Intersection Patterns of Set Systems on Manifolds with Slowly Growing Homological Shatter Functions

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Abstract

A theorem of Matoušek asserts that for any $k \geq 2$, any set system whose shatter function is $o(n^k)$ enjoys a fractional Helly theorem of order k: in the k-wise intersection hypergraph, positive density implies a linear-size clique. Kalai and Meshulam conjectured a generalization of that phenomenon to *homological shatter functions*. It was verified for set systems with bounded homological shatter functions and whose ground set has a *forbidden homological minor* (which includes $\mathbb{R}^d$ by a homological analogue of the van Kampen-Flores theorem). We present two contributions to this line of research:

- We study homological minors in certain manifolds (possibly with boundary), for which we prove analogues of the van Kampen-Flores theorem and of the Hanani-Tutte theorem.
- We introduce graded analogues of the Radon and Helly numbers of set systems and relate their growth rate to the original parameters. This allows to extend the verification of the Kalai-Meshulam conjecture to sufficiently slowly growing homological shatter functions.

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1 Introduction

A classical line of research in discrete geometry investigates generalizations of properties of convex sets beyond convexity, with a particular attention to topological conditions. At least three distinct lines of inquiry emerged:

(A) Generalizing families of convex sets into *acyclic/good covers*, meaning set systems in topological spaces such that every subfamily has empty or homologically/homotopically trivial intersection; an early example is Helly's topological theorem [15], see also the survey of Tancer [34] for an overview.

(B) Reformulating properties of convex sets of $\mathbb{R}^d$ as properties of linear maps into $\mathbb{R}^d$ and investigating their generalizations to *continuous* maps into $\mathbb{R}^d$, typically via theorems of Borsuk-Ulam type; an early example is the topological Radon theorem of Bajmóczy and Bárány [3] and more examples can be found *e.g.* in the survey of Bárány and Soberón [5].

(C) Analyzing set systems whose nerve enjoy properties of nerves of convex sets, like *d-collapsibility* or *d-Lerayness*; an early example is the sharpening of the Fractional Helly theorem by Kalai [18] and the work of Alon et al. [1] establishes several landmark results.

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Two decades ago, Kalai and Meshulam [19] proposed conjectures relating approaches (A) and (C), towards what they called a theory of *homological VC dimension*. First, in order to generalize the notion of *good cover*, let us measure the complexity of the intersection patterns of a set system $\mathcal{F}$ in a topological space by its *hth homological shatter function*

$$\phi_{\mathcal{F}}^{(h)}: \begin{cases} \mathbb{N} & \to & \mathbb{N} \cup \{\infty\} \\ k & \mapsto & \sup\left\{ \tilde{\beta}_i\left(\bigcap_{F\in\mathcal{G}} F; \mathbb{Z}_2\right) \;\middle|\; \mathcal{G} \subset \mathcal{F}, |\mathcal{G}| \le k, 0 \le i \le h \right\}. \end{cases} \tag{1}$$

Here h is some fixed parameter, and $\tilde{\beta}_i(\cdot;\mathbb{Z}_2)$ is the ith reduced Betti number with coefficients in $\mathbb{Z}_2$. (The set systems $\mathcal{F}$ with $\phi_{\mathcal{F}}^{(\infty)} \equiv 0$ are the acyclic covers and include good covers and convex sets.) The conjectures are about nerves of set systems whose intersection patterns have polynomially-growing topological complexity. In particular, their combination [19, Conjectures 6 and 7] (see also [13, Conjecture 1.9]) implies that polynomially-growing homological shatter functions give rise to a "*positive density implies big clique*" phenomenon:

▶ **Conjecture 1** (Kalai and Meshulam). *For any $d \in \mathbb{N}$ and any function $\Psi : \mathbb{N} \to \mathbb{N}$ such that $\Psi(n) = O(n^d)$, there exists $\beta : (0,1) \to (0,1)$ such that the following holds. For any $\alpha > 0$ and any set system $\mathcal{F}$ in $\mathbb{R}^d$ with $\phi_{\mathcal{F}}^{(d)} \le \Psi$, if a proportion α of the $(d+1)$-element subsets of $\mathcal{F}$ have nonempty intersection, then some $\beta(\alpha)|\mathcal{F}|$ members of $\mathcal{F}$ have a point in common.*

Such conditions that a positive density of d-faces in the nerve implies the existence of a linear-size face is called a *fractional Helly theorem* (see Section 1.1.1).

Conjecture 1 is a topological analogue of a theorem of Matoušek [25] which asserts that every set system with polynomial (combinatorial) shatter function enjoys a fractional Helly theorem.

Conjecture 1 was confirmed for functions Ψ that are bounded ([13, Corollary 1.3], building on [16, 17, 28]). The main contribution of the present paper is to extend that confirmation to some diverging homological shatter functions and to set systems on certain manifolds.

1.1 Context and motivation

Before we state our results precisely (in Section 1.2) let us provide some context and motivation, as well as introduce some necessary terminology.

1.1.1 Combinatorial background: convexity parameters of set systems

The classical theorems of Helly, Radon and Carathéodory have initiated a rich theory of the combinatorial properties of convexity, whose landmarks include the centerpoint theorem, Tverberg's theorem, the colorful Helly and Carathéodory theorems, the fractional Helly theorem, the selection lemma, the weak ε-net theorem, the (p,q)-theorem, etc. We refer the interested reader to the monograph of Bárány [4] and the textbook of Matoušek [22]. These classical convexity theorems have algorithmic consequences for instance in optimization and geometric data analysis [8, § 6 − 7] or in property testing [7], and one motivation for their extension beyond the convex setting is that several of these benefits generalize as well [24, 11].

We can associate to any set system $\mathcal{F}$ with ground set X some parameters inspired by convexity properties, for instance:

- The *Helly number* $\mathrm{h}_{\mathcal{F}}$ of $\mathcal{F}$ is the smallest integer h with the following property: If in a finite subfamily $\mathcal{G} \subset \mathcal{F}$, every h members of $\mathcal{G}$ intersect, then $\mathcal{G}$ has nonempty intersection. If no such h exists, we set $\mathrm{h}_{\mathcal{F}} = \infty$.

- The *Radon number* $\mathrm{r}_{\mathcal{F}}$ of $\mathcal{F}$ is the smallest integer r such that every r-element subset $S \subset X$ can be partitioned into two nonempty parts $S = P_1 \sqcup P_2$ such that $\mathrm{conv}_{\mathcal{F}}(P_1) \cap \mathrm{conv}_{\mathcal{F}}(P_2) \neq \emptyset$. (The set $\mathrm{conv}_{\mathcal{F}}(P)$, the *$\mathcal{F}$-convex hull* of a subset $P \subset X$, is the intersection of all the members of $\mathcal{F}$ that contain P.) If no such r exists, we set $\mathrm{r}_{\mathcal{F}} = \infty$.

Hence, letting $\mathcal{C}_d$ denote the set of all halfspaces in $\mathbb{R}^d$, Radon's lemma asserts that $\mathrm{r}_{\mathcal{C}_d} = d+2$ and Helly theorem that $\mathrm{h}_{\mathcal{C}_d} = d+1$. A classical result by Levi [21] asserts that for every set system $\mathcal{F}$ we have $\mathrm{h}_{\mathcal{F}} \leq \mathrm{r}_{\mathcal{F}} - 1$. (This is often stated for convexity spaces but it holds for set systems, see [2, App. C].) Similar relations between such parameters have been investigated over the years, like for instance the partition conjecture of Eckhoff [10] refuted by Bukh [6].

Conjecture 1 pertains to a parameter inspired by the *fractional Helly theorem* [20, 18], which asserts that in $(d+1)$-wise intersection hypergraphs of convex sets of $\mathbb{R}^d$, positive density implies a linear-size clique. Here is the associated parameter:

- The *fractional Helly number* $\mathrm{fh}_{\mathcal{F}}$ of $\mathcal{F}$ is the smallest integer s such that there exists a function $\beta_{\mathcal{F}} : (0,1) \to (0,1)$ with the following property: For every finite subfamily $\mathcal{F}' \subset \mathcal{F}$, whenever a fraction α of the s-tuples of $\mathcal{F}'$ have nonempty intersection, a subset $\mathcal{G}$ of $\mathcal{F}'$ of size $\beta_{\mathcal{F}}(\alpha)|\mathcal{F}'|$ has nonempty intersection.

Again, the fractional Helly theorem states that $\mathrm{fh}_{\mathcal{C}_d} = d+1$. The significance of the fractional Helly number was highlighted by Alon et al. [1], who proved that intersection-closed set systems with bounded fractional Helly number enjoy a *weak ε-net theorem*, a *Tverberg-type theorem*, a *selection lemma*, etc. It is tempting to reformulate Conjecture 1 as

> *For any function $\Psi : \mathbb{N} \to \mathbb{N}$ such that $\Psi(n) = O(n^d)$, every set set system $\mathcal{F}$ in $\mathbb{R}^d$ such that $\phi^{(d)}_{\mathcal{F}} \leq \Psi$ has fractional Helly number at most $d+1$.*

We note, however, that this statement is weaker than Conjecture 1 in that it does not assert that the function $\beta()$ underpinning the fractional Helly number depends only on Ψ and d.

1.1.2 Homological background: homological minors

A classical way to extend the theory of planar graphs is to consider embeddings of graphs into surfaces and of simplicial complexes into $\mathbb{R}^d$ or other topological spaces.

There is a rich theory of embedding of graphs on surfaces, both structural (a classic being the Heawood inequality [31]) and computational (*e.g.* the use of graph genus for parameterized complexity). Let us mention, in particular, the strong Hanani-Tutte theorem which asserts that a graph is planar if it can be drawn so that every pair of independent edges cross an even number of times. This statement generalizes to the projective plane [29, 9] but was found to fail in genus 4 [12].

Going to dimension higher than 2 changes the nature of the problems drastically, already because Fáry's theorem no longer holds. (For every $d \geq 3$ there are simplicial complexes that embed in $\mathbb{R}^d$ piecewise linearly but not linearly.) It is thus sometimes convenient to relax the notion of embedding and work with chain maps, and this was done in particular to analyze intersection patterns [14, 28, 13] using a notion of *homological minors* [35].

Formally, the *support of a singular chain* is the union of (the images of) the singular simplices with nonzero coefficient in that chain, and the *support of a simplicial chain* is the subcomplex induced by the simplices with nonzero coefficient in that chain. We write $\mathrm{supp}(\sigma)$ for the support of a (singular or simplicial) chain σ. A chain map $a : C_*(K) \to \mathrm{C}^{\mathrm{sing}}_*(X)$ (resp. $a : C_*(K) \to \mathcal{C}_*(T)$) is *nontrivial* if, for every vertex v of K, the support of $a(v)$ has odd size. Two faces in a simplicial complex K are *adjacent* if they have at least one vertex in common. A *homological almost-embedding* of a simplicial complex K into a topological space X (resp. into another simplicial complex L) is a nontrivial chain map $a : C_*(K) \to \mathrm{C}^{\mathrm{sing}}_*(X)$ (resp.

$a : C_*(K) \to C_*(L)$) such that any two non-adjacent faces $\sigma, \tau \in K$ have images with disjoint support, that is $\text{supp}(a(\sigma)) \cap \text{supp}(a(\tau)) = \emptyset$. In particular, for every embedding f the associated chain map $f_\#$ is a homological (almost) embedding.

Let Δ_N denote the N-dimensional simplex and for K a simplicial complex let $K^{(t)}$ denote its t-dimensional skeleton. It turns out that there is a homological version of the van Kampen-Flores theorem (see for instance [14, Corollary 14]):

▶ **Theorem 2.** *For any $d \geq 1$, $\Delta_{d+2}^{(\lceil d/2 \rceil)}$ does not homologically almost embed into $\mathbb{R}^d$.*

A simplicial complex K is a *homological minor* of a topological space or simplicial complex X if there is a homological almost-embedding of K into X. Theorem 2 thus asserts that $\Delta_{d+2}^{(\lceil d/2 \rceil)}$ is not a homological minor of $\mathbb{R}^d$, *i.e.*, that $\mathbb{R}^d$ has $\Delta_{d+2}^{(\lceil d/2 \rceil)}$ as *forbidden homological minor*.

1.1.3 Parameters of set systems with a forbidden homological minor

Matoušek [23] bounded the Helly number of topological set systems in $\mathbb{R}^d$ in which every subfamily intersects in a bounded number of connected components, all contractible. His approach starts from a set systems in $\mathbb{R}^d$, uses Ramsey theory to build a map from $\Delta_{d+2}^{(\lceil d/2 \rceil)}$ into $\mathbb{R}^d$ that is "constrained" by the known intersection patterns of $\mathcal{F}$ so that the intersection forced by the van Kampen-Flores theorem reveals a new intersection in $\mathcal{F}$.

This approach was generalized by Goaoc et al. [14] to set systems in $\mathbb{R}^d$ of bounded $\lceil d/2 \rceil$-level topological complexity, where the *h-level topological complexity* of $\mathcal{F}$ is the maximum over $\mathbb{N}$ of the homological shatter function $\phi_{\mathcal{F}}^{(h)}$, that is

$$\text{hc}_{\mathcal{F}}^{(h)} := \max \left\{ \max_{0 \leq i < h} \tilde{\beta}_i \left(\bigcap_{A \in \mathcal{G}} A; \mathbb{Z}_2 \right) \;\middle|\; \mathcal{G} \subset \mathcal{F} \right\}. \tag{2}$$

That generalization relied on homological minors and replaced the construction of the "constrained map" by the (simpler) construction of a "constrained chain map". The only aspect of the method that is specific to $\mathbb{R}^d$ is the use of the forbidden homological minor given by Theorem 2, so the approach readily generalizes to set systems whose ground set is a topological space with a forbidden homological minor, see the discussions in [28, §5.2] and [13, §2.2]. This method was refined and extended to analyze other parameters of set systems whose ground set has a forbidden homological minor [28, 26, 13].

1.1.4 Proof of Conjecture 1 for bounded homological shatter functions

Conjecture 1 was proven for bounded homological shatter functions in three steps:

- First, Holmsen and Lee [17, Theorem 1.1] proved that the fractional Helly number of *any* set system can be bounded by a function of its Radon number. Specifically, letting $\Psi_{\text{r}\to\text{fh}}(x)$ denote the supremum of the fractional Helly number of a set system with Radon number x, they proved that for every $r \geq 3$ we have $\Psi_{\text{r}\to\text{fh}}(r) \leq r^{r^{\lceil \log_2 r \rceil} + r\lceil \log_2 r \rceil}$.
- Second, Patáková [28, Theorem 2.1] proved that the Radon number of *any* set system whose ground set has K as forbidden homological minor can be bounded by a function of its $(\dim K)$-level topological complexity.
- Third, Goaoc, Holmsen and Patáková [13, Theorem 1.2] proved that for every set systems $\mathcal{F}$ whose ground set has K as forbidden homological minor, if the fractional Helly number $\text{fh}_{\mathcal{F}}$ is bounded then it is at most $\mu(K) + 1$, where $\mu(K)$ denotes the maximum sum of dimensions of two disjoint simplices in K.

1.2 Statement of the results

Our main contribution is to confirm Conjecture 1 for some diverging homological shatter functions and on some manifolds. This decomposes into five independent results.

Throughout the paper, we work with compact piecewise-linear (PL) manifolds (possibly with boundary); see [32, §1] for an introduction. We first generalize Theorem 2:

▶ **Theorem 3.** *For every integers $d \geq 3$ and b there exists $N = N(d, b)$ such that $\Delta_N^{(\lceil d/2 \rceil)}$ does not homologically almost embed in any compact, $(\lceil d/2 \rceil - 1)$-connected, d-dimensional PL manifold (possibly with boundary) $\mathcal{M}$ with $\beta_{\lceil d/2 \rceil}(\mathcal{M}; \mathbb{Z}_2) \leq b$.*

This partially answers [28, Problem 3], [13, Conjecture 1.7] and [26, Conjecture 2] and extends the previous confirmation of Conjecture 1 from set systems in $\mathbb{R}^d$ to set systems on manifolds. This also extends several results on set systems with bounded topological complexity (Helly's theorem, Radon's theorem, (p, q)-theorem, ...) from $\mathbb{R}^d$ to sufficiently connected manifolds (see [28, 13]). We can relax the connectivity assumption (see [2, App. B]) but not remove it.

One ingredient in the proof of Theorem 3 is the following analogue of the Hanani-Tutte theorem for homological almost-embeddings, which is of independent interest.

▶ **Theorem 4.** *Let K be a simplicial complex of dimension $k > 1$ and let $\mathcal{M}$ be a compact $2k$-dimensional PL manifold (possibly with boundary). If there exists a triangulation T of $\mathcal{M}$ and a non-trivial chain map $f : C_*(K; \mathbb{Z}_2) \to C_*(T; \mathbb{Z}_2)$ in general position such that the images of any two non-adjacent k-faces $\sigma, \tau \in K$ intersect in an even number of points, then K is a homological minor of $\mathcal{M}$.*

We formalize what we mean by general position in Section 2.

The homological shatter function $\Phi_{\mathcal{F}}^{(h)}$ defined in Equation (1) can be reformulated as a *graded* version of the topological complexity $\mathrm{hc}_{\mathcal{F}}^{(h)}$ defined in Equation (2), where the value for parameter t considers only intersections of subfamilies of size at most t: $\phi_{\mathcal{F}}^{(h)}(t) = \sup_{\substack{\mathcal{F}' \subset \mathcal{F} \\ |\mathcal{F}'| \leq t}} \mathrm{hc}_{\mathcal{F}'}^{(h)}$. We systematize this viewpoint and define graded analogues of other parameters of set systems. For instance here are the *graded Radon* and *graded Helly* numbers:

$$\mathrm{r}_{\mathcal{F}}(t) := \sup_{\substack{\mathcal{F}' \subset \mathcal{F} \\ |\mathcal{F}'| \leq t}} \mathrm{r}_{\mathcal{F}'} \quad \text{and} \quad \mathrm{h}_{\mathcal{F}}(t) := \sup_{\substack{\mathcal{F}' \subset \mathcal{F} \\ |\mathcal{F}'| \leq t}} \mathrm{h}_{\mathcal{F}'}. \tag{3}$$

It is straightforward to see that if the graded Helly numbers of a set system do not grow fast enough, then they are ultimately stationary and the set system has bounded Helly number (Lemma 10). We prove a similar condition for (graded) Radon numbers:

▶ **Theorem 5.** *Let $\mathcal{F}$ be a set system. If $\lim_{t \to \infty} \mathrm{r}_{\mathcal{F}}(t) - \log_2 t = -\infty$, then $\mathrm{r}_{\mathcal{F}} < \infty$.*

As a application, we extend Patáková's theorem from bounded to sufficiently slowly diverging homological shatter function (Corollary 12). This, in turns yields fractional Helly theorems for sufficiently slowly diverging homological shatter functions.

▶ **Corollary 6.** *For every simplicial complex K there exists a function $\Psi_K : \mathbb{N} \to \mathbb{N}$ with $\lim_{t \to \infty} \Psi_K(t) = +\infty$ such that the following holds. If $\mathcal{F}$ is a set system whose ground set has K as forbidden homological minor and such that $\phi_{\mathcal{F}}^{(\dim K)}(t) \leq \Psi_K(t)$ for t large enough, then $\mathcal{F}$ has fractional Helly number at most $(\mu(K) + 1)$.*

We then investigate more graded numbers and establish more relations between graded and ungraded numbers. As an application, we extend the Holmsen-Lee bound on $\Psi_{\mathrm{r} \to \mathrm{fh}}(\cdot)$. Let $\Xi : \mathbb{N} \to \mathbb{N}$ be the function defined by $\Xi(r) := r^{r^{\lceil \log_2 r \rceil} + r \lceil \log_2 r \rceil}$.

▶ **Theorem 7.** *Let $\Psi : \mathbb{N} \to \mathbb{N}$ and $t_0 \in \mathbb{N}$ such that $\Psi(t) < t+1$ for every $t \geq t_0$. If there exists an integer $t_1 \geq {t_0}^2$ such that $\Xi\left(\Psi(t_1)\right) < \frac{t_1}{t_0}$, then every set system whose graded Radon number function is bounded from above by Ψ has bounded fractional Helly number.*

With Theorem 7 we can strengthen Corollary 6, as a close inspection of the proof reveals that the function β associated to the fractional Helly number depends only on Ψ and t_0. (It also allows a slightly faster growth than Corollary 12.) We postpone this to the full version of the paper.

2 On homological minors

In this section we prove Theorems 3 and 4. For completeness, we start with a consequence of the simplicial approximation theorem that allows us to work purely in simplicial homology:

▶ **Lemma 8** ([2, App. A])**.** *A simplicial complex K is a homological minor of a compact PL manifold (possibly with boundary) $\mathcal{M}$ if and only if K is a homological minor of some triangulation of $\mathcal{M}$.*

When counting intersection points between chains, we focus on intersections that are stable under small perturbation. We therefore consider generic intersections, meaning intuitively that the chains intersect transversally. We formalize this in terms of linking numbers.[1]

Let X be a triangulation of $\mathbb{S}^{2k-1}$. Let S_1 and S_2 be two subcomplexes of X with $|S_1|$ and $|S_2|$ homeomorphic to $\mathbb{S}^{k-1}$. Suppose that the simplicial complex $X \setminus S_2$ induced by X on the vertices not in S_2 has a geometric realization homotopy equivalent to $\mathbb{S}^{2k-1} \setminus \mathbb{S}^{k-1}$ (this can always be ensured up to taking a subdivision of X). The *linking number* of S_1 and S_2 in X is 1 if the homology class $[S_1]$ generates $H_{k-1}(X \setminus S_2; \mathbb{Z}_2) \cong \mathbb{Z}_2$, and 0 if $[S_1]$ is trivial in $H_{k-1}(X \setminus S_2; \mathbb{Z}_2)$. (Exchanging S_1 and S_2 in this definition yields the same result.)

Let T be a triangulation of a compact PL manifold (possibly with boundary) $\mathcal{M}$. Two k-chains $z_1, z_2 \in C_k(T; \mathbb{Z}_2)$ *intersect generically* in vertex v if the closed star B of v in T satisfies: $B \cap z_1 \cap z_2 = \{v\}$, $D_1 := z_1 \cap B$ and $D_2 := z_2 \cap B$ are k-dimensional balls, and the spheres ∂D_1 and ∂D_2 have linking number 1 in ∂B. Two k-chains $z_1, z_2 \in C_k(T; \mathbb{Z}_2)$ are *in general position* if $z_1 \cap z_2$ consists of finitely many vertices, and z_1 and z_2 intersect generically in each of these vertices. Two k-chains $z_1, z_2 \in C_k(T; \mathbb{Z}_2)$ *intersect evenly* if they are in general position and $\mathrm{im}(z_1) \cap \mathrm{im}(z_2)$ has even size.

For K a k-dimensional complex, a simplicial chain map $f : C_\bullet(K; \mathbb{Z}_2) \to C_\bullet(T; \mathbb{Z}_2)$ is in *general position* if for every non-adjacent k-faces $\sigma, \tau \in K$, the chains $f(\sigma)$ and $f(\tau)$ are in general position and for each vertex $v \in f(\sigma) \cap f(\tau)$, σ and τ are the only faces of K whose images under f intersect the closed star of v in T. We say that a simplicial map $f : K \to T$ is in *general position* if the associated chain map $f_\#$ is.

For any compact PL manifold (possibly with boundary) $\mathcal{M}$ of dimension $2k$, there exists a map $\cap_{\mathcal{M}} : H_k(\mathcal{M}; \mathbb{Z}_2) \times H_k(\mathcal{M}; \mathbb{Z}_2) \to \mathbb{Z}_2$, called the *intersection form* of $\mathcal{M}$, such that $\cap_{\mathcal{M}}([z_1], [z_2]) \in \mathbb{Z}_2$ counts the intersection points of z_1 and z_2 modulo 2. We use no property of intersection forms besides their existence, and refer the interested reader to Prasolov [30, Chapter 2, §2.7] for a precise definition and to Paták and Tancer [27] for an brief account.

The next result is due to Paták and Tancer [27, Proposition 21] and formulated following the presentation of Skopenkov [33], which is better suited for our purpose. (More precisely, we reformulated [33, Theorem 1.1.5] using [33, Lemma 2.1.1].) Note that the proofs by Paták-Tancer [27] and by Skopenkov [33] use only PL maps.

[1] Intuitively, this generalizes the idea that in the plane, two curves cross at a point x if the branches of the curves alternate around x.

▶ **Theorem 9.** *Let L be a simplicial complex of dimension $k > 1$ and let $\mathcal{M}$ be a compact, $(k-1)$-connected, $2k$-dimensional PL manifold (possibly with boundary). The following statements are equivalent:*

(i) *There exists a triangulation T of $\mathcal{M}$ and a simplicial map $f : L \to T$ in general position such that the images of any two non-adjacent faces intersect evenly.*

(ii) *There exists a triangulation R of $\mathbb{R}^{2k}$ and a simplicial map $g : L \to R$ in general position and a map α that sends each k-face of L to an element of $H_k(\mathcal{M};\mathbb{Z}_2)$ such that any two non-adjacent k-faces $\sigma, \tau \in L$ have images that intersect in an even number of points if and only if $\cap_{\mathcal{M}}(\alpha(\sigma), \alpha(\tau)) = 0$.*

2.1 A homological Hanani-Tutte theorem

Let us now prove Theorem 4. Let K be a simplicial complex of dimension $k > 1$ and let T be a triangulation of a compact PL manifold (possibly with boundary) $\mathcal{M}$ of dimension $2k$. Let $f : C_*(K;\mathbb{Z}_2) \to C_*(T;\mathbb{Z}_2)$ be a non-trivial chain map in general position such that the images of any two non-adjacent k-faces $\sigma, \tau \in K$ intersect evenly.

Our goal is to prove that K is a homological minor of $\mathcal{M}$. This requires repeatedly subdividing the triangulation T. In what follows, every time we subdivide a triangulation T_i into T_{i+1}, all chain maps to T_i and subcomplexes of T_i are also subdivided to T_{i+1}. Also, throughout the proof we identify every pure ℓ-dimensional simplicial complex with the unique ℓ-chain with coefficients in $\mathbb{Z}_2$ it supports; we abuse the terminology and say that we add (pure) simplicial complexes over $\mathbb{Z}_2$ to mean that we add the corresponding chains.

If f is a homological almost-embedding we are done. Otherwise, there exist some non-adjacent k-faces $\sigma, \tau \in K$ such that $f(\sigma) \cap f(\tau)$ is non-empty. By assumption, this intersection is a set of vertices of even size, so let $x, y \in f(\sigma) \cap f(\tau)$ be two distinct such vertices. Recall that f is in general position, $f(\sigma)$ and $f(\tau)$ intersect generically in x and in y. We set out to construct a refinement T' of T and a new map $f' : C_*(K;\mathbb{Z}_2) \to C_*(T';\mathbb{Z}_2)$ that is also in general position, differs from f only on σ and τ, and satisfies $f'(\sigma) \cap f'(\tau) = (f(\sigma) \cap f(\tau)) \setminus \{x, y\}$ as well as $f'(\alpha) \cap f'(\beta) = f(\alpha) \cap f(\beta)$ for any $\alpha \in \{\sigma, \tau\}$ and any k-face $\beta \in K \setminus \{\sigma, \tau\}$. Iterating this procedure produces the announced homological almost-embedding of K into a triangulation of $\mathcal{M}$.

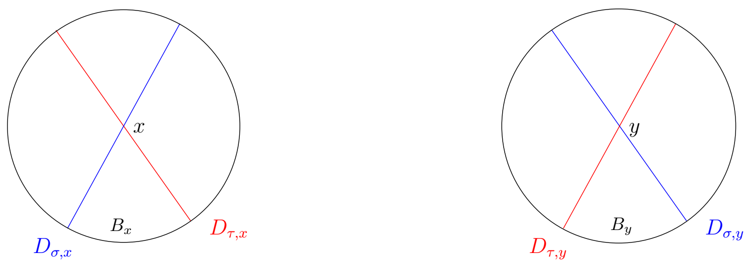

Figure 1 The setup for the construction of T' and f'.

Let B_x and B_y denote the closed stars of x and y in T. For $z \in \{x, y\}$ and $\alpha \in \{\sigma, \tau\}$ let $D_{\alpha,z} := \operatorname{supp}(f(\alpha)) \cap B_z$. Since $f(\sigma)$ and $f(\tau)$ intersect generically in x and y, the complexes $D_{\sigma,x}, D_{\sigma,y}, D_{\tau,x}$ and $D_{\tau,y}$ are k-dimensional balls, the $(k-1)$-spheres $\partial D_{\sigma,x}$ and $\partial D_{\tau,x}$ have linking number 1 in ∂B_x, and similarly $\partial D_{\sigma,y}$ and $\partial D_{\tau,y}$ have linking number 1 in ∂B_y.

We subdivide T into T_1 so that the closed star B'_x of x and B'_y of y in T_1 are disjoint from ∂B_x and ∂B_y, respectively. In particular, for $z \in \{x, y\}$ and $\alpha \in \{\sigma, \tau\}$, letting $D'_{\alpha,z} := \operatorname{supp}(f(\alpha)) \cap B'_z$, the pair $(\partial D'_{\alpha,z}, \partial B'_z)$ is homeomorphic to $(\mathbb{S}^{k-1}, \mathbb{S}^{2k-1})$. The genericity of x and y in $f(\sigma) \cap f(\tau)$ is preserved through the subdivision $T \to T_1$ so the four complexes $D'_{\bullet,\bullet}$ are k-dimensional balls and their bounding spheres have the same linking numbers in $\partial B'_x$ and $\partial B'_y$ as their counterparts on ∂B_x and ∂B_y.

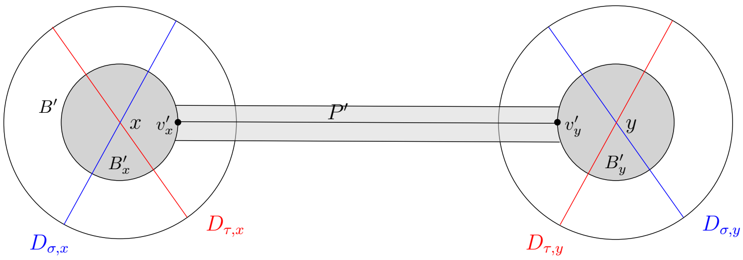

Figure 2 The construction of B'.

Up to subdividing T_1 into T_2, there exist vertices v'_x and v'_y in $\partial B'_x$ and $\partial B'_y$, respectively, and a path P' in the 1-skeleton of T_2 from v'_x to v'_y such that, letting $N(P')$ denote the closed star of P' in T_2, the closed star of $N(P')$ is disjoint from the image of f. The union $B' := B'_x \cup N(P') \cup B'_y$ is a ball. Observe that in $\partial B'$, the $(k-1)$-spheres $\partial D'_{\sigma,x}$ and $\partial D'_{\tau,x}$ retain the linking number 1 that they have on $\partial B'_x$. Similarly, in $\partial B'$, the $(k-1)$-spheres $\partial D'_{\sigma,y}$ and $\partial D'_{\tau,y}$ retain the linking number 1 that they have on $\partial B'_y$.

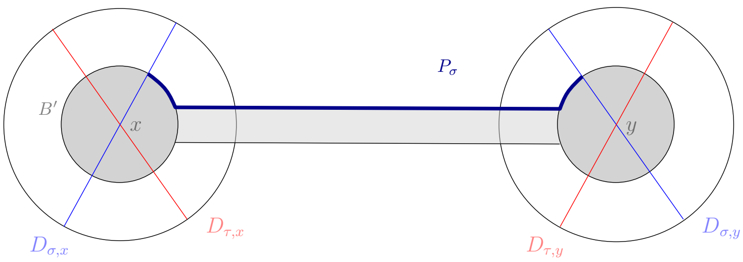

Figure 3 The path P_σ.

Thus, up to further subdividing T_2 into T_3, there exist a path P_σ in the 1-skeleton of $\partial B'$ that connects $\partial D'_{\sigma,x}$ to $\partial D'_{\sigma,y}$ and with relative interiors disjoint from the image of f. (Here we use that $\partial D'_{\alpha,z}$ is of codimension $k \geq 2$ in $\partial B'$.) Up to subdividing the triangulation further, we can pipe $D_{\sigma,x}$ and $D_{\sigma,y}$ together by a tube F_σ found in a neighborhood of the path P_σ [32, §5.10]. The tube $(F_\sigma, F_\sigma \cap D_{\sigma,x}, F_\sigma \cap D_{\sigma,y})$ is homeomorphic to $\left(\mathbb{D}^k \times [0,1], \mathbb{D}^k \times \{0\}, \mathbb{D}^k \times \{1\}\right)$. Moreover, by the (PL) general position theorem for embeddings [32, §5.3], we can take F_σ such that $\partial B'$ and ∂F_σ intersect in a generic way. By taking F_σ in a sufficiently small neighborhood U of P_σ so that $(U, \partial B' \cap U)$ is homeomorphic to $(\mathbb{R}^{2k}, \mathbb{R}^{2k-1} \times \{0\})$, and the triple $(F_\sigma \cap \partial B', F_\sigma \cap D_{\sigma,x} \cap \partial B', F_\sigma \cap D_{\sigma,y} \cap \partial B')$ is homeomorphic to $\left(\mathbb{D}^{k-1} \times [0,1], \mathbb{D}^{k-1} \times \{0\}, \mathbb{D}^{k-1} \times \{1\}\right)$.

Now, let C_σ be the sum $D_{\sigma,x} + \partial F_\sigma + D_{\sigma,y}$ (over $\mathbb{Z}_2$). Note that C_σ is contained in the union of $B_x \cup B_y$ and the closed star of $N(P')$, and therefore intersects the image of f only inside $B_x \cup B_y$. We note that $F_\sigma \cap \partial B'$ also pipes the $(k-1)$-spheres $D_{\sigma,x} \cap \partial B'$ and $D_{\sigma,y} \cap \partial B'$. Indeed, $(F_\sigma \cap \partial B', F_\sigma \cap D_{\sigma,x} \cap \partial B', F_\sigma \cap D_{\sigma,y} \cap \partial B') = (F_\sigma \cap \partial B', F_\sigma \cap \partial(D_{\sigma,x} \cap B'), F_\sigma \cap \partial(D_{\sigma,y} \cap B'))$ is homeomorphic to $\left(\mathbb{D}^{k-1} \times [0,1], \mathbb{D}^{k-1} \times \{0\}, \mathbb{D}^{k-1} \times \{1\}\right)$. It follows that $C_\sigma \cap \partial B'$ is homeomorphic to the connected sum of two $(k-1)$-spheres, and is therefore homeomorphic to a $(k-1)$-sphere.

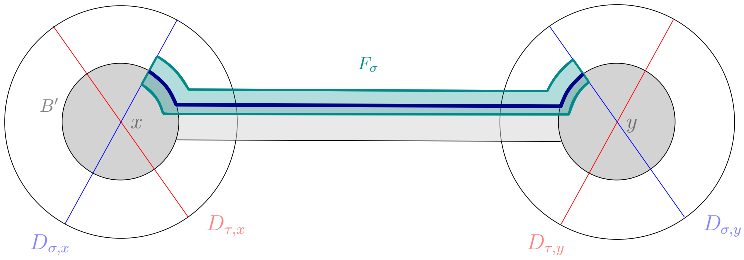

Figure 4 The piping F_σ between $D_{\sigma,x}$ and $D_{\sigma,y}$ in a neighborhood of P_σ.

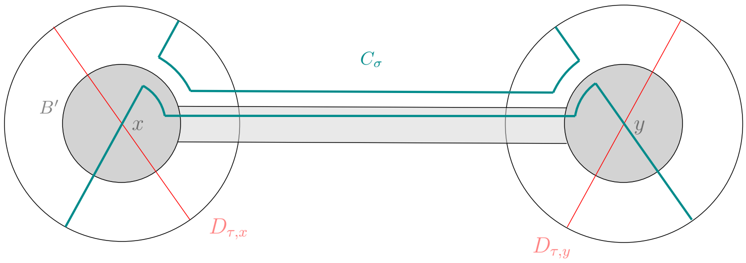

Figure 5 The chain $C_\sigma = D_{\sigma,x} + \partial F_\sigma + D_{\sigma,y}$ over $\mathbb{Z}_2$.

We claim that, in $\partial B'$, the linking number between $C_\sigma \cap \partial B'$ and $D_{\tau,x} \cap \partial B'$ equals the linking number between $D_{\sigma,x} \cap \partial B'$ and $D_{\tau,x} \cap \partial B'$, that is 1. This follows from the fact that the chain C_σ differs from $D_{\sigma,x}$ by $\partial F_\sigma + D_{\sigma,y}$, and that each of $\partial F_\sigma \cap \partial B'$ and $D_{\sigma,y} \cap \partial B'$ is a boundary in $\partial B' \setminus D_{\tau,x}$. The same claim holds if we exchange x for y.

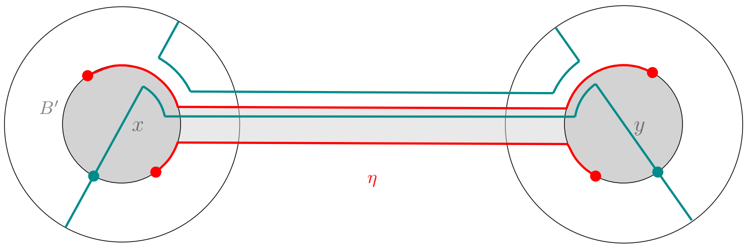

Figure 6 The chain $\eta \in C_k(\partial B' \setminus C_\sigma; \mathbb{Z}_2)$ such that $\partial \eta = D_{\tau,x} \cap \partial B' + D_{\tau,y} \cap \partial B'$ over $\mathbb{Z}_2$.

Altogether, we get that $D_{\tau,x} \cap \partial B'$ and $D_{\tau,y} \cap \partial B'$ are in the same homology class in $\partial B' \setminus C_\sigma$. Hence, their sum is a boundary, and there exists a chain $\eta \in C_k(\partial B' \setminus C_\sigma; \mathbb{Z}_2)$ such that the support of $\partial\eta$ is the sum over $\mathbb{Z}_2$ of $D_{\tau,x} \cap \partial B'$ and $D_{\tau,y} \cap \partial B'$. We finally set

$$f'(\sigma) = \underbrace{f(\sigma) + D_{\sigma,x} + D_{\sigma,y}}_{f(\sigma) \text{ with its restriction to } B_x \cup B_y \text{ removed}} + C_\sigma \quad \text{and} \quad f'(\tau) = \underbrace{f(\tau) + D'_{\tau,x} + D'_{\tau,y}}_{f(\tau) \text{ with its restriction to } B' \text{ removed}} + \eta$$

We set $f'(\omega) = f(\omega)$ for every other face $\omega \in K$. Extending f' linearly yields a chain map, since both $f'(\sigma) - f(\sigma)$ and $f'(\tau) - f(\tau)$ are cycles [2]. The chain map f' is as announced: $f'(\sigma) \cap f'(\tau) = f(\sigma) \cap f(\tau) \setminus \{x, y\}$ and every other intersection remains unchanged. This concludes the proof of Theorem 4.

[2] They are even boundaries, ensuring that f' is chain homotopic to f

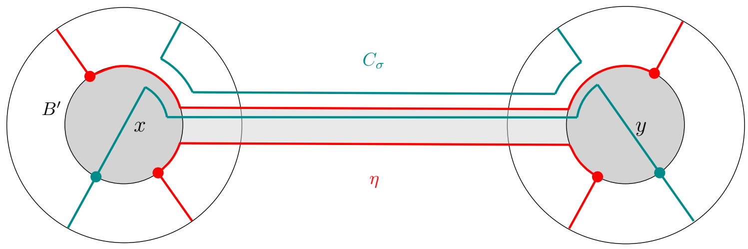

Figure 7 The chains C_σ and η reroute $f(\sigma)$ and $f(\tau)$ so as to remove intersections in B_x and B_y.

2.2 Forbidden homological minors for manifolds

We now prove Theorem 3. First, note that the odd-dimensional case reduces to the even-dimensional one. Indeed, for every $k \geq 2$, if $\mathcal{M}$ is a compact, $(k-1)$-connected, $(2k-1)$-dimensional PL manifold (possibly with boundary), then $\mathcal{M} \times [0,1]$ is a compact, $(k-1)$-connected, $2k$-dimensional PL manifold (possibly with boundary). Moreover, $\beta_k(\mathcal{M} \times [0,1]; \mathbb{Z}_2) = \beta_k(\mathcal{M}; \mathbb{Z}_2)$ and any homological minor of $\mathcal{M}$ is a homological minor of $\mathcal{M} \times [0,1]$. If the statement holds for d even and $b \in \mathbb{N}$ with some $N(b,d)$, then it holds with $d-1$ and b by putting $N(d-1,b) := N(d,b)$. So let us now consider the even-dimensional case.

Let $k \geq 2$ and $b \in \mathbb{N}$, and let $\mathcal{M}$ be a compact, $(k-1)$-connected, $2k$-dimensional PL manifold (possibly with boundary). Let us fix N and suppose that $K = \Delta_N^{(k)}$ is a homological minor of $\mathcal{M}$. By Lemma 8, there exist a triangulation T of $\mathcal{M}$ and a (simplicial) homological almost-embedding $C_*(K) \to C_*(T)$. Actually, letting $S := T^{(k)}$, we have that there exists a homological almost-embedding $a : C_*(K; \mathbb{Z}_2) \to C_*(S; \mathbb{Z}_2)$.

We apply Theorem 9 with $L = S$. Condition (i) holds with $T = S$ and f the identity, so Condition (ii) also holds. Hence, there exists a triangulation R of $\mathbb{R}^{2k}$, a simplicial map $g : S \to R$ in general position, and a map α that sends each k-face of S to an element of $H_k(\mathcal{M}; \mathbb{Z}_2)$ such that any two non-adjacent k-faces $\sigma, \tau \in S$ intersect evenly if and only if $\cap_{\mathcal{M}}(\alpha(\sigma), \alpha(\tau)) = 0$. We let $\tilde{\alpha} : C_k(S) \to H_k(\mathcal{M}; \mathbb{Z}_2)$ denote the linear extension of α.

Consider the chain map $b : C_*(K; \mathbb{Z}_2) \to C_*(R; \mathbb{Z}_2)$ defined by $b = g_\# \circ a$. Note that b is nontrivial since a is nontrivial and g is a simplicial map. Moreover, the fact that b is a chain map in general position follows from three observations:

- since g is a simplicial map in general position, $g_\#$ is a chain map in general position,
- since a is a homological almost-embedding, it is also a chain map in general position, and
- the composition of a homological almost-embedding and a chain map in general position is a chain map in general position.

Notice that for $N \geq 2k+3$, not every independent k-faces of K can have images under b that intersect evenly. Indeed Theorem 4 would then imply that K is a homological minor of $\mathbb{R}^{2k}$, which would contradict the homological van Kampen-Flores theorem (Theorem 2). We use Ramsey's theorem to show that this contradiction can be reached for some subcomplex of K.

So consider two non-adjacent k-faces $\sigma, \tau \in K$ and put $a(\sigma) = \sigma_1 + \sigma_2 + \ldots + \sigma_s$ and $a(\tau) = \tau_1 + \tau_2 + \ldots + \tau_t$. Since a is a homological almost-embedding, $\mathrm{supp}(a(\sigma))$ and $\mathrm{supp}(a(\tau))$ are disjoint, and σ_i and τ_j are thus non-adjacent for every $(i,j) \in [s] \times [t]$. Hence, given $(i,j) \in [s] \times [t]$, $g(\sigma_i)$ and $g(\tau_j)$ intersect evenly if and only if $\cap_{\mathcal{M}}(\alpha(\sigma_i), \alpha(\tau_j)) = 0$. We can thus count the intersections of $b(\sigma)$ and $b(\tau)$ (the following equalities are modulo 2 and Card () denotes the cardinal):

$$\begin{aligned}\operatorname{Card}\left(b(\sigma)\cap b(\tau)\right) &= \sum_{i\in[s],j\in[t]} \operatorname{Card}\left(g(\sigma_i)\cap g(\tau_j)\right)\\ &= \sum_{i\in[s],j\in[t]} \cap_{\mathcal{M}}(\alpha(\sigma_i),\alpha(\tau_j))\\ &= \sum_{i\in[s]} \cap_{\mathcal{M}}(\alpha(\sigma_i),\tilde{\alpha}(a(\tau))) = \cap_{\mathcal{M}}(\tilde{\alpha}(a(\sigma)),\tilde{\alpha}(a(\tau))).\end{aligned}$$

Let $r = \beta_k(\mathcal{M};\mathbb{Z}_2)$. The map $\beta : C_k(K) \to H_k(\mathcal{M};\mathbb{Z}_2)$ defined by $\beta = \tilde{\alpha}\circ a$ induces a coloring of the k-simplices of K by the (at most 2^r) elements of $H_k(M;\mathbb{Z}_2)$. Let N' denote the number of vertices of $\operatorname{sd}\Delta^{(k)}_{2k+2}$. By the hypergraph Ramsey theorem, for N large enough (as a function of r and k) there exists a subset W of N' vertices in K such that β is constant over all k-simplices of $K[W]$; let us denote by $\square$ this constant value. In particular, for every k-faces σ,τ of $K[W]$ such that σ and τ are not adjacent, we have, modulo 2, $\operatorname{Card}\left(b(\sigma)\cap b(\tau)\right) = \cap_{\mathcal{M}}(\beta(\sigma_1),\beta(\tau_1)) = \cap_{\mathcal{M}}(\square,\square)$. Let us fix a bijection from the vertices of $\operatorname{sd}\Delta^{(k)}_{2k+2}$ to W and extend it to a chain map $j : C_*\left(\operatorname{sd}\Delta^{(k)}_{2k+2}\right) \to C_*(K[W])$. Also, let $h : C_*\left(\Delta^{(k)}_{2k+2}\right) \to C_*\left(\operatorname{sd}\Delta^{(k)}_{2k+2}\right)$ denote the subdivision chain map, where each i-face of $\Delta^{(k)}_{2k+2}$ is mapped to the sum of the i-faces of $\operatorname{sd}\Delta^{(k)}_{2k+2}$ that it contains. In particular, for every k-face σ of $\Delta^{(k)}_{2k+2}$, the chain $h(\sigma)$ is supported on $k!$ k-faces of $\operatorname{sd}\Delta^{(k)}_{2k+2}$.

Let us examine the properties of $b\circ j\circ h$. First, it is a nontrivial chain map (because b, j and h are). Moreover, $j\circ h$ is a homological almost-embedding (since j and h are), and its composition with the chain map in general position b yields a chain map in general position. Furthermore, any two non-adjacent k-faces $\sigma,\tau \in \Delta^{(k)}_{2k+2}$ have images under $b\circ j\circ h$ that intersect evenly. To see this, let us put $j\circ h(\sigma) = \sigma_1+\sigma_2+\ldots+\sigma_s$ and $j\circ h(\tau) = \tau_1+\tau_2+\ldots+\tau_s$. Since $j\circ h$ is a homological almost-embedding, $\operatorname{supp}(j\circ h(\sigma))$ and $\operatorname{supp}(j\circ h(\tau))$ are disjoint. It follows that for every $i,j\in[s]$ the k-simplices σ_i and τ_j are non-adjacent. We therefore have, modulo 2,

$$\begin{aligned}\operatorname{Card}\left(b\circ j\circ h(\sigma)\cap b\circ j\circ h(\tau)\right) &= \sum_{i,j\in[s]} \operatorname{Card}\left(b(\sigma_i)\cap b(\tau_j)\right)\\ &= \sum_{i,j\in[s]} \cap_{\mathcal{M}}(\beta(\sigma_i),\beta(\tau_j)) = \sum_{i,j\in[s]} \cap_{\mathcal{M}}(\square,\square).\end{aligned}$$

To sum up, the cardinal of $b\circ j\circ h(\sigma)\cap b\circ j\circ h(\tau)$ has the same parity as $\cap_{\mathcal{M}}(\square,\square)s^2$. Since $s=k!$ is even, σ and τ have images under $b\circ j\circ h$ that intersect evenly.

To conclude, $b\circ j\circ h$ is a nontrivial chain map in general position from $\Delta^{(k)}_{2k+2}$ to R such that independent faces have images that intersect evenly. By Theorem 4 this means that $\Delta^{(k)}_{2k+2}$ is a homological minor of $\mathbb{R}^{2k}$, a contradiction with Theorem 2. Thus, for N large enough, the initial hypothesis that $\Delta^{(k)}_N$ is a homological minor of $\mathcal{M}$ cannot be true.

3 Graded parameters of set systems

In this section we present our contributions on set systems of parameters.

3.1 Graded Radon and Helly numbers

Each relation between parameters of a set system yields a relation between their graded analogues. From Levi's inequality we get the following inequality between the graded Helly and Radon numbers (defined in Section 1.1.1):

$$\forall t \in \mathbb{N}, \qquad \mathrm{h}_{\mathcal{F}}(t) = \sup_{\substack{\mathcal{F}' \subset \mathcal{F} \\ |\mathcal{F}'| \leq t}} \mathrm{h}_{\mathcal{F}'} \leq \sup_{\substack{\mathcal{F}' \subset \mathcal{F} \\ |\mathcal{F}'| \leq t}} (\mathrm{r}_{\mathcal{F}'} - 1) = \mathrm{r}_{\mathcal{F}}(t) - 1. \tag{4}$$

It follows from the definitions of graded parameters that each one is a non-decreasing function that converges to the ungraded parameter (possibly ∞). We notice that if a graded Helly number is asymptotically sublinear then it is bounded:

▶ **Lemma 10.** *Let $\mathcal{F}$ be a set system and $t_0 \in \mathbb{N}$. If $\mathrm{h}_{\mathcal{F}}(t) < t$ for all $t > t_0$, then $\mathrm{h}_{\mathcal{F}} \leq t_0$.*

Proof. By definition, we have $\mathrm{h}_{\mathcal{F}}(t) \leq t$ for every $t \in \mathbb{N}$. Moreover, $\mathrm{h}_{\mathcal{F}}(t) \neq \mathrm{h}_{\mathcal{F}}(t-1)$ if and only if $\mathrm{h}_{\mathcal{F}}(t) = t$. The assumption and a straightforward induction therefore implies that for every $t > t_0$ we have $\mathrm{h}_{\mathcal{F}}(t) = \mathrm{h}_{\mathcal{F}}(t_0) \leq t_0$. ◀

The growth of graded Radon numbers is at most linear [2, App. D] and rather constrained:

▶ **Lemma 11.** *Let $\mathcal{F}$ be a set system and $t \geq 2$ an integer. If $\mathrm{r}_{\mathcal{F}}(t) > \mathrm{r}_{\mathcal{F}}(t-1)$, then $\mathrm{r}_{\mathcal{F}}(t-1) \geq 1 + \log_2\left(1 + \frac{t}{\mathrm{h}_{\mathcal{F}}(t)}\right)$.*

Proof. Let X denote the ground set of $\mathcal{F}$ and let $n = \mathrm{r}_{\mathcal{F}}(t-1)$. Suppose that $\mathrm{r}_{\mathcal{F}}(t) > n$, so that there exist a subset $\mathcal{G} = \{G_1, G_2, \ldots, G_t\} \subset \mathcal{F}$ and a subset $S \subseteq X$ of size n such that

(i) there is no partition of S into two parts whose $\mathcal{G}$-convex hulls intersect,

(ii) for every $i \in [t]$, there exists a partition $\mathcal{P}_i$ of S into two parts whose $(\mathcal{G} \setminus \{G_i\})$-convex hulls intersect.

Recall that given $\mathcal{F}' \subseteq \mathcal{F}$, the $\mathcal{F}'$-convex hull $\mathrm{conv}_{\mathcal{F}'}(P)$ of a subset $P \subset X$ is the intersection of all the members of $\mathcal{F}'$ that contain P. In particular, for any $P \subseteq X$ such that $P \nsubseteq G_i$ we have $\mathrm{conv}_{\mathcal{G}}(P) = \mathrm{conv}_{\mathcal{G} \setminus \{G_i\}}(P)$. Conditions (i) and (ii) therefore imply that every $G_i \in \mathcal{G}$ contains one or the other part of $\mathcal{P}_i$.

There are at most $2^{n-1} - 1$ partitions of S in two nonempty parts. Let us assume that $t > (2^{n-1} - 1)h$ for some integer h, so that by the pigeonhole principle there exist $h+1$ indices $i_1, i_2, \ldots, i_{h+1}$ such that the partitions $\mathcal{P}_{i_1}, \mathcal{P}_{i_2}, \ldots, \mathcal{P}_{i_{h+1}}$ coincide. Let $\{P_1, P_2\}$ be that partition of S. Let us put $\mathcal{G}' = \{A \in \mathcal{G} \colon P_1 \subseteq A \text{ or } P_2 \subseteq A\}$. We make two observations:

- $\cap_{A \in \mathcal{G}'} A$ coincides with $\mathrm{conv}_{\mathcal{G}}(P_1) \cap \mathrm{conv}_{\mathcal{G}}(P_2)$ and is therefore empty.
- every choice of h elements in $\mathcal{G}'$ has nonempty intersection. Indeed, $\mathcal{G}'$ contains G_{i_1}, G_{i_2}, $\ldots$, $G_{i_{h+1}}$, so that any choice of h elements from $\mathcal{G}'$ is bound to miss G_{i_j} for at least one $j \in [h+1]$ and their intersection must then contain $\mathrm{conv}_{\mathcal{G} \setminus \{G_{i_j}\}}(P_1) \cap \mathrm{conv}_{\mathcal{G} \setminus \{G_{i_j}\}}(P_2) \neq \emptyset$.

For $h \geq \mathrm{h}_{\mathcal{F}}(t)$ these conditions are incompatible. We therefore have $t \leq (2^{n-1} - 1)\, \mathrm{h}_{\mathcal{F}}(t)$, and the statement follows. ◀

We can now prove that any set system with sufficiently slowly growing graded Radon numbers has finite Radon number.

Proof of Theorem 5. Let $\mathcal{F}$ be a set system with infinite Radon number. If the Helly number $\mathrm{h}_{\mathcal{F}}$ is also infinite, then by Lemma 10 there exists an increasing sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $\mathrm{h}_{\mathcal{F}}(t_i) = t_i$, and therefore $\mathrm{r}_{\mathcal{F}}(t_i) \geq t_i + 1$ by Inequality (4); this prevents $\mathrm{r}_{\mathcal{F}}(t) - \log_2 t$ from going to $-\infty$ as $t \to \infty$. So suppose that the Helly number $\mathrm{h}_{\mathcal{F}}$ is finite. The assumption that $\mathrm{r}_{\mathcal{F}} = \infty$ ensures that there exists an increasing sequence $\{t_i\}_{i\in\mathbb{N}}$ such that $\mathrm{r}_{\mathcal{F}}(t_i) > \mathrm{r}_{\mathcal{F}}(t_i - 1)$. Lemma 11 implies that $\mathrm{r}_{\mathcal{F}}(t_i) > \mathrm{r}_{\mathcal{F}}(t_i - 1) \geq \log_2 t_i - \log_2 \mathrm{h}_{\mathcal{F}}$. Again, this prevents $\mathrm{r}_{\mathcal{F}}(t) - \log_2 t$ from going to $-\infty$ as $t \to \infty$. The statement follows by contraposition. ◀

3.2 Consequences for topological set systems

Let us finally consider topological set systems with slowly growing homological shatter function and ground set with a forbidden homological minor.

▶ **Corollary 12.** *For every simplicial complex K there exists a function $S_K : \mathbb{N} \to \mathbb{N}$ with $\lim_{t\to\infty} S_K(t) = +\infty$ such that the following holds. Any set system $\mathcal{F}$ whose ground set has K as forbidden homological minor and satisfies $\phi_{\mathcal{F}}^{(\dim K)}(t) \leq S_K(t)$ for t large enough has finite Radon number.*

Proof. Recall that $\Psi^{(K)}_{\mathrm{hc}\to\mathrm{r}}(x)$ denotes the supremum of the Radon number of a set system with $(\dim K)$-level topological complexity at most x and whose ground set has K as forbidden homological minor. Patáková [28] proved that $\Psi^{(K)}_{\mathrm{hc}\to\mathrm{r}}(x)$ is finite for every K and x.

For $t \in \mathbb{N}$, we define $S_K(t) = \max\{x \in \mathbb{N} \mid \Psi^{(K)}_{\mathrm{hc}\to\mathrm{r}}(x) \leq \frac{1}{2}\log_2 t\}$. This ensures that $\Psi^{(K)}_{\mathrm{hc}\to\mathrm{r}}(S_K(t)) \leq \frac{1}{2}\log_2 t$ for every $t \in \mathbb{N}$. Observe that $\lim_{t\to\infty} S_K(t) = \infty$ since $\Psi^{(K)}_{\mathrm{hc}\to\mathrm{r}}(x)$ is finite for every $x \in \mathbb{N}$.

Now consider a set system $\mathcal{F}$ with function $\phi_{\mathcal{F}}^{(\dim K)} \leq S_K$ and whose ground set has K as forbidden homological minor. Let $t \in \mathbb{N}$ and consider a subset $\mathcal{F}' \subseteq \mathcal{F}$ of size t. The ground set of $\mathcal{F}'$ also has K as forbidden homological minor. Moreover, $\mathcal{F}'$ has $(\dim K)$-level topological complexity at most $\phi_{\mathcal{F}}^{(\dim K)}(t) \leq S_K(t)$. It follows that $\mathrm{r}_{\mathcal{F}'} \leq \Psi^{(K)}_{\mathrm{hc}\to\mathrm{r}}(S_K(t)) \leq \frac{1}{2}\log_2 t$. This holds for every $\mathcal{F}' \subseteq \mathcal{F}$ of size t, so $\mathrm{r}_{\mathcal{F}}(t) \leq \frac{1}{2}\log_2 t$. This inequality holds for every $t \in \mathbb{N}$ so Theorem 5 implies that $\mathcal{F}$ has bounded Radon number. ◀

We can finally prove a fractional Helly theorem for diverging homological shatter functions.

Proof of Corollary 6. Recall that $\Psi_{\mathrm{r}\to\mathrm{fh}}(y)$ denotes the supremum of the fractional Helly number of a set system with Radon number y. Holmsen and Lee [17, Theorem 1.1] proved that $\Psi_{\mathrm{r}\to\mathrm{fh}}(y)$ is finite for every y. Let S_K denote the function from Corollary 12. Now consider a set system $\mathcal{F}$ whose ground set has K as forbidden homological minor and satisfies $\phi_{\mathcal{F}}^{(\dim K)} \leq S_K$. Corollary 12 ensures that $\mathrm{r}_{\mathcal{F}}$, the Radon number of $\mathcal{F}$, is finite. It follows that the fractional Helly number of $\mathcal{F}$ is at most $\Psi_{\mathrm{r}\to\mathrm{fh}}(\mathrm{r}_{\mathcal{F}})$, and is therefore finite. From there, [13, Theorem 1.2] ensures that this fractional Helly number is at most $\mu(K) + 1$. ◀

3.3 Other graded parameters and relations

With the intersection hypergraph of $\mathcal{F}$ in mind, we say that a set $\mathcal{G}$ in a set system $\mathcal{F}$ is a *clique* if the intersection of all members of $\mathcal{G}$ is non-empty. The colorful Helly theorem suggests the following parameter:

- The *colorful Helly number* $\mathrm{ch}_{\mathcal{F}}$ of $\mathcal{F}$ is the smallest number of colors m such that for every coloring of a subfamily $\mathcal{F}' \subset \mathcal{F}$ with m colors, if every subfamily that contains exactly one element of each color is a clique, then at least one color class is a clique.

We say that a set $\mathcal{G}$ in a set system $\mathcal{F}$ is a *c-wise clique* if every c-element subset of $\mathcal{G}$ is a clique. Clearly a clique is a c-wise clique, and when $c \geq \mathrm{h}_{\mathcal{F}}$ the converse is true. To analyze set systems with large, infinite or unknown Helly numbers, it is useful to consider variants of the colorful and fractional Helly numbers where cliques are replaced by c-wise cliques:

- The *cth colorful Helly number* $\mathrm{ch}_{\mathcal{F}}^{(c)}$ of $\mathcal{F}$ is the smallest number of colors $m \geq c$ such that for every coloring of a subfamily $\mathcal{F}' \subset \mathcal{F}$ with m colors, if every subfamily of $\mathcal{F}'$ that contains exactly one element of each color forms a c-wise clique, then at least one color class is a c-wise clique.
- The *cth fractional Helly number* $\mathrm{fh}_{\mathcal{F}}^{(c)}$ of $\mathcal{F}$ is the smallest integer s such that there exists a function $\beta_{\mathcal{F}} : (0,1) \to (0,1)$ with the following property: For every finite subfamily $\mathcal{F}' \subset \mathcal{F}$, whenever a fraction α of the s-tuples of $\mathcal{F}'$ forms a c-wise clique, a subset $\mathcal{G}$ of $\mathcal{F}'$ of size $\beta_{\mathcal{F}}(\alpha)|\mathcal{F}'|$ forms a c-wise clique.

Obviously for every set system $\mathcal{F}$, if $c \geq \mathrm{h}_{\mathcal{F}}$, then $\mathrm{fh}_{\mathcal{F}}^{(c)} = \mathrm{fh}_{\mathcal{F}}$ and $\mathrm{ch}_{\mathcal{F}}^{(c)} = \mathrm{ch}_{\mathcal{F}}$. Holmsen [16, Theorem 1.2] proved that $\mathrm{fh}_{\mathcal{F}}^{(c)} \leq \mathrm{ch}_{\mathcal{F}}^{(c)}$ for every set system $\mathcal{F}$ and every $c \in \mathbb{N}$. A close inspection of that proof provides another bridge between the graded and ungraded parameters:

▶ **Lemma 13.** *Let $\ell > c$ be integers. Every set system $\mathcal{F}$ such that $\mathrm{ch}_{\mathcal{F}}^{(c)}(c\ell) \leq \ell$ satisfies $\mathrm{fh}_{\mathcal{F}}^{(c)} \leq \mathrm{ch}_{\mathcal{F}}^{(c)}(c\ell)$.*

Proof. Let $c < \ell$, let $\mathcal{F}$ be a set system and let $\ell' := \mathrm{ch}_{\mathcal{F}}^{(c)}(c\ell)$. By definition of $\mathrm{ch}_{\mathcal{F}}^{(c)}(\cdot)$, the c-uniform hypergraph recording which c-element subsets of $\mathcal{F}$ form cliques cannot contain a certain pattern on $c\ell'$ vertices, namely the *complete ℓ'-tuples of missing edges* [17, §3]. This is the only property needed to ensure that $\mathrm{fh}_{\mathcal{F}}^{(c)} \leq \ell'$ [16, Theorem 1.2]. ◀

Let $\Psi_{\mathrm{r}\to\mathrm{ch}}^{(c)}(x)$ denote the supremum of the cth colorful Helly number of a set system with Radon number x. Holmsen and Lee [17, Theorem 2.2] proved that $\Psi_{\mathrm{r}\to\mathrm{ch}}^{(c)}(x) \leq \max(\Xi(x), c)$ for every $c \geq x-1$, where $\Xi : \mathbb{N} \to \mathbb{N}$ is the function defined by $\Xi(r) := r^{r^{\lceil \log_2 r \rceil} + r\lceil \log_2 r \rceil}$. Applying this inequality to subsets of size t we get:

$$\mathrm{ch}_{\mathcal{F}}^{(c)}(t) \leq \max\left(\Xi\left(\mathrm{r}_{\mathcal{F}}(t)\right), c\right) \qquad \text{for every } \mathcal{F}, c, t \text{ such that } c \geq \mathrm{h}_{\mathcal{F}}(t). \tag{5}$$

Proof of Theorem 7. Let $\Psi : \mathbb{N} \to \mathbb{N}$ and $t_0 \in \mathbb{N}$ such that $\Psi(t) < t+1$ for every $t \geq t_0$. Also suppose that there exists an integer $t_1 \geq {t_0}^2$ such that $\Xi\left(\Psi(t_1)\right) < \frac{t_1}{t_0}$. We now consider a set system $\mathcal{F}$ such that $\mathrm{r}_{\mathcal{F}}(t) \leq \Psi(t)$ for every $t \in \mathbb{N}$ and argue that $\mathrm{fh}_{\mathcal{F}}$, the fractional Helly number of $\mathcal{F}$, is bounded.

By Levi's inequality (4) we have $\mathrm{h}_{\mathcal{F}}(t) \leq \mathrm{r}_{\mathcal{F}}(t) - 1 \leq \Psi(t) - 1$. It follows that $\mathrm{h}_{\mathcal{F}}(t) < t$ for every $t \geq t_0$, so by Lemma 10 we have $\mathrm{h}_{\mathcal{F}} \leq t_0$. Hence, the graded version (5) of the Holmsen-Lee inequality applies with $c = t_0$ and every t, that is $\mathrm{ch}_{\mathcal{F}}^{(t_0)}(t) \leq \max\left(\Xi\left(\Psi(t)\right), t_0\right)$.

Let us apply Lemma 13 with $c = t_0$ and $\ell = \frac{t_1}{t_0}$. Observe that $\ell > c$ holds because $t_1 > {t_0}^2$, and that $\mathrm{ch}_{\mathcal{F}}^{(c)}(c\ell) \leq \ell$ follows from the assumption that $\Xi\left(\Psi(t_1)\right) \leq \frac{t_1}{t_0}$, as

$$\mathrm{ch}_{\mathcal{F}}^{(c)}(c\ell) = \mathrm{ch}_{\mathcal{F}}^{(t_0)}(t_1) \leq \max\left(\Xi\left(\Psi(t_1)\right), t_0\right) \leq \frac{t_1}{t_0} = \ell.$$

Hence, $\mathrm{fh}_{\mathcal{F}}^{(t_0)} \leq \mathrm{ch}_{\mathcal{F}}^{(t_0)}(t_1)$. Since $\mathrm{h}_{\mathcal{F}} \leq t_0$ we have $\mathrm{fh}_{\mathcal{F}}^{(t_0)} = \mathrm{fh}_{\mathcal{F}}$ and the statement follows. ◀

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