On the Size of k-Irreducible Triangulations

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Abstract

A triangulation of a surface is k-irreducible if every non-contractible curve has length at least k and any edge contraction breaks this property. Equivalently, every edge belongs to a non-contractible curve of length k and there are no shorter non-contractible curves. We prove that a k-irreducible triangulation of an orientable surface of genus g has $O(k^2g)$ triangles, which is optimal. This is an improvement over the previous best bound $k^{O(k)}g^2$ of Gao, Richter and Seymour [*Journal of Combinatorial Theory, Series B, 1996*].

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1 Introduction

The main objects of study in this article are **(surface) triangulations**. There are many inequivalent definitions of those, and for us they will be simplicial complexes homeomorphic to surfaces. Rephrased in a graph-theoretical language, this means that a triangulation is a graph embedded in a surface where all the faces are topological disks and have degree three, and the graph is **simple**: multiple edges and loops are forbidden.

In 1922, Steinitz [41] proved that all triangulations of the 2-dimensional sphere could be obtained from the tetrahedron by *splitting vertices*. Nowadays, it is more common to consider the inverse operation of **contracting an edge**, which consists in identifying the endpoints of an edge and collapsing the two adjacent triangles. Such a contraction must preserve the properties of triangulations, and in particular must not induce multiple edges and loops. Thus, Steinitz proved that the tetrahedron is the unique **irreducible triangulation** of the sphere: every triangulation of the sphere can be reduced to the tetrahedron by iteratively contracting edges, and the tetrahedron cannot be contracted further.

Barnette initiated the study of the analogous problem on the projective plane [3], which led to the classification of irreducible triangulations of surfaces up to Euler genus 4 by Sulanke [42] by means of the computer program `surftri`. It is now known that for every

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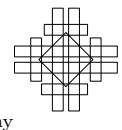

surface of Euler genus g, the number of irreducible triangulations is finite and that they have at most $13g - 4$ vertices, see Joret and Wood [29]. The latter improves previous bounds by Barnette and Edelson [2] and by Nakamoto and Ota [38]. This bound has been extended to surfaces with boundaries by Boulch, Colin de Verdière and Nakamoto [6].

A closed curve on a surface is **non-contractible** if it is not homotopic (intuitively, if it cannot be deformed continuously) to a point. Any closed walk on the edges of a triangulation determines a closed curve on the underlying surface. The **edge-width** of a triangulation is the length of the shortest (by the number of edges) non-contractible closed walk. A triangulation is **k-irreducible** if it has edge-width at least k and it is irreducible for this property: any edge contraction reduces this edge-width. Equivalently, every edge belongs to a non-contractible curve of length k and there are no shorter non-contractible curves. We refer to the PhD thesis of Melzer for a proof of this equivalence [36, Lemma 3.1] and for a thorough introduction to k-irreducible triangulations, in particular the different motivations for which they have been introduced.

Since loops and multiple edges are forbidden, triangulations of surfaces of positive Euler genus have edge-width at least 3. From the definitions, it is easy to see that irreducible triangulations and 3-irreducible triangulations match. Therefore, k-irreducible triangulations generalize irreducible triangulations, and one is naturally led to wonder about their combinatorial properties.

This problem was first studied by Malnič and Nedela [34] who proved that the number of k-irreducible triangulations of a surface is finite (they also observed that this follows from a variant of Robertson-Seymour theory), see also Juvan, Malnič and Mohar [30]. The current best known bound, due to Gao, Richter and Seymour [25] is that a k-irreducible triangulation of a surface of Euler genus g has $k^{O(k)}g^2$ edges. In that paper, the authors say that they expect the correct bound to depend linearly on g instead of quadratically. In his aforementioned PhD thesis, Melzer [36, p. 18] states "Our expectation would be something like $O(k^2g)$, but we lack any evidence at all to conjecture". Our main result confirms the expectations of Gao, Richter, Seymour and Melzer for orientable surfaces.

▶ **Theorem 1.** *Any k-irreducible triangulation of an orientable surface S of genus g has at most $966k^2g = O(k^2g)$ edges.*

The bound $O(k^2g)$ in Theorem 1 is tight up to the value of the constant which we have not tried to optimize: intuitively it can be achieved by taking g connected sums of tori resembling k by k grids. We refer to Melzer [36, Section 6.2] for precise descriptions of families of k-irreducible triangulations of size $\Theta(k^2g)$. One interesting follow-up question would be to determine the correct constant in some asymptotic regimes in g and k.

Note that an immediate application of the Euler formula yields similar bounds for the number of vertices and faces.

Irreducible systems of curves. Our proof of Theorem 1 is actually a consequence of a stronger result on system of curves that we introduce now. We investigate k-irreducible triangulations from the powerful perspective initiated by (de Graaf and) Schrijver in a series of papers (see e.g., [39, 40, 19, 20]), which consists in encoding the metric structure of an embedded graph using a system of curves. Note that this approach was already implicit in Steinitz's proof of his aforementioned theorem, as explained in Chang [10].

For the case of k-irreducible triangulations, this is done as follows. One first observes that instead of considering shortest non-contractible closed walks, one could consider instead shortest non-contractible closed curves crossing the triangulation exactly at the vertices.

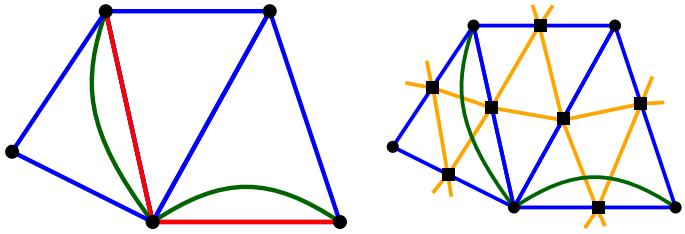

Figure 1 The triangulation in blue, a walk in red, its corresponding noose in green and the medial graph in orange.

Such curves are commonly called **nooses** in structural graph theory. If one defines the length of those curves as this number of intersections, one sees that any non-contractible closed walk in a triangulation can be deformed to a non-contractible noose of the same length and vice-versa, see Figure 1, left. The length of a shortest non-contractible noose is called the **face-width** (also called **representativity** in, e.g. [25]), and therefore we can equivalently define k-irreducibility as having face-width at least k and being irreducible for this property.

The **medial graph** M of a triangulation T of a surface S is the embedded graph in S which has a vertex in the middle of each edge of T and in each face f of T we add edges between vertices of M that belong to consecutive edges of f. See Figure 1, right for an illustration. Note that M is a simple face-bipartite 4-regular graph and that the length of a noose c for T is equal to half the number of transverse intersections it has with M. Because M is 4-regular, it can be considered as a **system of curves** $\mathcal{C} = (c_1, \dots, c_n)$. The **crossing number** of a system of curves $\mathcal{C}$, denoted by $\mathrm{cr}(\mathcal{C})$ is its number of intersections. The **length** of a curve c with respect to $\mathcal{C}$ is the number of intersections of c and $\mathcal{C}$, denoted by $\mathrm{cr}(c, \mathcal{C})$.

The strength of this perspective is that this allows to understand metric deformations of T (e.g., edge contractions) under the lens of topological deformations (homotopies) of $\mathcal{C}$. For instance, it follows from the work of Schrijver [40] that on an orientable surface, the system of curves corresponding to a minor-minimal graph of face-width k is necessarily in **minimal position**: it contains no contractible curves and its crossing number $\mathrm{cr}(\mathcal{C})$ is minimized over all families $\mathcal{C}'$ homotopic to $\mathcal{C}$ (we refer to Section 2 for the background on homotopies and a precise statement of Theorem 4 by Schrijver).

A **smoothing** of a system of curves $\mathcal{C}$ is the system of curves obtained from $\mathcal{C}$ by removing a vertex v and reconnecting the incident strands in one of two ways as in Figure 2. Note that a smoothing may change the number of components.

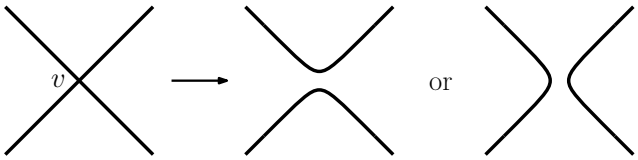

Figure 2 A smoothing at v.

A system of curves $\mathcal{C}$ on a surface S is a **k-irreducible** system of curves for some $k \geqslant 1$ if it is connected and:

- Every non-contractible closed curve c on S transverse to $\mathcal{C}$ satisfies $\mathrm{cr}(c, \mathcal{C}) \geqslant k$, and
- for every system of curves $\mathcal{C}'$ obtained by a smoothing of $\mathcal{C}$, there is a non-contractible closed curve c such that $\mathrm{cr}(c, \mathcal{C}') < k$.

Finally, a system of curves on a surface is **face-bipartite** if there is a 2-coloration of its faces (connected components of the complement) such that any two adjacent faces have different colors. We now have all the ingredients to state our second theorem.

▶ **Theorem 2.** *Let $\mathcal{C}$ be a face-bipartite k-irreducible system of curves on an orientable surface of genus g. The crossing number of $\mathcal{C}$ is at most $120k^2g = O(k^2g)$.*

Connections to systolic geometry. Theorems 1 and 2 provide discrete counterparts to results in systolic geometry. This subfield of differential geometry investigates topological spaces endowed with continuous metrics (typically Riemannian) and aims at understanding the structure and properties of their shortest non-contractible curves, called **systoles**. We refer to the book of Katz [31] for an introduction.

The famed systolic inequality of Gromov [26] states that for any surface of genus g and area A, the systole has length $O(\sqrt{A/g}\log g)$. This is known to be tight: Brooks [8] and Buser and Sarnak [9] provided families of hyperbolic surfaces achieving this bound. There are discrete analogues of these results for embedded graphs [28, 16, 33], showing that the edge-width of a triangulation with n triangles is similarly bounded by $O(\sqrt{n/g}\log g)$ and that this is asymptotically tight. Equivalently, this shows that any triangulation of edge-width k has $\Omega(k^2g/\log^2 g)$ triangles[1]. Bounding the size of k-irreducible triangulations then amounts to looking for a *reverse systolic inequality*, bounding how far from this lower bound a locally extremal object for the systolic inequality can be. From this perspective, it is noteworthy that the logarithmic factor, which is known to be required for the systolic inequality, disappears in our bounds in Theorems 1 and 2, even though our bounds are also tight.

We are not aware of any works investigating continuous versions of such a reverse systolic inequality. One difficulty is that the space of Riemannian metrics on a surface is not compact, and extremal surfaces for the systolic inequality in negative Euler characteristic are expected to exhibit singularities, see for example Gromov [26, pp.62-65]. This can be addressed by adding additional assumptions on the curvature, restricting to hyperbolic surfaces as in Fortier Bourque and Rafi [7], or nonpositively curved surfaces as in Katz and Sabourau [32]. Since the discrete setting of triangulations does not suffer from these analytic issues, it provides a convenient testbed for the study of locally extremal surfaces. The framework of geodesic currents of Bonahon [5] could provide an interesting middle-ground between these two worlds, keeping a continuous flavour but with strong compactness properties.

Related works. Many articles in computational geometry in the last twenty years have been targeted at understanding topologically meaningful curves on surfaces, from the perspectives of both algorithms and combinatorics. We refer to the dedicated chapter by Éric Colin de Verdière in the handbook of computational geometry [15] for a survey of this active field of research. Our proof techniques deal specifically with families of curves in minimal position on surfaces, a problem for which polynomial-time algorithms have been recently provided by Chang and de Mesmay [12], and Dubois [23]. The study of systolic properties of surfaces through the lens of such systems of curves has been instrumental in recent works of Cossarini [17] and Cossarini and Sabourau [18]. The medial graph construction connects homotopy of curves with electrical moves, as studied in Chang and Erickson [13], Chang, Cossarini and Erickson [11] and Aranguri, Chang and Fridman [1]. Very recently, Delecroix, Fontaine and Lazarus [22] have made algorithmic Schrijver's concept of *kernel* [40], which generalizes our k-irreducible triangulations to metric structures which are extremal with respect to the entire set of lengths of non-contractible curves (the *length spectrum*).

[1] Note that the growth is sensibly different if one considers the number of vertices instead of the number of triangles, see Melzer [36, Section 2.2]

Organization of the paper. After some preliminaries in Section 2, we first explain how Theorem 2 implies Theorem 1 in Section 3. Then the main ingredient is Proposition 7 proven in Section 4. Namely given a k-irreducible system of curves, it shows the existence of $O(g)$ curves of length at most $2k$ filling S. This immediately yields a first bound (Theorem 13), albeit one not linear in g. Section 5 introduces additional combinatorial tools to obtain the tight bound of Theorem 2. Due to line restrictions, some proofs are deferred to the full version [21], which provides slightly more general results and a treatment of one case of non-orientable surface.

2 Preliminaries

In this section we provide the main definitions used throughout the paper. The central objects of study are topological surfaces and curves on them whose definitions are recalled in Section 2.1. A (discrete) geometry on a surface allowing to measure lengths can then be defined in two related ways: either by the mean of an embedded graph also called combinatorial surface (see Section 2.2), or using a system of curves as in Section 2.3.

2.1 Surfaces and curves

In this paper, we use the term **surface** for a closed orientable topological surface. It is always denoted by S and is sometimes implicit. We refer to Massey [35] for topological background.

Curves. A **curve** on a surface S is a continuous map $c : [0, 1] \to S$ and a **closed curve** on S is a continuous map $c : \mathbb{S}^1 \to S$, where $\mathbb{S}^1$ is the one-dimensional circle $\mathbb{R}/\mathbb{Z}$. Since all the curves in this article are closed, we will often just write curve instead of closed curve. Two closed curves c and c' on S are **(freely) homotopic** if there is a continuous map $h : [0, 1] \times \mathbb{S}^1 \to S$ such that $h(0, t) = c(t)$ and $h(1, t) = c'(t)$. Being freely homotopic is an equivalence relation on closed curves on S and we refer to **(free) homotopy classes** for an equivalence class of closed curves with respect to free homotopy. We always denote curves with latin letters and homotopy classes with greek letters. A closed curve is **contractible** if it is homotopic to a constant curve. A curve is **simple** if it does not self-intersect. A closed curve is **primitive** if it is not homotopic to the k-fold concatenation of a curve c for some $k \geqslant 2$. In particular, a primitive curve is not contractible.

2.2 Combinatorial surfaces

Combinatorial surfaces. A **combinatorial surface** is an embedding $\phi : G \to S$ of a graph $G = (V, E)$, possibly with multiple edges and loops, in a surface S, such that $S \setminus \phi(G)$ is a disjoint union of open disks. These disks are called the **faces** of G. The counterclockwise sequence of arcs (i.e., oriented edges) in the boundary of a face is called a **face boundary** or **facial walk**. The **degree of a face** is the length of its facial walk.

Two combinatorial surfaces $\phi : G \to S$ and $\phi' : G' \to S'$ are **isomorphic** if there exists a graph isomorphism $\theta : G \to G'$ and a homeomorphism $\psi : S \to S'$ such that $\phi' \circ \theta = \psi \circ \phi$. The isomorphism class of a combinatorial surface can be conveniently encoded by a **rotation system** describing the circular orderings of the (oriented) edges around each vertex. See Mohar and Thomassen [37] for more details on this structure. Most of the time, the embedding ϕ is implicit and we identify G and the corresponding combinatorial surface.

Let $\phi : G \to S$ be a combinatorial surface. The **length (with respect to G)** of a curve c in S that crosses only the vertices of G and not its edges is the number of times this curve passes through a vertex of G. It is denoted $\mathrm{cr}(c, G)$. The **systole** of G is the minimum length

over all non-contractible closed curves c on S passing only through vertices of G of $\mathrm{cr}(c, G)$. We will use the common abuse of language to also call systole such a non-contractible closed curve of minimal length when it leads to no confusion.

Triangulation. A combinatorial surface (S, G) is a **triangulation** if G is simple (no loops nor multiple edges) and every face has degree 3. A triangulation T on a surface S is **k-irreducible** if

1. the systole of T is equal to k
2. for every edge $e = (uv)$ of T, there is a non-contractible curve c of length k on T crossing consecutively u and v.

▶ Remark 3. In line with the introduction, we define our triangulations to be simplicial, but note that this is implied by k-irreducibility for $k \geqslant 3$. Indeed, if there is a loop or a bigon formed of two parallel edges, it has to be contractible and thus bounds a disk, but then the edges inside that disk cannot belong to a systole as there would be a shortcut, violating k-irreducibility.

2.3 Systems of curves

Systems of curves. A **system of curves** on a surface S is a family $\mathcal{C} = (c_1, \ldots, c_p)$ of non-contractible closed curves on S in *general position*, i.e., they only cross transversely and at most two curves cross at a point.

An **intersection** of $\mathcal{C}$ is a pair $(i, s) \neq (j, t)$ such that $c_i(s) = c_j(t)$. Because a system of curves is in general position, the set of intersections is an isolated subset of points in S. The **crossing number of $\mathcal{C}$** is its number of intersections. Note that $\mathcal{C}$ defines a 4-regular embedded graph on S whose vertex set is the intersections of $\mathcal{C}$ and its edges are segments of the curves c_i. Conversely, a 4-regular embedded graph $\phi : G \to S$ determines a system of curves by "going straight" at each crossing. We will freely use this identification and use the vocabulary of embedded graphs, i.e., vertices, edges and faces for a system of curves $\mathcal{C}$. Note however that in this context the embedded graph might not be a combinatorial surface because its complement could have topology. a system of curves is **connected** if the corresponding graph is connected.

Two systems of curves $\mathcal{C} = \{c_1, \ldots, c_p\}$ and $\mathcal{C}' = \{c'_1, \ldots, c'_p\}$ are **homotopic** if for all i in $[1, p]$, c_i is freely homotopic to c'_i. We say that a system of curves $\mathcal{C}$ is in **minimal position** if it contains no contractible curves and has the minimal crossing number over all systems of curves homotopic to $\mathcal{C}$. We say that two curves **intersect essentially** if they intersect at least once in minimal position.

Length of curves. Let $\mathcal{C}$ be a system of curves on S and c be a curve transverse to $\mathcal{C}$. The **length** of c with respect to $\mathcal{C}$ is the number of crossings of c with $\mathcal{C}$. We recall that this quantity is denoted $\mathrm{cr}(c, \mathcal{C})$. The **length** of a free homotopy class γ is the minimal number of crossings of a representative c of γ with $\mathcal{C}$. We use the notation $\mathrm{cr}(\gamma, C)$ as a shortcut for $\min_{c \in \gamma} \mathrm{cr}(c, C)$. For $\mathcal{C}$ in minimal position, we define the **systole** of $\mathcal{C}$ to be $\min_{\gamma} \mathrm{cr}(\gamma, \mathcal{C})$ where γ runs over all non-trivial free homotopy classes of closed curves in S. Here again, we sometimes also call a closed curve achieving this minimum length a systole.

We say that a system of curves $\mathcal{C}$ in minimal position on a surface S is **filling** if for any non-trivial homotopy class γ we have $\mathrm{cr}(\gamma, \mathcal{C}) > 0$. Equivalently, a system of curves $\mathcal{C}$ in minimal position on a surface S is filling if and only if it corresponds to a combinatorial surface (i.e. the complement of the curves is a disjoint union of topological disks).

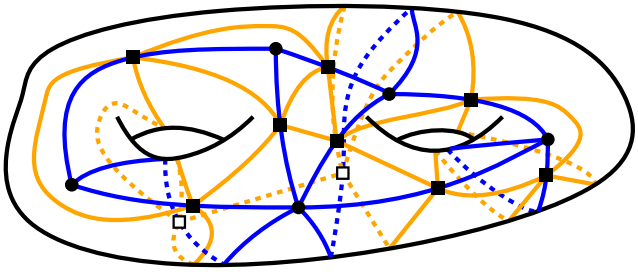

Figure 3 A graph G in blue on a genus 2 surface and its medial graph in orange.

k-irreducible systems of curves. Recall from the definition that a system of curves is k-irreducible for some $k \geqslant 1$ if it is connected and:

- for every non-contractible closed curve c on S we have $\mathrm{cr}(c, \mathcal{C}) \geqslant k$, and
- for every system of curves $\mathcal{C}'$ obtained by a smoothing (pictured in Figure 2) of $\mathcal{C}$, there is a non-contractible closed curve c such that $\mathrm{cr}(c, \mathcal{C}') < k$.

A system of curves $\mathcal{C}$ is **tight** if it does not contain a contractible curve disjoint from the other curves and if for any smoothing $\mathcal{C}'$ of $\mathcal{C}$ there is a curve in minimal position with respect to $\mathcal{C}$ and $\mathcal{C}'$ which is shorter with respect to $\mathcal{C}'$ than with respect to $\mathcal{C}$. This is an equivalent reformulation of a notion introduced by Schrijver, who proved the following theorem:

▶ **Theorem 4** ([39, Theorem 5]). *A system of curves is tight if and only if it is made of primitive curves in minimal position.*

It is immediate that a k-irreducible system of curves is tight, and therefore it is in minimal position. Note also that a k-irreducible system of curves is necessarily filling.

▶ Remark 5. In this article, we often alternate between different metric models. In order to minimize confusion, we use the following guidelines:

- in a triangulation T, we consider curves which are walks on the edges of the triangulation, and their lengths are the number of edges that they use.
- in a combinatorial surface G, we consider curves which cross G only at vertices, and their lengths are the number of vertices that they cross.
- in a system of curves $\mathcal{C}$, we consider curves which are in general position with respect to $\mathcal{C}$, and their lengths are their number of intersections with $\mathcal{C}$.

3 From triangulations to system of curves

The goal of this section is to prove how Theorem 1 on k-irreducible triangulations follows from Theorem 2 on k-irreducible system of curves.

Let T be a k-irreducible triangulation on S with vertex set V. We first explain how to associate a system of curves to T, and we then study its properties. Let G be a subgraph of T with the same vertex set V and minimal (with respect to edge inclusion) for the property that every non-contractible curve c intersecting G only at V satisfies $\mathrm{cr}(c, G) \geqslant k$. Let M be the medial graph of G: recall that it is the graph with a vertex at the midpoint of every edge of G and two midpoints are connected by an edge of M whenever their supporting edges in G are consecutive in a face of G, see Figure 3. The graph M is 4-regular and is thus naturally associated to a system of curves $\mathcal{C}$ on S. We call $\mathcal{C}$ a **system of curves associated to T** (note that there can be many of them, depending on the choice of G). As proven in [40], for any homotopy class γ of curves in S we have $\mathrm{cr}(\gamma, \mathcal{C}) = 2\,\mathrm{cr}(\gamma, G)$.

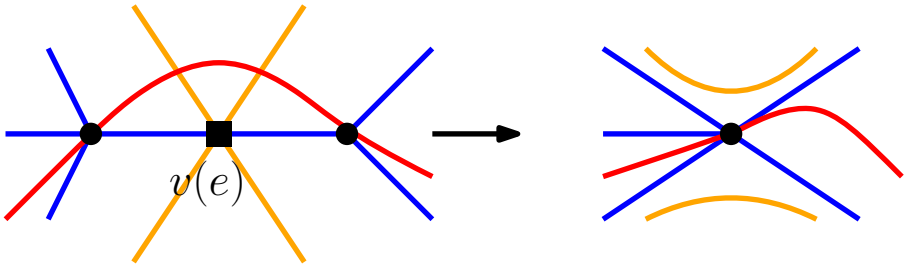

Figure 4 The contraction of an edge in G. The blue graph is G, the orange one is $\mathcal{C}$ and the red curve is c.

▶ **Proposition 6.** *Let T be a k-irreducible triangulation on S. Then any system of curves associated to T is face-bipartite, $2k$-irreducible and in minimal position.*

Proof. Let us denote as before by G an (edge inclusion-wise) minimal subgraph of T such that the length of a shortest non-contractible noose on G is still k, and by $\mathcal{C}$ the system of curves associated to G. As $\mathrm{cr}(c, G) \geqslant k$ for every non-contractible curve c, $\mathrm{cr}(\gamma, \mathcal{C}) = 2\,\mathrm{cr}(\gamma, G) \geqslant 2k$ for every non-trivial homotopy class γ.

Let $\mathcal{C}'$ be a smoothing of $\mathcal{C}$. As shown in [40], a smoothing of $\mathcal{C}$ either corresponds to an edge deletion or an edge contraction in G. We work separately on these two cases.

1. Assume that $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a smoothing corresponding to deleting the edge e. Let $G' = G \setminus \{e\}$. Then by definition of G, there is a non-contractible curve c such that $\mathrm{cr}(c, G') < k$. Thus $\mathrm{cr}(\gamma, \mathcal{C}') \leqslant 2 \cdot \mathrm{cr}(c, G') < 2k$ where γ is the homotopy class of c.
2. Assume that $\mathcal{C}'$ is obtained from $\mathcal{C}$ by a smoothing corresponding to contracting the edge $e = (uv)$. As T is a k-irreducible triangulation, there is a systole c of T going through u and v consecutively. Thus this systole is also a systole of G with the same property and is also a systole of $\mathcal{C}$ crossing two consecutive edges of $\mathcal{C}$ around the vertex $v(e)$ (see Figure 4). Thus, the smoothing reduces the length of c and $\mathrm{cr}(\gamma, \mathcal{C}') < 2k$ where γ denotes the homotopy class of c.

Since G is connected, so is $\mathcal{C}$. Therefore the system of curves $\mathcal{C}$ is $2k$-irreducible. It follows from Theorem 4 that it is in minimal position. Finally, it is face-bipartite by construction: the faces of the medial graph are either faces or vertices of T. ◀

We are now ready to prove Theorem 1 assuming Theorem 2. We restate it for convenience.

▶ **Theorem 1.** *Any k-irreducible triangulation of an orientable surface S of genus g has at most $966k^2g = O(k^2g)$ edges.*

Proof. As before, let G be a minimal subgraph of T such that the length of a shortest non-contractible noose on G is k, and let $\mathcal{C}$ be its associated system of curves. By Proposition 6, any system of curves associated to T is a $2k$-irreducible system of curves. Thus by Theorem 2, the crossing number $\mathrm{cr}(\mathcal{C})$ of $\mathcal{C}$ is at most $120(2k)^2g = 480k^2g$.

Now let M be the medial graph of G from which is built the system of curves $\mathcal{C}$. It is such that $|V(M)| = \mathrm{cr}(\mathcal{C})$ and because it is quartic $|E(M)| = 2\,\mathrm{cr}((C))$. The graph M is face bipartite and the classes of the bipartition are in bijection with the vertices $V(G)$ and the faces $F(G)$ of G. These bijections preserve the degree and hence $\sum_{v \in V(G)} \deg_G(v) = |E(M)| = 2\,\mathrm{cr}(\mathcal{C})$. Now, by Proposition 6, $\mathcal{C}$ is in minimal position. In particular, $\mathcal{C}$ has

neither monogon nor bigon. In other words, each face of M has degree at least 3. Thus $\sum_{v \in V(G)} \deg_G(v) \geqslant 3|V(G)|$. We conclude that $|V(T)| = |V(G)| \leqslant \frac{2}{3}\mathrm{cr}(\mathcal{C}) \leqslant \frac{2}{3}480k^2g = 320k^2g$, and $|E(T)| \leqslant 3|V(T)| + 6g - 6 \leqslant 966k^2g$. ◀

4 Short filling systems for k-irreducible systems of curves

The main result of this section is the following proposition, which extracts from k-irreducibility a more topological property: the existence of a filling family of short curves.

▶ **Proposition 7.** *Let $\mathcal{C}$ be a face-bipartite k-irreducible system of curves. Then there is a set of curves of lengths at most $2k$ with respect to $\mathcal{C}$ which is in minimal position and filling.*

The proof of this proposition relies partially on the notion of universal cover. We refer to Massey [35, Chapter 5] for an introduction to this classical object in algebraic topology and to Farb and Margalit [24, Section 1.2] for the specifics in the case of curves on surfaces. Given a curve $c : \mathbb{S}^1 \to S$ in a surface S, a *lift* $\tilde{c} : \mathbb{R} \to \tilde{S}$ of c is a map to the universal cover $\tilde{S}$ that commutes with the projections $\mathbb{R} \to S^1$ and $\tilde{S} \to S$. When the lift $\tilde{c}$ is simple we call it a *line*.

Proof. We first observe that we can assume that k is even. Indeed, if this is not the case, since $\mathcal{C}$ is k-irreducible and face-bipartite, $\mathcal{C}$ is $(k-1)$-irreducible. We will build a set of curves of lengths at most $2k$ made of systoles and some curves that are obtained by concatenation of two systoles, and then show that these curves are filling.

The following claim is standard in geometry of curves on surfaces. As it is used many times throughout our proof, we provide a standalone proof.

▷ Claim 8. Let c and c' be non-contractible curves on S in minimal position. Then any two lifts $\tilde{c}$ and $\tilde{c'}$ in the universal cover $\tilde{S}$ are lines that intersect at most once.

Proof. Assume that the lift $\tilde{c}$ has a self-intersection. Then it forms a monogon: there is a disk in $\tilde{S}$ bounded by a subline of $\tilde{c}$. By homotopy, one can remove this monogon contradicting the fact that c was in minimal position on S. Therefore, $\tilde{c}$ and $\tilde{c'}$ are lines in $\tilde{S}$.

Let us assume for contradiction that two lifts $\tilde{c}$ and $\tilde{c'}$ intersect twice. Then, since $\tilde{S}$ is contractible, they form a bigon: there is a disk in $\tilde{S}$ bounded by one subline α of $\tilde{c'}$ and one subline β of $\tilde{c}$. Now, the homotopy moving α to β projects to a homotopy on the surface, and leads to a new pair of curves intersecting less. This contradicts the hypothesis that c and c' were in minimal position. ◁

The next claim is the first step in defining the systoles we will work with.

▷ Claim 9. Let v be a crossing of $\mathcal{C}$ and e, e' two edges adjacent to v that are consecutive for the counter-clockwise ordering of edges around v. Then there is a systole that passes consecutively through e and e'.

Proof. Let $\mathcal{C}'$ be the system of curves obtained by smoothing v and connecting e with e'. As $\mathcal{C}$ is a k-irreducible system, there is a non-contractible curve c in minimal position with respect to both $\mathcal{C}$ and $\mathcal{C}'$ and such that $\mathrm{cr}(c, \mathcal{C}') < k$. Since $\mathcal{C}$ and $\mathcal{C}'$ differ from each other only at v, the curve c goes from e to e' in a neighborhood of v. If we pick c with the property that $\mathrm{cr}(c, \mathcal{C}') < k$ and $\mathrm{cr}(c, \mathcal{C})$ minimal, then it passes only once consecutively through e and e' and $\mathrm{cr}(c, \mathcal{C}') = \mathrm{cr}(c, \mathcal{C}) - 2$.

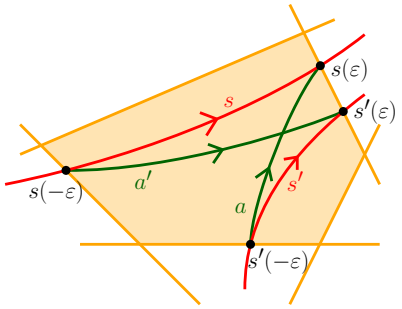

Figure 5 Wedding of two systoles s and s'.

As $\mathcal{C}$ is face-bipartite, all crossing numbers of curves with respect to $\mathcal{C}$ are even. Since k is even, we must have $\mathrm{cr}(c, \mathcal{C}') = k - 2$ and $\mathrm{cr}(c, \mathcal{C}) = k$. The curve c is therefore a systole, this concludes the proof of the claim. ◁

Let e_1, e_2, e_3 and e_4 be the four consecutive edges around a vertex v of $\mathcal{C}$. By Claim 9 there is a systole s_1 crossing consecutively e_1 and e_2. Similarly, there is a systole s_2 crossing consecutively e_2 and e_3. We denote by $\mathcal{S}$ a set of curves containing a pair of such systoles s_1 and s_2 for each vertex of $\mathcal{C}$. Note that by definition a systole is in minimal position with respect to $\mathcal{C}$.

We first handle the case where the surface is a torus.

▷ Claim 10. If the surface S is a torus and v is a vertex of $\mathcal{C}$ and s_1 and s_2 are the two corresponding systoles of $\mathcal{S}$, then s_1 and s_2 intersect essentially.

The proof is provided in the full version [21]. Since on a torus, any pair of curves intersecting essentially is filling, Claim 10 concludes the proof for the torus with the family $\mathcal{S}$. In the rest of the proof we assume that the surface has negative Euler characteristic. Therefore its universal cover is homeomorphic to an open disk and can be endowed with a boundary at infinity $\partial\tilde{S}$ which is a circle. The lifts of curves in $\tilde{S}$ are lines with endpoints on this boundary, and two curves are homotopic if and only if there exists lifts of the two curves having the same endpoints. Furthermore, no two lifts have exactly one endpoint in common. We refer to [24, Section 1].

We now define curves obtained by concatenations of two systoles. They will be used to fill the surface S. Let s and s' be two systoles in $\mathcal{S}$ that start in the same face f of $\mathcal{C}$ and such that $\mathcal{C} \cup \{s, s'\}$ is in minimal position. Up to homotopy we can further assume that the first crossings with the boundary of f occurs for both s and s' at time $\pm\varepsilon$ and that s and s' do not intersect in this initial face. Namely $s([-\varepsilon, \varepsilon]) \cap s'([-\varepsilon, \varepsilon]) = \emptyset$ and that the points $s(\varepsilon)$, $s(-\varepsilon)$, $s'(-\varepsilon)$ and $s'(\varepsilon)$ appear on the boundary of f in that counterclockwise order. Let a and a' be arcs in the face f that are in minimal position and go respectively from $s'(-\varepsilon)$ to $s(\varepsilon)$ and $s(-\varepsilon)$ to $s'(\varepsilon)$. These two arcs intersect exactly once. We define a new curve t by the concatenation $t := a \cdot s|_{[\varepsilon,-\varepsilon]} \cdot a' \cdot s'|_{[\varepsilon,-\varepsilon]}$, see Figure 5. We say that t is obtained by **wedding** s and s' if t is in minimal position with respect to itself. Equivalently, we can make a wedding if in the universal cover $\widetilde{S}$ of S the two lifts of s and s' starting from the same lift $\widetilde{f}$ of f do not intersect. By definition, a wedding has length $2k$ with respect to $\mathcal{C}$. If such a wedding is put in minimal position with respect to $\mathcal{C}$ by a homotopy, it might even get shorter.

We define the set $\mathcal{S}_2$ to be a set of representatives of the homotopy classes of all possible weddings of systoles in $\mathcal{S}$ such that $\mathcal{C} \cup \mathcal{S} \cup \mathcal{S}_2$ is in minimal position. The following claim proves that the curves in $\mathcal{S} \cup \mathcal{S}_2$ intersect every possible simple closed curve on the surface.

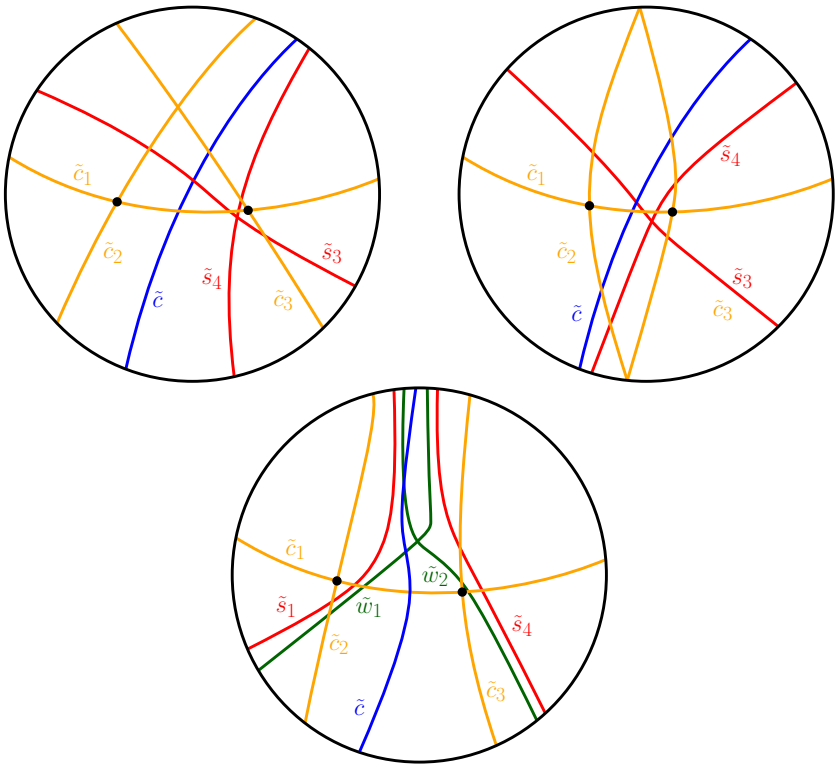

Figure 6 The three different cases leading to the proof of claim 11.

▷ Claim 11. Let c be a non-contractible simple closed curve. Then there exists $s \in (\mathcal{S} \cup \mathcal{S}_2)$ such that c and s intersect essentially.

Proof. We first put c in minimal position with respect to $\mathcal{C} \cup \mathcal{S} \cup \mathcal{S}_2$. Since $\mathcal{C}$ is filling, there is an edge $e = (uv)$ of $\mathcal{C}$ that intersects c. Let c_1 be the curve of $\mathcal{C}$ containing the edge e and c_2 and c_3 be the curve crossing c_1 at u and v respectively. We denote by s_1 and s_2 the two systoles in $\mathcal{S}$ corresponding to u and by s_3 and s_4 the ones corresponding to v. We will prove that c intersects either s_1, s_2, s_3, s_4 or one of their pairwise weddings.

In the universal cover, our local picture lifts as in Figure 6, yielding lines $\tilde{c_1}$, $\tilde{c_2}$, $\tilde{c_3}$, $\tilde{s_1}$, $\tilde{s_2}$, $\tilde{s_3}$, $\tilde{s_4}$ and $\tilde{c}$. The lines $\tilde{c_1}$,$\tilde{c_2}$, $\tilde{c_3}$ intersect pairwise at most once and are simple because $\mathcal{C}$ is in minimal position. We do a case by case analysis depending on the topology of $\tilde{c_2}$ and $\tilde{c_3}$. The only ingredient we use is the fact that each pair of these curves intersect at most once.

If $\tilde{c_2}$ and $\tilde{c_3}$ intersect, then it is immediate that $\tilde{c}$ intersects at least one of the curves $\tilde{s_1}$, $\tilde{s_2}$, $\tilde{s_3}$ or $\tilde{s_4}$, see top left Figure 6. The same happens if c_2 and c_3 are homotopic since in that case $\tilde{c_2}$ and $\tilde{c_3}$ have the same endpoints in $\partial\tilde{S}$. Since these intersections are forced by the endpoints on $\partial\tilde{S}$ and homotopies do not move these endpoints, these intersections on $\tilde{S}$ project to essential intersections on S. See top right Figure 6.

The last case is if $\tilde{c_2}$ and $\tilde{c_3}$ are disjoint (including at their endpoints) as in bottom Figure 6. Then the wedding w of s_1 and s_4 based at the added crossing lifts to two lines $\tilde{w_1}$ and $\tilde{w_2}$ which cross the lines $\tilde{c_1}$ and $\tilde{c_2}$, respectively $\tilde{c_2}$ and $\tilde{c_3}$ the same way that $\tilde{s_1}$ and $\tilde{s_4}$ do. In particular, this wedding exists since $\tilde{w_1}$ and $\tilde{w_2}$ cannot cross again due to $\tilde{c_1}, \tilde{c_2}$ and $\tilde{c_3}$ acting as barriers for $\tilde{s_1}$ and $\tilde{s_4}$. Since $\tilde{c}$ intersects $\tilde{e}$, it must intersect either $\tilde{w_1}$ or $\tilde{w_2}$ and thus c intersects w essentially. This concludes the proof. ◁

Therefore, the family $\mathcal{S} \cup \mathcal{S}_2$, intersects every possible simple non-contractible curve in the surface: it is filling. This claim concludes the proof. ◀

We encapsulate this last result in the following definition: we say that a system of curves $\mathcal{C}$ is **k-saturating** if there exists another system of curves $\mathcal{S}$ such that:

- $\mathcal{C} \cup \mathcal{S}$ is in minimal position,
- Each curve in $\mathcal{S}$ has length at most k, and
- The system of curves $\mathcal{S}$ is filling.

Therefore, Proposition 7 shows that a face-bipartite k-irreducible system of curves is $2k$-saturating. The following lemma shows that one can extract a filling family of size $O(g)$ from any filling family. This is certainly folklore, see for example Chen [14, Theorem 3.15]. We include a proof in the full version [21] for completeness.

▶ **Lemma 12.** *Let $\mathcal{C}$ be a filling system of curves in minimal position on a closed orientable surface S of genus $g \geqslant 1$. Then there is a sub-system $\mathcal{C}' \subset \mathcal{C}$ of at most $3g-1$ curves which is also filling.*

We can now combine our tools to obtain a first polynomial upper bound.

▶ **Theorem 13.** *Let $\mathcal{C}$ be a k-saturating system of curves. Then the crossing number of $\mathcal{C}$ is at most $2((3g-1)\cdot k)^2$.*

Proof. By the definition of a k-saturating system of curves and thanks to Lemma 12, there exists a filling family $\mathcal{S} = (s_i)_{i\in I}$ of closed curves of length at most k with respect to $\mathcal{C}$ and of size at most $3g-1$. In particular, for every $i \in I$, $\mathrm{cr}(s_i, \mathcal{C}) \leqslant k$.

Recall that by the definition of k-saturating, the system $\mathcal{C} \cup \mathcal{S}$ is in minimal position. We think of $\mathcal{S}$ as a 4-valent graph that is a combinatorial surface. In other words:

- Every curve of $\mathcal{C}$ intersect transversely the curves s_i, and
- Every intersection vertex of $\mathcal{C}$ is inside a face of $\mathcal{S}$.

We denote by F the faces of $\mathcal{S}$. For such a face f, let $p(f)$ denote the perimeter of f i.e. the number of crossings of $\mathcal{C}$ with the boundary of f. Then, by double counting, $\sum_{f\in F} p(f) = 2\,\mathrm{cr}(\mathcal{S}, \mathcal{C}) \leqslant 2 \cdot k \cdot \#\mathcal{S} \leqslant 2(3g-1)k$.

Similarly let $n(f)$ denote the number of vertices of $\mathcal{C}$ in the face f. Then $\mathrm{cr}(\mathcal{C}) = \sum_{f\in F} n(f)$. As $\mathcal{C}$ is in minimal position, the restriction of $\mathcal{C}$ to any face f is also in minimal position. This implies that any two curves in $\mathcal{C}$ cross at most once in f, since otherwise they would form a bigon and thus would not be in minimal position. Thus $n(f) \leqslant \dfrac{p(f)\cdot(p(f)-1)}{2}$. So

$$\mathrm{cr}(\mathcal{C}) = \sum_{f\in F} n(f) \leqslant \sum_{f\in F} \frac{p(f)^2}{2} - \frac{p(f)}{2} \leqslant \frac{1}{2}\left(\sum_{f\in F} p(f)\right)^2 \leqslant \frac{(2(3g-1)k)^2}{2}. \qquad \blacktriangleleft$$

Combining Proposition 7 and Theorem 13 provides us with a first bound of $O(k^2g^2)$ on the number of vertices in a k-irreducible system of curves, and thus on the number of edges in a k-irreducible triangulation via the reduction in Section 3. In the next section we add another ingredient to strengthen this bound to $O(k^2g)$.

▶ Remark 14. Note that the family $\mathcal{C}$ in Theorem 13 is not assumed to be k-irreducible, and thus this theorem could be of independent interest. It is easy to see that Theorem 13 is tight up to a constant factor, by considering a genus g orientable surface obtained by identifying opposite edges in a $2g$-gon, and a system of curves made of k parallel curves between each pair of opposite edges.

5 Upper bounds for k-irreducible systems of curves

In this section, we show:

▶ **Theorem 2.** *Let $\mathcal{C}$ be a face-bipartite k-irreducible system of curves on an orientable surface of genus g. The crossing number of $\mathcal{C}$ is at most $120k^2g = O(k^2g)$.*

To prove this result, we first establish a preliminary lemma on configuration of arcs in a disk. An **arc** in a disk is a curve with endpoints on the boundary of the disk. For a system of arcs $\mathcal{A}$ in minimal position in a disk, we denote by d the largest distance between a face of $\mathcal{A}$ and the boundary of the disk (where we measure lengths in terms of crossings with $\mathcal{A}$) and by n the number of intersections between pairs of arcs of $\mathcal{A}$. The following inequality allows us to bound n in terms of d and p.

▶ **Lemma 15.** *Let $\mathcal{A}$, p, d, n be as above. Then $(6d+7)p \geqslant n$.*

Our proof relies on a recent structural result of Hickingbotham, Illingworth, Mohar and Wood [27] and the fact that a graph of treewidth k with n vertices has less than kn edges (see, e.g., Baste, Noy and Sau [4]).

Proof. Let G be the intersection graph of $\mathcal{A}$: it has one vertex per arc of $\mathcal{A}$ and an edge between vertices corresponding to intersecting pair of arcs. By definition, G has p vertices and n edges.

Let M_G be the map graph of $\mathcal{A}$ which has one vertex for each face of $\mathcal{A}$, including the outer face, and edges between vertices corresponding to faces that share at least one vertex. Let $\mathrm{rad}(M_G)$ be its radius, that is, the minimum radius of a ball that covers M_G. Then, by [27, Theorem 5], we have $\mathrm{tw}(G) \leqslant 6 \cdot \mathrm{rad}(M_G) + 7$, where $\mathrm{tw}(G)$ denotes the tree-width of G.

Let v_∞ be the vertex of M_G corresponding to the external face. Then v_∞ is at distance at most d from every face of $\mathcal{A}$ in the dual graph of $\mathcal{A}$. It follows from the definitions that the distances in the map graph are smaller than the one in the dual graph, so v_∞ is at distance at most d in M_G from every point of M_G. Thus $\mathrm{rad}(M_G) \leqslant d$ and $\mathrm{tw}(G) \leqslant 6d+7$.

Since a graph of tree-width t and v vertices has strictly less than tv edges, $(6d+7)p \geqslant n$. ◀

We can now conclude the proof of Theorem 2.

Proof of Theorem 2. The proof starts like the proof of Theorem 13. By Proposition 7 and Lemma 12, a k-irreducible system of curves is $2k$-saturating and thus there is a filling family $\mathcal{S}$ of at most $3g-1$ closed curves of length at most $2k$ with respect to $\mathcal{C}$. We think of $\mathcal{S}$ as a 4-valent combinatorial surface with a family of faces F and denote by $n(f)$ the number of vertices of $\mathcal{C}$ in a face f, and by $p(f)$ the perimeter of f, i.e., the number of intersections of $\mathcal{C}$ and the boundary of f.

▷ Claim 16. For any face f of F, $n(f) \leqslant (3k+7) \cdot p(f)$

Proof. Assume that there is a face f such that $n(f) > (3k+7)p(f)$. By Lemma 15, $(6d+7)p(f) > (3k+7)p(f)$ and thus $d > k/2$. In this case, there is a point R in f so that any path from R to the boundary of f crosses more than $k/2$ curves of $\mathcal{C}$. This point R lies in a face of $\mathcal{C}$, and this implies that smoothing one of the vertices on the boundary of that face cannot change the length of any non-contractible closed curve of length k, since any such curve going through (two consecutive edges adjacent to) that vertex (as in Claim 9) must be longer than k. See Figure 7. Thus $\mathcal{C}$ is not a k-irreducible system of curves. ◁

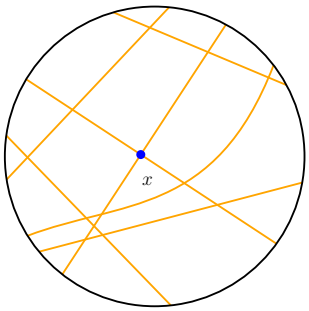

Figure 7 A system of arcs in a disk with a face x at distance 2 and a blue vertex that may be smoothed without changing the length of any curve of length smaller than 2.

We conclude using $\sum_{f\in F} n(f) = \mathrm{cr}(\mathcal{C})$ and $\sum_{f\in F} p(f) = 2\,\mathrm{cr}(\mathcal{S},\mathcal{C}) \leqslant 2(3g-1)2k$, which gives

$$\mathrm{cr}(\mathcal{C}) = \sum_{f\in F} n(f) \leqslant \sum_{f\in F} (3k+7)\cdot p(f) \leqslant (3k+7)2(3g-1)2k \leqslant 120k^2 g. \qquad \blacktriangleleft$$

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