Cauchy's Surface Area Formula in the Funk Geometry

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Abstract

Cauchy's surface area formula expresses the surface area of a convex body as the average area of its orthogonal projections over all directions. While this tool is fundamental in Euclidean geometry, with applications ranging from geometric tomography to approximation theory, extensions to non-Euclidean settings remain less explored. In this paper, we establish an analog of Cauchy's formula for the Funk geometry induced by a convex body K in $\mathbb{R}^d$, for the Holmes–Thompson surface area. The formula is based on central projections to boundary points of K. We show that when K is a convex polytope, the formula reduces to a weighted sum of contributions associated with the vertices of K. Finally, as a consequence of our analysis, we derive a generalization of Crofton's formula for surface areas in the Funk geometry. By viewing Euclidean, Minkowski, Hilbert, and hyperbolic geometries as limiting or special cases of the Funk setting, our results provide a unified framework for these classical surface area formulas.

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1 Introduction

Cauchy's surface area formula is a surprising and elegant result in Euclidean geometry. It states that the surface area of any convex body is a constant multiple of the average $(d-1)$-dimensional measure of its orthogonal projections, or "shadows", taken over all possible directions. To state the result formally in any dimension $d \geq 1$, let S^d denote the d-dimensional unit sphere, λ_d denote the standard d-dimensional Lebesgue measure, ω_d denote the volume of a d-dimensional Euclidean unit ball, and $(\cdot \mid \cdot)$ denote orthogonal projection. Cauchy's formula states that for any convex body K in $\mathbb{R}^d$, its surface area is

$$\frac{1}{\omega_{d-1}} \int_{S^{d-1}} \lambda_{d-1}(K \mid u^{\perp})\, du, \tag{1}$$

where $u^{\perp}$ is the linear subspace orthogonal to u. The related *Crofton formula* allows us to express the surface area of a convex body K as the average number of lines that intersect K, assuming that the lines are sampled from an appropriate measure.

Cauchy's formula is computationally fundamental, as it underpins algorithms in applications where estimating global geometric properties from lower-dimensional measurements is a recurring task, including geometric tomography [18], stereology [10], and surface area estimation for digitized 3D objects [24, 22]. Integral formulas of this form are the essential first steps in efficient sampling processes, such as computing ε-nets [3, 14], since they implicitly define the probability distribution upon which to base the sampling.

Modern applications have increasingly explored non-Euclidean spaces, such as the *Hilbert geometry* and its close relative, the *Funk geometry.* (See Section 2 for definitions.) These geometries arise naturally whenever the domain of interest is a convex set. Examples include the analysis of discrete probability distributions in machine learning, where the domain is the probability simplex [12, 25], analysis of networks through hyperbolic geometry [21], analysis of positive definite matrices in deep learning [20, 23], and lattice-based cryptosystems [8]. Hilbert is more widely studied than Funk, but Faifman argues that for many applications, including computing volumes and areas, the Funk geometry is more natural and yields cleaner results [16].

While the metric properties of these spaces are well-understood in differential geometry [16, 26, 27], computational aspects of these geometries are only beginning to emerge. Recent work has employed the Hilbert geometry for polytope approximation [7, 8], clustering [25], and approximate membership queries [2]. Despite this growing algorithmic interest, the integral geometry of these spaces lacks the computational primitives required for efficient implementations. A major barrier is that the standard definition of surface area (the Holmes–Thompson area [19]) relies on symplectic forms or global integrals involving the polar body, making direct numerical evaluation prohibitively complex for high-dimensional or large-scale settings.

In this paper, we bridge this gap by establishing explicit Cauchy and Crofton formulas for the Funk geometry. Our formulas transform the abstract definitions of Finsler area into simple, concrete geometric quantities, which are readily accessible to discrete computation. Beyond the specific interest in Funk geometry, our results reveal it as a unifying bridge connecting diverse geometric settings.

Our main result is a Funk analog of Cauchy's formula. Let K be a convex body in $\mathbb{R}^d$, and let G be a convex set contained in the interior of K. Our objective is to compute the Holmes–Thompson surface area of G in the Funk geometry induced by K, which we denote by $\operatorname{area}^F_K(G)$. For simplicity, let us assume for now that K's boundary is strictly convex, implying that for each direction $u \in S^{d-1}$, there is a unique boundary point $v_K(u)$ whose supporting hyperplane is orthogonal to u (see Figure 1(a)). Define the *central shadow* $S_K(G, u)$ as the slice of the cone subtended by G at $v_K(u)$, orthogonal to the direction u:

$$S_K(G, u) \;=\; -u^* \cap \operatorname{cone}(G - v_K(u)),$$

where $-u^*$ is the hyperplane tangent to S^{d-1} at $-u$ and $\operatorname{cone}(G - v_K(u))$ is the translation of the subtended cone to the origin (see Figure 1(b)). The measure of this section captures the "visual size" of G as seen from the boundary point $v_K(u)$. Just as the Euclidean formula averages orthogonal shadows, our Funk formula averages these central shadows. We show that G's surface area is proportional to the average area of these shadows, taken over all directions u.

▶ **Theorem 1.1.** *Let G and K be two convex bodies in $\mathbb{R}^d$ such that $G \subset \operatorname{int}(K)$. Then*

$$\operatorname{area}^F_K(G) \;=\; \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \lambda_{d-1}(S_K(G, u))\, d\sigma(u),$$

where σ denotes the rotation-invariant surface measure on S^{d-1}.

This offers a "tomographic interpretation" in the sense that the intrinsic Funk surface area can be recovered solely from these central slices, which capture the visibility of G from the boundary of K. It is interesting to note that, while the Holmes–Thompson surface area

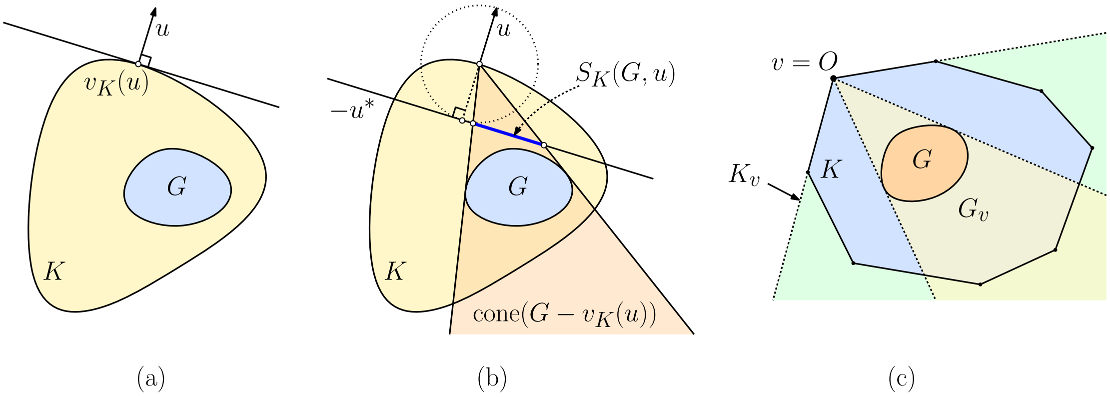

Figure 1 (a) The boundary point $v_K(u)$, (b) the central shadow $S_K(G, u)$ (translated so that $v_K(u)$ coincides with the origin), and (c) vertex decomposition.

of G (presented in Section 2.2) is defined in terms of the areas of the polars of Finsler balls in the Funk geometry induced by K, the formula of Theorem 1.1 involves only simple Euclidean quantities and the standard Lebesgue measure.

Our formulation connects the Funk geometry to several classical settings. When K is expanded to infinity, the central shadows become parallel projections, and our result converges to a Cauchy-type formula for the surface area in Minkowski spaces. In the special case where K is a Euclidean ball, this limit yields the classical Euclidean Cauchy formula. Furthermore, for a finite Euclidean ball K, our formula yields the exact surface area in the Beltrami-Klein model of hyperbolic geometry [13], generalizing planar observations by Alexander *et al.* [5]. Finally, for arbitrary convex bodies, our Funk surface area serves as a constant-factor approximation for the Hilbert surface area, with factors depending only on the dimension d. These results have been omitted due to space restrictions, but they appear in the full version of the paper [9].

Crucially for algorithmic applications, we derive a *discrete surface area formula* when K is a polytope. We show that the Funk surface area of a body G nested within K can be decomposed into a sum of local terms associated with the *vertices of* K. Each vertex v of K is naturally associated with a pointed cone $K_v = \mathrm{cone}(K - v)$, which is the cone subtended by K after translating v to the origin (see Figure 1(c)). The body G is similarly associated with a cone $G_v = \mathrm{cone}(G - v)$. In Section 2.3, we define a simple shadow-based notion of the Funk volume of G_v with respect to K_v, which we denote by $\mathrm{vol}^F_{K_v}(G_v)$. We show that the total surface area of G can be expressed as the sum of these cone-based Funk volumes over the vertices of K.

▶ **Theorem 1.2** (Vertex Decomposition for Polytopes)**.** *Let K be a convex polytope in $\mathbb{R}^d$, and let $G \subset \mathrm{int}(K)$ be a convex body. For each vertex $v \in V(K)$, let $K_v = \mathrm{cone}(K - v)$ and $G_v = \mathrm{cone}(G - v)$. Then*

$$\mathrm{area}^F_K(G) \;=\; \sum_{v \in V(K)} \mathrm{vol}^F_{K_v}(G_v).$$

While the cone-based Funk volume defined in Section 2.3 involves integration, it can be readily estimated through random sampling. Furthermore, this decomposition scales linearly with the vertex count of K (see the discussion following the proof of Theorem 1.1 in Section 4). This is in contrast to prior Crofton measures for Hilbert geometries, which involve quadratic enumerations over face pairs [30].

Finally, we establish a *Funk–Crofton formula* (Theorem 4.4), expressing the Funk surface area as the K-dependent measure of the set of line parameters whose associated lines intersect the body. This result provides the basis for Monte-Carlo estimation of surface area, similar to methods used in Euclidean geometry [1, 22]. That is, one can approximate the Funk surface area simply by sampling from the associated parameter distribution and counting intersections of the corresponding lines with the body, bypassing the need for complex analytical evaluation.

1.1 Related Work

The study of integral geometry in projective Finsler spaces has a rich history in differential geometry [6, 29, 31]. Schneider [29] established the existence of Crofton measures for general projective Finsler spaces using the dual Holmes–Thompson volume. For the specific case of *polytopal Hilbert geometries*, he derived an explicit Crofton measure involving a combinatorial decomposition over pairs of complementary faces of the polytope [30]. Our work differs by focusing on Funk geometry and providing a decomposition based on the *vertices* of the ambient polytope, which as noted above offers significant computational advantages.

In the planar setting, Alexander [4] and later Alexander, Berg, and Foote [5] derived elegant Cauchy-type perimeter formulas for the Hilbert geometry using trigonometric integrals. Note that in this case, Hilbert and Funk surface areas coincide. Like ours, their formulas are explicit and admit a shadow interpretation in the Beltrami-Klein disk model of hyperbolic geometry. However, their techniques are specialized to the plane and do not seem to extend to arbitrary convex domains in higher dimensions. In contrast, our approach unifies these perspectives by showing that the "average of shadows" principle holds for general convex bodies in any dimension, with integrands that are directly interpretable as measures of central projections.

2 Preliminaries

Let us begin by presenting notation and terminology that will be used throughout the paper. We use $\langle \cdot, \cdot \rangle$ for the standard inner (dot) product on $\mathbb{R}^d$, and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ for the Euclidean norm. Let O denote the origin, let B^d be the Euclidean unit ball centered at O, and let $S^{d-1} = \partial B^d$ be the unit sphere. Given a linear subspace $E \subset \mathbb{R}^d$ and a set $K \subset \mathbb{R}^d$, let $K \,\big|\, E$ denote the orthogonal projection of K onto E.

A *convex body* in $\mathbb{R}^d$ is a compact convex set with nonempty interior. Given a convex set K, we denote its boundary and interior by ∂K and $\mathrm{int}(K)$, respectively. For $\alpha \geq 0$, αK denotes the dilation of K by factor α about the origin, and for $x \in \mathbb{R}^d$, $K + x$ denotes the translate of K by x. Given a convex body K and a direction $u \in S^{d-1}$, the *support function* of K is $h_K(u) = \sup\{\langle u, x \rangle : x \in K\}$, and the corresponding supporting hyperplane is

$$H_K(u) \;=\; \{x \in \mathbb{R}^d : \langle u, x \rangle = h_K(u)\}.$$

For $1 \leq k \leq d$, let λ_k denote the k-dimensional Hausdorff measure on $\mathbb{R}^d$. In particular, λ_d is the Lebesgue measure, and on any C^1 k-dimensional submanifold of $\mathbb{R}^d$, λ_k agrees with the usual k-dimensional surface area. Let σ denote the standard rotation-invariant surface measure on S^{d-1}. Finally, let $\omega_d = \lambda_d(B^d)$.

Given two C^1 hypersurfaces $\Phi, \Psi \subset \mathbb{R}^d$ and a differentiable map $f : \Phi \to \Psi$, the Jacobian of f at $x \in \Phi$ is the local area-stretch factor induced by the differential $Df(x)$ on the tangent space of Φ at x. Equivalently, it is the factor appearing in the change-of-variables formula for surface integrals (see, for example, [15]).

Given two convex sets $A, B \subset \mathbb{R}^d$, their *Minkowski sum* is

$$A + B \;=\; \{a + b \,:\, a \in A,\; b \in B\}.$$

Given a convex body $K \subset \mathbb{R}^d$, its *difference body* is

$$\Delta(K) \;=\; K + (-K),$$

which is centrally symmetric (see Figure 2(a)).

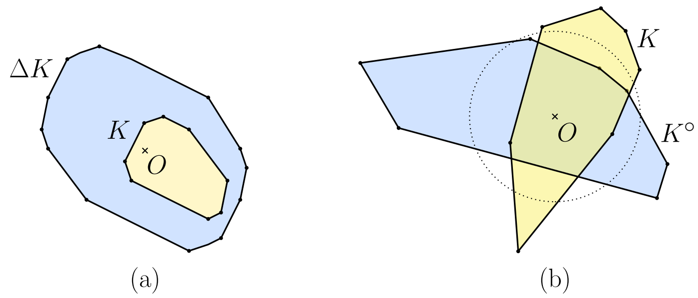

Figure 2 (a) The difference body of a convex body and (b) the polar of a convex body.

For any nonempty set $K \subset \mathbb{R}^d$, its *polar*, denoted K°, is defined by

$$K^\circ \;=\; \{y \in \mathbb{R}^d \,:\, \langle x, y\rangle \le 1 \text{ for all } x \in K\}$$

(see Figure 2(b)). The polar has several standard properties (see, for example, Barvinok [11]). It is always closed, convex, and contains the origin. If $O \in \mathrm{int}(K)$, then K° is bounded. Moreover, if K is a convex body with $O \in \mathrm{int}(K)$, then K° is also a convex body, and $(K^\circ)^\circ = K$. Finally, polarity reverses inclusion: if $K_1 \subseteq K_2$, then $K_2^\circ \subseteq K_1^\circ$.

For a linear subspace $E \subseteq \mathbb{R}^d$ and a nonempty set $G \subseteq E$, define the polar of G in E by

$$G^{\circ_E} \;=\; \{y \in E : \langle x, y\rangle \le 1, \text{ for all } x \in G\}.$$

The next lemma is the standard duality between sections and projections, stated in the form needed later. A proof may be found in [33, Theorem 2.2.9 and Corollary 2.2.10].

▶ **Lemma 2.1** (Projection-Section Duality)**.** *Let $K \subseteq \mathbb{R}^d$ be a convex body such that $O \in \mathrm{int}(K)$, and let E be a linear subspace of $\mathbb{R}^d$. Then the polar in E of the section $K \cap E$ equals the orthogonal projection of K° onto E. That is,*

$$(K \cap E)^{\circ_E} \;=\; K^\circ \,\big|\, E.$$

2.1 The Funk and Hilbert Geometries

In this section, we introduce the Funk and Hilbert geometries and the related Finsler structures used to define volume and surface area. Let K be a convex body in $\mathbb{R}^d$. For any two distinct points $x, y \in \mathrm{int}(K)$, let y' denote the point where the ray directed from x through y intersects ∂K (see Figure 3(a)). The *Funk distance* with respect to K, denoted $\mathrm{dist}^F_K(\cdot,\cdot)$, is defined to be

$$\mathrm{dist}^F_K(x, y) \;=\; \ln \frac{\|x - y'\|}{\|y - y'\|},$$

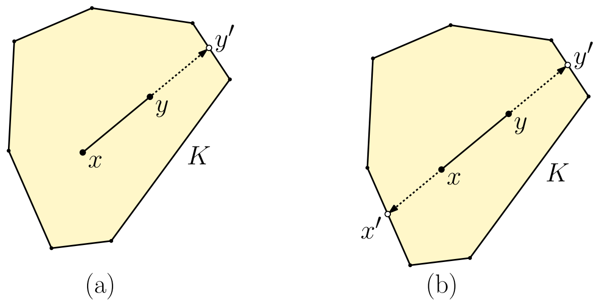

Figure 3 (a) The Funk distance and (b) the Hilbert distance.

where $\mathrm{dist}_K^F(x,y) = 0$ if $x = y$. The Funk distance is nonnegative, asymmetric, satisfies the triangle inequality, and is invariant under invertible affine transformations [27]. Because of its asymmetry, this is often referred to as a weak metric or quasi-metric.

The *Hilbert distance* with respect to K, denoted $\mathrm{dist}_K^H(\cdot,\cdot)$, is the symmetrization of the Funk distance. Letting x' denote the point where the directed ray from y through x intersects ∂K, it is defined to be

$$\mathrm{dist}_K^H(x,y) \;=\; \frac{1}{2}\left(\mathrm{dist}_K^F(x,y) + \mathrm{dist}_K^F(y,x)\right) \;=\; \frac{1}{2}\ln\frac{\|y-x'\|}{\|x-x'\|}\frac{\|x-y'\|}{\|y-y'\|}$$

(see Figure 3(b)). It is well known that this is a metric (symmetric, positive except when $x = y$, and satisfying the triangle inequality). Observe that the quantity in the logarithm is the cross ratio $(x, y; y', x')$, and hence the Hilbert distance is invariant under invertible projective transformations.

Before defining volume and surface areas, let us first introduce some related concepts. Given a convex body D in $\mathbb{R}^d$ with $O \in \mathrm{int}(D)$ and $u \in \mathbb{R}^d$, the *Minkowski functional* (or *gauge*) induced by D is defined by

$$\|u\|_D \;=\; \inf\{\lambda > 0 \,:\, u \in \lambda D\}.$$

This functional is positively homogeneous and subadditive. If D is centrally symmetric, then $\|\cdot\|_D$ is a norm in the usual sense; in general, it is the Minkowski functional (or gauge) of D. A *Finsler metric* on a manifold is a continuous function on its tangent bundle, whose restriction to each tangent space is such a gauge. It is well known that both the Funk and Hilbert geometries induce a Finsler structure [34].

To define the Finsler structure for Funk, consider any $x \in \mathrm{int}(K)$. Identify the tangent space at x, denoted T_x, with $\mathbb{R}^d$. For each nonzero $v \in T_x$, let x^+ denote the point where the ray emanating from x in direction v intersects ∂K (see Figure 4(a)), and let t be a positive scalar such that $x + tv = x^+$. The resulting Finsler gauge at x is defined as

$$\mathrm{Finsler}_K^F(x,v) \;=\; \frac{1}{t} \quad\text{and}\quad \mathrm{Finsler}_K^F(x,0) \;=\; 0.$$

This defines a gauge whose unit ball, denoted $B_K^F(x)$, is $K - x$. Thus, the ball is just a translate of K such that x coincides with the origin.

The Finsler structure for Hilbert is a symmetric variant of the Funk Finsler structure. For each $x \in \mathrm{int}(K)$ and each nonzero vector $v \in T_x$, let x^+ be as before, and let x^- be the point where the ray emanating from x in direction $-v$ intersects ∂K (see Figure 4(a)). Let t^+ and t^- be positive scalars such that $x + t^+v = x^+$ and $x - t^-v = x^-$. Then

$$\mathrm{Finsler}_K^H(x,v) \;=\; \frac{1}{2}\left(\frac{1}{t^+} + \frac{1}{t^-}\right) \quad\text{and}\quad \mathrm{Finsler}_K^H(x,0) \;=\; 0.$$

Since this gauge is symmetric, it is a norm. Its unit ball, denoted $B_K^H(x)$, is centrally symmetric (see Figure 4(b)). If K is a polytope, then so is $B_K^H(x)$.

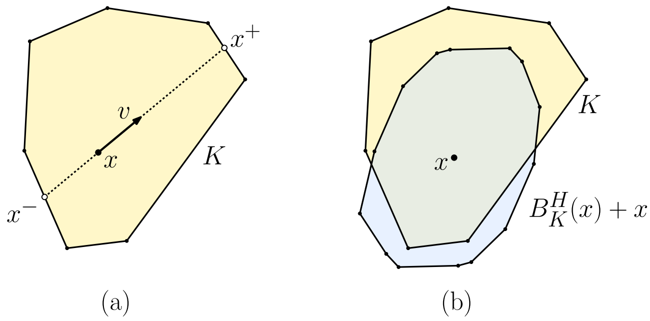

Figure 4 (a) The Finsler structure and (b) the Finsler ball for Hilbert (recentered about x).

The following lemma relates the polar of the Finsler ball in the Hilbert metric to the difference body of $(K - x)^\circ$. The lemma is a straightforward consequence of the Finsler interpretation of the Hilbert metric (see, e.g., [26, 31]).

▶ **Lemma 2.2.** *Given a convex body $K \subset \mathbb{R}^d$ and $x \in \operatorname{int}(K)$, $B_K^H(x)^\circ = \frac{1}{2}\Delta((K - x)^\circ)$.*

2.2 Holmes–Thompson Volume and Surface Area

There are several ways to define volume and surface area in Finsler spaces. The volume of a subset $G \subset \operatorname{int}(K)$ arises by integrating the weighted contributions of volume elements over G, where the weight of each element depends on the local geometry, as expressed through the Finsler ball. Intuitively, as the size of the Finsler ball decreases, the relative importance of the associated Lebesgue volume element increases. Due to the reciprocal nature of polarity, as the volume of the Finsler ball decreases, the volume of its polar increases. This gives rise to the following definitions, due to Holmes and Thompson [19].

Let us first consider the Funk geometry. Given a convex body K and $x \in \operatorname{int}(K)$, the Holmes–Thompson volume element is obtained by scaling the Lebesgue volume element $d\lambda_d(x)$ by the ratio of the Lebesgue volume of the polar of the Funk Finsler ball, $\lambda_d\left(B_K^F(x)^\circ\right)$, and the Lebesgue volume of the unit Euclidean ball, ω_d [17, 19]. Define the *Funk volume element* to be

$$d \operatorname{vol}_K^F(x) \;=\; \frac{1}{\omega_d} \lambda_d\left(B_K^F(x)^\circ\right) d\lambda_d(x).$$

Recall that $B_K^F(x) = K - x$. The *Funk volume* of any convex body $G \subset \operatorname{int}(K)$ is

$$\operatorname{vol}_K^F(G) \;=\; \int_{x \in G} d \operatorname{vol}_K^F(x).$$

The Holmes–Thompson surface area in the Funk geometry is defined analogously. At each smooth boundary point $x \in \partial G$, let T_x denote the tangent space to ∂G at x, viewed as a linear subspace of $\mathbb{R}^d$ and equipped with the induced Hausdorff measure. The *Funk surface-area element* is given by

$$d \operatorname{area}_K^F(x) \;=\; \frac{1}{\omega_{d-1}} \lambda_{d-1}\left(\left(B_K^F(x) \cap T_x\right)^{\circ_{T_x}}\right) d\lambda_{d-1}(x).$$

By Lemma 2.1, we can express this alternatively in terms of the projected polar as

$$d \operatorname{area}_K^F(x) \;=\; \frac{1}{\omega_{d-1}} \lambda_{d-1}\left(B_K^F(x)^\circ \,\middle|\, T_x\right) d\lambda_{d-1}(x).$$

Since $B^F_K(x) = K - x$, this is exactly

$$d\,\mathrm{area}^F_K(x) \;=\; \frac{1}{\omega_{d-1}} \lambda_{d-1}\left((K-x)^\circ \,\middle|\, T_x\right)\, d\lambda_{d-1}(x). \tag{2}$$

The *Funk surface area* of $G \subset \mathrm{int}(K)$ is

$$\mathrm{area}^F_K(G) \;=\; \int_{x \in \partial G} d\,\mathrm{area}^F_K(x).$$

The Holmes–Thompson volume and surface area in the Hilbert geometry are defined analogously, replacing the Funk Finsler ball $B^F_K(x)$ by the Hilbert Finsler ball $B^H_K(x)$. We denote the resulting quantities by $\mathrm{vol}^H_K(G)$ and $\mathrm{area}^H_K(G)$.

Faifman showed that the Hilbert- and Funk-based volumes and surface areas are related up to factors that depend on dimension.

▶ **Lemma 2.3** (Faifman [16]). *Let $K \subset \mathbb{R}^d$ be a convex body, let $G \subset \mathrm{int}(K)$ be a convex body, and let $\beta(d) = \binom{2d}{d}/2^d$. Then*

$$\begin{aligned} \mathrm{vol}^F_K(G) \;&\leq\; \mathrm{vol}^H_K(G) \;\leq\; \beta(d) \cdot \mathrm{vol}^F_K(G) \qquad \textit{and} \\ \mathrm{area}^F_K(G) \;&\leq\; \mathrm{area}^H_K(G) \;\leq\; \beta(d-1) \cdot \mathrm{area}^F_K(G). \end{aligned}$$

2.3 Cones

The standard Cauchy formula relies on orthogonal projections. We shall see that in the context of the Funk geometry induced by a convex body, the appropriate generalization utilizes central projections towards the boundary of the body. Such projections naturally involve cones. Throughout this paper, a *cone* in $\mathbb{R}^d$ denotes a full-dimensional convex set closed under positive scaling (i.e., if x is in the set, then λx is in the set for all $\lambda \geq 0$). We assume our cones are *pointed*, meaning that they contain no lines. This implies that each cone has a unique *apex* at the origin.

The *conical hull* of a convex set $G \subset \mathbb{R}^d$, denoted $\mathrm{cone}(G)$, is the smallest cone containing G, that is,

$$\mathrm{cone}(G) \;=\; \{\gamma x \,:\, x \in G \text{ and } \gamma \geq 0\}$$

(see Figure 5(a)). Let $K \subset \mathbb{R}^d$ be a closed, full-dimensional convex set. For any boundary point $v \in \partial K$, the *normal cone* at v consists of all outer normal vectors to K at v:

$$\{u \in \mathbb{R}^d : \langle u, x - v\rangle \leq 0, \text{ for all } x \in K\}.$$

(see Figure 5(b)). In the cases of primary interest to us, such as when v is a vertex of a polytope or the apex of a pointed cone, the tangent cone $K_v = \mathrm{cone}(K - v)$ is pointed. In these settings, the normal cone is full-dimensional and coincides with the polar cone K°_v.

For any nonzero vector $z \in \mathbb{R}^d$, we denote the associated *dual hyperplane* by

$$z^* = \{u \in \mathbb{R}^d \,:\, \langle u, z\rangle = 1\}.$$

For a pointed cone $K \subset \mathbb{R}^d$ and a vector $x \in \mathrm{int}(K)$, we define the *dual cross-section* $F_K(x)$ to be the intersection of the polar cone K° with the dual hyperplane associated with $-x$ (see Figure 5(c)). That is,

$$F_K(x) \;:=\; K^\circ \cap -x^* \;=\; K^\circ \cap (-x)^* \;=\; \{y \in K^\circ : \langle x, y\rangle = -1\}. \tag{3}$$

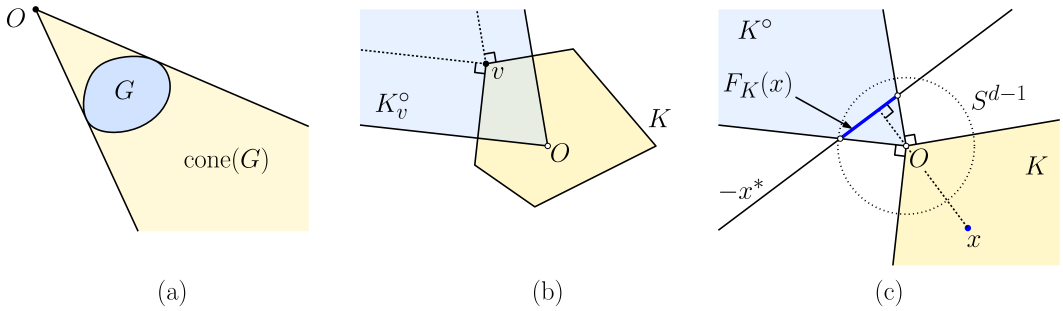

Figure 5 (a) The cone cone(G), (b) the normal cone K_v°, and (c) $F_K(x)$.

Geometrically, this is the slice of the polar cone by a hyperplane orthogonal to the direction x. Since x lies in the interior of K, and every vector in K° has nonpositive inner product with every vector in K, this hyperplane intersects K° in a bounded set.

Next, we establish a simple technical lemma relating the polar of a body to the polar of its subtended cones. This result allows us to characterize the facets of the polar polytope using local cone geometry.

▶ **Lemma 2.4.** *Let $Q \subset \mathbb{R}^d$ be a convex body containing the origin in its interior. Let $v \in \mathbb{R}^d \setminus \mathrm{int}(Q)$. Let $Q_v = \mathrm{cone}(Q - v)$ be the cone subtended by Q at v (see Figure 6(a)). Then the intersection of the polar cone Q_v° with the dual hyperplane $v^* = \{y \in \mathbb{R}^d : \langle v, y\rangle = 1\}$ coincides with the slice of the polar body Q° by v^*, that is, $Q_v^\circ \cap v^* = Q^\circ \cap v^*$ (see Figure 6(b) and (c)).*

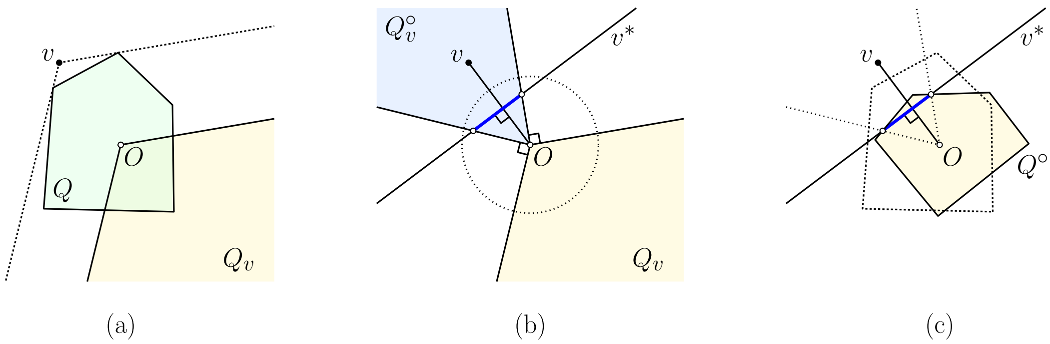

Figure 6 (a) The cone Q_v, (b) $Q_v^\circ \cap v^*$, and (c) $Q^\circ \cap v^*$.

Proof. By definition, a vector $y \in Q_v^\circ$ if and only if $\langle z, y\rangle \le 0$ for all $z \in Q_v$. Since Q_v is the conical hull of $Q - v$, this condition is equivalent to:

$$\langle x - v, y\rangle \ \le \ 0, \quad \forall x \in Q.$$

Restricting our attention to the hyperplane v^*, where $\langle v, y\rangle = 1$, this inequality becomes:

$$\langle x, y\rangle - \langle v, y\rangle \le 0 \iff \langle x, y\rangle - 1 \le 0 \iff \langle x, y\rangle \le 1, \quad \forall x \in Q.$$

The condition $\langle x, y\rangle \le 1$ for all $x \in Q$ is exactly the definition of the polar body Q°. Thus, restricted to the hyperplane v^*, the condition for membership in the polar cone is identical to the condition for membership in the polar body. ◀

When applied to the vertices of a polytope, this lemma provides a functional characterization of the boundary of the polar body. Recall that for a polytope K with $O \in \mathrm{int}(K)$, the facets of the polar body K° are in one-to-one correspondence with the vertex set $V(K)$ [35]. Specifically, for each vertex $v \in V(K)$, the dual facet is the intersection of K° with the supporting hyperplane v^*. Lemma 2.4 implies that this facet is exactly $K_v^\circ \cap v^*$, which matches the definition of the dual cross-section $F_{K_v}(-v)$.

▶ **Corollary 2.5.** *Let $K \subset \mathbb{R}^d$ be a convex polytope with $O \in \mathrm{int}(K)$. For each vertex $v \in V(K)$, the facet of K° dual to v, denoted F_v, is given by*

$$F_v \;=\; K_v^\circ \cap v^* \;=\; F_{K_v}(-v).$$

Furthermore, $\partial K^\circ = \bigcup_{v \in V(K)} F_v$.

Extending the definition of Funk volume from bounded convex bodies to unbounded cones requires care. If G and K are pointed cones in $\mathbb{R}^d$ with $(G \setminus \{O\}) \subset \mathrm{int}(K)$, the standard Funk distance is degenerate, and the classical volume is infinite. However, the projective nature of Funk geometry allows us to define a meaningful volume for G relative to K by considering cross-sections. Faifman [16] demonstrated that the Funk geometry is essentially projective. A key consequence is that the Holmes–Thompson volume of the section $G \cap H$, measured with respect to the ambient section $K \cap H$, is invariant to the choice of the hyperplane H, provided $K \cap H$ is bounded. We call such a hyperplane *admissible*.

This invariance implies that the volume is intrinsic to the nested cone structure itself. We define the *cone Funk volume* of G relative to K by the integral

$$\mathrm{vol}_K^F(G) \;:=\; \frac{1}{\omega_{d-1}} \int_{\Sigma(G)} \lambda_{d-1}(F_K(s))\, d\sigma(s), \qquad \text{where } \Sigma(G) := S^{d-1} \cap G \tag{4}$$

(see Figure 7(a)). In the full version of the paper [9], we show that this integral coincides with $\mathrm{vol}_{K\cap H}^F(G \cap H)$ for any admissible hyperplane H, thus justifying this terminology.

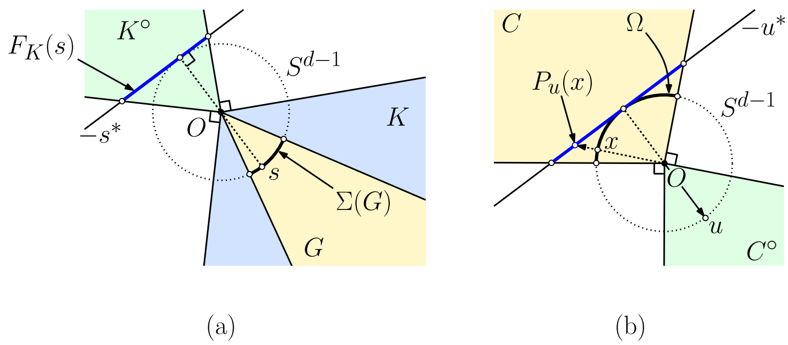

Figure 7 (a) Defining $\mathrm{vol}_K^F(G)$ and (b) the gnomonic projection.

When dealing with spherical cross-sections of cones, it is useful to relate their spherical measures to the Euclidean measures of corresponding hyperplane sections. Let $C \subset \mathbb{R}^d$ be a pointed cone, let $\Omega = C \cap S^{d-1}$, and let $u \in S^{d-1}$ satisfy $\langle x, u \rangle < 0$ for all $x \in \Omega$. The associated gnomonic projection maps Ω onto the hyperplane section $-u^* \cap C$ (see Figure 7(b)). A straightforward Jacobian computation yields the following lemma (see, e.g., [16, Lemma 3.10]).

▶ **Lemma 2.6** (Cone gnomonic projection). *Let $C \subset \mathbb{R}^d$ be a pointed cone, let $\Omega = C \cap S^{d-1}$, and let $u \in S^{d-1}$ satisfy $\langle x, u\rangle < 0$, for all $x \in \Omega$ (equivalently, $u \in \mathrm{int}(C^\circ)$). Define*

$$P_u(x) \;:=\; -\frac{x}{\langle x, u\rangle} \qquad (x \in \Omega).$$

Then $P_u(\Omega) = -u^* \cap C$, and

$$\lambda_{d-1}(P_u(\Omega)) \;=\; \int_\Omega |\langle x, u\rangle|^{-d}\, d\sigma(x).$$

3 Vertex Decomposition for Convex Polytopes

In this section, we establish a discrete surface area formula for the case where K is a convex polytope. Our result provides a vertex-based decomposition of the Funk surface area similar in spirit to that of Alexander *et al.* [5] in the planar Hilbert geometry, but derived by a distinct approach that applies in arbitrary dimensions.

Our proof proceeds by decomposing the Funk surface area of G into local contributions associated with the vertices of the ambient polytope K. For each vertex $v \in V(K)$, let $K_v = \mathrm{cone}(K - v)$ and $G_v = \mathrm{cone}(G - v)$ denote the local cones subtended at v. We will show that the total Funk surface area of G is the sum of the Funk volumes of the cones G_v relative to K_v. We begin with a lemma that expresses the Funk volume of $\mathrm{cone}(G)$ relative to a pointed cone K as a boundary integral involving the projected areas of the dual cross-sections $F_K(x)$. In the statement below, T_x denotes the tangent space of G at x.

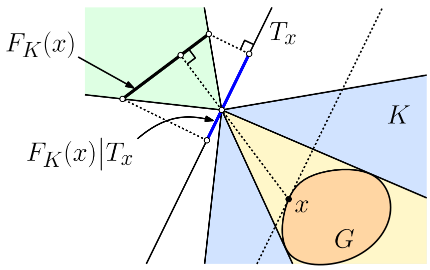

Figure 8 Lemma 3.1.

▶ **Lemma 3.1.** *Let $K \subset \mathbb{R}^d$ be a pointed cone and let $G \subset \mathrm{int}(K)$ be a convex body. Then*

$$2 \cdot \mathrm{vol}^F_K(\mathrm{cone}(G)) \;=\; \frac{1}{\omega_{d-1}} \int_{\partial G} \lambda_{d-1}\big(F_K(x) \,\big|\, T_x\big)\, d\lambda_{d-1}(x)$$

(see Figure 8).

The next step in our derivation utilizes the dual facet characterization (Corollary 2.5) to decompose the projection of the polar body into a sum over its facets.

▶ **Lemma 3.2.** *Let $K \subset \mathbb{R}^d$ be a convex polytope. For any $x \in \mathrm{int}(K)$ and any $(d-1)$-dimensional linear subspace T,*

$$2\lambda_{d-1}\big((K - x)^\circ \,\big|\, T\big) \;=\; \sum_{v \in V(K)} \lambda_{d-1}\big(F_{K_v}(x - v) \,\big|\, T\big)$$

(see Figures 9(a) and (b)).

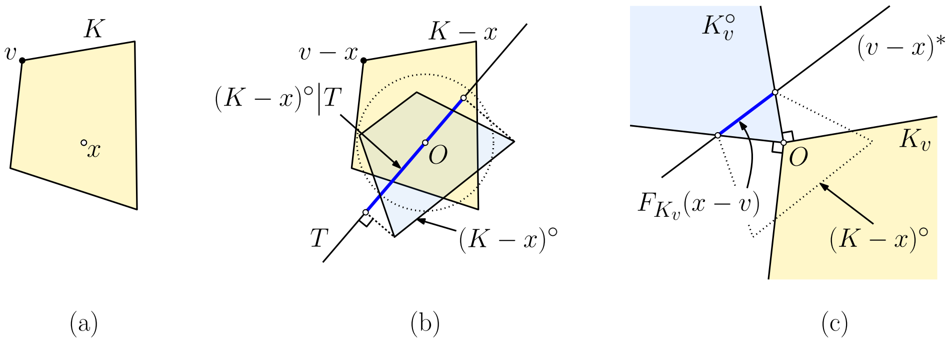

Figure 9 Lemma 3.2 and its proof.

The proofs of Lemmas 3.1 and 3.2 appear in the full version of the paper [9].

We are now ready to prove the vertex decomposition theorem from Section 1.

Proof. (Of Theorem 1.2) For each $v \in V(K)$, applying Lemma 3.1 to the pair $(K_v, G - v)$ yields

$$2 \cdot \mathrm{vol}^F_{K_v}(G_v) \;=\; \frac{1}{\omega_{d-1}} \int_{\partial(G-v)} \lambda_{d-1}\left(F_{K_v}(x') \,\middle|\, T_{x'}\right) \, d\lambda_{d-1}(x').$$

Perform the change of variables $x = x' + v$. Then $\partial(G - v)$ maps to ∂G, and the tangent spaces are identified as $T_{x'} = T_x$, yielding

$$2 \cdot \mathrm{vol}^F_{K_v}(G_v) \;=\; \frac{1}{\omega_{d-1}} \int_{\partial G} \lambda_{d-1}\left(F_{K_v}(x - v) \,\middle|\, T_x\right) \, d\lambda_{d-1}(x).$$

Summing over $v \in V(K)$ and using linearity of integration, we have

$$2 \sum_{v \in V(K)} \mathrm{vol}^F_{K_v}(G_v) \;=\; \frac{1}{\omega_{d-1}} \int_{\partial G} \left[\sum_{v \in V(K)} \lambda_{d-1}\left(F_{K_v}(x - v) \,\middle|\, T_x\right) \right] d\lambda_{d-1}(x).$$

By Lemma 3.2, the term in brackets is equal to $2\,\lambda_{d-1}((K - x)^\circ \mid T_x)$. Substituting this and dividing by 2 yields

$$\sum_{v \in V(K)} \mathrm{vol}^F_{K_v}(G_v) \;=\; \frac{1}{\omega_{d-1}} \int_{\partial G} \lambda_{d-1}\left((K - x)^\circ \,\middle|\, T_x\right) \, d\lambda_{d-1}(x).$$

By Eq. (2), the right-hand side is exactly $\mathrm{area}^F_K(G)$. ◀

4 Cauchy-Type Formulas for General Convex Bodies

We now derive the general Cauchy formula from the polytope case. The key additional ingredient is a representation of the Funk volume of a cone as an average of the areas of its central shadows. Combined with Theorem 1.2, this yields the formula for polytopal K, and the general case then follows by approximation.

▶ **Lemma 4.1** (Funk Volume of a Cone). *Let K and G be pointed cones in $\mathbb{R}^d$ with $(G \setminus \{O\}) \subset \mathrm{int}(K)$. Let $U = K^\circ \cap S^{d-1}$ (the spherical image of the polar cone), and for $u \in U$, let $S(u) = -u^* \cap G$ (see Figure 10(a)). Then*

$$\mathrm{vol}^F_K(G) \;=\; \frac{1}{\omega_{d-1}} \int_U \lambda_{d-1}(S(u)) \, d\sigma(u). \tag{5}$$

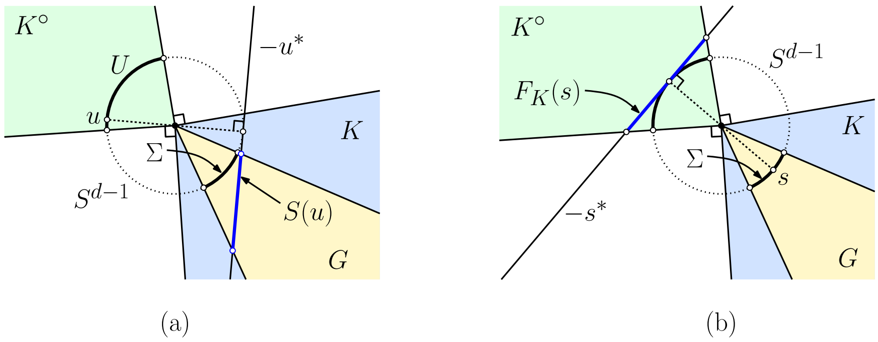

Figure 10 Lemma 4.1 and its proof.

Proof. Let $\Sigma = S^{d-1} \cap G$ and define the double integral

$$I \;=\; \frac{1}{\omega_{d-1}} \int_U \int_\Sigma |\langle s, u\rangle|^{-d}\, d\sigma(s)\, d\sigma(u).$$

Since the integrand is nonnegative, by Tonelli's theorem [28, Theorem 8.8(a)], we may integrate in either order. We will show that one order leads to the right-hand side of Eq. (5), and the other leads to the left-hand side.

First fix $u \in U$ and consider the inner integral in s. Since $u \in K^\circ$ and $(G \setminus \{O\}) \subset \operatorname{int}(K)$, we have $\langle s, u\rangle < 0$ for all $s \in \Sigma$. Applying Lemma 2.6 to the cone G with center u, we obtain

$$\int_\Sigma |\langle s, u\rangle|^{-d}\, d\sigma(s) \;=\; \lambda_{d-1}(-u^* \cap G) \;=\; \lambda_{d-1}(S(u))$$

(see Figure 10(a)). Substituting into I yields the right-hand side of Eq. (5).

Next fix $s \in \Sigma$ and consider the inner integral in u. Since $s \in \Sigma \subset \operatorname{int}(K)$ and $U = K^\circ \cap S^{d-1}$, we have $\langle u, s\rangle < 0$ for all $u \in U$. Applying Lemma 2.6 to the cone K° with center s, we obtain

$$\int_U |\langle u, s\rangle|^{-d}\, d\sigma(u) \;=\; \lambda_{d-1}(-s^* \cap K^\circ) \;=\; \lambda_{d-1}(F_K(s))$$

(see Figure 10(b)). Substituting into I, we obtain

$$I \;=\; \frac{1}{\omega_{d-1}} \int_\Sigma \lambda_{d-1}(F_K(s))\, d\sigma(s).$$

By the definition of Funk volume (Eq. (4)), this expression is exactly $\mathrm{vol}_K^F(G)$, which matches the left-hand side of Eq. (5). ◀

With the cone formula established, we now derive the Cauchy formula for arbitrary convex bodies. Let $K \subset \mathbb{R}^d$ be a convex body and let $G \subset \operatorname{int}(K)$ be a convex body. For almost every direction $u \in S^{d-1}$, the support set $K \cap H_K(u)$ is a singleton; denote its unique point by $v_K(u)$ [32, Theorem 2.2.11]. On the exceptional null set, choose $v_K(u)$ arbitrarily in $K \cap H_K(u)$. For each $u \in S^{d-1}$, define the central shadow by

$$S_K(G, u) \;:=\; -u^* \cap \operatorname{cone}(G - v_K(u))$$

(recall Figure 1). We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Assume first that K is a convex polytope. For each vertex $v \in V(K)$, let

$$U_v \;:=\; K_v^\circ \cap S^{d-1}$$

denote the spherical image of the normal cone at v. The collection $\{U_v\}_{v\in V(K)}$ forms a partition of S^{d-1} up to a set of σ-measure zero. For almost every $u \in U_v$, the support point of K in direction u is unique and equals v. Recalling that $G_v = \operatorname{cone}(G - v)$, it follows that for almost every $u \in U_v$,

$$S_{K_v}(G_v, u) \;=\; S_K(G, u).$$

Therefore, by Theorem 1.2 and Lemma 4.1, we obtain

$$\begin{aligned}
\operatorname{area}^F_K(G) \;&=\; \sum_{v\in V(K)} \operatorname{vol}^F_{K_v}(G_v)\\
&=\; \sum_{v\in V(K)} \frac{1}{\omega_{d-1}} \int_{U_v} \lambda_{d-1}(S_{K_v}(G_v,u))\, d\sigma(u)\\
&=\; \frac{1}{\omega_{d-1}} \sum_{v\in V(K)} \int_{U_v} \lambda_{d-1}(S_K(G,u))\, d\sigma(u)\\
&=\; \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \lambda_{d-1}(S_K(G,u))\, d\sigma(u).
\end{aligned}$$

This establishes the result when K is a polytope. The general case follows by approximating K in the Hausdorff metric by convex polytopes K_m, applying the polytopal identity to K_m, and passing to the limit using the continuity of polarity and the Dominated Convergence Theorem; details appear in the full version [9]. ◀

From a computational perspective, when K is a polytope, the preceding proof yields a natural Monte-Carlo estimator. The identity

$$\operatorname{area}^F_K(G) \;=\; \sum_{v\in V(K)} \frac{1}{\omega_{d-1}} \int_{U_v} \lambda_{d-1}(S_{K_v}(G_v,u))\, d\sigma(u)$$

decomposes the surface area into local contributions indexed by the vertices of K. The sampling distribution depends only on the spherical normal-cone decomposition $\{U_v\}_{v\in V(K)}$, while the sampled quantity is the $(d-1)$-dimensional volume of the corresponding central shadow. Thus one may sample a vertex v with probability proportional to $\sigma(U_v)$ and then sample a direction u uniformly from U_v, for example by triangulating the normal cone into simplicial cones. This gives an unbiased estimator whose evaluation uses only geometry local to the chosen vertex. In particular, it avoids direct use of the Holmes–Thompson definition through Eq. (2), which would require integrating

$$\lambda_{d-1}\big((K-x)^\circ \,\big|\, T_x\big)$$

over all $x \in \partial G$.

Using Lemma 2.6, we can rewrite the shadow integral in Theorem 1.1 in terms of the spherical cross-sections of the subtended cones. For any $v \in \partial K$, let

$$\Sigma(G, v) \;:=\; S^{d-1} \cap \operatorname{cone}(G - v)$$

denote the spherical cross-section of the cone subtended by G at v. The following corollary gives a double-integral representation of the Funk surface area.

▶ **Corollary 4.2** (Double-Integral Formula for Funk Area). *Let G and K be convex bodies in $\mathbb{R}^d$ with $G \subset \mathrm{int}(K)$. Then*

$$\mathrm{area}_K^F(G) \;=\; \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \int_{\Sigma(G, v_K(u))} |\langle s, u\rangle|^{-d} \, d\sigma(s)\, d\sigma(u).$$

Proof. By Theorem 1.1,

$$\mathrm{area}_K^F(G) \;=\; \frac{1}{\omega_{d-1}} \int_{S^{d-1}} \lambda_{d-1}(S_K(G,u))\, d\sigma(u).$$

For each $u \in S^{d-1}$,

$$S_K(G,u) \;=\; -u^* \cap \mathrm{cone}(G - v_K(u)).$$

Since $v_K(u)$ is a support point of K in direction u and $G \subset \mathrm{int}(K)$, we have $\langle s, u\rangle < 0$ for all $s \in \Sigma(G, v_K(u))$. Applying Lemma 2.6 with $C = \mathrm{cone}(G - v_K(u))$ and center u, we obtain

$$\lambda_{d-1}(S_K(G,u)) \;=\; \int_{\Sigma(G, v_K(u))} |\langle s, u\rangle|^{-d} \, d\sigma(s).$$

Substituting this into the preceding formula completes the proof. ◀

We now recast Corollary 4.2 as a Crofton formula on oriented lines. For each $u \in S^{d-1}$, choose $v_K(u) \in K \cap H_K(u)$ as above, and let

$$\mathcal{P} \;:=\; \{(u,s) \in S^{d-1} \times S^{d-1} ST \ \langle s, u\rangle < 0\}.$$

For $(u,s) \in \mathcal{P}$, let $\ell_K^+(u,s)$ denote the oriented line through $v_K(u)$ in direction s. For fixed $u \in S^{d-1}$, a direction $s \in S^{d-1}$ with $\langle s, u\rangle < 0$ belongs to $\Sigma(G, v_K(u))$ if and only if the ray from $v_K(u)$ in direction s meets G. Thus Corollary 4.2 can be rewritten as follows.

▶ **Lemma 4.3** (Funk–Crofton formula for oriented lines). *Let G and K be convex bodies in $\mathbb{R}^d$ with $G \subset \mathrm{int}(K)$. Then*

$$\mathrm{area}_K^F(G) \;=\; \frac{1}{\omega_{d-1}} \int_{\mathcal{P}} \mathbf{1}_{\{\ell_K^+(u,s) \cap G \neq \emptyset\}} \, |\, \langle s, u\rangle \,|^{-d} \, d\sigma(s)\, d\sigma(u).$$

Replacing each oriented line by its underlying unoriented line, we obtain the following K-dependent parameter-space Crofton formula.

▶ **Theorem 4.4** (Funk–Crofton formula for unoriented lines). *Let G and K be convex bodies in $\mathbb{R}^d$ with $G \subset \mathrm{int}(K)$. Then*

$$\mathrm{area}_K^F(G) \;=\; \frac{1}{\omega_{d-1}} \int_{\mathcal{P}} \mathbf{1}_{\{L_K(u,s) \cap G \neq \emptyset\}} \, |\, \langle s, u\rangle \,|^{-d} \, d\sigma(s)\, d\sigma(u),$$

where $L_K(u,s)$ denotes the unoriented line underlying $\ell_K^+(u,s)$.

Proof. Since intersection with G is independent of orientation, the indicator in Lemma 4.3 is unchanged when $\ell_K^+(u,s)$ is replaced by its underlying unoriented line $L_K(u,s)$. ◀

Concluding Remarks and Open Problems

We established an explicit Cauchy-type formula for the Holmes–Thompson surface area in the Funk geometry induced by a convex body $K \subset \mathbb{R}^d$. This formula expresses the surface area of a convex body G as the average, over directions $u \in S^{d-1}$, of the $(d-1)$-dimensional measures of the corresponding central shadows of G (Theorem 1.1). For polytopal K, this identity admits a discrete vertex-based decomposition (Theorem 1.2), and the same framework yields a Crofton-type representation in terms of an explicit measure on the space of unoriented lines intersecting G (Theorem 4.4).

Taken together, these results provide a concrete shadow-averaging principle for surface area in a projective Finsler setting. From a computational perspective, our formulas involve Euclidean $(d-1)$-dimensional volumes of explicitly defined slices or projections, thereby avoiding direct evaluation of the Holmes–Thompson definitions through polars of pointwise Finsler balls. In the polytopal case, the resulting decomposition is especially simple, replacing more combinatorial Crofton descriptions based on pairs of faces by a sum over vertices and normal cones.

Our work raises several natural open problems. A first direction is to develop provably efficient randomized algorithms for estimating Funk surface area from the Cauchy and Crofton formulas, ideally with explicit variance bounds and high-probability guarantees. For polytopal K, this includes efficient sampling from the spherical normal-cone decomposition and efficient evaluation or approximation of the associated central shadows. It would also be interesting to determine to what extent these formulas can serve as practical primitives for geometric computation in Funk- and Hilbert-type domains, in a manner analogous to the role of Cauchy and Crofton formulas in Euclidean stereology, tomography, and randomized surface area estimation.

A second direction is to investigate whether this shadow-based approach extends beyond surface area to higher-order intrinsic quantities, such as the Holmes–Thompson analogs of quermassintegrals or curvature measures. A third is to better understand the relationship with Hilbert geometry, especially by identifying conditions on K under which the Funk formula yields the exact Hilbert surface area beyond the known cases of $d = 2$ and ellipsoids. More generally, it would be interesting to determine how geometric properties of K govern the approximation gap between Hilbert and Funk surface areas.

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