

# Upward Book Embeddings of Partitioned Digraphs

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## Abstract

In 1999, Heath, Pemmaraju, and Trenk [SIAM J. Comput. 28(4), 1999] extended the classic notion of book embeddings to digraphs, introducing the concept of *upward book embeddings*, in which the vertices must appear along the spine in a topological order and the edges are partitioned into pages, so that no two edges in the same page cross. For a partitioned digraph  $G = (V, \bigcup_{i=1}^k E_i)$ , that is, a digraph whose edge set is partitioned into  $k$  subsets, an upward book embedding is required to assign edges to pages as prescribed by the given partition. In a companion paper, Heath and Pemmaraju [SIAM J. Comput. 28(5), 1999] proved that the problem of testing the existence of an upward book embedding of a partitioned digraph is linear-time solvable for  $k = 1$  and recently Akitaya, Demaine, Hesterberg, and Liu [GD, 2017] have shown the problem NP-complete for  $k \geq 3$ . In this paper, we study upward book embeddings of partitioned digraphs and focus on the unsolved case  $k = 2$ . Our first main result is a novel characterization of the *upward embeddings* that support an upward book embedding in two pages. We exploit this characterization in several ways, and obtain a rich picture of the complexity landscape of the problem. First, we show that the problem remains NP-complete when  $k = 2$ , thus closing the complexity gap for the problem. Second, we show that, for an  $n$ -vertex partitioned digraph with a prescribed planar embedding, the existence of an upward book embedding that respects the given planar embedding can be tested in  $O(n \log^3 n)$  time. Finally, leveraging the SPQ(R)-tree decomposition of biconnected graphs into triconnected components, we present a cubic-time testing algorithm for biconnected directed partial 2-trees.

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## 1 Introduction

Book embeddings are a classic and influential topic in combinatorial and algorithmic graph theory. The notion of a book as a topological space was introduced in the late 60s by Persinger [48] and Atneosen [5], and later developed in its current and more popular form by the seminal work of Ollmann [47]. In a *book embedding* of a graph  $G = (V, E)$ , all vertices lie along a line – referred to as the *spine* – while edges are placed into distinct half-planes bounded by the spine, known as the *pages* of the book. Therefore, constructing such an embedding for  $G$  amounts to computing a pair  $(\pi, \sigma)$ , where  $\pi : V \leftrightarrow \{1, \dots, |V|\}$  is a linear ordering of the vertices and  $\sigma : E \rightarrow \{1, \dots, k\}$  is a partition of the edges into  $k$  pages so that no two edges in the same page cross according to  $\pi$ , i.e., their end-vertices do not alternate in  $\pi$ ; see Fig. 1a for an example. The minimum value of  $k$  for which this is possible is the *book thickness* of  $G$  (also called *stack number* or *page number*).



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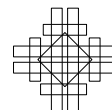
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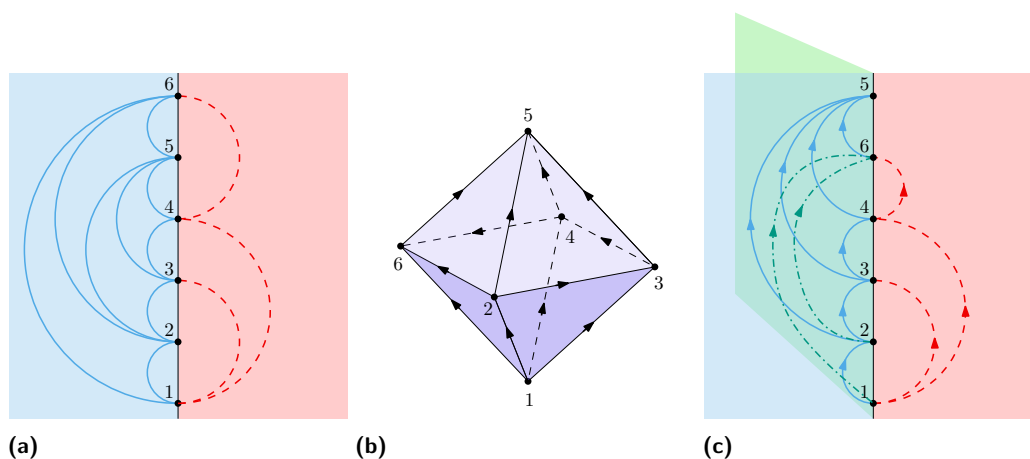
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Research on book embeddings originated from problems in VLSI circuit design [20], and has since found applications in sorting permutations [49, 52], fault-tolerant processing [50], compact graph encodings [40, 46], graph drawing [14, 25, 29, 33], computational origami [1, 45], parallel process scheduling [13], parallel matrix computations [36], and other domains. For a more comprehensive overview of applications, see [28]. The notion of book embedding was extended to digraphs by Heath, Pemmaraju, and Trenk [37] by introducing the natural requirement that in a book embedding  $(\pi, \sigma)$  of a digraph  $G$  the ordering  $\pi$  must be a topological ordering of  $G$ ; see Figs. 1b and 1c for an example. Such book embeddings are called *upward book embeddings* as they are naturally depicted with vertices placed on a vertical line and edges drawn as arcs monotonically increasing in the  $y$ -direction in their page.

Next, we provide an overview of the major algorithmic results on book embeddings; in the full version [21], we also discuss combinatorial results. Concerning undirected graphs, already in 1979, Bernhart and Kainen [10] showed that the graphs of book thickness 1 are the outerplanar graphs and that the graphs of book thickness 2 are the sub-Hamiltonian (planar) graphs. Whereas the former are recognizable in linear time [54], recognizing the latter is NP-complete, even for planar triangulations [55]. Recently, Ganian et al. [31] presented a sub-exponential-time algorithm for testing sub-Hamiltonicity, which is asymptotically tight under ETH. Concerning digraphs, the problem of testing the existence of an upward book embedding in  $k$  pages is called UPWARD BOOK EMBEDDING. For more than two decades, the only known NP-completeness result for the problem was the one shown by Heath and Pemmaraju [35] when  $k = 6$ . Recently, in two subsequent papers, Binucci et al. [16] and Bekos et al. [8, 9] closed the computational gap by showing NP-completeness for  $k \geq 3$  and  $k = 2$ , respectively. These results, together with the linear-time algorithm for  $k = 1$  [35], completely characterize the complexity of the UPWARD BOOK EMBEDDING problem with respect to  $k$ . For  $k = 2$ , efficient algorithms have been devised for special classes [15, 42, 43].

**Partitioned book embeddings.** Due to the NP-hardness of testing the existence of a book embedding  $(\pi, \sigma)$ , it is natural to study the complexity of the problem if  $\pi$  or  $\sigma$  is given as part of the input. If the vertex ordering  $\pi$  is fixed, the problem becomes the  $k$ -coloring problem for circle graphs. The problem is clearly linear-time solvable for  $k \leq 2$ . Unger [53] showed that it is NP-complete for  $k \geq 4$ . The complexity of the case  $k = 3$  is still open [6, 7], although a



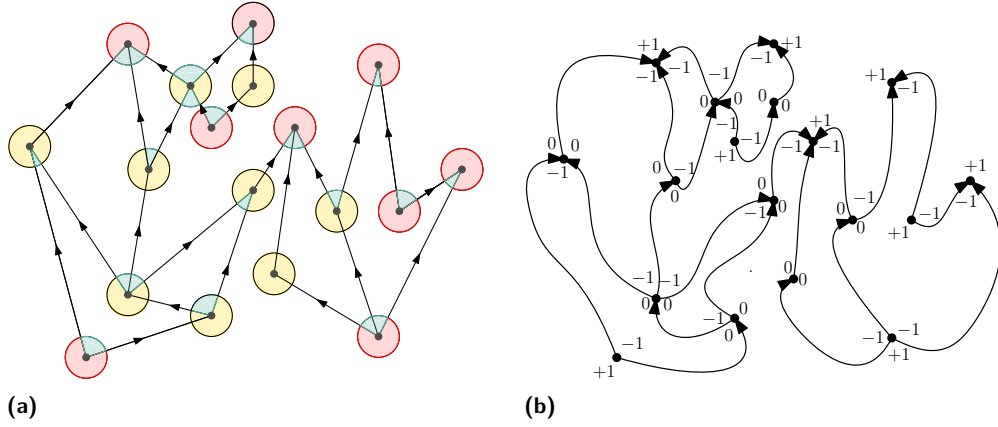
■ **Figure 1** (a) A book embedding of the octahedron in 2 pages. (b) An orientation of the octahedron. (c) An upward book embedding of the directed octahedron in (b) in 3 pages, which is optimal.

■ **Table 1** Complexity of the upward book embedding problem. Results marked ✓ are trivial.

|                          |          | vertex order $\pi$                                                              |                                                                                                 |
|--------------------------|----------|---------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
|                          |          | fixed                                                                           | variable                                                                                        |
| page assignment $\sigma$ | fixed    | $O(n+m)$ time ✓                                                                 | 1 page: $O(n)$ time [35]<br>2 pages: NP-complete (Theorem 4)<br>$\geq 3$ pages: NP-complete [1] |
|                          | variable | 2 pages: $O(n)$ time ✓<br>3 pages: OPEN [7]<br>$\geq 4$ pages: NP-complete [53] | 2 pages: NP-complete [9]<br>$\geq 3$ pages: NP-complete [16]                                    |

quasi-polynomial-time algorithm for  $k = 3$  has recently been proposed [51]. Conversely, if the partition  $\sigma$  into  $k$  pages is given, for undirected graphs, there are polynomial-time algorithms for  $k = 1$  [35] and  $k = 2$  [4, 38, 39], whereas the problem is NP-complete for  $k \geq 3$  [3]. For directed graphs, in which the vertex ordering has to be a topological ordering, the problem is called PARTITIONED UPWARD BOOK EMBEDDING. For  $k = 1$  it coincides with the “unpartitioned” case, solved in [35, 37]. For  $k > 1$ , it was first studied by Akitaya et al. [1]. They connected the problem to applications in map folding [45] and attributed it to Edmonds, who, in 1997, posed the question specifically for  $k = 4$  when the edges assigned to each page form a matching. They showed NP-completeness for  $k = 3$  and for  $k \geq 4$  even if the edges in each page form a matching. For  $k = 2$ , they gave a linear-time algorithm for the special case when the edges in each page form a matching. However, they left the complexity of the general case for  $k = 2$  open. See Table 1 for an overview of the complexity landscape.

**Our Contributions.** We study the PARTITIONED UPWARD BOOK EMBEDDING problem in the unsolved case  $k = 2$ . As every upward book embedding corresponds to an upward planar drawing, i.e., the edges are drawn as non-crossing  $y$ -monotone curves, we make use of combinatorial descriptions of such drawings, called *upward embeddings* [11]. Our first main result is a characterization of the upward embeddings that support an upward book embedding in two pages. We exploit this characterization in several ways and obtain a rich picture of the complexity landscape of the problem. First, we show that the problem remains NP-complete when  $k = 2$ , thus closing the complexity gap and exhibiting a sharp contrast with the undirected case, where the problem is linear-time solvable [39]. Our proof also implies that the problem is W[1]-hard with respect to the treewidth. Second, we show that, when the planar embedding is prescribed, the problem can be solved in  $O(n \log^3 n)$  time. Our algorithm is inspired by the network-flow approach of Bertolazzi et al. [11] and requires the use of several non-trivial ingredients arising from our characterization. Finally, in the variable-embedding setting, we present a cubic-time algorithm for biconnected directed partial 2-trees. Our algorithm exploits a compact representation, called *descriptor pair*, of the features of upward embeddings of subgraphs that admit an extension to an upward embedding, satisfying our characterization, of the entire graph. This allows us to compute, for the subgraph associated with each node of the SPQ(R)-tree, its set of realizable descriptor pairs by dynamic programming. While we focus on the decision problem, all our algorithms are constructive and can be modified to compute an upward book embedding, if one exists. A full version of the paper with complete proofs can be found in [21].



■ **Figure 2** (a) An upward planar drawing  $\Gamma$  and its big, small, and flat angles depicted as red, green, and yellow sectors, respectively. (b) The upward-consistent angle assignment  $\lambda_\Gamma$  defined by  $\Gamma$ .

## 2 Preliminaries

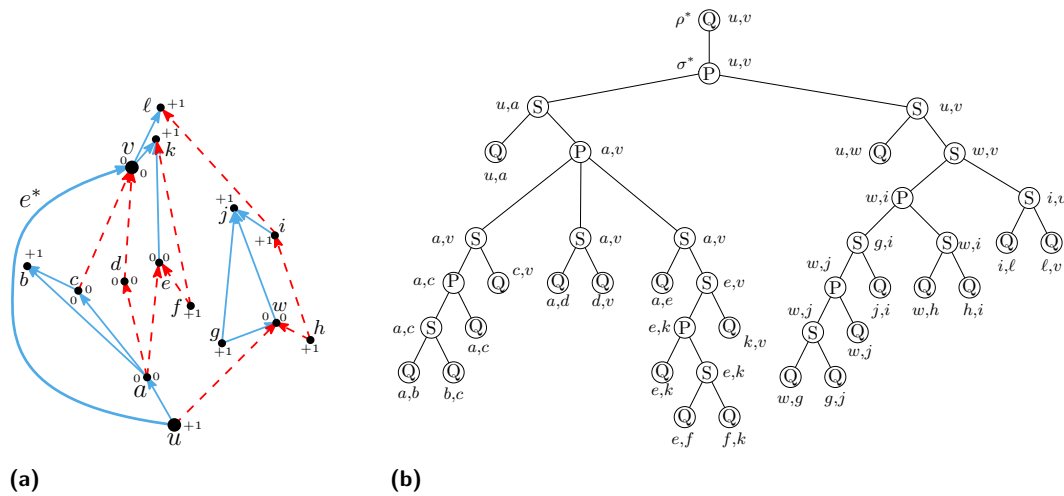
For basic definitions on graphs and their drawings see [22, 27] or the full version of the paper.

Two planar drawings of a connected digraph are *topologically equivalent* if they have the same clockwise order of the edges around each vertex and the same clockwise order of the edges along the boundary of the outer face. A *planar embedding* is a class of topologically equivalent planar drawings. A *plane digraph* is a planar digraph together with a planar embedding. A planar embedding of a digraph is *bimodal* if, for every vertex  $v$ , the edges that have their tail at  $v$  are consecutive in the clockwise order of the edges incident to  $v$ .

**Upward planarity.** A *switch* in a digraph  $G$  is a source or a sink. A drawing of  $G$  is *upward* if every edge is represented by a Jordan arc that is strictly increasing in the  $y$ -direction from its tail to its head. If  $G$  admits an upward planar drawing, it is an *upward planar digraph*. Let  $\mathcal{E}$  be a bimodal planar embedding of  $G$ . An *angle* at a vertex  $v$  is an ordered pair  $(e_1, e_2)$  of edges incident to  $v$  where  $e_2$  immediately follows  $e_1$  in clockwise order around  $v$ . An angle  $(e_1, e_2)$  is a *switch angle* if  $v$  is the head or the tail of both  $e_1$  and  $e_2$ , otherwise it is a *flat angle*. The set of angles of  $\mathcal{E}$  is denoted as  $\mathcal{A}_\mathcal{E}$ . An *angle assignment for  $\mathcal{E}$*  is a function  $\lambda: \mathcal{A}_\mathcal{E} \rightarrow \{-1, 0, 1\}$ , and it is *upward-consistent* if it satisfies the following conditions: (C1) For each angle  $a$  in  $\mathcal{A}_\mathcal{E}$ , we have  $\lambda(a) = 0$  if and only if  $a$  is flat. (C2) For each vertex  $v$  of  $G$ , we have  $\sum_a \lambda(a) = 2 - \deg(v)$ , where the sum is over the angles at  $v$ . (C3) For each face  $f$  of  $\mathcal{E}$ , we have  $\sum_a \lambda(a) = -2$  if  $f$  is an internal face and  $\sum_a \lambda(a) = 2$  if  $f$  is the outer face, where the sum is over the angles in  $f$ .

Let  $\Gamma$  be an upward planar drawing with planar embedding  $\mathcal{E}$ , see Fig. 2. Clearly,  $\mathcal{E}$  is bimodal. Also, it defines an angle assignment  $\lambda_\Gamma$  as follows. Let  $a = (e_1, e_2)$  be an angle of  $\mathcal{E}$ . If  $a$  is flat, then  $\lambda_\Gamma(a) = 0$ . Otherwise, consider the geometric angle  $\alpha$  in  $\Gamma$  corresponding to  $a$ . We define  $\lambda_\Gamma(a) = -1$  if  $\alpha < \pi$  and  $\lambda_\Gamma(a) = 1$  if  $\alpha > \pi$ . Note that  $\lambda_\Gamma$  is upward-consistent.

A pair  $(\mathcal{E}, \lambda)$  of a planar embedding  $\mathcal{E}$  and an angle assignment  $\lambda$  is an *upward embedding* if there exists an upward planar drawing  $\Gamma$  with embedding  $\mathcal{E}$  such that  $\lambda = \lambda_\Gamma$ . Angles  $a$  with  $\lambda(a) = 0$  are flat; also, we say that an angle  $a$  is *large* if  $\lambda(a) = 1$  and *small* if  $\lambda(a) = -1$ . An angle assignment is determined by *assigning* each switch to an incident angle, which corresponds to making that angle large; a switch angle to which no switch is assigned is small. A digraph equipped with an upward embedding is an *upward plane digraph*.



**Figure 3** (a) An upward planar drawing  $\Gamma$  of a biconnected partitioned directed partial 2-tree  $G$ . The labeling  $\lambda_\Gamma$  of the angles determined by  $\Gamma$  is shown; the missing labels are equal to  $-1$ . (b) The SPQ-tree of  $G$  rooted at the Q-node corresponding to the edge  $e^*$  of  $G$ .

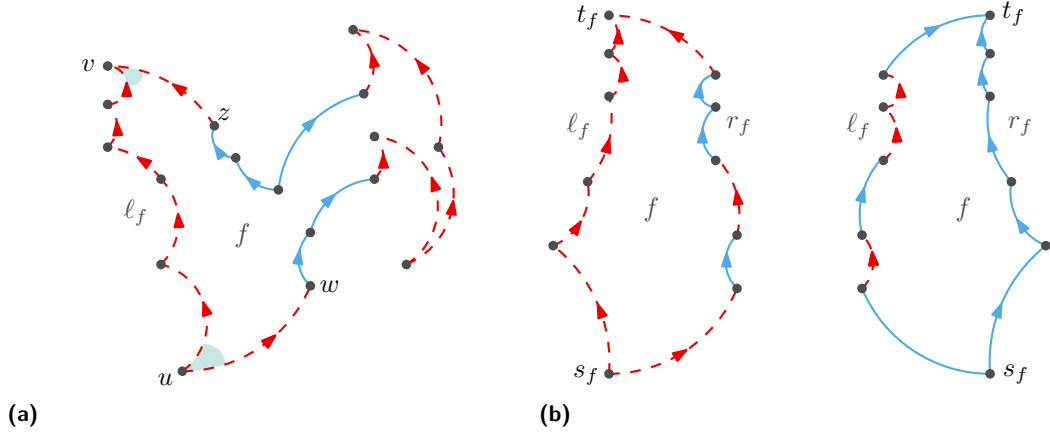
► **Theorem 1** ([11, 26]). *Let  $\mathcal{E}$  be a planar embedding and  $\lambda$  be an angle assignment for  $\mathcal{E}$ . The pair  $(\mathcal{E}, \lambda)$  is an upward embedding if and only if  $\mathcal{E}$  is bimodal and  $\lambda$  is upward-consistent.*

**Upward Book Embeddings.** An  $n$ -vertex **partitioned digraph** is a digraph  $G = (V, \bigcup_{i=1}^k E_i)$  whose edge set is partitioned into  $k$  sets. An **upward book embedding** (in  $k$  pages) of  $G$  is a bijection  $\pi: V \leftrightarrow \{1, \dots, n\}$  such that: (i) for each edge  $e = (u, v)$ , it holds that  $\pi(u) < \pi(v)$ , and (ii) for any  $i \in \{1, \dots, k\}$ , no two edges  $(u, v), (w, x) \in E_i$  cross, where  $(u, v)$  and  $(w, x)$  **cross** if  $\pi(u) < \pi(w) < \pi(v) < \pi(x)$  or  $\pi(w) < \pi(u) < \pi(x) < \pi(v)$ . We focus on the case  $k = 2$  and let  $L := E_1$  and  $R := E_2$ . We often omit that our upward book embeddings are in two pages and call **left** and **right** the edges in  $L$  and  $R$ , respectively, since an upward book embedding  $(\pi, \sigma)$  of  $G = (V, L \cup R)$  determines an upward planar drawing of  $G$  where the vertices lie along a vertical line, called **spine**, so that the  $y$ -coordinate of each vertex  $v$  is  $\pi(v)$ , and each edge  $(u, v)$  in  $L$  (resp. in  $R$ ) is drawn as a semi-circle to the left (resp. to the right) of the spine. In all the illustrations, the edges in  $L$  are blue and solid, while the edges in  $R$  are red and dashed.

An upward planar drawing **respects** a planar embedding  $\mathcal{E}$  if it belongs to the equivalence class  $\mathcal{E}$ . An upward book embedding **respects** a planar embedding  $\mathcal{E}$  if the associated upward planar drawing  $\Gamma$  does and **respects** an upward embedding  $(\mathcal{E}, \lambda)$  if it respects  $\mathcal{E}$  and  $\lambda_\Gamma = \lambda$ .

In an upward embedding  $(\mathcal{E}, \lambda)$ , a vertex  $v$  is **4-modal** if: (i) If  $v$  is not a switch, then in clockwise order around  $v$  in  $\mathcal{E}$  we have all outgoing left edges, all outgoing right edges, all incoming right edges, and all incoming left edges; one of the former two sets and/or one of the latter two sets might be empty. (ii) If  $v$  is a source (resp. a sink), then in clockwise order around  $v$  in  $(\mathcal{E}, \lambda)$  we have the large angle at  $v$ , all outgoing left edges, and all outgoing right edges (resp. the large angle at  $v$ , all incoming right edges, and all incoming left edges); one of the two sets might be empty. Also,  $(\mathcal{E}, \lambda)$  is **4-modal** if all vertices are 4-modal.

► **Property 2.** *Let  $\Gamma$  be an upward book embedding of a partitioned digraph  $G = (V, L \cup R)$  and let  $(\mathcal{E}, \lambda)$  be the upward embedding of  $G$  defined by  $\Gamma$ . Then  $(\mathcal{E}, \lambda)$  is 4-modal.*



■ **Figure 4** Impossible faces in (a) an upward plane digraph and (b) a plane  $st$ -graph.

**Partial 2-trees.** A graph is an (undirected) *partial 2-tree* if, equivalently, it has treewidth at most two, or it excludes  $K_4$  as a minor, or it is a subgraph of a 2-tree, which is a graph that can be obtained from an edge by repeatedly inserting a degree-2 vertex adjacent to two adjacent vertices. Notably, the class of partial 2-trees includes the *series-parallel graphs*.

Let  $G$  be a biconnected partial 2-tree and let  $e^* = (u^*, v^*)$  be an edge of  $G$ , see Fig. 3. The *SPQ-tree*  $T$  of  $G$  with respect to  $e^*$  is a rooted tree that describes a recursive decomposition of  $G$  into smaller partial 2-trees; it is a specialization of the well-known *SPQR-tree*, which is defined for general biconnected planar graphs [24, 34], and can be computed in linear time.

The root of  $T$  is a Q-node  $\rho^*$  associated with  $G$  and with a single child  $\sigma^*$ . Define  $G - e^*$  as the *pertinent graph* of  $\sigma^*$ , and let  $u^*$  and  $v^*$  be the *poles* of  $\sigma^*$ . We proceed recursively. Suppose we are given a quadruple  $\langle \mu, u, v, G_\mu \rangle$ , where  $\mu$  is a node of  $T$  with poles  $u$  and  $v$ , and  $G_\mu$  is its pertinent graph (initially, this is  $\langle \sigma^*, u^*, v^*, G - e^* \rangle$ ). Three cases can occur:

- If  $G_\mu$  is a single edge  $(u, v)$ , then  $\mu$  is a Q-node representing that edge;  $\mu$  is a leaf of  $T$ .
- If  $G_\mu$  is not biconnected, then  $\mu$  is an S-node. Let  $w$  be an arbitrary cut-vertex of  $G_\mu$ . Then  $\mu$  has two children  $\nu_1$  and  $\nu_2$  in  $T$  corresponding to the two subgraphs  $G_{\nu_1}$  and  $G_{\nu_2}$ , respectively, obtained by splitting  $G_\mu$  at  $w$ . The poles of  $\nu_1$  are  $u$  and  $w$ , and those of  $\nu_2$  are  $w$  and  $v$ . The construction of  $T$  recurses on  $\langle \nu_1, u, w, G_{\nu_1} \rangle$  and on  $\langle \nu_2, w, v, G_{\nu_2} \rangle$ .
- If  $G_\mu$  is biconnected, then  $\mu$  is a P-node. Then  $\mu$  has  $k$  children  $\nu_1, \dots, \nu_k$ , where  $k$  is the number of subgraphs  $G_{\nu_1}, \dots, G_{\nu_k}$  obtained by splitting  $G_\mu$  at  $u$  and  $v$ . For  $i = 1, \dots, k$ , the poles of  $\nu_i$  are  $u$  and  $v$ , and the construction of  $T$  recurses on  $\langle \nu_i, u, v, G_{\nu_i} \rangle$ .

A *directed partial 2-tree* is a digraph whose underlying graph is a partial 2-tree, where the underlying graph is the undirected graph obtained by ignoring the edge directions. An SPQ-tree of a biconnected directed partial 2-tree  $G$  is an SPQ-tree of its underlying graph, although the edges of the pertinent graph of each node are oriented as in  $G$ .

### 3 Characterization for Upward Embeddings

In this section we characterize the upward embeddings  $(\mathcal{E}, \lambda)$  of a partitioned upward plane digraph  $G = (V, L \cup R)$  that allow for the construction of an upward book embedding. For a face  $f$  of  $\mathcal{E}$ , let  $\mathcal{L}_f$  (resp.  $\mathcal{R}_f$ ) be the set of maximal directed paths in the boundary of  $f$  that consist of edges with  $f$  to their right (resp. to their left). The face  $f$  is *impossible* if it satisfies one of the following two conditions (see Fig. 4a):



- (i)  $\mathcal{L}_f$  contains a path  $\ell_f$  with the following properties: a)  $\ell_f$  consists of edges in  $R$ ; b) the rest of the boundary of  $f$  is not a single edge; and c) let  $e_f^u$  and  $e_f^v$  be the edges of  $\ell_f$  incident to the extremes  $u$  and  $v$  of  $\ell_f$ , respectively; then the angle in  $f$  incident to  $u$  and to the right of  $e_f^u$  and the angle in  $f$  incident to  $v$  and to the right of  $e_f^v$  are small.
- (ii)  $\mathcal{R}_f$  contains a path  $r_f$  with the following properties: a)  $r_f$  consists of edges in  $L$ ; b) the rest of the boundary of  $f$  is not a single edge; and c) let  $e_f^u$  and  $e_f^v$  be the edges of  $r_f$  incident to the extremes  $u$  and  $v$  of  $r_f$ , respectively; then the angle in  $f$  incident to  $u$  and to the left of  $e_f^u$  and the angle in  $f$  incident to  $v$  and to the left of  $e_f^v$  are small.

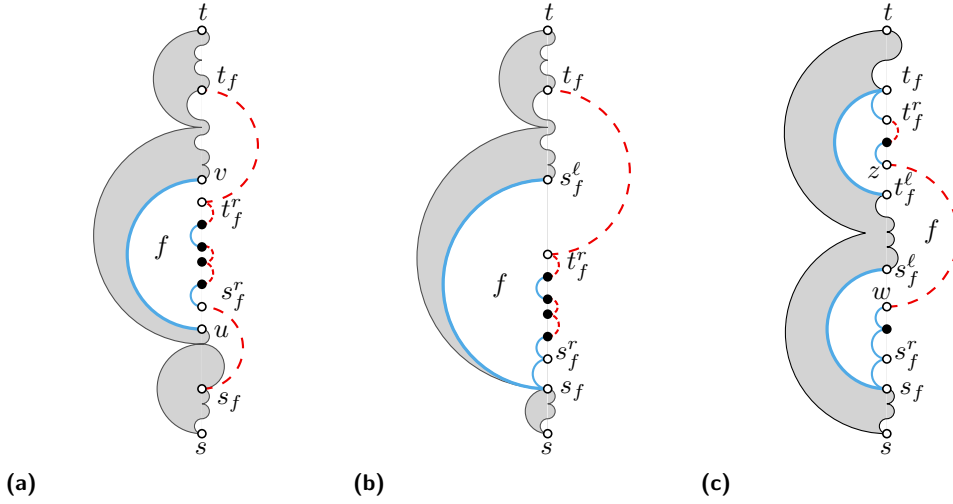
A **good embedding** is an upward embedding  $(\mathcal{E}, \lambda)$  that is 4-modal and that is such that no face of  $\mathcal{E}$  is impossible. We now prove our characterization.

► **Theorem 3.** *Let  $G$  be a partitioned upward plane digraph with upward embedding  $(\mathcal{E}, \lambda)$ . Then  $G$  admits an upward book embedding respecting  $(\mathcal{E}, \lambda)$  if and only if  $(\mathcal{E}, \lambda)$  is good.*

**Sketch.** We sketch the proof for the case where  $G$  is connected; the general case is discussed in the full version. For the necessity, let  $\Gamma$  be an upward book embedding of  $G$  respecting  $(\mathcal{E}, \lambda)$ . Then  $\mathcal{E}$  is 4-modal by Property 2. Suppose, for a contradiction, that a face  $f$  of  $\mathcal{E}$  is impossible. W.l.o.g., assume that  $\mathcal{L}_f$  contains a path  $\ell_f$  satisfying condition (i) and let  $u, v, e_f^u, e_f^v$  be defined as above. Let further  $w$  and  $z$  be the vertices adjacent to  $u$  and  $v$ , respectively, such that the edge  $(u, w)$  follows  $e_f^u$  in clockwise order around  $u$  and the edge  $(z, v)$  follows  $e_f^v$  in counter-clockwise order around  $v$ . Since  $\Gamma$  is upward planar, the order of the vertices of  $\ell_f$  along the spine is the same as their order in  $\ell_f$ , with  $u$  below  $v$ ; also,  $w$  lies above  $u$  and  $z$  below  $v$ . If  $w$  or  $z$  lies in the strip  $S_{uv}$  delimited by the horizontal lines through  $u$  and  $v$ , then  $(u, w)$  or  $(z, v)$  would cross an edge of the path  $\ell_f$ , because the small angles in  $(\mathcal{E}, \lambda)$  force the edges  $(u, w)$  and  $(v, z)$  to be to the right of  $e_f^u$  and  $e_f^v$ , respectively. However, no point of the spine lies to the right of  $\ell_f$  and in the strip  $S_{uv}$ . It follows that  $w$  has to lie above  $v$  and  $z$  below  $u$ . This, however, implies that the edges  $(u, w)$  and  $(z, v)$  cross each other, given that they are both drawn to the right of the spine. This proves the necessity.

For the sufficiency, we build on a procedure by Bertolazzi et al. [12], who showed that any upward plane digraph can be augmented to a plane  $st$ -graph by adding edges, while maintaining the upward embedding. A **plane  $st$ -graph**  $G$  is an upward plane digraph with one source  $s$  and one sink  $t$  that are incident to the outer face. The boundary of each face  $f$  of  $G$  consists of two directed paths  $\ell_f$  and  $r_f$ , called **left** and **right path**, respectively. In the full version, we extend the technique of Bertolazzi et al. and show that, by suitably adding paths and carefully assigning their edges to  $L$  and  $R$ , we can augment a good embedding of the given partitioned upward plane digraph  $G$  to a good embedding of a partitioned plane  $st$ -graph. For this reason, we now assume that  $G$  is a partitioned plane  $st$ -graph. In this case, a face  $f$  is impossible if  $\ell_f$  only consists of edges in  $R$  and  $r_f$  is not a single edge, or if  $r_f$  only consists of edges in  $L$  and  $\ell_f$  is not a single edge; see Fig. 4b.

To show that a partitioned plane  $st$ -graph  $G$  with a good embedding  $(\mathcal{E}, \lambda)$  admits an upward book embedding  $\Gamma$ , we use the fact that it can be constructed starting from the left path  $\ell_{f_O}$  of the outer face  $f_O$  by repeatedly adding the right path of an internal face whose left path already belongs to the graph, see, e.g., [2, 23, 30, 44]. We start by drawing in  $\Gamma$  the edges of  $\ell_{f_O}$  as semi-circles in their assigned side of the spine. When we draw the right path  $r_f$  of a face  $f$  whose left path  $\ell_f$  already belongs to the subgraph of  $G$  drawn in  $\Gamma$ , we distinguish two cases. First, if  $r_f$  is a single edge  $e$ , then, since  $G$  is simple,  $\ell_f$  contains an internal vertex. Since  $f$  is not impossible,  $e \in R$ . Then we just draw  $e$  as a semi-circle to



■ **Figure 5** Illustrations for the proof sketch of Theorem 3.

the right of the spine. Second, if  $r_f$  is not a single edge, as in Fig. 5, then  $\ell_f$  contains a left edge  $(u, v)$ , since  $f$  is not impossible. If both the first and the last edge of  $r_f$  are in  $R$ , we place all the internal vertices of  $r_f$  on the spine between  $u$  and  $v$  in the order in which they appear along  $r_f$  and draw the edges of  $r_f$  as semi-circles in their assigned side of the spine; see Fig. 5a. Otherwise, if the first edge of  $r_f$  is in  $L$  and its last edge is in  $R$ , then also the first edge of  $\ell_f$  is in  $L$  due to the 4-modality of  $(\mathcal{E}, \lambda)$ . We can then choose  $(u, v)$  as the first edge of  $\ell_f$  and proceed as discussed in the previous case; see Fig. 5b. The case where the first edge of  $r_f$  is in  $R$  and its last edge is in  $L$  is analogous. Finally, if both the first and the last edge of  $r_f$  are in  $L$ , then so are the first edge  $(s_f, s_f^\ell)$  and the last edge  $(t_f^\ell, t_f)$  of  $\ell_f$ , by the 4-modality of  $(\mathcal{E}, \lambda)$ . If  $\ell_f$  is a single edge, then it plays the role of  $(u, v)$  and we place the internal vertices of  $r_f$  and draw its edges as above. Otherwise, since  $f$  is not impossible,  $r_f$  contains an edge  $(w, z) \in R$ . We embed the internal vertices of the subpath of  $r_f$  from its start to  $w$ , including  $w$ , between  $s_f$  and  $s_f^\ell$  in the order as they appear along  $r_f$ , and we embed the internal vertices of the subpath of  $r_f$  from  $z$  to its end, including  $z$ , between  $t_f^\ell$  and  $t_f$  in the order as they appear along  $r_f$ , and draw the edges of  $r_f$  as semi-circles in their assigned side of the spine; see Fig. 5c. ◀

#### 4 Computational Complexity with Variable Embedding

In this section we study the complexity of testing whether a partitioned digraph  $G = (V, L \cup R)$  admits an upward book embedding in two pages. We show that the problem is NP-complete by exploiting Theorem 3 and the NP-hardness of UPWARD PLANARITY TESTING.

► **Theorem 4.** *It is NP-complete to decide whether a partitioned planar digraph admits an upward book embedding.*

**Proof.** Clearly, the problem is in NP. In order to prove NP-hardness, we give a reduction from UPWARD PLANARITY TESTING, which was proved to be NP-hard by Garg and Tamassia [32]. Given a planar digraph  $G = (V, E)$ , we construct a partitioned digraph  $G' = (V', L \cup R)$  as follows. Subdivide each edge  $e \in E$  with a new vertex  $v_e$ ; these subdivision vertices, together with the vertices in  $V$ , form  $V'$ . The edge set  $L$  consists of the edges outgoing from vertices



in  $V$  (and incoming into vertices in  $V' \setminus V$ ), and the edge set  $R$  consists of the remaining edges. Clearly,  $G'$  can be constructed from  $G$  in polynomial time. It remains to show that  $G'$  admits an upward book embedding if and only if  $G$  admits an upward planar drawing.

For the necessity, observe that an upward book embedding  $\Gamma'$  of  $G'$  is an upward planar drawing of  $G'$ . Then an upward planar drawing  $\Gamma$  of  $G$  is obtained from  $\Gamma'$  by (i) placing each vertex of  $G$  in  $\Gamma$  as in  $\Gamma'$  and by (ii) drawing in  $\Gamma$  each edge  $e = (u, w)$  of  $G$  as the Jordan arc formed by the union of the drawings of  $(u, v_e)$  and of  $(v_e, w)$  in  $\Gamma'$ .

For the sufficiency, suppose that  $G$  admits an upward planar drawing  $\Gamma$ . An upward planar drawing  $\Gamma'$  of  $G'$  can be constructed from  $\Gamma$  by placing each subdivision vertex  $v_e$  at any internal point of the Jordan arc representing the edge  $e$ . Let  $(\mathcal{E}', \lambda')$  be the upward embedding corresponding to  $\Gamma'$ . We prove that  $(\mathcal{E}', \lambda')$  is 4-modal. Consider any vertex  $v$  of  $G'$ . If  $v \notin V$ , then  $v$  has one incoming and one outgoing edge in  $G'$ , hence it is trivially 4-modal. If  $v \in V$  then, by construction, all the edges outgoing from  $v$  (incoming into  $v$ ), if any, are in  $L$  (resp. are in  $R$ ) and are consecutive in the clockwise order of the edges incident to  $v$ , due to the bimodality of  $\Gamma$ . It follows that  $v$  is 4-modal. Finally,  $(\mathcal{E}', \lambda')$  has no impossible face, since by construction any maximal directed path in the boundary of any face contains both an edge in  $L$  and an edge in  $R$ . Hence,  $(\mathcal{E}', \lambda')$  is a good embedding. By Theorem 3, we have that  $G'$  admits an upward book embedding respecting  $(\mathcal{E}', \lambda')$ . ◀

UPWARD PLANARITY TESTING is known to be  $W[1]$ -hard with respect to the treewidth [41]. Also, the reduction shown in Theorem 4 constructs a graph which is a subdivision of the original instance of UPWARD PLANARITY TESTING. Since any two graphs, one of which is a subdivision of the other one, have the same treewidth, we get the following.

► **Corollary 5.** *It is  $W[1]$ -hard with respect to the treewidth to decide whether a partitioned digraph admits an upward book embedding.*

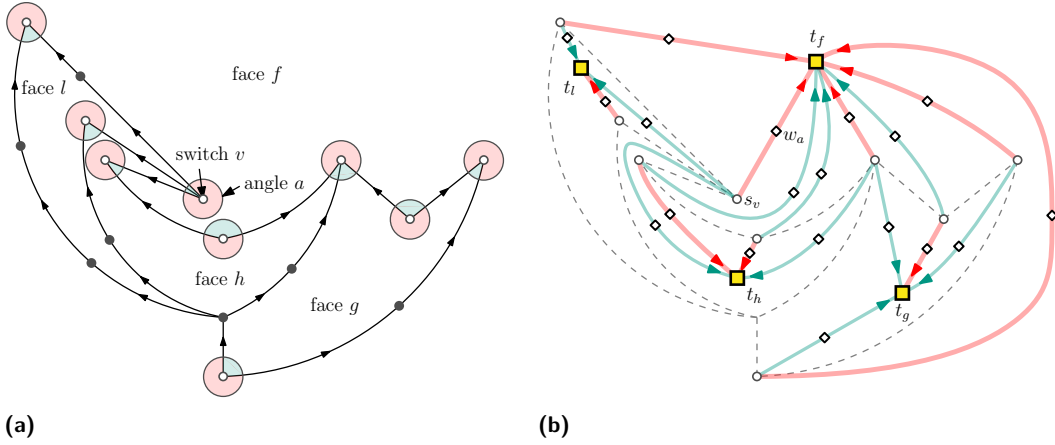
Furthermore, by subdividing the edges twice, rather than once, so that the edges incident to vertices in  $V$  are in  $L$  and the other edges in  $R$ , the reduction gives an instance where one edge part induces a matching and the other one a forest of stars. This is in sharp contrast with the fact that the problem is linear-time solvable when both edge parts are matchings [1].

## 5 Test for Graphs with a Fixed Planar Embedding

In this section we show how to exploit the characterization of Theorem 3 in order to prove that, for an  $n$ -vertex partitioned plane digraph  $G$  with a given planar embedding  $\mathcal{E}$ , it can be tested in  $O(n \log^3 n)$  time whether  $G$  admits an upward book embedding respecting  $\mathcal{E}$ . We assume that  $G$  is connected and refer to the full version for the case in which it is not.

We review a tool for testing whether  $G$  admits an upward planar drawing  $\Gamma$  respecting  $\mathcal{E}$ . By Theorem 1, the bimodality of  $\mathcal{E}$  is necessary for the existence of  $\Gamma$  and can be tested in  $O(n)$  time. Thus, the existence of  $\Gamma$  corresponds to the existence of an upward-consistent angle assignment  $\lambda$  for  $\mathcal{E}$ , which can be decided by the following strategy of Bertolazzi et al. [11]. Starting from the plane digraph  $G$  with planar embedding  $\mathcal{E}$ , one can construct a planar flow network  $\mathcal{N}$ , as illustrated in Fig. 6, where:

- for each switch  $v$  of  $G$ , the network  $\mathcal{N}$  contains a source  $s_v$  supplying a single unit of flow;
- for each face  $f$  of  $\mathcal{E}$ , the network  $\mathcal{N}$  contains a sink  $t_f$  that demands a number of units of flow equal to  $n_f/2 - 1$  or  $n_f/2 + 1$ , depending on whether  $f$  is an internal face or the outer face of  $\mathcal{E}$ , respectively (where  $n_f$  is the number of switch angles incident to  $f$ );
- for each angle  $\alpha$  in  $\mathcal{E}$  at a switch of  $G$ , the network  $\mathcal{N}$  contains a node  $w_\alpha$ ; and



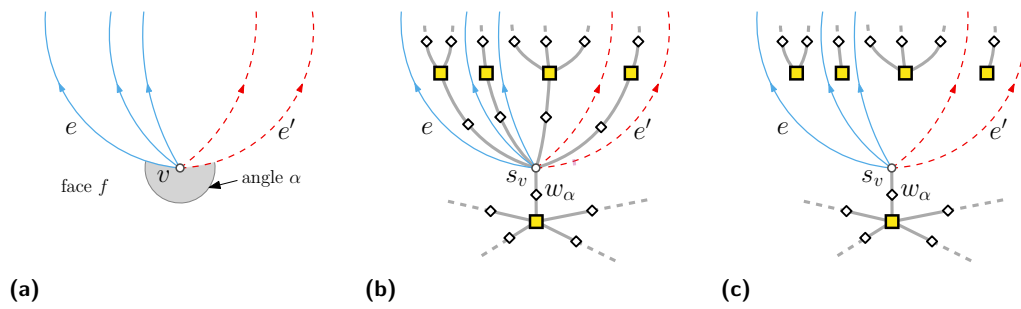
■ **Figure 6** (a) An upward planar drawing of a digraph  $G$ ; large and small angles at switches of  $G$  are depicted as red and green sectors of discs, respectively. (b) The network  $\mathcal{N}$  constructed from the planar embedding of  $G$ ; the edges of  $G$ , which are not part of  $\mathcal{N}$ , are drawn as gray dashed curves. Arcs of  $\mathcal{N}$  traversed by the flow are red and thick, whereas arcs with no flow are green and thin.

- the network  $\mathcal{N}$  contains an arc from each source  $s_v$  to each node  $w_\alpha$  such that the angle  $\alpha$  is incident to  $v$  in  $\mathcal{E}$  and an arc from each node  $w_\alpha$  to each sink  $t_f$  such that the angle  $\alpha$  is incident to  $f$  in  $\mathcal{E}$ ; all such arcs have a capacity of a single unit of flow.

By [11], there exists an upward-consistent angle assignment  $\lambda$  for  $\mathcal{E}$  if and only if  $\mathcal{N}$  admits a feasible flow whose value is the sum  $d_T$  of the demands of the sinks in  $\mathcal{N}$ . Indeed, upward-consistent angle assignments and (integral) feasible flows with value  $d_T$  are in bijection. Namely, from an upward-consistent angle assignment  $\lambda$ , one gets a feasible flow for  $\mathcal{N}$  with value  $d_T$  by assigning flow 1 only to arcs incident to a node  $w_\alpha$  where  $\alpha$  is assigned a large angle by  $\lambda$ , and from a feasible flow for  $\mathcal{N}$  with value  $d_T$ , one gets an upward-consistent angle assignment  $\lambda$  by assigning a large angle to each switch angle  $\alpha$  such that the arcs incident to node  $w_\alpha$  are assigned one unit of flow, and a small angle to all other switch angles. Testing whether  $G$  admits an upward planar drawing respecting  $\mathcal{E}$  then becomes equivalent to testing whether  $\mathcal{N}$  admits a flow whose value is  $d_T$ , which can be done in  $O(n \log^3 n)$  time [17].

Our idea is to modify  $\mathcal{N}$  so that there is an upward-consistent angle assignment  $\lambda$  for  $\mathcal{E}$  such that  $(\mathcal{E}, \lambda)$  is a good embedding if and only if  $\mathcal{N}$  has a feasible flow whose value is  $d_T$ . The correspondence is actually stronger: As we show in the full version, the integral feasible flows with value  $d_T$  for the modified network are in bijection with the upward-consistent angle assignments  $\lambda$  for  $\mathcal{E}$  such that  $(\mathcal{E}, \lambda)$  is a good embedding. We show an algorithm, called  $\mathcal{N}$ -MODIFIER, that performs a sequence of modifications to  $\mathcal{N}$ . Along the way,  $\mathcal{N}$ -MODIFIER might stop and conclude that  $G$  has no upward book embedding respecting  $\mathcal{E}$ .

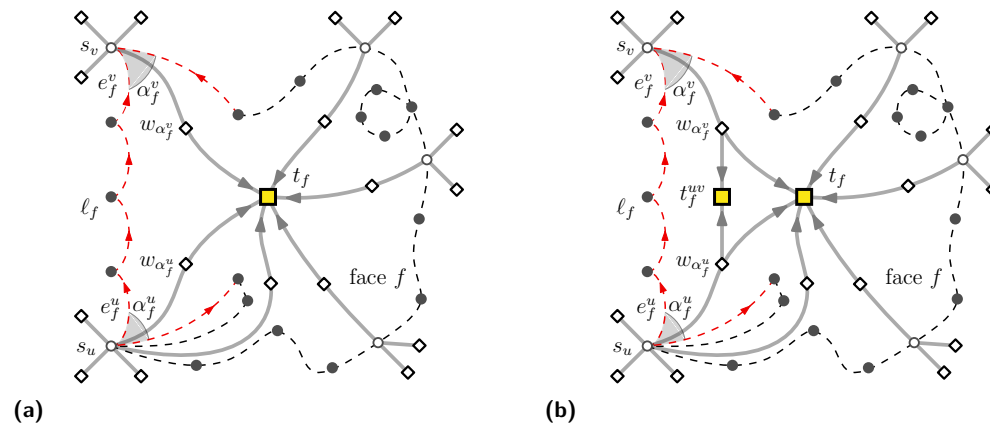
**THE  $\mathcal{N}$ -MODIFIER ALGORITHM.** We start by describing the modifications to  $\mathcal{N}$  to ensure 4-modality. Consider any vertex  $v$  of  $G$ . If  $v$  is not a switch, then its 4-modality does not depend on the angle assignment, thus we check whether in  $\mathcal{E}$ , in clockwise order around  $v$ , we have outgoing left edges, outgoing right edges, incoming right edges, and incoming left edges. If the test is negative, then  $G$  admits no upward book embedding respecting  $\mathcal{E}$ . If  $v$  is a switch, say a source, then we check whether in  $\mathcal{E}$  the outgoing left edges are consecutive. If the test is negative, then  $G$  admits no upward book embedding respecting  $\mathcal{E}$ . If the test is positive and  $v$  has no outgoing left edges or no outgoing right edges, the processing of  $v$  is concluded. If the test is positive and  $v$  has both outgoing left and outgoing right edges, we modify  $\mathcal{N}$  as follows; see Fig. 7. Let  $e$  be the left edge outgoing from  $v$  such that the edge  $e'$



■ **Figure 7** Modifying  $\mathcal{N}$  to ensure 4-modality. (a) A vertex  $v$  with both outgoing left edges and outgoing right edges. (b) The part of  $\mathcal{N}$  close to  $v$  (the edges of  $G$  are not in  $\mathcal{N}$ , but they are shown to maintain a visual reference with (a)). (c) Removal of the neighbors of  $s_v$  different from  $w_\alpha$ .

preceding  $e$  in clockwise order around  $v$  is a right edge, let  $\alpha$  be the angle  $(e, e')$ . For each angle  $\beta \neq \alpha$  incident to  $v$  in  $\mathcal{E}$ , the algorithm  $\mathcal{N}$ -MODIFIER removes  $w_\beta$  and its incident arcs from  $\mathcal{N}$ . This forces the arc  $(s_v, w_\alpha)$  to be assigned one unit of flow, hence making  $\alpha$  large.

We next describe the modifications  $\mathcal{N}$ -MODIFIER applies to  $\mathcal{N}$  in order to ensure the absence of impossible faces. Consider any face  $f$  of  $\mathcal{E}$  and any maximal directed path  $\ell_f$  in the boundary of  $f$  with  $f$  to its right (the treatment of the maximal directed paths with  $f$  to their left is analogous). If  $\ell_f$  contains a left edge or if the rest of the boundary of  $f$  is a single edge, then the processing of  $\ell_f$  is concluded. Otherwise, let  $u$  and  $v$  be the extremes of  $\ell_f$ , let  $e_f^u$  and  $e_f^v$  be the edges of  $\ell_f$  incident to  $u$  and  $v$ , let  $\alpha_f^u$  be the angle in  $f$  incident to  $u$  and to the right of  $e_f^u$ , and let  $\alpha_f^v$  be the angle in  $f$  incident to  $v$  and to the right of  $e_f^v$ . By Theorem 3, at least one of the angles  $\alpha_f^u$  and  $\alpha_f^v$  has to be large. By the maximality of  $\ell_f$ , both  $\alpha_f^u$  and  $\alpha_f^v$  are switch angles. We say that  $\alpha_f^u$  (resp.  $\alpha_f^v$ ) is **enlargeable** if the node  $w_{\alpha_f^u}$  (resp. the node  $w_{\alpha_f^v}$ ) is in  $\mathcal{N}$ . Note that, even if  $u$  (resp.  $v$ ) is a switch of  $G$ ,  $w_{\alpha_f^u}$  (resp.  $w_{\alpha_f^v}$ ) might not be in  $\mathcal{N}$ , because of some previous modification of  $\mathcal{N}$ . We distinguish three cases.



■ **Figure 8** Modifying  $\mathcal{N}$  to avoid impossible faces determined by  $\ell_f$ , when both  $\alpha_f^u$  and  $\alpha_f^v$  are enlargeable. The part of  $\mathcal{N}$  associated with  $f$  before (a) and after (b) the modification.

- If neither  $\alpha_f^u$  nor  $\alpha_f^v$  is enlargeable, then  $G$  admits no upward book embedding respecting  $\mathcal{E}$ .
- If just one of  $\alpha_f^u$  and  $\alpha_f^v$ , say  $\alpha_f^u$ , is enlargeable, we modify  $\mathcal{N}$  as follows: For each angle  $\beta \neq \alpha_f^u$  incident to  $u$  in  $\mathcal{E}$ , we remove  $w_\beta$  and its incident arcs from  $\mathcal{N}$ , thus forcing the arc  $(s_u, w_{\alpha_f^u})$  to be assigned one unit of flow and making  $\alpha_f^u$  large.

- Finally, suppose that both  $\alpha_f^u$  and  $\alpha_f^v$  are enlargeable, as in Fig. 8. If there exists no angle  $\beta \neq \alpha_f^u$  incident to  $u$  whose corresponding node  $w_\beta$  is in  $\mathcal{N}$ , or no angle  $\beta \neq \alpha_f^v$  incident to  $v$  whose corresponding node  $w_\beta$  is in  $\mathcal{N}$ , then the processing of  $\ell_f$  is concluded, since it is already guaranteed that one of  $\alpha_f^u$  and  $\alpha_f^v$  will be a large angle. Otherwise, we add to  $\mathcal{N}$  a sink  $t_f^{uv}$  with demand 1, as well as arcs  $(w_{\alpha_f^u}, t_f^{uv})$  and  $(w_{\alpha_f^v}, t_f^{uv})$  with capacity 1, and decrease by 1 the demand of  $t_f$ . This forces one of  $(w_{\alpha_f^u}, t_f^{uv})$  and  $(w_{\alpha_f^v}, t_f^{uv})$  (and consequently one of  $(s_u, w_{\alpha_f^u})$  and  $(s_v, w_{\alpha_f^v})$ ) to be assigned one unit of flow, hence making large  $\alpha_f^u$  or  $\alpha_f^v$ , respectively. Note that  $\mathcal{N}$  remains a planar flow network.

The described algorithm is the main ingredient for the proof of the following theorem.

► **Theorem 6.** *Let  $G$  be an  $n$ -vertex partitioned plane digraph with planar embedding  $\mathcal{E}$ . It is possible to test in  $O(n \log^3 n)$  time whether  $G$  admits an upward book embedding respecting  $\mathcal{E}$ .*

**Sketch.** In the full version, we prove the mentioned bijection between the feasible flows with value  $d_T$  of the network  $\mathcal{N}$  constructed by  $\mathcal{N}$ -MODIFIER and the upward-consistent angle assignments  $\lambda$  such that  $(\mathcal{E}, \lambda)$  is a good embedding. Then Theorem 3 implies that  $G$  has an upward book embedding respecting  $\mathcal{E}$  if and only if  $\mathcal{N}$ -MODIFIER did not conclude the opposite and  $\mathcal{N}$  has a feasible flow with value  $d_T$ . Since  $\mathcal{N}$ -MODIFIER can be easily implemented to run in  $O(n)$  time and the algorithm by Borradaile et al. [17] to test whether  $\mathcal{N}$  has a feasible flow with value  $d_T$  runs in  $O(n \log^3 n)$  time, the theorem follows. ◀

## 6 Test for Biconnected Partitioned Directed Partial 2-Trees

In this section we show how to test whether an  $n$ -vertex biconnected partitioned directed partial 2-tree  $G = (V, L \cup R)$  admits an upward book embedding in two pages. We leverage tools introduced in [18, 19] to efficiently test whether a directed partial 2-tree is upward planar. Note that, lacking a characterization such as the one in Theorem 3, it would be prohibitive to lift such tools to work for our problem. Let  $e^*$  be an edge of  $G$ . We test in  $O(n^2)$  time whether  $G$  admits a good embedding with  $e^*$  on the outer face. Repeating this test for all  $O(n)$  choices of  $e^*$  yields an  $O(n^3)$ -time algorithm to decide whether  $G$  admits a good embedding. By Theorem 3, this is equivalent to testing whether  $G$  admits an upward book embedding.

Let  $T$  be an SPQ-tree of  $G$ , rooted at the Q-node  $\rho^*$  corresponding to  $e^*$ . Let  $\mu$  be a node of  $T$  with poles  $u$  and  $v$  and with pertinent graph  $G_\mu$ . A *uv-external good embedding* of  $G_\mu$  is a good embedding of  $G_\mu$  in which  $u$  and  $v$  are incident to the outer face. The requirement that  $e^*$  is incident to the outer face of any upward embedding  $(\mathcal{E}, \lambda)$  of  $G$  implies that the restriction of  $(\mathcal{E}, \lambda)$  to  $G_\mu$  is indeed a *uv-external good embedding* of  $G_\mu$ .

A *uv-external good embedding*  $(\mathcal{E}_\mu, \lambda_\mu)$  of  $G_\mu$  can be succinctly represented by a *descriptor pair*, which consists of a *shape descriptor*  $\langle \tau_l, \tau_r, \lambda^u, \lambda^v, \rho_l^u, \rho_r^u, \rho_l^v, \rho_r^v \rangle$  and of a *pbe descriptor* (short for partitioned book embedding descriptor)  $\langle p_l^u, p_r^u, p_l^v, p_r^v, \chi_l, \chi_r, \alpha_l^u, \alpha_r^u, \alpha_l^v, \alpha_r^v \rangle$ , with the following meaning. Let the *left outer path*  $P_l$  (resp. the *right outer path*  $P_r$ ) be the path obtained by traversing the boundary of the outer face  $f_\mu$  of  $\mathcal{E}_\mu$  from  $u$  to  $v$  in clockwise (resp. counterclockwise) direction. The label  $\tau_l$  ( $\tau_r$ ) is the sum of the labels assigned by  $\lambda_\mu$  to the angles in  $f_\mu$  at the vertices of  $P_l$  (resp.  $P_r$ ), excluding  $u$  and  $v$ . The values  $\lambda^u$  and  $\lambda^v$  are the labels of the angles at  $u$  and  $v$  in  $f_\mu$ , respectively. The label  $\rho_l^u$  is in or out depending on whether the edge of  $P_l$  incident to  $u$  is incoming or outgoing at  $u$ , respectively; the values  $\rho_r^u$ ,  $\rho_l^v$ , and  $\rho_r^v$  are defined analogously. The label  $p_l^u$  is  $L$  or  $R$  depending on whether the edge of  $P_l$  incident to  $u$  belongs to  $L$  or  $R$ , respectively; the values  $p_r^u$ ,  $p_l^v$ , and  $p_r^v$  are defined analogously. The label  $\chi_l$  is 1 if  $P_l$  is a directed path from  $u$  to  $v$  and all its edges belong to  $L$  or if  $P_l$  is a directed path from  $v$  to  $u$  and all its

edges belong to  $R$ , it is 0 otherwise. The label  $\chi_r$  is defined analogously for  $P_r$ . The label  $\alpha_l^u$  is 1 if  $P_l$  contains a directed path from  $u$  to a vertex  $w \notin \{u, v\}$  whose edges belong to  $L$  or a directed path from a vertex  $w \notin \{u, v\}$  to  $u$  whose edges belong to  $R$  and if  $\lambda_\mu$  assigns a small angle at  $w$  in  $f_\mu$ ; the label  $\alpha_l^u$  is 0 otherwise. The labels  $\alpha_r^u$ ,  $\alpha_l^v$ , and  $\alpha_r^v$  are defined analogously, with respect to  $P_r$  rather than  $P_l$  and/or with respect to  $v$  rather than  $u$ .

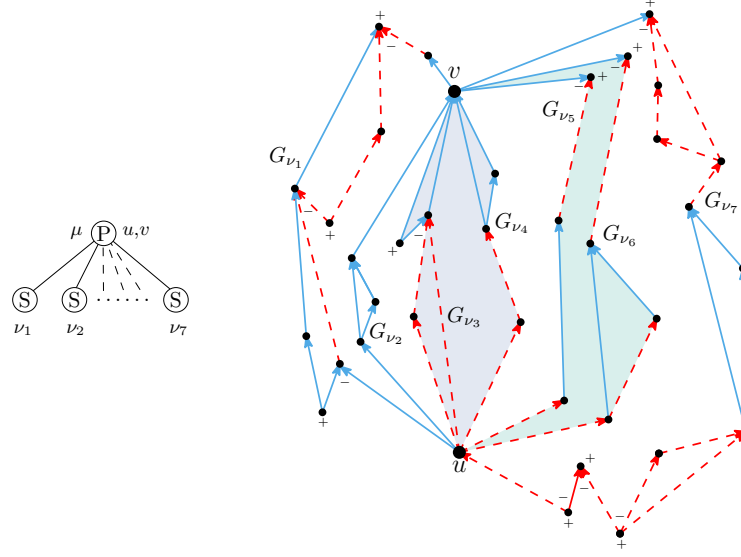
The information in a descriptor pair fully describes how a  $uv$ -external good embedding of  $G_\mu$  interfaces with the rest of  $G$  for the construction of a good embedding of  $G$  with  $e^*$  on the outer face. That is, for a good embedding  $(\mathcal{E}, \lambda)$  of  $G$  with  $e^*$  on the outer face, let  $(\mathcal{E}_\mu, \lambda_\mu)$  be the  $uv$ -external good embedding of  $G_\mu$  in  $(\mathcal{E}, \lambda)$ , and let  $(\sigma, \omega)$  be the descriptor pair of  $(\mathcal{E}_\mu, \lambda_\mu)$ . Replacing  $(\mathcal{E}_\mu, \lambda_\mu)$  with *any* other  $uv$ -external good embedding of  $G_\mu$  with descriptor pair  $(\sigma, \omega)$  still results in a good embedding of  $G$  with  $e^*$  on the outer face. Even more, for a  $uv$ -external good embedding  $(\mathcal{E}_\mu, \lambda_\mu)$  of  $G_\mu$  with descriptor pair  $(\sigma, \omega)$ , let  $\nu$  be a child of  $\mu$  with poles  $w$  and  $z$ , and let  $(\sigma', \omega')$  be the descriptor pair of the  $wz$ -external good embedding  $(\mathcal{E}_\nu, \lambda_\nu)$  of  $G_\nu$  in  $(\mathcal{E}_\mu, \lambda_\mu)$ . Replacing  $(\mathcal{E}_\nu, \lambda_\nu)$  with *any* other  $wz$ -external good embedding of  $G_\nu$  with descriptor pair  $(\sigma', \omega')$  results in a  $uv$ -external good embedding of  $G_\mu$  with descriptor pair  $(\sigma, \omega)$ . This allows us to only keep track of the descriptor pairs  $(\sigma, \omega)$  that are “realizable” by  $G_\mu$ , rather than of the actual  $uv$ -external good embeddings of  $G_\mu$ . This is formalized by the notion of **feasible set**  $\mathcal{F}_\mu$  for  $\mu$ , which contains all the descriptor pairs  $(\sigma, \omega)$  such that  $G_\mu$  admits a  $uv$ -external good embedding with descriptor pair  $(\sigma, \omega)$ .

The core of our algorithm consists of a bottom-up computation of the feasible set  $\mathcal{F}_\mu$  for each node  $\mu$  of  $T$ . If  $\mu$  is a **non-root Q-node**, then  $G_\mu$  has a unique  $uv$ -external good embedding, hence  $\mathcal{F}_\mu$  contains a unique descriptor pair and is computed in  $O(1)$  time.

If  $\mu$  is an **S-node**, let  $\nu_1$  and  $\nu_2$  be the children of  $\mu$  in  $T$  and let  $w$  be the unique vertex shared by  $G_{\nu_1}$  and  $G_{\nu_2}$ . We combine every descriptor pair  $(\sigma_1, \omega_1)$  in  $\mathcal{F}_{\nu_1}$  with every descriptor pair  $(\sigma_2, \omega_2)$  in  $\mathcal{F}_{\nu_2}$ ; for every such combination, we assign the two angles at  $w$  in the outer face with every possible label in  $\{-1, 0, 1\}$ . We test in  $O(1)$  time whenever the combination and the assignment result in a descriptor pair  $(\sigma, \omega)$  of a good embedding of  $G_\mu$ , by checking whether the properties of Theorems 1 and 3 are satisfied. In the positive case, we add  $(\sigma, \omega)$  to  $\mathcal{F}_\mu$ . Thus, the computation of  $\mathcal{F}_\mu$  takes time  $O(|\mathcal{F}_{\nu_1}| \cdot |\mathcal{F}_{\nu_2}|)$ , which is in  $O(|V(G_{\nu_1})| \cdot |V(G_{\nu_2})|)$ . This sums up to  $O(n^2)$  time over all S-nodes of  $T$ .

If  $\mu$  is a **P-node** with children  $\nu_1, \dots, \nu_k$ , we cannot just combine the descriptor pairs in  $\mathcal{F}(\nu_1), \dots, \mathcal{F}(\nu_k)$ , as the number of combinations might be super-polynomial. Instead, we consider every descriptor pair that *might* describe a  $uv$ -external good embedding of  $G_\mu$  and test whether it is in  $\mathcal{F}_\mu$ . This is done as follows. A set  $\mathcal{U}_n$  of descriptor pairs is  **$n$ -universal** if: (i) for every  $h$ -vertex biconnected partitioned directed partial 2-tree  $H$  with  $h \leq n$ , for every two vertices  $u_H$  and  $v_H$  of  $H$ , and for every  $u_H v_H$ -external good embedding  $(\mathcal{E}_H, \lambda_H)$  of  $H$ , the descriptor pair of  $(\mathcal{E}_H, \lambda_H)$  is in  $\mathcal{U}_n$ , and (ii) for every descriptor pair  $(\sigma, \omega)$  in  $\mathcal{U}_n$ , there exists a biconnected partitioned directed partial 2-tree  $H$  that contains two vertices  $u_H$  and  $v_H$  and admits a  $u_H v_H$ -external good embedding  $(\mathcal{E}_H, \lambda_H)$  with descriptor pair  $(\sigma, \omega)$ . Note that  $\mathcal{F}_\mu \subseteq \mathcal{U}_n$ . We can construct an  $n$ -universal set  $\mathcal{U}_n$  with  $|\mathcal{U}_n| \in O(n)$  in  $O(n)$  time.

Consider a  $uv$ -external good embedding  $(\mathcal{E}_\mu, \lambda_\mu)$  of  $G_\mu$  with descriptor pair  $(\sigma, \omega)$ . Assume, w.l.o.g. up to a change of the indices, that the clockwise order around  $u$  in  $\mathcal{E}_\mu$  of the pertinent graphs of the children of  $\mu$  starting at the outer face is  $G_{\nu_1}, \dots, G_{\nu_k}$ . The **descriptor sequence** of  $(\mathcal{E}_\mu, \lambda_\mu)$  is the sequence  $\mathcal{S}_\mu = [(\sigma_1, \omega_1), \dots, (\sigma_k, \omega_k)]$  of the descriptor pairs of the  $uv$ -external good embeddings of  $G_{\nu_1}, \dots, G_{\nu_k}$  in  $(\mathcal{E}_\mu, \lambda_\mu)$ . The **contracted descriptor sequence** of  $(\mathcal{E}_\mu, \lambda_\mu)$  is obtained from  $\mathcal{S}_\mu$  by identifying equal consecutive descriptor pairs, see Fig. 9. Finally, the **generating set**  $\mathcal{G}(\sigma, \omega)$  for  $(\sigma, \omega)$  is the set of contracted descriptor



■ **Figure 9** A  $uv$ -external good embedding  $(\mathcal{E}_\mu, \lambda_\mu)$  of  $G_\mu$ . Large and small angles on the outer faces of the embeddings of  $G_{\nu_1}, G_{\nu_2}, \dots, G_{\nu_7}$  are marked  $+$  and  $-$ , respectively. The descriptor pair  $(\sigma, \omega)$  of  $(\mathcal{E}_\mu, \lambda_\mu)$  has  $\sigma = \langle 1, 2, 0, -1, \text{out}, \text{in}, \text{out}, \text{out} \rangle$  and  $\omega = \langle L, R, L, L, 0, 0, 1, 0, 0, 0 \rangle$ . The descriptor sequence of  $(\mathcal{E}_\mu, \lambda_\mu)$  is  $[(\sigma_1, \omega_1), (\sigma_2, \omega_2), \dots, (\sigma_7, \omega_7)]$ , where  $\sigma_1 = \langle 1, -1, 1, 1, \text{out}, \text{out}, \text{out}, \text{out} \rangle$ ,  $\sigma_2 = \langle 0, 0, 1, 1, \text{out}, \text{out}, \text{in}, \text{in} \rangle$ ,  $\sigma_3 = \sigma_4 = \langle 0, 0, 1, 1, \text{out}, \text{out}, \text{in}, \text{in} \rangle$ ,  $\sigma_5 = \sigma_6 = \langle -1, 1, 1, 1, \text{out}, \text{out}, \text{out}, \text{out} \rangle$ ,  $\sigma_7 = \langle -2, 2, 1, 1, \text{in}, \text{in}, \text{out}, \text{out} \rangle$ ,  $\omega_1 = \langle L, L, L, L, 0, 0, 1, 0, 0, 0 \rangle$ ,  $\omega_2 = \langle L, L, L, L, 1, 0, 0, 0, 0, 0 \rangle$ ,  $\omega_3 = \omega_4 = \omega_5 = \omega_6 = \langle R, R, L, L, 0, 0, 0, 0, 0, 0 \rangle$ , and  $\omega_7 = \langle R, R, L, L, 0, 0, 1, 0, 0, 0 \rangle$ . The contracted descriptor sequence of  $(\mathcal{E}_\mu, \lambda_\mu)$  is  $[(\sigma_1, \omega_1), (\sigma_2, \omega_2), (\sigma_3, \omega_3), (\sigma_5, \omega_5), (\sigma_7, \omega_7)]$ .

sequences that a P-node with poles  $w$  and  $z$  can have in a  $wz$ -external good embedding with descriptor pair  $(\sigma, \omega)$ . We can prove that  $\mathcal{G}(\sigma, \omega)$  has size  $O(1)$ , can be constructed in  $O(1)$  time, and that each contracted descriptor sequence in  $\mathcal{G}(\sigma, \omega)$  has length  $O(1)$ .

In order to compute  $\mathcal{F}_\mu$ , we construct the generating set  $\mathcal{G}(\sigma, \omega)$  of  $(\sigma, \omega)$ , for each descriptor pair  $(\sigma, \omega)$  in  $\mathcal{U}_n$ , in overall  $O(n)$  time. For each contracted descriptor sequence  $\mathcal{C}$  in  $\mathcal{G}(\sigma, \omega)$ , we test whether  $\mathcal{C}$  is **realizable** by  $G_\mu$ , i.e., whether there exists a  $uv$ -external good embedding of  $G_\mu$  whose contracted descriptor sequence  $\mathcal{C}'$  is a subsequence of  $\mathcal{C}$  containing the first and last elements of  $\mathcal{C}$ , which ensures that  $\mathcal{C}'$  belongs to  $\mathcal{G}(\sigma, \omega)$ . We decide whether  $\mathcal{C}$  is realizable by  $G_\mu$  in  $O(k)$  time, by constructing an auxiliary graph with vertex set  $(\{\nu_1, \dots, \nu_k\}, \{(\sigma_1, \omega_1), \dots, (\sigma_\ell, \omega_\ell)\})$  and with an edge between  $\nu_i$  and  $(\sigma_j, \omega_j)$  if  $(\sigma_j, \omega_j)$  belongs to  $\mathcal{F}_{\nu_i}$ , and by then testing whether each vertex in  $\{\nu_1, \dots, \nu_k\}$  can be assigned to a neighbor, so that both  $(\sigma_1, \omega_1)$  and  $(\sigma_\ell, \omega_\ell)$  are assigned to at least one vertex. We add  $(\sigma, \omega)$  to  $\mathcal{F}_\mu$  if some  $\mathcal{C}$  in  $\mathcal{G}(\sigma, \omega)$  is realizable by  $G_\mu$ . Since  $|\mathcal{U}_n| \in O(n)$ ,  $|\mathcal{G}(\sigma, \omega)| \in O(1)$ , and the realizability of a contracted descriptor sequence can be decided in  $O(k)$  time, the computation of  $\mathcal{F}_\mu$  takes  $O(nk)$  time, which sums up to  $O(n^2)$  time over all P-nodes of  $T$ .

Finally, if  $\mu$  is the **root Q-node**  $\rho^*$ , we treat it as a P-node with two children, corresponding to  $e^*$  and  $G - e^*$ , and compute  $\mathcal{F}_{\rho^*}$  in  $O(n)$  time. Then  $G$  has a good embedding with  $e^*$  on the outer face if and only if  $\mathcal{F}_{\rho^*}$  is non-empty. We thus get the following.

► **Theorem 7.** *Let  $G$  be an  $n$ -vertex biconnected partitioned directed partial 2-tree. It is possible to test in  $O(n^3)$  time whether  $G$  admits an upward book embedding.*



## 7 Conclusions

We studied upward book embeddings of partitioned digraphs, focusing on the unsolved case of two pages – one of the final open algorithmic book-embedding problems. We closed the complexity gap for the problem by proving its NP-hardness. We provided a characterization of the upward embeddings that admit such layouts and combined this with several algorithmic tools, including flow techniques, SPQ-decompositions, and compact embedding representations, obtaining efficient testing and embedding algorithms for digraphs with a fixed planar embedding and for biconnected partial 2-trees with variable embedding. We aim to (i) remove the linear overhead in our algorithm for biconnected partial 2-trees caused by rerouting the SPQ-tree at each Q-node, and (ii) extend the method to arbitrary simply connected partial 2-trees. The latter is challenging due to nested biconnected components in the block-cut-vertex tree, which must be arranged while keeping cut vertices 4-modal and avoiding impossible faces. Additional research directions include:

- studying the complexity of the problem for partitioned digraphs with few sources;
- determining whether the problem is NP-complete for bounded-treewidth instances; and
- developing FPT algorithms for parameters more restrictive than treewidth.

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