduction from any finite construction on $[n] \times [n]$, like that of Theorem 8, to a construction on $\mathbb{Z} \times \mathbb{Z}$, yielding Theorem 7.

4 Experimental Findings in the Squishy Grid

If Question 2 seems too daunting, a natural idea is to simplify the problem by considering only *monotone* paths, that is, paths that use the fewest number of edges.

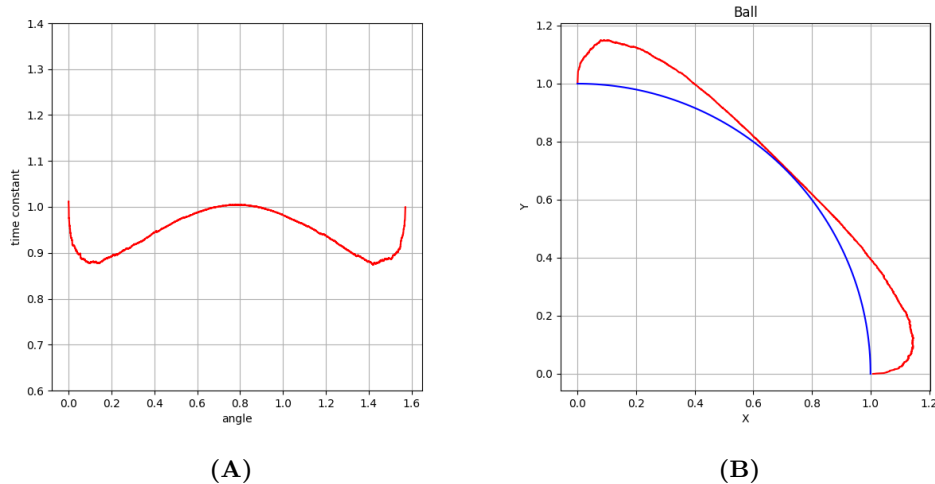
► **Question 3.** Define $\text{monodist}_w(u, v)$ to be the length of the shortest u - v path that uses $\|u - v\|_1$ edges. Is there a distribution $\mathcal{D}^*$ over $\mathbb{R}_{\geq 0}$ such that if $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}^*]$ is a randomly weighted graph, for all $u, v \in V(\text{Grid})$, $\mathbb{E}(\text{monodist}_w(u, v)) = (1 \pm o(1))\|u - v\|_2$?

At first glance this problem may seem easier, or more plausible, than Question 2. Whereas it is an open problem finding a distribution $\mathcal{D}$ satisfying Equation (1) (the time constant in the 0° and 45° directions are 1), this is nearly trivial when we consider monodist . Refer to [7] for proof.

► **Lemma 12.** *There exists a distribution $\mathcal{D}$ on $\mathbb{R}_{\geq 0}$ such that*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\text{monodist}_w(\mathbf{0}, ne_0))}{n} = \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\text{monodist}_w(\mathbf{0}, ne_{45}))}{n} = 1.$$

Let $B_{\text{mono}}(t) = \{u \in \mathbb{Z}^2 \mid \text{monodist}_w(\mathbf{0}, u) \leq t\}$ and $\mathcal{B}_{\text{mono}}(\mathcal{D})$ be the limiting shape of $B_{\text{mono}}(t)/t$ in $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}]$ as $t \rightarrow \infty$. Thus, $\mathcal{B}_{\text{mono}}(\mathcal{D})$ coincides with the L_2 -ball in the eight intercardinal directions N, E, S, W, NE, SE, SW, NW. If $\mathcal{B}_{\text{mono}}(\mathcal{D})$ were convex, then it would have to be quite close to the L_2 -ball. Unfortunately, our experiments show that $\mathcal{B}_{\text{mono}}(\mathcal{D})$ is *not* convex, which casts serious doubt on Question 3 having an affirmative answer. See Figure 6.



■ **Figure 6** (A) The stretch of $\text{monodist}_w(\mathbf{0}, ne_\theta) / \|ne_\theta\|_2$, as a function of the angle $\theta \in [0, \frac{\pi}{2}]$, expressed in radians. (B) The shape of $\mathcal{B}_{\text{mono}}(\mathcal{D}_{\epsilon^*})$.

In retrospect, Question 3 is less likely than Question 2 to be answered in the affirmative since $\text{monodist}_w(\mathbf{0}, (n, m))$, $m \leq n$, is much more sensitive to small deviations in m than $\text{dist}_w(\mathbf{0}, (n, m))$. Considering the cases when $m = 0$, $m = 0.1n$, and $m = n$, $\text{monodist}_w(\mathbf{0}, (n, m))$ is the minimum of $\binom{n+0}{0} = 1$, $\binom{n+0.1n}{0.1n} \approx 1.39^n$, and $\binom{2n}{n} \approx 4^n$ different paths, respectively. This sharp jump from constant to exponential in the vicinity $m = 0$ does not exist in Question 2. Assuming the variance of $\mathcal{D}$ is sufficiently large, $\text{dist}_w(\mathbf{0}, ne_0)$ is the minimum of an exponential number of plausible shortest paths.

4.1 Discrete Distributions for First Passage Percolation

When dealing with discrete distributions the most natural measure of complexity is *support size*. Therefore, we study Question 2 experimentally by considering the space of 2- and 3-point distributions. For a fixed integer k , the k -point distribution $\mathcal{D}(\{(p_i, x_i)\}_{i=1}^k)$, is such that $\Pr_{w_0 \sim \mathcal{D}}(w_0 = x_i) = p_i$. It is determined by $2k - 1$ parameters, as $p_k = 1 - (p_1 + \dots + p_{k-1})$.

4.1.1 Experimental Methodology

To identify locally optimal distributions among k -point discrete distributions of the form $\mathcal{D}(\{(p_i, x_i)\}_{i=1}^k)$, we employ a two-layer iterative strategy:

- We first perturb the probability vector $(p_1, \dots, p_k)$.
- For fixed probabilities $(p_1, \dots, p_k)$, we generate k random initial values $x_1, x_2, \dots, x_k$ and then alternate between the following two update steps:

Perturbation Step. We perturb each value x_i and compute the estimated directional stretch for both $\theta = 0$ and $\theta = \frac{\pi}{4}$. Let μ_θ be the empirical ratio $\frac{\text{dist}_w(\mathbf{0}, ne_\theta)}{n}$, obtained from this round of simulation. Here $n \approx 30,000$.

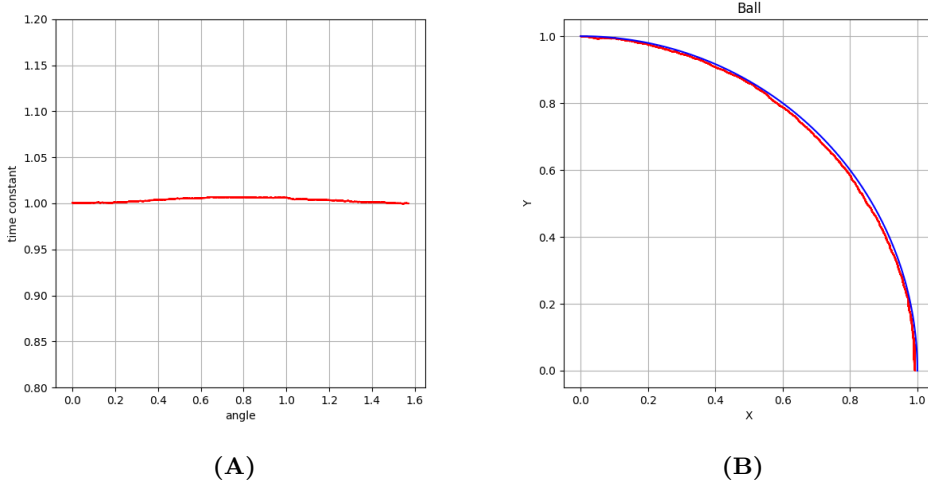
Normalization Step. We normalize the values $\{x_i\}_{i=1}^k$ by setting $x_i \leftarrow \frac{x_i}{\sqrt{\mu_0 \cdot \mu_{\frac{\pi}{4}}}}$. This scaling ensures that the average stretch along the cardinal and intercardinal directions remains close to 1, thereby facilitating comparisons between distributions.

4.1.2 2-Point Distributions

The best 2-point distribution identified with this method is $\mathcal{D}_2$, given below. Roughly speaking, every edge weight is either 0.41 or 4.75, 44% and 56% of the time, respectively.

$$\mathcal{D}_2 = \{(0.44273, 0.41401), (0.55727, 4.75309)\}.$$

We find that $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}_2]$ empirically approximates Euclidean distances up to stretch 1.00750, i.e., up to $(3/4)\%$ error. Figure 7(A) plots the observed stretch $\text{dist}_w(\mathbf{0}, ne_\theta) / \|ne_\theta\|_2$ as a function of the angle $\theta \in [0, \pi/2]$. Figure 7(B) shows the set of all grid points whose empirical graph distance from the origin first exceeds n . The resulting boundary is visually close to a Euclidean circle, suggesting that $\mathcal{D}_2$ induces an approximately isotropic metric in expectation.



■ **Figure 7** Results on the 2-point distribution $\mathcal{D}_2$. (A) Stretch $\text{dist}_w(\mathbf{0}, ne_\theta) / \|ne_\theta\|_2$, as a function of the angle $\theta \in [0, \frac{\pi}{2}]$. (B) The empirical distance- n ball in $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}_2]$.

Note that $\mathbb{E}(w_0 \sim \mathcal{D}_2) \approx 2.83$, meaning that $\text{dist}_w(\mathbf{0}, ne_0) \approx n$ is likely to be realized by a highly non-monotone path, consistent with the observations in Figure 6.

4.1.3 3-Point Distributions

$\mathcal{D}_2$ does not leave much room for improvement, but we are able to eke out a slightly better empirical stretch of 1.00622 with a 3-point distribution $\mathcal{D}_3$.

$$\mathcal{D}_3 = \mathcal{D}(\{(0.34809, 0.20647), (0.25735, 2.51586), (0.39456, 9.32215)\}).$$

See Figure 8 for visual representations of the empirical stretch of $\mathcal{D}_3$.

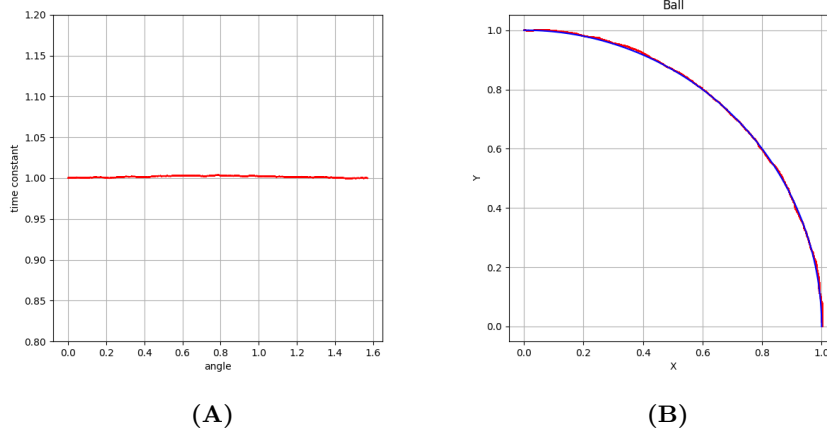


Figure 8 Results on the 3-point distribution $\mathcal{D}_3$. (A) Stretch $\frac{\text{dist}_w(\mathbf{0}, ne_\theta)}{\|ne_\theta\|_2}$, as a function of $\theta \in [0, \frac{\pi}{2}]$. (B) The empirical distance- n ball in $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}_3]$.

5 Conclusion

In this paper we asked how well the integer grid graph Grid can approximate Euclidean distances, if weighted appropriately. We gave two deterministic weighting schemes that answer Question 1, the best one achieving a **polynomial** additive stretch, that is, $\mathbb{E}[\text{dist}_w(u, v)] = \|u - v\|_2 + O(\|u - v\|_2^{1-\delta})$ for $\delta = 1/9$. Improving the additive stretch to something **subpolynomial** $\|u - v\|_2^{o(1)}$ seems to require a new approach to the problem. Our “highway” method seems incapable of achieving subpolynomial additive error, and even if Question 2 is answered affirmatively (choosing weights i.i.d. from some distribution $\mathcal{D}^*$), the tail bounds here only give polynomial additive error [2].

We conjecture that no weighting achieves subpolynomial additive error.

► **Conjecture 13.** *The additive error $\text{dist}_w(u, v) - \|u - v\|_2$ of $w : E(\text{Grid}) \rightarrow \mathbb{R}_{\geq 0}$ is a function of $d = \|u - v\|_2$.*

Weak Conjecture. *There is no w with constant additive error $O(1)$, independent of d .*

Strong Conjecture. *There is no w with subpolynomial additive error $d^{o(1)}$.*

In the randomized setting, Question 2 is equivalent to an old problem in *first passage percolation* [2, 11] that is unsolved, but has some empirical evidence in its favor. Alm and Deijfen [1] showed that when $\mathcal{D}$ is a certain Fisher distribution, that $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}]$

approximates Euclidean distances to less than 1% error. In this paper, we demonstrated that a simple 2-point distribution $\mathcal{D}_2$ achieves 0.75% error, and that a 3-point distribution $\mathcal{D}_3$ achieves 0.622% error. It is an interesting open problem to prove that in $\text{Grid}[w] \sim \text{Grid}[\mathcal{D}_2]$, the expected error of dist_w is less than 1%.

References

- 1 Sven Erick Alm and Maria Deijfen. First passage percolation on $\mathbb{Z}^2$: A simulation study. *J. Stat. Phys.*, 161:657–678, 2015. doi:10.1007/s10955-015-1356-0.
- 2 Antonio Auffinger, Michael Damron, and Jack Hanson. *50 Years of First-Passage Percolation*. University Lecture Series. American Mathematical Society, 2017.
- 3 Alan M. Baker. The theory of linear forms in logarithms. In *Transcendence theory: advances and applications* (A. Baker, D. W. Masser, Eds.), pages 1–27. Academic Press, London, 1976.
- 4 Glencora Borradaile and David Eppstein. Near-linear-time deterministic plane Steiner spanners for well-spaced point sets. *Comput. Geom.*, 49:8–16, 2015. doi:10.1016/J.COMGEO.2015.04.005.
- 5 Yann Bugeaud. *Distribution Modulo One and Diophantine Approximation*, pages 1–14. Cambridge University Press, 2012.
- 6 Dmitri Burago and Sergei Ivanov. Uniform approximation of metrics by graphs. *Proc. Amer. Math. Soc.*, 143:1241–1256, 2015.
- 7 Zixi Cai, Kuowen Chen, Shengquan Du, Arnold Filtser, Seth Pettie, and Daniel Skora. The squishy grid problem. *CoRR*, abs/2507.23105, 2025. doi:10.48550/arXiv.2507.23105.
- 8 Hsien-Chih Chang, Robert Krauthgamer, and Zihan Tan. Almost-linear ϵ -emulators for planar graphs. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (STOC)*, pages 1311–1324, 2022. doi:10.1145/3519935.3519998.
- 9 John H. Conway and Charles Radin. Quaquaversal tilings and rotations. *Inventiones Mathematicae*, 132:179–188, 1998.
- 10 J. Theodore Cox and Richard Durrett. Some limit theorems for percolation with necessary and sufficient conditions. *Annals of Probability*, 9:583–603, 1981. doi:10.1214/aop/1176994364.
- 11 John M. Hammersley and Dominic J. A. Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Bernoulli 1713, Bayes 1763, Laplace 1813: Anniversary Volume, Proceedings of an International Research Seminar, Statistical Laboratory, University of California, Berkeley, 1963*, pages 61–110. Springer, 1965.
- 12 Harry Kesten. Aspects of first passage percolation. In *École d’Été de Probabilités de Saint Flour XIV, Lecture Notes in Mathematics, 1180*. Springer, 1986.
- 13 Lauwerens Kuipers and Harald Niederreiter. *Uniform Distribution of Sequences*, pages 118–132. Wiley, 1974.
- 14 János Pach, Richard Pollack, and Joel Spencer. Graph distance and Euclidean distance on the grid. In *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, pages 555–559. Springer, 1990.
- 15 Charles Radin. The pinwheel tilings of the plane. *Annals of Mathematics*, 139:661–702, 1994.
- 16 Charles Radin and Lorenzo Sadun. The isoperimetric problem for pinwheel tilings. *Commun. in Math. Phys.*, 177:255–263, 1996.