Polychromatic 2-Colorings with Bounded Discrepancy for Triangulations

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Abstract

A polychromatic 2-coloring of a triangulation is a 2-coloring of the vertices such that no face is monochromatic. The discrepancy of a coloring is the maximum difference between the sizes of the color classes. Asayama and Matsumoto (Graphs and Combinatorics, 2022) proved that every triangulation admits a polychromatic 2-coloring with discrepancy at most $\frac{5n-16}{9}$, and that there exists a class of triangulations for which every polychromatic 2-coloring has discrepancy at least $\frac{n}{3}-2$, where n is the number of vertices. We improve the upper bound, showing that every triangulation admits a polychromatic 2-coloring with discrepancy at most $\frac{3n-16}{7}$ and such a 2-coloring can be computed in quadratic time. We also show a discrepancy of at most $n-\frac{4M}{3}$ for triangulations with a matching of size M. This implies, for example, that Delaunay triangulations admit a discrepancy of at most $\frac{n}{3}$. We provide a linear-time algorithm to compute a 2-coloring whose discrepancy is at most $\frac{5n-24}{7}$.

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1 Introduction

Colorings of planar graphs play a central role in both combinatorial and algorithmic graph theory. The study of colorings that ensure coverage or visibility conditions has received growing attention in recent years, particularly in contexts where each region of a planar subdivision must contain representatives from multiple categories or sensors. For instance, such colorings arise naturally in network coverage problems, geometric guarding, and frequency assignment, where it is desirable that every bounded region has access to all available resources. This motivated the study of *polychromatic colorings*, colorings in which every face of a planar embedding contains at least one vertex of each color. For a survey on polychromatic colorings see [8].

For the case of two colors, a polychromatic 2-coloring of a triangulation assigns one of two colors to each vertex such that no face is monochromatic. Unlike proper colorings, which aim to separate adjacent vertices, polychromatic colorings of planar graphs emphasize inclusion: every face must *see* both colors. Although such colorings always exist, the balance between the sizes of the two color classes is not guaranteed. The *discrepancy* of a specific polychromatic 2-coloring of G is defined as the absolute difference between the sizes of its two color classes. It measures how unevenly the two colors are distributed. For a graph G, we define $\mathrm{disc}(G)$ as the minimum discrepancy over all possible polychromatic 2-colorings of G. Thus, $\mathrm{disc}(G)$ captures the best achievable balance between the two colors while maintaining the polychromatic property. A polychromatic coloring of G is said to be *balanced* when its discrepancy is at most one.

Asayama and Matsumoto [1] initiated the systematic study of balanced polychromatic 2-colorings in planar triangulations. Among other results, they proved that every triangulation on n vertices admits a polychromatic 2-coloring with discrepancy at most $\frac{5n-16}{9}$, and they constructed infinite families of triangulations where every such coloring has discrepancy at least $\frac{n}{3} - 2$. They conjectured that $\frac{n}{3}$ is the best possible asymptotic upper bound.

In this work we make progress toward that conjecture. We show that every triangulation on n vertices admits a polychromatic 2-coloring satisfying $\mathrm{disc}(G) \leq \frac{3n-16}{7}$, which improves the previous bound of $\frac{5n-16}{9}$. Our proof refines the discharging arguments used in earlier analyses and integrates structural constraints derived from the Four Color Theorem. Moreover, the proof is constructive and leads to an $O(n^2)$ algorithm to compute such a coloring.

We further establish that if a triangulation G admits a matching of size M, then G has a polychromatic 2-coloring with discrepancy at most $\frac{n-4M}{3}$. This bound is tight for certain triangulations and immediately implies that Delaunay triangulations satisfy $\mathrm{disc}(G) \leq n/3$, since they always satisfy having a perfect matching. In addition, we provide a linear-time recoloring procedure that guarantees a polychromatic 2-coloring with discrepancy at most $\frac{5n-24}{7}$. This yields the first efficient algorithm that achieves a nontrivial upper bound on discrepancy in planar triangulations.

Our results strengthen the connection between structural graph properties and color balance in planar settings, advancing the quantitative understanding of how evenly a triangulation can be 2-colored while ensuring that every face remains non-monochromatic.

2 Preliminaries

Most of the basic notions about graph coloring can be found in [11] and [17]. For general concepts of graph theory, see [4] and [9]. We consider graphs that are finite and simple; that is, they have no loops or multiple edges. We denote a graph by $G = (V, E)$, where V is the set of vertices, and E is the set of edges. For planar graphs, we denote by F the set of faces. When needed, we denote by $V(G)$, $E(G)$ and $F(G)$ the sets of vertices, edges and faces to specify the graph to which we refer. We use n, m, and l to denote the number of vertices, edges and faces, respectively. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v, and it is denoted by $d(v)$. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degree of the vertices of G, respectively. The degree of a face $f \in F$, denoted by $d(f)$, is the number of edges (or vertices) on its boundary. A *vertex coloring* χ of G is a function $\chi : V \to \{c_1, c_2, ..., c_k\}$. We call it a *k-coloring* to emphasize the number of colors.

▶ **Definition 1.** *Let G be a graph with n vertices, let χ be a vertex coloring of G, and let $\{V_1, V_2, \ldots, V_k\}$ be the partition induced by χ. We define the* discrepancy *of χ, denoted as $\mathrm{disc}(\chi)$, to be the difference between the sizes of the largest and the smallest color class.*

$$\mathrm{disc}(\chi) = \max\{|V_i| - |V_j| : i, j \in \{1, 2, \ldots, k\}\}$$

We say that a coloring χ is *balanced* when $\text{disc}(\chi) \leq 1$. Balanced proper colorings are also called *equitable colorings* [13].

The following are some known results that we use in the following sections.

▶ **Proposition 2** (Euler's Formula). *Let G be a connected planar graph with n vertices, m edges, and l faces. Then $n - m + l = 2$.*

▶ **Proposition 3** (Diestel [9]). *Every planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges and at most $2n - 4$ faces. Every planar triangulation with $n \geq 3$ vertices has exactly $3n - 6$ edges and $2n - 4$ faces. Every planar quadrangulation with $n \geq 3$ vertices has exactly $2n - 4$ edges and $n - 2$ faces.*

An *independent set I* of G is a subset of vertices such that no two elements of I are adjacent. Observe that in a proper coloring, every color class is an independent set. The *independence number* of G, denoted by $\alpha(G)$, is the size of the maximum independent set.

▶ **Proposition 4** (Caro and Roditty [7]). *Let G be a planar triangulation with $n \geq 4$ vertices and minimum degree $\delta(G)$. Then $\alpha(G) \leq \frac{2n-4}{\delta(G)}$.*

Given a planar graph G and its planar embedding, the *dual graph* of G, denoted by G^* is such that the set of vertices of G^* corresponds to the set of faces of G, and two vertices are adjacent if and only if the corresponding two faces share an edge of their boundary. The following claim is implied by Tait's reformulation of the Four Color Theorem [16].

▶ **Proposition 5.** *Let G be a planar triangulation. The dual graph G^* has a proper 3-edge-coloring.*

A *matching M* of G is a subset of edges such that no two elements of M share a common vertex. A *maximum matching* is a matching with maximum cardinality. A *perfect matching* is a matching that includes all vertices of G, and a *near-perfect matching* includes all of them, but one.

▶ **Proposition 6** (Nishizeki and Baybars [14]). *Every planar triangulation G with n vertices has a matching of size at least $\frac{n}{3}$.*

3 Upper Bounds

In this section, we prove that every planar triangulation has a polychromatic 2-coloring with discrepancy at most $\frac{3n-16}{7}$. We first show some bounds related to other parameters of G.

In the first subsection, given a planar triangulation G with a maximum matching M, we show how to find a polychromatic 2-coloring whose discrepancy is at most $n - \frac{4|M|}{3}$. In particular, the discrepancy is at most $\frac{n}{3}$ when G has a perfect matching.

In the second subsection, given a proper 4-coloring of the triangulation G we show how to partition the four color classes into two sets so that the resulting coloring is polychromatic and its discrepancy is bounded depending on the size of the largest color class in the 4-coloring. In particular, we guarantee the existence of a polychromatic 2-coloring with discrepancy at most $\frac{n}{3}$ for graphs with independent number at most $\frac{n}{2}$. In the last subsection, we extend this idea to improve the general upper bound.

3.1 Discrepancy and Matchings

Polychromatic 2-colorings of triangulations are closely related to spanning quadrangulations and matchings as stated in [1], [2], and [5]. It is known that every triangulation has a spanning quadrangulation, where a *quadrangulation* is a plane graph such that each of its faces is a quadrilateral. It is also known that every quadrangulation is bipartite, and hence 2-colorable.

Let G be a planar triangulation, and let Q be a spanning quadrangulation of G. Every face of Q is the union of two adjacent faces of G, with their shared edge removed. Moreover, Q contains exactly two boundary edges of each face of G. Observe that the unique proper 2-coloring of $V(Q)$ corresponds to a polychromatic 2-coloring of $V(G)$. We now show how the size of the discrepancy relates to the size of a matching of Q in the following lemma.

▶ **Lemma 7.** *Let ρ be the unique proper 2-coloring of Q and let M_Q be a matching in Q. Then, ρ is a polychromatic 2-coloring in G with $\operatorname{disc}(\rho) \leq n - 2|M_Q|$.*

Proof. Let R and B be the color classes in ρ. Every edge in the matching M_Q has one endpoint in each color class. Let $U_R \subseteq R$ and $U_B \subseteq B$ be the unmatched vertices. Then $|R| = |M_Q| + |U_R|$ and $|B| = |M_Q| + |U_B|$. Without loss of generality, assume that $|R| \geq |B|$. Then, $\operatorname{disc}(\chi) = |R| - |B| = |U_R| - |U_B| \leq |U_R| + |U_B| = n - 2|M_Q|$. ◀

Now, to define an appropriate spanning quadrangulation with the appropriate matching, the strategy is to start with a matching M in the triangulation G, and then show that there exists a spanning quadrangulation containing a large fraction of edges from M.

There is a well-known equivalence between the 4-coloring of the vertices of a triangulation G and the 3-coloring of the edges of its dual G^*. Such edge-coloring induces a partition of the edges of G^* into 3 perfect matchings of G^*. Let E_1, E_2 and E_3 be the sets of edges of G corresponding to each of the 3 perfect matchings in G^*. Let $Q_i = G \setminus \{E_i\}$ for $i \in \{1, 2, 3\}$. We note that each Q_i is a spanning quadrangulation of G. (see e.g. [1], [5], [6], and [12]).

▶ **Lemma 8.** *Let M be a matching in G. There exists a spanning quadrangulation of G that has a matching of size at least $\frac{2}{3}|M|$.*

Proof. Every edge of M appears by construction in exactly 2 of the Q_i's, which implies that one of the Q_i's contains at least $\frac{2}{3}|M|$ edges of M. ◀

The following is the main result of this subsection.

▶ **Theorem 9.** *Every triangulation G with n vertices has a polychromatic 2-coloring with discrepancy at most $n - \frac{4}{3}|M|$, where M is a matching in G.*

Proof. By Lemma 8, G has a a spanning quadrangulation Q that has a matching M_Q of size at least $\frac{2}{3}|M|$. By Lemma 7 the proper 2-coloring of Q is a polychromatic 2-coloring of G with discrepancy at most $n - 2|M_Q| \leq n - \frac{4}{3}|M|$. ◀

▶ **Corollary 10.** *Every planar triangulation G has a polychromatic 2-coloring whose discrepancy is at most $\frac{5n-16}{9}$.*

Proof. Follows from Theorem 9 with the matching that is guaranteed by Proposition 6. ◀

A relevant application of Theorem 9 is to obtain a better upper bound for the discrepancy of polychromatic 2-colorings in Delaunay triangulations, stated in the following Corollary.

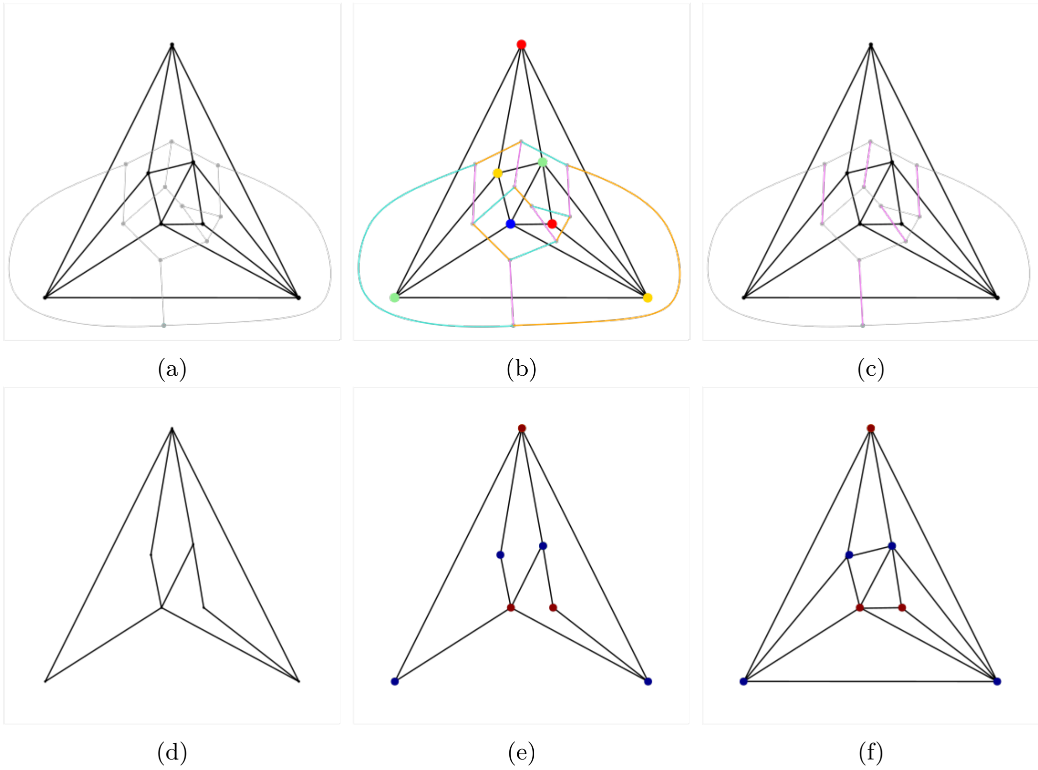

Figure 1 (a) A triangulation G and its dual graph G^*. (b) Equivalence between a 4-coloring on $V(G)$ and a 3-coloring on $E(G)$. (c) Perfect matching on G^*, corresponding to one color class in the 3-edge-coloring. (d) Spanning quadrangulation of G induced by deleting the dual edges of the perfect matching on G^*. (e) Proper 2-coloring on $V(Q)$. (f) Polychromatic 2-coloring on $V(G)$.

▶ **Corollary 11.** *Let D a graph on n vertices corresponding to the Delaunay triangulation of a set of n points in the plane. D has a polychromatic 2-coloring whose discrepancy is at most $\frac{n}{3}$.*

Proof. It is known that Delaunay triangulations have perfect matchings [3, 10]. Then by Theorem 9, D has a polychromatic 2-coloring whose discrepancy is at most $n - \frac{4}{3}\left(\frac{n}{2}\right)$. ◀

Observe that the bound $\frac{n}{3}$ is true for any triangulation with a perfect matching.

3.2 Discrepancy and Independent Sets

In this section, the strategy is to start with a proper 4-coloring of the triangulation, and define a 2-coloring that is polychromatic whose discrepancy is bounded according to the sizes of the color classes in the 4-coloring. This idea was used in [1] to prove the general bound. We extend the idea to relate the discrepancy to the size of the maximum independent set, which generalizes two of their main results and Theorem 9.

Let G be a planar triangulation with n vertices. Let $\chi : V(G) \to \{1, 2, 3, 4\}$ be a 4-proper coloring of $V(G)$. Let V_1, V_2, V_3, V_4 be the respective color classes and let n_1, n_2, n_3, n_4 be their respective sizes. We note that $n_1 + n_2 + n_3 + n_4 = n$. Without loss of generality, assume $n_1 \geq n_2 \geq n_3 \geq n_4$. Since $n_2 + n_3 + n_4 = n - n_1$ we get the following inequality:

▶ **Observation 12.** $n_4 \leq \frac{n-n_1}{3}$.

We describe a 2-coloring ρ of $V(G)$, whose color classes are R, B (sometimes referred to as the colors red and blue), and r, b their respective sizes. The discrepancy of ρ is defined as $\mathrm{disc}(\rho) = |r - b|$ with $r + b = n$. Therefore, $\mathrm{disc}(\rho) = |2r - n|$. We consider two cases to eliminate the absolute value.

▶ **Observation 13.** *If $r \geq \frac{n}{2}$, then $\mathrm{disc}(\rho) = 2r - n$. If $r < \frac{n}{2}$, then $\mathrm{disc}(\rho) = n - 2r$.*

▶ **Lemma 14.** *If $n_1 + n_4 \leq \frac{2n}{3}$ or $n_1 \leq \frac{n}{2}$, then G has a polychromatic 2-coloring whose discrepancy is at most $\frac{n}{3}$.*

Proof. Let ρ be the 2-coloring that assigns $R = V_1 \cup V_4$ and $B = V_2 \cup V_3$. Then $r = n_1 + n_4$ and $b = n_2 + n_3$. We claim that ρ is a polychromatic 2-coloring. Since every face of G is triangular, it contains vertices from 3 different color classes of the 4-coloring χ. Therefore, in ρ every face contains at least one red and one blue vertex.

Since $n_1 + n_4 \geq n_2 \geq n_3$, we have that $r = n_1 + n_4 \geq \frac{n}{3}$. Thus, if $n_1 + n_4 \leq \frac{2n}{3}$ then $\mathrm{disc}(\rho) = |2r - n| \leq \frac{n}{3}$.

If $n_1 \leq \frac{n}{2}$ then by applying Observation 12 we get $n_2 + n_3 = n - n_1 - n_4 \geq \frac{2n-2n_1}{3} \geq \frac{n}{3}$. Once again we get then $\mathrm{disc}(\rho) = |2r - n| \leq \frac{n}{3}$. ◀

▶ **Lemma 15.** *If $n_1 + n_4 \geq \frac{n}{2}$, then G has a polychromatic 2-coloring whose discrepancy is at most $\frac{4n_1-n}{3}$.*

Proof. Let ρ be the polychromatic 2-coloring defined in the proof of Lemma 14. By observation 12, we know that $r = n_1 + n_4 \leq \frac{n+2n_1}{3}$. Since $r = n_1 + n_4 \geq \frac{n}{2}$, we have $\mathrm{disc}(\rho) = 2r - n$. We conclude that

$$\mathrm{disc}(\rho) \leq 2\left(\frac{n + 2n_1}{3}\right) - n \leq \frac{4n_1 - n}{3}.$$ ◀

▶ **Theorem 16.** *Let G be a planar triangulation with n vertices. If G has a proper 4-coloring χ whose largest color class has size at most s, then G has a polychromatic 2-coloring ρ such that $\mathrm{disc}(\rho) \leq \max\left(\frac{n}{3}, \frac{4s-n}{3}\right)$.*

Proof. The sizes of the color classes in χ meet the conditions of Lemma 14 or 15. ◀

▶ **Observation 17.** *By the Four Color Theorem, every triangulation has a proper 4-coloring. Since the color classes are independent sets, Theorem 16 can always be applied with $s = \alpha(G)$, the independence number of G.*

▶ **Corollary 18.** *Let G be a planar triangulation with $n \geq 4$ vertices. Then G has a polychromatic 2-coloring whose discrepancy is at most $\frac{5n-16}{9}$. Moreover, if $\delta(G) \geq 4$ then G has a polychromatic 2-coloring whose discrepancy is at most $\frac{n}{3}$.*

Proof. By the Four Color Theorem, G has a proper 4-coloring χ. Let n_1 be the size of the largest color class of χ. By Observation 17 and Proposition 4, we have $n_1 \leq \alpha(G) \leq \frac{2n-4}{\delta(G)}$ and we can apply Theorem 16 with $s = \frac{2n-4}{\delta(G)}$. Since every planar triangulation with $n \geq 4$ vertices has minimum degree at least 3, it follows that:

$$\mathrm{disc}(\rho) \leq \max\left(\frac{n}{3}, \frac{4\left(\frac{2n-4}{3}\right) - n}{3}\right) = \frac{5n - 16}{9}.$$

If $\delta(G) \geq 4$ then

$$\mathrm{disc}(\rho) \leq \max\left(\frac{n}{3}, \frac{4\left(\frac{2n-4}{\delta(G)}\right) - n}{3}\right) = \frac{n}{3}.$$ ◀

There is a relation between the size of a matching and the size of the maximum independent set in G.

▶ **Observation 19.** *Given a matching M of G, any subset of more than $n - |M|$ vertices includes an edge of M and then it is not independent. Therefore, $\alpha(G) \leq n - |M|$ for any matching M. Based on this fact and Observation 17, we can apply Theorem 16 with $s = n - |M|$ and get Theorem 9.*

We have provided an alternative proof for two theorems from [1]. Moreover, we have generalized these results by establishing an upper bound on the discrepancy of a polychromatic 2-coloring in terms of an upper bound on the size of the largest color class in a proper 4-coloring.

3.3 Improving the General Discrepancy Bounds

In this section we prove the following theorem:

▶ **Theorem 20.** *Let G be a planar triangulation with $n \geq 6$ vertices. Then G has a polychromatic 2-coloring whose discrepancy is at most $\frac{3n-16}{7}$.*

Recall the two color classes from the previous section (that are obtained as a partition of the colors of a proper 4-coloring). In view of Lemma 14 we may assume that $n_1 + n_4 \geq n_2 + n_3$ as otherwise the discrepancy is at most $n/3$. The main proof idea is to start by coloring all vertices in V_1 red, all vertices in $V_2 \cup V_3$ blue, and possibly splitting the vertices in V_4 between red and blue to get a smaller discrepancy.

Let $F_1 \subset F(G)$ be the set of faces that contain a vertex from V_1 and let $F_{234} = F(G) \setminus F_1$ be the set of faces that do not. If $V_1 \subseteq R$ and $V_2 \cup V_3 \subseteq B$ then all faces in F_1 are guaranteed to be polychromatic, regardless of the color of the vertices in V_4. In addition, every face $f \in F_{234}$ contains a vertex from V_4 that we can assign to R to make ρ polychromatic. Let $S_4 \subseteq V_4$ be the set of vertices in V_4 belonging to at least one face in F_{234}. Let $s_4 = |S_4|$. We call the vertices in S_4 *essential*, since they are required to be in the color class R to make the coloring polychromatic. The rest of the vertices in V_4 are called *non-essential*, and we denote the set as $\bar{S}_4 = V_4 \setminus S_4$. Let ρ be the 2-coloring that assigns $R = V_1 \cup S_4$ and let $B = V_2 \cup V_3 \cup \bar{S}_4$. The sizes of the color classes of ρ are $r = n_1 + s_4$ and $b = n_2 + n_3 + n_4 - s_4$. Observe that ρ is a polychromatic 2-coloring and $disc(\rho) = 2r - n$, by Observation 13 and Lemma 14. As a warm-up, in the following proposition we bound the discrepancy of ρ by analyzing the properties of S_4; this result is independent of the rest of the section.

▶ **Proposition 21.** $disc(\rho) \leq \frac{n-4}{2}$.

Proof. We give an upper bound for $r = n_1 + s_4$ based on the following observations:

1. Every vertex in V_1 is in at least 3 faces from F_1 because $\delta(G) \geq 3$. Every face in F_1 has exactly one vertex from V_1. Then $3n_1 \leq |F_1|$.
2. Every vertex in S_4 is in at least one face from F_{234} by definition. Every face in F_{234} has exactly one vertex in S_4. Then $s_4 \leq |F_{234}|$.

By Proposition 3, the total number of faces of G is $|F_1| + |F_{234}| = 2n - 4$, then we conclude that:

$$3n_1 + s_4 \leq |F_1| + |F_{234}| = 2n - 4 \tag{1}$$

We know that $S_4 \subseteq V_4$, then $s_4 \leq n_4$. By Observation 12, we have $n_4 \leq \frac{n-n_1}{3}$, which implies that:

$$n_1 + 3s_4 \leq n_1 + 3\left(\frac{n - n_1}{3}\right) = n \tag{2}$$

By adding up the equations 1 and 2, we get $4n_1 + 4s_4 \leq 3n - 4$. Then $r = n_1 + s_4 \leq \frac{3n-4}{4}$ and $\text{disc}(\rho) = 2r - n \leq \frac{n-4}{2}$. ◀

Some extra observations help to improve this bound. Let G be a planar triangulation with n vertices. Let χ_{min} be a proper 4-coloring of $V(G)$ such that the difference between the sizes of the largest and the smallest color class, i.e. $n_1 - n_4$, is minimum over all proper 4-colorings of $V(G)$. From the results in Subsection 3.2 (Lemma 14), we know that if $n_1 + n_4 \leq \frac{2n}{3}$ or $n_1 \leq \frac{n}{2}$, then the coloring that assigns $R = V_1 \cup V_4$ and $B = V_2 \cup V_3$ is polychromatic and has a discrepancy at most $\frac{n}{3}$. Thus, we may assume that $n_1 > \frac{n}{2}$ and $n_1 + n_4 > \frac{2n}{3}$ in the following. Then, Observation 12 implies that $n_4 \leq \frac{n}{6}$. Therefore, we get

▶ **Observation 22.** $n_1 - n_4 \geq \frac{n}{3}$.

Let $V_1^{d=3}$ be the set of vertices in V_1 whose degree equals 3 and let $V_1^{d\geq 4} = V_1 \setminus V_1^{d=3}$ be those with degree at least 4.

We refer to the faces in F_{234} as *good faces*. We classify every essential vertex as *type I* if it is in exactly one good face and *type II* if it is in at least two good faces. For any vertex v in G, let $N_i(v)$ be the set of neighbors of v in V_i for all $i \in \{1, 2, 3, 4\}$.

▶ **Lemma 23.** *Every essential vertex x of type I has an odd degree $d(x) \geq 3$ and $|N_1(x)| = \frac{d(x)-1}{2}$.*

Proof. Let $x \in S_4$ be an essential vertex of type I with degree $d(x)$. Since x is in exactly one good face, there is exactly one vertex in $N_2(x)$ and one vertex in $N_3(x)$ that are adjacent to each other. The other $d(x) - 2$ vertices are alternating between elements in $N_1(x)$ and elements in $N_2(x) \cup N_3(x)$ starting and ending with color 1 adjacent to the good face, otherwise we contradict that vertex x is of type I. This implies that $\frac{d(x)-1}{2}$ of the neighbors have color 1. ◀

We say that a connected subgraph of G is *i-j-alternating* if it is properly 2-colored with colors i and j.

▶ **Lemma 24.** *Let $n > 4$. If $x \in S_4$ is an essential vertex and every vertex in $N_1(x)$ has degree 3, then the vertices in $N_2(x) \cup N_3(x)$ form a 2-3-alternating separating cycle C in G. Moreover, if x is type I, then x has odd degree $d(x) \geq 7$.*

Proof. Let $y \in N_1(x)$. Since $d(y) = 3$, y is inside a separating triangle formed by x and the endpoints of a 2-3-edge. Then there is a 2-3-edge for each vertex in $N_1(x)$ and a 2-3 edge for each good face incident to x, all those edges together form a 2-3-alternating separating cycle C whose inside vertices are $x \in V_4$ and all the vertices in $N_1(x)$. Let $\ell(C)$ be the length of C. Observe that $\ell(C)$ is even and at least 4, so in the case where x is of type I, exactly one of the edges in C is part of a good face containing x and $\ell(C) - 1$ edges are part of a separating triangle that contains a vertex of color 1. Then $|N_1(x)| = \ell(C) - 1$. Note that $d(x) = |N_1(x)| + \ell(C) = 2\ell(C) - 1 \geq 2(4) - 1 = 7$. ◀

▶ **Observation 25.** *For every vertex x satisfying Lemma 24, $|N_1(x)| < |N_2(x) \cup N_3(x)|$ as shown in the proof of the lemma. Since we assume that $n_2 + n_3 < \frac{n}{3}$, we have that $|N_1(x)| < \frac{n}{3}$.*

▶ **Observation 26.** *Let C be a 2-3-alternating separating cycle. If we swap the colors 1 and 4 of the vertices inside C, we obtain another valid 4-coloring of $V(G)$.*

▶ **Lemma 27.** *Every essential vertex x of type I is adjacent to at least one vertex in $V_1^{d\geq 4}$.*

Proof. Let $x \in S_4$ be an essential vertex of type I. By Lemma 23, since $d(x) \geq 3$, $|N_1(x)| \geq 1$. Assume for a contradiction that every vertex in $N_1(x)$ has degree 3. By Lemmas 23 and 24, x has odd degree $d(x) \geq 7$, $|N_1(x)| = \frac{d(x)-1}{2} \geq 3$ and there is a 2-3-separating alternating

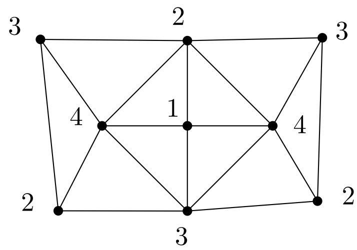

Figure 2 Illustration for Observation 28.

cycle C containing exactly one vertex of color 4, namely x, and $|N_1(x)|$ vertices of color 1 in its interior. By swapping colors 1 and 4 in C (Observation 26) we obtain a valid proper 4-coloring χ' whose color classes that has $n_1 - |N_1(x)| + 1$ vertices of color 1 and $n_4 - 1 + |N_1(x)|$ vertices of color 4. This would decrease the difference between n_1 and n_4 and hence contradicts our choice of χ_{min}.

We have to be careful here as color 4 may now be the largest color class in χ'. Hence the difference would be

$$(n_4 - 1 + |N_1(x)|) - (n_1 - |N_1(x)| + 1) = (n_4 - n_1) + 2|N_1(x)| - 2 < \frac{n}{3},$$

where the inequality is valid by Observation 22 and Observation 26. This still contradicts our choice of χ_{min} in view of Observation 26. ◀

Similar to the partition of V_4 in essential and non-essential vertices, we partition V_1 according to the faces they belong to. Let $S_1 \subseteq V_1$ be the set of vertices in V_1 belonging to at least one face that does not contain a vertex from S_4. Let $s_1 = |S_1|$. Let $\bar{S}_1 = V_1 \setminus S_1$ and call these vertices *non-essential* since they can be colored red or blue without violating the polychromatic 2-coloring property.

▶ **Observation 28.** *If a vertex $y \in V_1$ has degree 4 and two neighbors in S_4, then the four faces that contain y contain a vertex in S_4 and thus y is non-essential.*

In the following Lemma we state an upper bound for $s_1 + s_4$, which we later use to bound the discrepancy of a polychromatic 2-coloring.

▶ **Lemma 29.** $s_1 + s_4 \leq \frac{5n-8}{7}$.

Proof. We use a discharging argument. We assign 1 unit of charge to every face in F_{234}. For every vertex $y \in V_1$, we assign a charge equal to $d(y)$. Thus, the total initial charge is $|F_{234}| + |F_1| = 2n - 4$. Apply the transfer of charge as follows: Each face in F_{234} transfers its charge to its unique vertex in V_4. For every vertex $y \in V_1$, if $d(y) \geq 4$, then transfer 1 unit of charge to each of its neighbors that is in S_4. The total final charge is equal to the total initial charge. We analyze the final charge in the vertices of S_1 and S_4.

▷ Claim 1. Every vertex in S_1 has final charge at least 3.

Let $y \in S_1$. We know $d(y) \geq 3$. If $d(y) = 3$, then y has final charge equal to 3. If $d(y) = 4$, then y has at most one neighbour in S_4, otherwise it contradicts Observation 28. Therefore, the final charge is at least 3. If $d(y) \geq 5$, then y has at most $\lfloor \frac{d(y)}{2} \rfloor$ neighbors in S_4, since no two color 4 vertices are adjacent. Thus, the final charge of y is at least $d(y) - \lfloor \frac{d(y)}{2} \rfloor = \lceil \frac{d(y)}{2} \rceil \geq \lceil \frac{5}{2} \rceil = 3$.

▷ Claim 2. Every vertex in S_4 has final charge at least 2.

Let $x \in S_4$ be an essential vertex. If x is of type II, then it gets at least 2 units of charge from the two faces in F_{234}. Assume x is type I, then it gets 1 unit of charge from its incident F_{234} face. By Lemma 27, x is adjacent to a vertex $y \in V_1$ such that $d(y) \geq 4$, from which it gets an additional unit of charge. Thus, every vertex in S_4 has final charge at least 2.

Since the total final charge is equal to the total initial charge, the final charge in $S_1 \cup S_4$ is at most the total initial charge.

$$3s_1 + 2s_4 \leq 2n - 4 \tag{3}$$

We know that $S_1 \subseteq V_1$ and $S_4 \subseteq V_4$, then $s_1 \leq n_1$ and $s_4 \leq n_4$. By Observation 12, we have $n_4 \leq \frac{n-n_1}{3}$, which implies that:

$$s_1 + 3s_4 \leq n_1 + 3\left(\frac{n - n_1}{3}\right) = n \tag{4}$$

Multiplying by 2 both sides of (3) and adding it to (4), we get $7s_1 + 7s_4 \leq 5n - 8$. Then $s_1 + s_4 \leq \frac{5n-8}{7}$. ◀

We know that $S_1 \cup S_4 \subseteq V_1 \cup V_4$, and $|V_1 \cup V_4| = n_1 + n_4 > \frac{n}{2}$. If $s_1 + s_4 \geq \frac{n}{2}$, then let ρ be the 2-coloring that assigns $R = S_1 \cup S_4$. Otherwise, add some vertices from $\bar{S_1} \cup \bar{S_4}$ to R in order to guarantee $r = |R| \geq \frac{n}{2}$. The sizes of the color classes of ρ are $r = \max(s_1 + s_4, \frac{n}{2})$ and $b = n - r$. Observe that ρ is a polychromatic 2-coloring and $disc(\rho) = 2r - n$. By Lemma 29, $\mathrm{disc}(\rho) = 2r - n \leq 2\left(\frac{5n-8}{7}\right) - n = \frac{3n-16}{7}$. This finishes the proof of Theorem 20.

▶ Remark. The only place that we use the minimality of our 4-coloring is the contradictory argument in the proof of Lemma 27 where a color-4 vertex x and its color-1 neighbors are inside a 2-3-separating cycle formed by its color-2 and color-3 neighbors. Any such cycle is identified by the neighbors of a color-4 vertex. To get an algorithm, we could start from an arbitrary 4-coloring (that can be computed in $O(n^2)$ time [15]) and then for each such color-4 vertex x we swap the color of x with its color-1 neighhbors. This reduces $n_1 - n_4$. This extra step takes a total of $O(n)$ time for all such x's.

4 Linear-time Algorithm to Compute a Polychromatic 2-Coloring with Bounded Discrepancy for Planar Triangulations

In this section, we show how to compute, for a given planar triangulation, a polychromatic 2-coloring whose discrepancy is at most $\frac{5n-24}{7}$, in linear time. We use the linear-time algorithm by Bose et al. [5] to compute an initial polychromatic 2-coloring. Their algorithm uses a linear time algorithm to compute a maximum matching in the dual graph of a triangulation by Biedl et al. [2]. Thus, for any given planar graph, a polychromatic 2-coloring can be computed in linear time. However, this polychromatic 2-coloring may not have a guarantee on the discrepancy. We show how to extend this algorithm to guarantee a discrepancy of at most $\frac{5n-24}{7}$, without increasing the time complexity. The main approach is to make local modifications. We start by showing that the discrepancy of any polychromatic 2-coloring on the vertices of a triangulation G has an upper bound in terms of the maximum degree of G.

▶ **Theorem 30.** *Let G be a planar triangulation with $n \geq 3$ vertices and maximum degree $\Delta(G)$. Every polychromatic 2-coloring of G has discrepancy at most $\frac{(\Delta(G)-4)n+8}{\Delta(G)}$.*

Proof. Let G be a planar triangulation with n vertices, and $\chi : V(G) \to \{c_1, c_2\}$ a polychromatic 2-coloring of G. By Proposition 3 the number of faces of G is exactly $2n-4$. Since χ is a polychromatic coloring, every face contains at least one vertex of each color class. The number of faces that contain a vertex v is the degree of the vertex, $d(v)$. Then, any vertex belongs to at most $\Delta(G)$ faces. Therefore, every color class of χ has size at least $\frac{2n-4}{\Delta(G)}$. This implies that $\mathrm{disc}(\chi) \leq n - 2\left(\frac{2n-4}{\Delta(G)}\right) = \frac{(\Delta(G)-4)n+8}{\Delta(G)}$. ◀

To bound the discrepancy after a recoloring process, we define a class of colorings in which there is no single vertex to recolor that decreases the discrepancy and preserves polychromaticity. A polychromatic 2-coloring ρ of the vertices of a triangulation G is *locally minimal* if there is no vertex in $V(G)$ that can be recolored to reduce the discrepancy of ρ without producing a monochromatic face. This naturally raises the question: How large can the discrepancy of a locally minimal coloring be?

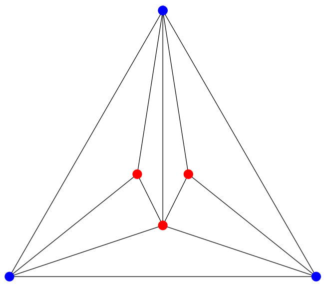

Figure 3 Construction that can be repeated in every face of a triangulation to get a locally minimal coloring. Blue points represent vertices in B and red points represent vertices in R.

▶ **Theorem 31.** *Let G be a planar triangulation with $n \geq 6$ vertices. Every polychromatic 2-coloring of G that is locally minimal has discrepancy at most $\frac{5n-24}{7}$.*

Proof. We start with the upper bound on the discrepancy. Consider an arbitrary planar triangulation G and a locally minimal polychromatic 2-coloring $\hat{\rho}$ as described before. Let R, B be the respective color classes, and let r, b be their respective sizes. Without loss of generality, assume $r \geq b$. If $r \leq b+1$ then the discrepancy is at most 1. Assume that $r \geq b+2$. Due to minimality of $\hat{\rho}$, every vertex $v \in R$ is incident to a face with an edge $e = (x, y)$ such that both x and y are in R, because otherwise we could recolor v to reduce discrepancy. Each such edge e is in exactly two faces of G, and hence count for at most two vertices in R. That means that the number of monochromatic edges of color B is at least $\frac{r}{2}$. Let G_2 be the subgraph of G induced by B. G_2 is a planar graph with b vertices, then by Proposition 3 it has at most $3b-6$ edges. Therefore, there are no more than $3b-6$ monochromatic edges of color B. It follows that $\frac{r}{2} \leq 3b-6$. Since $r+b=n$, we have that $7b-12 \geq n$, or $b \geq \frac{n+12}{7}$. Thus $r = n-b \leq n - \frac{n+12}{7} = \frac{6n-12}{7}$. The discrepancy $r-b$ is at most $\frac{6n-12}{7} - \frac{n+12}{7} = \frac{5n-24}{7}$.

We show that this bound is tight by constructing an infinite family of triangulations with a locally minimal polychromatic 2-coloring whose discrepancy is $\frac{5n-24}{7}$. We start with G_2 equal to a planar triangulation on b vertices of color B. Into each triangular face of G_2, we place three vertices of color R, connected as shown (for one face) in Figure 3. Note that no R-vertex can be changed to a B-vertex in any instance of this construction, as each is adjacent to an edge of G_2; thus the construction is minimal. With b vertices, G_2 has $2b-4$ triangular faces, by Proposition 3. This means that $r = 3(2b-4) = 6b-12$, and so the construction has $r > b$ for each $b \geq 3$. Since $r + b = n$, we have $7b - 12 = n$, or $b = \frac{n+12}{7}$. Thus, $r = \frac{6n-12}{7}$, as in the upper bound. The discrepancy $r - b$ is then $\frac{5n-24}{7}$. So we meet the upper bound by starting with any triangulation of any size. ◀

▶ **Theorem 32.** *Let G be a planar triangulation with $n \geq 6$ vertices. A polychromatic 2-coloring on the vertices of G whose discrepancy is at most $\frac{5n-24}{7}$ can be computed in $O(n)$ time complexity.*

Proof. The proof follows from the following algorithm.

Algorithm 1 Algorithm to compute a minimal polychromatic 2-coloring.

Require: A planar triangulation G, with n vertices.
Ensure: A locally minimal polychromatic 2-coloring on the vertices of G.
  Compute a polychromatic 2-coloring. Let R and B be the two color classes. Without loss of generality assume $|R| \geq |B|$.
  while there is a vertex $v \in R$, such that v is not in a face with two vertices from B **do**
    Recolor v from red to blue and update the color classes to $R \backslash \{v\}$ and $B \cup \{v\}$.
  end while
  Output the two color classes.

Correctness. The first step is to run the algorithm provided in [5], to compute a coloring with no monochromatic faces. The algorithm always maintains a valid polychromatic 2-coloring, since it only recolors a vertex when it does not produce monochromatic faces. We only recolor vertices from red to blue and every vertex is recolored at most once. Then the algorithm terminates, and the discrepancy decreases in every recoloring step until the coloring is locally minimal.

Running Time. Each iteration of the **while** loop examines a vertex and possibly changes its color. We say a vertex $v \in V$ is *recolorable* if $v \in R$, and v is not in a face with two vertices from B. We can process the vertices in any given order, since a vertex that is not recolorable never becomes recolorable. For every vertex v, we can verify if it is recolorable in $O(d(v))$ because we evaluate all the faces that contain v. Since every face is triangular, during the whole process we check every face three times, and since G is planar, the number of faces is $3n - 6$. Therefore, the total run of the algorithm takes $O(n)$ assuming constant-time adjacency and face incidence queries. ◀

5 Conclusions

We studied the discrepancy of polychromatic 2-colorings for planar triangulations and presented improved bounds and algorithms. We conclude with a few open questions for further investigation. The main conjecture in this area remains open: does every triangulation on n vertices admit a polychromatic 2-coloring with discrepancy at most $\frac{n}{3}$? From an algorithmic perspective, it remains open whether a polychromatic 2-coloring with discrepancy less than $\frac{5n-24}{7}$ can be computed in linear time.

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