

How Many Slopes Does Polynomial Area Cost?

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Abstract

In this work, we study the interplay between the number of slopes, the number of bends per edge, and the area requirements for planar drawings of bounded-degree graphs. Our motivation stems from the fact that, while numerous algorithms produce planar drawings with few slopes for graphs of relatively small degree in polynomial area, existing approaches for higher-degree graphs often require super-polynomial area. We address this gap in the literature by presenting new constructions that yield polynomial-area drawings with few bends per edge while slightly increasing the required number of slopes, thereby providing the first systematic study of slopes, bends and area trade-offs.

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1 Introduction

Producing drawings of graphs while minimizing the number of slopes is a well-studied problem in Graph Drawing, motivated by both theoretical interest and practical applications such as network visualization and VLSI design [2, 19, 41]. From an algorithmic perspective, given a graph, the goal is to determine its slope number, that is, the minimum number of pairwise distinct slopes used by its edge segments, taken over all polyline drawings of the graph with a prescribed number of bends per edge. Wade and Chu [42] were among the first to investigate this problem by showing that the slope number of the complete graph K_n in the straight-line setting is n . The problem has been extensively studied since then; several results are known for general graphs of bounded degree [15, 26, 34–36], and beyond-planar graphs [11, 28].

When the input graph is planar, the output drawing is additionally required to be crossing-free. The problem of finding planar drawings with few slopes has also been extensively studied [4, 6, 7, 12–14, 18, 25, 29–31, 33], as it traces back to orthogonal graph drawing [8, 17, 40], where edges are represented as polygonal chains consisting of alternating horizontal and vertical segments, that is, using only two slopes. In this context, a central result by Biedl and Kant [5] guarantees that every planar graph of maximum degree 4, except for the octahedron, admits an orthogonal drawing on an $n \times n$ grid in which each edge has at most two bends. This result is tight in the sense that there exist planar graphs of maximum degree 4 that do not admit planar orthogonal drawings with one bend per edge [40].



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■ **Table 1** Summary of our results for planar graph drawings with at most k bends per edge.

k	Degree	Connectivity	Drawing area	No. of Slopes	Ref.
1	$\Delta \geq 5$	3	$\mathcal{O}(\Delta n^2) \times \mathcal{O}(\Delta n^3)$	$3\Delta - 8$	Thm. 1
1	$\Delta \geq 5$	–	$\mathcal{O}(\Delta n^2) \times \mathcal{O}(\Delta n^3)$	$\lceil \frac{9}{2}\Delta \rceil + 1$	Cor. 2
1	5	3	$\mathcal{O}(n^3) \times \mathcal{O}(n^4)$	5	Thm. 3
2	$\Delta \geq 3$	2	$\mathcal{O}(n) \times \mathcal{O}(\Delta n^2)$	$\lceil \frac{\Delta}{2} \rceil$	Thm. 4
2	Δ	–	$\mathcal{O}(n) \times \mathcal{O}(\Delta n^2)$	$\lceil \frac{\Delta}{2} \rceil + 1$	Cor. 5
4	Δ	–	$\mathcal{O}(n) \times \mathcal{O}(n)$	Δ	Thm. 6

A natural extension of the orthogonal drawing model is the octilinear, which additionally supports diagonal segments at $\pm 45^\circ$, yielding a total of four slopes. In this model, every planar graph with maximum degree at most 3 admits a bendless planar drawing [12] on a $O(n) \times O(n)$ grid, while every planar graph with maximum degree at most 4 (and 5, respectively) admits a planar drawing with at most one bend per edge on a $O(n^2) \times O(n)$ grid (and a super-polynomial grid, respectively) [3].

For graphs of higher degree, Keszegh, Pach, and Pálvölgyi [25] extended the algorithm of Biedl and Kant [5] and showed that every planar graph of maximum degree $\Delta \geq 3$, with the exception of the octahedron, admits a planar drawing with at most two bends per edge using segments of at most $\lceil \Delta/2 \rceil$ distinct equidistant slopes; this bound on the number of slopes is clearly optimal. Improving previous related results [25, 32], Angelini, Bekos, Liotta, and Montecchiani [1] demonstrated that every planar graph of maximum degree $\Delta \geq 4$ admits a planar drawing with at most one bend per edge using segments from any arbitrary set of $\Delta - 1$ pairwise distinct slopes. In the straight-line setting, Keszegh, Pach, and Pálvölgyi [25] showed that every planar graph with maximum degree Δ admits a straight-line drawing using segments of $2^{O(\Delta)}$ distinct slopes.

Our contribution. In this work, we identify and close a critical gap in the literature. While almost all aforementioned algorithms for planar graphs of small fixed degree produce drawings of polynomial area [3, 5, 12, 17, 40] (and are therefore practically applicable), the corresponding algorithms that have been proposed for planar graphs of higher maximum degree require super-polynomial area [1, 25, 32], which limits significantly their practical applicability. Thus, in this paper, we study for first time the interplay between the number of slopes, the number of bends per edge, and the corresponding area requirements; see Table 1. Our focus is on algorithms that produce planar drawings with few bends per edge on polynomial-size grids while only slightly increasing the number of slopes used. More precisely:

- In Section 3, we prove that every 3-connected n -vertex planar graph of maximum degree Δ admits a planar grid drawing with at most one bend per edge, using at most $3\Delta - 8$ slopes on a $O(\Delta n^2) \times O(\Delta n^3)$ grid. Our approach builds upon the incremental construction by Angelini, Bekos, Liotta and Montecchiani [1], which produces such drawings on *arbitrary* sets of $\Delta - 1$ slopes. In contrast, we fix the slope set in advance and increase its size from $\Delta - 1$ to $3\Delta - 8$ in order to guarantee polynomial area. As a consequence, for general planar graphs (i.e., not necessarily 3-connected) the number of slopes becomes $\lceil \frac{9}{2}\Delta \rceil + 1$.
- For the special case of planar graphs of maximum degree 5, we decrease the number of required slopes from $3\Delta - 8 = 7$ to 5 at the cost of increasing the drawing area by a factor of $O(n^2)$; see Theorem 3. Compared with the best-known algorithm in [3], our construction achieves polynomial-area while increasing the number of slopes by one.

- In Section 4, we prove that every planar graph G of maximum degree $\Delta \geq 3$ admits a 2-bend planar drawing on a $O(n) \times O(\Delta n^2)$ grid using at most $\lceil \Delta/2 \rceil$ slopes if G is biconnected, and at most $\lceil \Delta/2 \rceil + 1$ slopes otherwise. In contrast to the algorithm by Keszegh, Pach, and Pálvölgyi [25], which uses *equidistant* slopes to support rotations and scalings of biconnected components around cut vertices, our algorithm uses a different slope set to guarantee polynomial-drawing area. However, it requires one additional slope for general (i.e., non-biconnected) planar graphs, as it can rely neither on rotations (because of the non-equidistant slopes) nor on scaling (because of the area requirement).
- In Section 5, we prove that, regardless of the maximum degree Δ of the input planar graph, quadratic area in the number of vertices of the graph suffices to obtain a planar drawing with at most Δ slopes, where each edge has at most four bends. If the graph is additionally subhamiltonian, then three bends suffice.

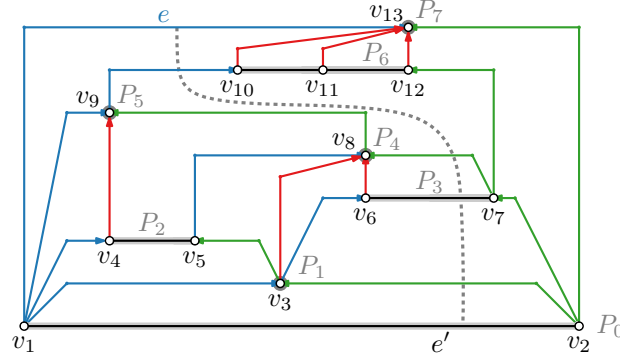
Note that a fundamental requirement of our grid drawings is that both vertices and edge bends lie on grid points. Furthermore, we stress that the algorithms that we present in this work can be implemented to run in time linear in the size of the input graphs.

2 Preliminaries

In this section, we introduce preliminary definitions and notation used throughout the paper. Unless stated otherwise, all graphs considered are simple and undirected. The *degree* of a vertex is the number of its neighbors. A graph has *maximum degree* Δ if it contains a vertex of degree Δ and no vertex of degree greater than Δ . A graph is *connected* if every pair of vertices is joined by a path. More generally, for $k \geq 1$, a graph is *k-connected* if the removal of any set of at most $k - 1$ vertices leaves the graph connected. In particular, 2- and 3-connected graphs are also referred to as *biconnected* and *triconnected*, respectively.

A *drawing* of a graph maps each vertex of the graph to a point of the Euclidean plane and each of its edges to a Jordan arc connecting its endpoints. A drawing is *planar* if no two edges intersect except possibly at common endpoints. Such a drawing partitions the plane into connected regions called *faces*; the unbounded one is the *outer face*. A graph is *planar* if it admits a planar drawing. A *planar embedding* of a planar graph is an equivalence class of planar drawings that define the same set of faces and the same outer face. A planar drawing is *k-bend* if each of its edges is a polygonal chain composed of at most $k + 1$ straight-line segments. The point where two such segments meet is called a *bend*. Unless otherwise specified, we consider *grid drawings*, that is, drawings in which each vertex and each bend lies on a point of the Euclidean plane with integer coordinates. Given such a drawing Γ , we denote by $W(\Gamma)$ and by $H(\Gamma)$ the width and the height of the minimum rectangle enclosing Γ , respectively.

The *slope* of a line measures its steepness and direction. It is defined as the ratio of the vertical change (rise) to the horizontal change (run) between any two points on the line. This equivalently corresponds to the tangent of the counter-clockwise angle through which a horizontal line must be rotated to coincide with the given line. The *horizontal (vertical) slope* is the slope of a line parallel (perpendicular) to the x -axis. The slope of an edge segment is the slope of the line containing it. Given a set of slopes S , a *k-bend* planar drawing is said to be *on S* if each of its edge segments has a slope belonging to S . For a vertex v in a *k-bend* planar drawing, each slope s of S determines two distinct rays emanating from v with slope s , which we call *ports*. If s is the horizontal slope, these rays are called *horizontal*. The upward (downward) directed ports are called *top (bottom)* ports. We say that a port ρ_v incident to v is *free* if no edge incident to v is drawn along ρ_v ; otherwise, ρ_v is *occupied*.



■ **Figure 1** Illustration of a canonical order and of a 4-coloring of a 3-connected planar graph. The dotted curve is a cut (dashed) at edge e .

Given an edge e at the outer face of a k -bend planar drawing Γ , a *cut at edge e* is a strictly y -monotone curve that (i) starts at a point on a horizontal segment of e , (ii) ends at a point on a horizontal segment of an edge e' incident to the outer face of Γ , with $e' \neq e$, and (iii) intersects only horizontal segments of Γ ; see Figure 1. Such a cut allows to *stretch* Γ horizontally by translating all vertices and edges on one side of the cut horizontally by an arbitrary distance $d > 0$, thereby increasing the horizontal distance between the two resulting parts without introducing crossings. Since the stretching is purely horizontal, the slopes of all non-horizontal segments remain unchanged, while the lengths of the horizontal segments crossed by the cut increase. Hence, if Γ is on S before the stretching, it remains on S afterward. When we say that we *stretch an edge e* , we refer precisely to this operation.

Let G be a 3-connected n -vertex plane graph and let $\Pi = (P_0, \dots, P_m)$ be a partition of its vertex set into paths such that $P_0 = \{v_1, v_2\}$, $P_m = \{v_n\}$, the edges (v_1, v_2) and (v_1, v_n) exist and belong to the outer face of G . For $i = 0, \dots, m$, let G_i be the subgraph induced by $P_0 \cup \dots \cup P_i$ and denote by C_i the contour of G_k defined as follows: If $i = 0$, then C_0 is the edge (v_1, v_2) of P_0 , while if $i > 0$, then C_i is the path from v_1 to v_2 obtained by removing (v_1, v_2) from the cycle delimiting the outer face of G_i . We say that Π is a *canonical order* [9, 21] of G if for each $i = 1, \dots, m - 1$ the following hold (see Figure 1): (i) G_i is biconnected, internally 3-connected and embedded with $C_i \cup \{(v_1, v_2)\}$ as its outer face; (ii) all neighbors of P_i in G_{i-1} are on C_{i-1} ; (iii) P_i either consists of a single vertex (called *singleton*), or the degree of each of its vertices is 2 in G_i (called *chain*); (iv) every vertex in P_i has at least one neighbor in P_j with $j > i$. A canonical order of a 3-connected planar graph can be computed in linear time [21].

Given a 3-connected plane graph G and a canonical order Π of it, it is possible to compute a 4-edge-coloring of G similar to the one by Schnyder [16, 39]. The edge (v_1, v_2) of G_0 is colored black. For $i = 1, \dots, m$, a 4-coloring of G_{i-1} is extended to one of G_i as follows (see, e.g., Figures 1 and 3). First, consider the edges of G_i that do not belong to G_{i-1} and lie on the contour C_i . The first (last) such edge encountered on a traversal of C_i from v_1 to v_2 is colored blue (green, respectively), while all remaining ones (i.e., those whose endpoints both belong to P_i , when P_i is a chain) are colored black. The remaining edges of G_i that do not belong to G_{i-1} are colored red; these are precisely the edges incident to P_i that are not part of C_i (i.e., when P_i is a singleton of the canonical order). Finally, we treat all black edges as undirected, and all remaining edges as directed where the orientation of an edge connecting a vertex $u \in P_i$ to a vertex $v \in P_j$ with $0 \leq i < j \leq m$ is from u to v .

For an n -vertex graph with two designated vertices s and t , an st -ordering $v_1, \dots, v_n$ is a permutation of its vertices such that $v_1 = s$ and $v_n = t$, and every vertex v_j with $1 < j < n$ has at least two neighbors v_i and v_k with $i < j < k$. It is known that every biconnected graph admits such an st -ordering [38]. In the case of a planar input, one may additionally guarantee that the vertices v_1, v_2, v_n all lie on the outer face of the graph, such that the edge (v_1, v_2) is incident to this face [5].

In our algorithms, we occasionally augment the input graph to make it biconnected or triconnected. To augment a connected planar graph G to a simply biconnected planar graph G' , we utilize an algorithm by Kant and Bodlaender [22]. This algorithm runs in linear time and ensures that the maximum degree $\Delta(G')$ of the resulting graph satisfies $\Delta(G') \leq \Delta(G) + 2$. Furthermore, we utilize an algorithm by Kant [20] to augment a biconnected planar graph G to a triconnected planar graph G' . This augmentation is performed in linear time, and the resulting maximum degree $\Delta(G')$ is bounded by $\Delta(G') \leq \max\{2, \lceil \frac{3}{2}\Delta(G) \rceil\}$.

3 1-bend planar drawings of 3-connected degree- Δ planar graphs

In this section, we seek to prove that every 3-connected n -vertex planar graph with maximum degree Δ admits a 1-bend planar grid drawing with at most $3\Delta - 8$ slopes on a $O(\Delta n^2) \times O(\Delta n^3)$ grid. Since for $\Delta \leq 4$ our results is superseded by [3, 10], we assume w.l.o.g. that $\Delta \geq 5$. Our approach builds upon the incremental construction by Angelini, Bekos, Liotta and Montecchiani [1], which uses a canonical ordering of the input 3-connected planar graph G to produce a 1-bend planar drawing Γ of it on an *arbitrary* set of $\Delta - 1$ slopes. In contrast, we fix the slope set in advance. Moreover, to obtain polynomial area, we enlarge the number of available slopes from $\Delta - 1$ to $3\Delta - 8$.

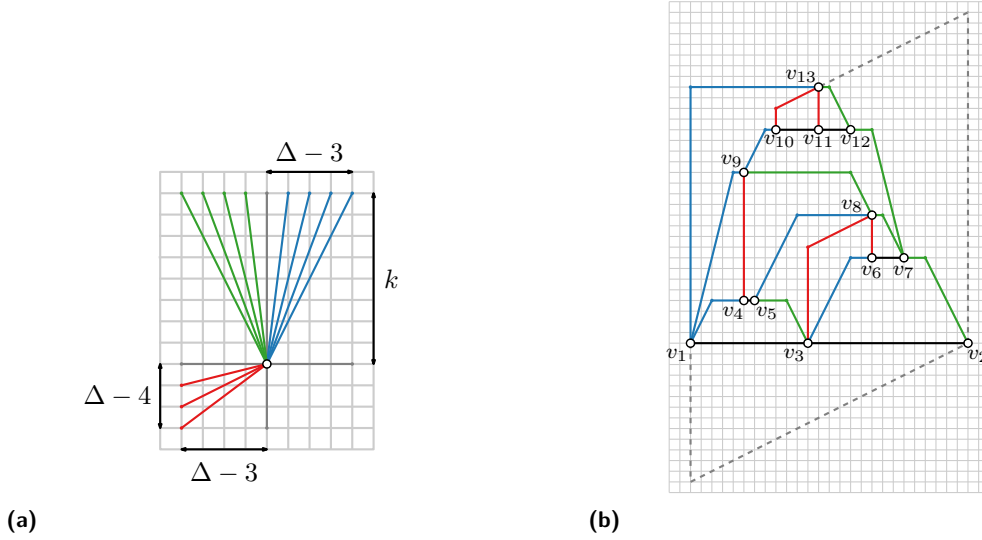
► **Theorem 1.** *Every 3-connected planar n -vertex graph G with maximum degree $\Delta \geq 5$ admits a 1-bend planar grid drawing with at most $3\Delta - 8$ slopes on a $12\Delta n^2 \times 18\Delta n^3$ grid.*

Proof. The slope set S used by our algorithm is defined with respect to a parameter $k > \Delta n^2$ that we will specify later, and is the union of the following sets. S_v consists only of the vertical slope. S_h consists only of the horizontal slope. S_{ls} consists of $\Delta - 3$ *left steep* slopes $-\frac{k}{1}, \dots, -\frac{k}{\Delta-3}$ (green in Figure 2a). S_{rs} consists of $\Delta - 3$ *right steep* slopes $\frac{k}{1}, \dots, \frac{k}{\Delta-3}$ (blue in Figure 2a). Finally, S_f consists of $\Delta - 4$ *flat* slopes $\frac{1}{\Delta-3}, \dots, \frac{\Delta-4}{\Delta-3}$ (red in Figure 2a). Hence, the cardinality of S is $3\Delta - 8$, as desired.

Let $\Pi = (P_0, \dots, P_m)$ be a canonical order of G . For the drawing algorithm, we first remove the edge (v_1, v_2) from the graph and color the edges incident to v_1 and v_2 in G_1 black. After the drawing algorithm is completed, we will insert the edge (v_1, v_2) below the constructed drawing with a vertical segment incident to v_1 and a segment with slope $\frac{1}{\Delta-3} \in S_f$ incident to v_2 (see Figure 2b). For now, we also assume that v_n has degree strictly less than Δ (note that if $\Delta \geq 6$, then this assumption is w.l.o.g. since we can choose v_n such that its degree is at most 5). We will describe how to handle the other case later.

Assume that for $0 < i \leq m$, we have already constructed a 1-bend planar grid drawing Γ_{i-1} of G_{i-1} on S that satisfies the following invariants.

- I.1 The contour C_{i-1} of Γ_{i-1} is drawn strictly x -monotone.
- I.2 There exists a cut at every edge belonging to the contour C_{i-1} of Γ_{i-1} .
- I.3 Every vertex of C_{i-1} has at least as many unoccupied top ports in each of S_{rs} and S_{ls} incident to the outer face of Γ_{i-1} as it has neighbors in $G \setminus G_{i-1}$ minus one, and the port corresponding to S_v is unoccupied if the vertex has at least one neighbor in $G \setminus G_{i-1}$ or its degree is strictly less than Δ .
- I.4 All vertices of G_{i-1} are at y -coordinates that are multiples of k in Γ_{i-1} .



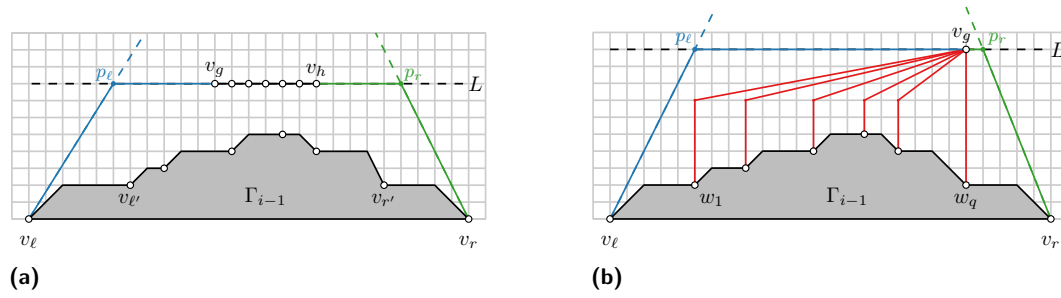
■ **Figure 2** (a) Illustration of the slopes used in Theorem 1. (b) The drawing created by the algorithm in Theorem 1 with $k = 4$ for the graph in Figure 1.

I.5 Based on their colors, the edges of G_{i-1} have been drawn as follows in Γ_{i-1} :

- Each black edge of G_{i-1} consists of a single horizontal segment (i.e., its slope is in S_h).
- Each blue edge of G_{i-1} consists of two segments; the one incident to its source has a slope in $S_v \cup S_{rs}$, while the one incident to its target is in S_h .
- Each red edge of G_{i-1} consists of at most two segments; the one incident to its source is vertical (i.e., its slope is in S_v), while the one incident to its target has a slope in $S_f \cup S_v$, i.e, if the slope of the second segment is in S_v , then the edge consists of one segment.
- Each green edge of G_{i-1} consists of two segments; the one incident to its source has a slope in $S_v \cup S_{ls}$, while the one incident to its target is in S_h .

Note that Invariants I.1 and I.2 are inherited from [1]; Invariant I.3 is adapted to our setting, and Invariants I.4 and I.5 are specific to our construction. The base case of our recursive algorithm is the graph G_1 , which consists of the vertices v_1, v_2 , and all vertices in P_1 . We place these vertices on a horizontal line: v_1 is positioned at $(0, 0)$, v_2 at $(|P_1| + 1, 0)$, and the vertices of P_1 are placed between them at unit distance. All edges are then drawn as horizontal segments. This obviously satisfies the invariants. So, we may assume that $i > 1$. To derive a drawing Γ_i of G_i maintaining Invariants I.1–I.5, we introduce the vertices of path P_i into the drawing Γ_{i-1} by distinguish two cases depending on whether P_i is a chain or a singleton of degree 2 (Case 1) and a singleton of degree greater than 2 (Case 2). At a high level, Invariant I.4 will allow us to determine a grid point for each vertex of P_i and for each bend of the blue and green edges incident to it, while Invariant I.2 will support stretching the drawing Γ_{i-1} to create additional space to accomplish this placement, if needed.

Case 1: P_i is a chain or a singleton of degree 2. Suppose that $P_i = \{v_g, \dots, v_h\}$ is a chain or a singleton of degree 2 in G_i . Note that in the latter case, $g = h$ holds. Let v_ℓ and v_r be the neighbors of v_g and v_h in G_{i-1} , respectively. Refer to Figure 3a. W.l.o.g., we may assume that v_ℓ appears before v_r along the contour C_{i-1} in a traversal of it from v_1 to v_2 . Let also ρ_ℓ (ρ_r) be the first unoccupied port at v_ℓ (v_r) encountered in a counter-clockwise (clockwise) traversal

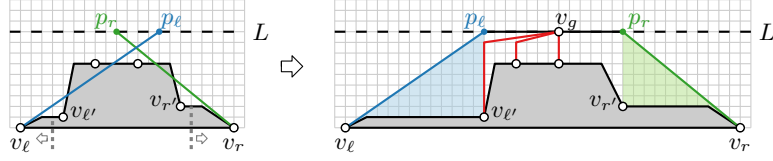


■ **Figure 3** Illustration of the case where P_i is (a) a chain or a singleton of degree 2, and (b) a singleton of degree greater than 2 in Theorem 1.

of its top ports in $S_{rs} \cup S_v$ ($S_{ls} \cup S_v$) starting from the rightward (leftward) horizontal port, which exist by Invariant I.3. To satisfy Invariant I.4, we set the y -coordinate of each of $v_g, \dots, v_h$ to $H(\Gamma_{i-1}) + k$. The x -coordinates of $v_g, \dots, v_h$ will be determined by the constraints arising from the way that $(v_\ell, v_g), (v_g, v_{g+1}), \dots, (v_{h-1}, v_h), (v_h, v_r)$ must be drawn. Since (v_ℓ, v_g) and (v_r, v_h) are incoming edges to v_g and v_h in G_i , we satisfy Invariants I.5.b and I.5.d by drawing (v_ℓ, v_g) and (v_r, v_h) with a horizontal segment incident to v_g and v_h and a second segment attached to p_ℓ and p_r , respectively, which maintains Invariant I.3 for v_ℓ and v_r . The remaining edges of P_i (if any) will be drawn as unit-length horizontal segments satisfying Invariant I.5.a. Each of the vertices $v_g, \dots, v_h$ has at most $\Delta - 2$ neighbors in $G \setminus G_i$ and all $\Delta - 3$ ports unoccupied in each of S_{rs} and S_{ls} that are incident to the outer face of Γ_i , which ensures Invariant I.3 for them.

Let p_ℓ and p_r be the points where the rays that correspond to ρ_ℓ and ρ_r intersect the horizontal line L through v_g ; see Figure 4. We have to ensure that p_ℓ lies at least $h - g + 2$ units to the left of p_r such that $v_g, \dots, v_h$ can be placed between them in Γ_i . Furthermore, the rays at ρ_ℓ and ρ_r are not allowed to cross any part of Γ_{i-1} . To guarantee these conditions, we horizontally stretch Γ_{i-1} as follows. Let $(v_\ell, v_{\ell'})$ be the first edge of C_{i-1} that is encountered when traversing C_{i-1} from v_ℓ to v_2 . Analogously, $(v_r, v_{r'})$ is the first edge of C_{i-1} that is encountered when traversing C_{i-1} from v_r to v_1 . (Note that $v_{\ell'} = v_r$ and $v_{r'} = v_\ell$ is possible.) Since both these edges lie on C_{i-1} , by Invariant I.2 there is a cut at each of them. This allows us to stretch $(v_\ell, v_{\ell'})$ until p_ℓ lies to the left of $v_{\ell'}$, and symmetrically, stretch $(v_r, v_{r'})$ until p_r lies to the right of $v_{r'}$. This further guarantees that the horizontal distance between p_ℓ and p_r is at least 2. If needed, we further stretch one of the edges $(v_\ell, v_{\ell'})$ and $(v_{r'}, v_r)$, say w.l.o.g. the former, by up to $h - g$ additional units, so as to ensure that the horizontal distance between p_ℓ and p_r is at least $h - g + 2$. Hence, it is possible to place each of $v_g, \dots, v_h$ at a grid point along L that lies between p_ℓ and p_r , and draw the edges (v_ℓ, v_g) and (v_r, v_h) with one bend each at p_ℓ and p_r , respectively. The remaining edges of P_i are drawn as unit-length horizontal edge segments, as we initially sought; see Figure 3. This completes the drawing of Γ_i .

It remains to show that Γ_i is planar and satisfies Invariant I.1. For the former, observe that the introduction of $v_g, \dots, v_h$ into Γ_{i-1} yields $h - g + 4$ new edge segments in Γ_i . The two segments connecting p_ℓ with v_g and p_r with v_h , as well as all edge segments connecting internal vertices of P_i (if any) lie completely above Γ_{i-1} . So they cannot cross any edge of Γ_{i-1} . The port ρ_ℓ that is used by the edge segment connecting v_ℓ with p_ℓ is the next available port in S_{rs} , i.e., the one that follows the corresponding port used by the edge $(v_\ell, v_{\ell'})$. Furthermore, the only edge of C_{i-1} that lies between the x -coordinates of v_ℓ and p_ℓ is the edge $(v_\ell, v_{\ell'})$. So no edge of C_{i-1} and thus no edge of Γ_{i-1} can be crossed by the



■ **Figure 4** Illustration of using cuts to maintain planarity in Theorem 1.

edge segment connecting v_ℓ and p_ℓ . A symmetric argument applies to the edge segment connecting v_r and p_r . Since these two edge segments cannot cross each other, it follows that Γ_i is planar, as desired. Having ensured this property, the fact that the contour C_i of Γ_i is x -monotone is implied by the choice of ρ_ℓ and ρ_r and by the fact that $\rho_\ell \in S_{rs}$ and $\rho_r \in S_{ls}$. Hence, Γ_i satisfies Invariant I.1.

Before considering the case where P_i is a singleton of degree greater than 2 in G_i , we establish an upper bound on how much the drawing Γ_{i-1} must be stretched horizontally to accommodate $v_g, \dots, v_h$ in Γ_i . To this end, consider the edge (v_ℓ, v_g) . Let x_ℓ be the x -coordinate of p_ℓ before applying any stretching. Let also $s_\ell \in S_{rs}$ be the slope of the edge segment connecting v_ℓ with p_ℓ ; symmetrically, $s_r \in S_{ls}$ is defined. It follows that $x_\ell = x(v_\ell) + \frac{y(v_g) - y(v_\ell)}{s_\ell} \leq x(v_\ell) + \frac{y(v_g)}{s_\ell}$. After the stretching of the edge $(v_\ell, v_{\ell'})$, the x -coordinate of p_ℓ should be at most $x(v_{\ell'}) - 1$. To achieve this, the edge $(v_\ell, v_{\ell'})$ must be stretched by at most $x_\ell - (x(v_{\ell'}) - 1) \leq x(v_\ell) + \frac{y(v_g)}{s_\ell} - x(v_\ell) = \frac{y(v_g)}{s_\ell}$ units of length. Symmetrically, the edge $(v_r, v_{r'})$ must be stretched by at most $\frac{y(v_g)}{|s_r|}$ units of length. Now the x -coordinate of p_ℓ is at most the old x -coordinate of v_ℓ and the x -coordinate of p_r is at least the old x -coordinate of v_r , so p_ℓ and p_r are at least one unit apart. To make space for $v_g, \dots, v_h$ between them, we might have to stretch the edge $(v_\ell, v_{\ell'})$ by $h - g + 1$ more units. Since $|P_i| = h - g + 1$, it follows that the total stretch applied is at most

$$\begin{aligned} \frac{y(v_g)}{s_\ell} + \frac{y(v_g)}{|s_r|} + |P_i| &\leq \frac{H(\Gamma_{i-1}) + k}{\frac{k}{\Delta-3}} + \frac{H(\Gamma_{i-1}) + k}{\frac{k}{\Delta-3}} + |P_i| \\ &= 2 \cdot \frac{H(\Gamma_{i-1}) + k}{k} \cdot (\Delta - 3) + |P_i| \leq 2\Delta \cdot \frac{H(\Gamma_{i-1}) + k}{k} + |P_i|. \end{aligned} \quad (1)$$

Case 2: P_i is a singleton of degree more than 2. Suppose that $P_i = \{v_g\}$ is a singleton of degree greater than 2 in G_i . Let $v_\ell, w_1, \dots, w_q, v_r$, with $q \geq 1$, be the neighbors of v_g in Γ_{i-1} as they appear from left to right along C_{i-1} . We will place v_g above w_q , such that the edge connecting them is vertical; see Figure 3b. Equivalently, this corresponds to setting the x -coordinate of v_g to the one of w_q . The y -coordinates of v_g will be determined by the constraints arising from the way that $(v_\ell, v_g), (v_g, w_1), \dots, (v_g, w_q), (v_g, v_r)$ must be drawn (see Invariants I.5.b, I.5.c and I.5.d). More precisely, each edge $(w_j, v_g), 1 \leq j \leq q$ will be drawn with a vertical segment incident to w_j , which is unoccupied by Invariant I.3, and, if $j \neq q$, with a second segment of slope $s_j = \frac{j}{\Delta-4} \in S_f$. Clearly, if $q \leq \Delta - 3$, then Invariant I.5.c is satisfied. This condition always holds when $i \neq m$, or when $i = m$ and the degree of v_n is strictly less than Δ . We will discuss the remaining case later. Furthermore, the edges (v_ℓ, v_g) and (v_g, v_r) will be drawn afterwards as in Case 1, thereby satisfying Invariants I.2, I.3, I.5.b, and I.5.d.

To ensure that the bend point of each edge $(w_j, v_g), 1 \leq j < q$ lies on a grid point, we first ensure that the horizontal distance between w_j and v_g is a multiple of $\Delta - 3$. To this end, let (w_j, w'_j) be the first edge of C_{i-1} that is encountered when traversing C_{i-1} from w_j to v_2 (possibly $w'_j = w_{j+1}$). We leverage Invariant I.2 to stretch (w_j, w'_j) by at most $\Delta - 4$

units so as to guarantee the required distance between w_j and v_g . Overall, this increases the width by at most $(q-1) \cdot (\Delta-4)$ units. To satisfy Invariant I.4, we set the y -coordinate of v_g to $H(\Gamma_{i-1}) + \alpha k$, where $\alpha \in \mathbb{N}^*$ is a parameter that is chosen such that, for every neighbor w_j of v_g , the bend point of the edge (w_j, v_g) lies above Γ_{i-1} ; see Figure 3b. Since the height of the flat segment of the edge (w_j, v_g) is $s_j \cdot (x(v_g) - x(w_j))$, where $x(v_g) - x(w_j)$ is its width, the y -coordinate of v_g must be at least $H(\Gamma_{i-1}) + s_j \cdot (x(v_g) - x(w_j))$. We choose $y(v_g)$ as the smallest multiple of k that fulfills all these bounds. Thus, we have

$$y(v_g) \leq H(\Gamma_{i-1}) + \max_{j \in \{1, \dots, q-1\}} \{s_j \cdot (x(v_g) - x(w_j))\} + k \leq H(\Gamma_{i-1}) + W(\Gamma_{i-1}) + k. \quad (2)$$

Having determined the x - and y -coordinates of v_g , the drawing Γ_i is completed by drawing each edge incident to v_g in G_i as described above, thereby guaranteeing Invariants I.2-I.5. By an argument analogous to that of Case 1, the edges (v_ℓ, v_g) and (v_r, v_g) are crossing-free in Γ_i . We still have to argue that the edges (w_j, v_g) are also crossing free in Γ_i . The flat slopes of the edges (w_j, v_g) , with $j < q$, have been assigned in counterclockwise order around v_g . Also, the left-to-right order of the bend points of these edges matches the left-to-right order of $w_1, \dots, w_{q-1}$ along the contour C_{i-1} ; see Figure 3b. Therefore, no two edges (w_j, v_g) and $(w_{j'}, v_g)$ with $0 \leq j < j' \leq q$ can cross each other. Furthermore, the horizontal segments of the edges (v_ℓ, v_g) and (v_r, v_g) lie above the flat segments of the edges (w_j, v_g) , while the stretching performed along the edge $(v_\ell, v_{\ell'})$ and $(v_r, v_{r'})$ ensures that the steep segments of (v_ℓ, v_g) and (v_r, v_g) lie completely to the left and to the right of these flat segments; see Figure 4. Since the flat segments of the edges (w_i, v_g) are drawn completely above Γ_{i-1} , they cannot cross any edge of G_{i-1} , thereby implying that Γ_i is planar, as desired. Having ensured this property, the fact that the contour C_i of Γ_i is x -monotone is implied by the fact that the slopes of the two segments of the edge (v_ℓ, v_g) are in $S_{rs} \cup S_h$, while the ones of (v_g, v_r) in $S_h \cup S_{ls}$ (i.e., symmetrically to Case 1). Hence, Γ_i also satisfies Invariant I.1.

We now establish an upper bound on how much the drawing Γ_{i-1} must be stretched horizontally to accommodate v_g in Γ_i . As in Case 1, the edges $(v_\ell, v_{\ell'})$ and $(v_r, v_{r'})$ have been stretched by at most $\frac{y(v_g)}{s_\ell}$ and $\frac{y(v_g)}{|s_r|}$ units, respectively. Furthermore, the edges (w_j, w'_j) have been stretched by at most $(q-1) \cdot (\Delta-4)$ unit in total. Thus, the total stretch applied is at most

$$\begin{aligned} & \frac{y(v_g)}{s_\ell} + \frac{y(v_g)}{|s_r|} + (q-1) \cdot (\Delta-4) \\ & \stackrel{(2)}{\leq} \frac{H(\Gamma_{i-1}) + W(\Gamma_{i-1}) + k}{s_\ell} + \frac{H(\Gamma_{i-1}) + W(\Gamma_{i-1}) + k}{|s_r|} + (\deg(v_g) - 1) \cdot (\Delta-4) \\ & \leq 2 \cdot \frac{H(\Gamma_{i-1}) + W(\Gamma_{i-1}) + k}{\frac{k}{\Delta-3}} + \deg(v_g) \cdot \Delta \\ & \leq 2\Delta \cdot \frac{H(\Gamma_{i-1}) + W(\Gamma_{i-1}) + k}{k} + \Delta n. \end{aligned} \quad (3)$$

To complete the description of our drawing algorithm, it remains to consider the case in which v_n is of degree exactly Δ . Let $w_1, \dots, w_\Delta$ be the neighbors of v_n as they appear along C_{m-1} from v_1 to v_2 . Before applying our drawing algorithm as described so far, we remove the edge (w_Δ, v_n) from the graph and recolor the edge $(w_{\Delta-1}, v_n)$ green. After the last step of the drawing algorithm, we reinsert the edge (w_Δ, v_n) . Since w_Δ lies on C_m and its degree is strictly less than Δ (in the absence of $(w_{\Delta-1}, v_n)$), the vertical top port of w_Δ is unoccupied by Invariant I.3. By Invariant I.1, we can draw the edge $(w_{\Delta-1}, v_n)$ with a vertical segment at w_Δ and a segment of slope $\frac{1}{\Delta-3}$ at v_n (see Figure 2b). This completes the drawing of the input graph G .

6:10 How Many Slopes Does Polynomial Area Cost?

It remains to analyze the width and the height of the drawing. For ease of notation, we denote by W_i and H_i , $0 \leq i \leq m$ the width and height of the drawing Γ_i , that is, $W_i = W(\Gamma_i)$ and $H_i = H(\Gamma_i)$. In the base of our recursive algorithm, it holds that $W_0 = 1$ and $H_0 = 0$. By Equation (2), we obtain $H_i \leq H_{i-1} + W_{i-1} + k$. So,

$$H_m \leq \sum_{i=0}^{m-1} (W_i + k) \leq m \cdot (W_m + k) \leq n \cdot W_m + kn. \quad (4)$$

By Equation (1) and Equation (3), we obtain

$$\begin{aligned} W_i &\leq W_{i-1} + 2\Delta \cdot \frac{H_{i-1} + W_{i-1} + k}{k} + |P_i| + \Delta n \\ &= \frac{2\Delta}{k} \cdot H_{i-1} + \left(1 + \frac{2\Delta}{k}\right) \cdot W_{i-1} + (n+2)\Delta + |P_i|. \end{aligned}$$

By series expansion, we obtain

$$\begin{aligned} W_m &\leq \sum_{i=0}^{m-1} \left(1 + \frac{2\Delta}{k}\right)^{m-i} \left(\frac{2\Delta}{k} H_i + (n+2)\Delta + |P_i|\right) \\ &\leq \left(1 + \frac{2\Delta}{k}\right)^m \cdot \sum_{i=0}^{m-1} \left(\frac{2\Delta}{k} H_i + (n+2)\Delta + |P_i|\right) \\ &\leq \left(1 + \frac{2\Delta}{k}\right)^m \cdot \left(m \cdot \frac{2\Delta}{k} \cdot H_m + (n+2)\Delta m + n\right) \quad m \leq n, n \geq 6 \\ &\leq \left(1 + \frac{2\Delta}{k}\right)^n \cdot \left(n \cdot \frac{2\Delta}{k} \cdot H_m + 2\Delta n^2\right). \end{aligned}$$

If we choose $k = 4\Delta n^2$, then we have $\left(1 + \frac{2\Delta}{k}\right)^n = \left(1 + \frac{1}{2n^2}\right)^n$. Since $\ln(1+x) \leq x$ for $0 < x < 1$, we obtain $\left(1 + \frac{1}{2n^2}\right)^n \leq e^{1/(2n)} \leq e^{1/6} < 1.19$, so

$$W_m \leq 1.19 \left(n \cdot \frac{2\Delta}{k} \cdot H_m + 2\Delta n^2\right). \quad (5)$$

Plugging Equation (5) into Equation (4), we obtain

$$\begin{aligned} H_m &\leq 1.19 \cdot \left(n \cdot \frac{2\Delta}{k} \cdot H_m + 2\Delta n^2\right) + kn \leq \frac{2 \cdot 1.19 \cdot k\Delta n^3 + k^2 n}{k - 2 \cdot 1.19 \cdot \Delta n^2} \\ &= \frac{2 \cdot 1.19 \cdot 4\Delta^2 n^5 + 16\Delta^2 n^5}{4\Delta n^2 - 2 \cdot 1.19 \cdot \Delta n^2} = \frac{25.52\Delta n^3}{1.62} \leq 15.76\Delta n^3 \in O(\Delta n^3). \end{aligned}$$

Plugging this back into Equation (5), we can bound the width by

$$\begin{aligned} W_m &\leq 1.19 \left(n \cdot \frac{2\Delta}{4\Delta n^2} \cdot H_m + 2\Delta n^2\right) \leq \frac{1.19}{2n} \cdot H_m + 2.38\Delta n^2 \leq \frac{18.76\Delta n^3}{2n} + 2.38\Delta n^2 \\ &\leq 11.76\Delta n^2 \in O(\Delta n^2). \end{aligned}$$

Reinserting (v_1, v_2) and (w_Δ, v_n) at the end increases the height by at most $2W_m/(\Delta-3) \leq 11.76\Delta n^2 \leq 1.96\Delta n^3$, since $\Delta \geq 5$ and $n \geq 6$. Thus, our drawing has area $O(\Delta^2 n^5)$. ◀

Using the algorithms in [20, 22] (see Section 2), we can augment any planar graph with maximum degree $\Delta \geq 5$ to a triconnected planar graph of maximum degree at most $\lceil 3\Delta/2 \rceil + 3$.

► **Corollary 2.** *Every planar n -vertex graph with maximum degree $\Delta \geq 5$ admits a 1-bend planar grid drawing with at most $\lceil \frac{9}{2}\Delta \rceil + 1$ slopes on a $O(\Delta n^2) \times O(\Delta n^3)$ grid.*

For the special case of planar graphs of maximum degree 5, we can slightly improve the number of slopes while increasing the required area by a factor of $O(n^2)$.

► **Theorem 3.** *Every 3-connected planar graph G with maximum degree $\Delta = 5$ admits a 1-bend planar grid drawing Γ using a fixed set of 5 slopes. Such a drawing can be constructed on a grid of size $O(n^3) \times O(n^4)$.*

Sketch. To reduce the number of slopes, we relax the requirement that the contour of the drawing is x -monotone. Instead, at vertices which do not have remaining outgoing edges non-monotone parts are allowed. This allows to construct a drawing using only five slopes, namely, $\{0, \infty, 1, k, -k\}$, where $k = 5n^2$. The horizontal stretch needed at each step increases to $O(k) + O(W/k) + O(H/k)$, while the height to $O(k) + O(W) + O(H/k)$. Solving the resulting recurrences yields width $O(n^3)$ and height $O(n^4)$. ◀

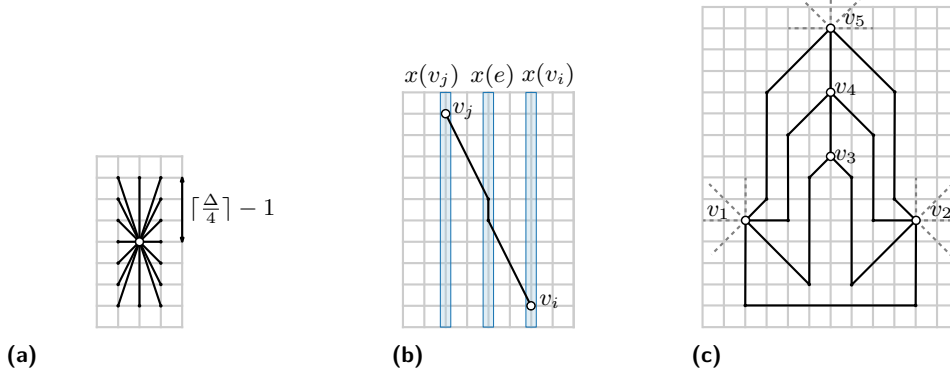
4 2-bend planar drawings of degree- Δ graphs

In this section, we extend a result of Keszegh, Pach, and Pálvölgyi [24, 25] by showing that every planar graph G with maximum degree $\Delta \geq 3$ admits a 2-bend planar drawing on a grid of polynomial size using at most $\lceil \Delta/2 \rceil$ slopes, if G is biconnected and at most $\lceil \Delta/2 \rceil + 1$ slopes otherwise. We first address the former case.

► **Theorem 4.** *Every biconnected planar graph G with maximum degree $\Delta \geq 3$ admits a 2-bend planar drawing with at most $\lceil \Delta/2 \rceil$ distinct slopes on a grid of size $O(n) \times O(n^2\Delta)$. The only exception is the octahedron graph, which requires 3 slopes.*

Proof. Without loss of generality, we may assume that Δ is even. Since planar graphs with $\Delta = 4$ always admit an orthogonal drawing with at most two bends per edge on a grid of size $O(n) \times O(n)$ [5, 37], we may further assume $\Delta \geq 6$. We choose a vertex v_n of G with degree at most 5 (which exists in every planar graph) and consider a planar embedding $\mathcal{E}$ of G with v_n on its outer face. The fact that G is biconnected implies that it admits an st -ordering $v_1, \dots, v_n$, such that (v_1, v_2) is an edge of the outer face of $\mathcal{E}$. We will use this st -ordering to construct incrementally a 2-bend planar drawing Γ of G on the following set of $\frac{\Delta}{2}$ slopes: $S = \{-\lfloor \frac{\Delta}{4} \rfloor + 1, \dots, 0, \dots, \lceil \frac{\Delta}{4} \rceil - 1, \infty\}$; see Figure 5a. In particular, each edge (v_i, v_j) with $i < j$ will be drawn with a *first non-vertical segment* incident to v_i (of possibly zero length), followed by a vertical segment (of non-zero length) and a *last non-vertical segment* incident to v_j (of possibly zero length); see Figure 5b.

We first compute a *sketch* drawing Γ' of G using the algorithm of [24, 25] and taking $v_1, \dots, v_n$ as the input st -ordering. In the resulting drawing Γ' , each edge is drawn with at most two bends and contains a vertical segment. These vertical segments lie in consecutive columns of Γ' that also contain the vertices of G . Given a vertex v of G , we denote by $x(v)$ the index of the column of Γ' that contains v in the left-to-right ordering of the columns of Γ' . Accordingly, given an edge e , with slight abuse of notation, we denote by $x(e)$ the index of the column of Γ' that contains the vertical segment of e in the left-to-right ordering of the columns of Γ' ; see Figure 5a. Let $S' = \{s'_{-\lfloor \Delta/4 \rfloor + 1}, \dots, s'_0, \dots, s'_{\lceil \Delta/4 \rceil - 1}, s'_\infty\}$ be the slopes used in Γ' such that s'_∞ is the vertical slope and $s'_i < s'_j$ if $i < j$. We produce a drawing Γ of G on a grid of size $O(n) \times O(n^2\Delta)$ such that every edge segment that is drawn with slope s'_i in Γ' is drawn with slope i in Γ . Hence, Γ and Γ' share the same planar embedding. Furthermore, the x -coordinate of each vertex v of G will be $x(v)$ in Γ , while the x -coordinate of the vertical segment of each edge e of G will be $x(e)$. Hence, the width of Γ will be $O(n)$.



■ **Figure 5** (a) The set of $\lceil \Delta/2 \rceil$ slopes used in Theorem 4. (b) Example construction of a 2-bend edge $e = (v_i, v_j)$. (c) A sketch of our drawing.

Let G_i with $1 \leq i \leq n$ be the subgraph of G induced by $\{v_1, \dots, v_i\}$, and let $\mathcal{E}_i$ be the restriction of $\mathcal{E}$ to G_i . We say that an edge (v_j, v_k) of G is a *pending* edge of G_i if and only if $j \leq i < k$. For $2 \leq i \leq n$, we denote by Γ_i a drawing of G_i which additionally contains the first non-vertical segment (if any) of each pending edge of G_i . In the following, we will first describe how to compute drawing Γ_2 . Then, assuming that we have recursively computed a drawing Γ_{i-1} with $2 \leq i < n$, we describe how to compute drawing Γ_i .

To obtain drawing Γ_2 , we begin by placing v_1 at $(0, 0)$ and v_2 at $(x(v_2), 0)$, as in the original algorithm. We connect v_1 to v_2 with a 2-bend edge drawn below both vertices. This edge consists of two vertical segments, each of length $\lfloor \frac{\Delta}{4} \rfloor (2m - n) + 1$, incident to v_1 and v_2 , and a horizontal segment connecting their lower endpoints; see Figure 5c. To complete drawing Γ_2 , we next draw the first non-vertical segment of each pending edge that is incident to v_1 and v_2 preserving $\mathcal{E}_2$ as follows. For $j \in \{1, 2\}$, consider a pending edge $e = (v_j, v_k)$ with $k > 2$ and suppose that this edge has a non-vertical segment incident to v_j in Γ' . Let s'_ℓ be the slope of this segment in Γ' . We draw a segment starting at v_j with slope ℓ until its x -coordinate equals $x(e)$. This point serves as the first bend of e and is a grid point, since S consists of integer slopes. Furthermore, the vertical extent of this segment is at most $\lfloor \Delta/4 \rfloor \cdot |x(e) - x(v_j)|$. This guarantees that this segment does not cross (v_1, v_2) , since the width of the drawing is bounded by $2m - n$, as we will shortly show, and thus, $|x(e) - x(v_j)| \leq 2m - n$.

We now describe how to compute drawing Γ_i , assuming that we have recursively constructed drawing Γ_{i-1} . We place v_i at x -coordinate $x(v_i)$ and y -coordinate $(i - 1) \cdot (\Delta/2 \cdot (2m - n) + 1)$, such that v_i lies above one of its median predecessors in G_i , as in the original algorithm. Then, we complete the drawing of the edges connecting v_i to its neighbors in G_{i-1} while preserving $\mathcal{E}_i$. Consider such an edge $e = (v_j, v_i)$ with $i > j$ and assume that s'_ℓ is the slope of its last non-vertical segment in Γ' . In drawing Γ_i , we draw a segment starting at v_i with slope ℓ until its x -coordinate equals $x(e)$ followed by a vertical segment that connects it to the endpoint of the first non-vertical segment of the edge e (which has already been drawn, when Γ_j was computed). To complete the drawing of Γ_i , we draw the first non-vertical segments of the pending edges incident to v_i in the same way as described for Γ_2 . It follows that the vertical extent of any non-vertical segment incident to v_i belonging to edge e is bounded by $\lfloor \Delta/4 \rfloor \cdot |x(e) - x(v_i)|$.

Since each vertex v_i and its incident edges occupy at most $\deg(v_i) - 1$ columns, the total width of the resulting drawing Γ of G is bounded by $\sum_{i=1}^n (\deg(v_i) - 1) = 2m - n \in O(n)$. To estimate the height of Γ , let H_i be the maximum y -coordinate in Γ_i . For each $i > 2$, H_i is

determined by the y -coordinate of v_i plus the vertical extent of the first non-vertical segment of each of the pending edges of v_i plus the vertical extent of edge (v_1, v_2) . Thus,

$$\begin{aligned} H_{i-1} &\leq y(v_{i-1}) + \lfloor \frac{\Delta}{4} \rfloor \cdot W_{i-2} + \lfloor \frac{\Delta}{4} \rfloor (2m - n) + 1 \\ &\leq (i - 2) \cdot \left(\frac{\Delta}{2} \cdot (2m - n) + 1 \right) + \frac{1}{2} \cdot \frac{\Delta}{2} \cdot (2m - n) + \lfloor \frac{\Delta}{4} \rfloor (2m - n) + 1 \\ &< \left(\frac{\Delta}{2} \cdot (2m - n) + 1 \right) \left(i - \frac{3}{2} \right) + \lfloor \frac{\Delta}{4} \rfloor (2m - n) + 1. \end{aligned}$$

The lowest point of any segment of a pending edge of v_i in Γ_i is at least

$$y(v_i) - \lfloor \frac{\Delta}{4} \rfloor \cdot W_{i-1} + \lfloor \frac{\Delta}{4} \rfloor (2m - n) + 1 > \left(\frac{\Delta}{2} \cdot (2m - n) + 1 \right) \left(i - \frac{3}{2} \right) + \lfloor \frac{\Delta}{4} \rfloor (2m - n) + 1.$$

This ensures that all segments of the pending edge of v_i lie above drawing Γ_{i-1} . Hence Γ_i is planar. The total height of the drawing is $O(n \cdot \Delta \cdot n) = O(\Delta n^2)$. ◀

Since any simply connected planar graph can be augmented to a biconnected planar graph by adding auxiliary edges such that every vertex receives at most two augmenting incident edges [20, 22], we obtain the following corollary.

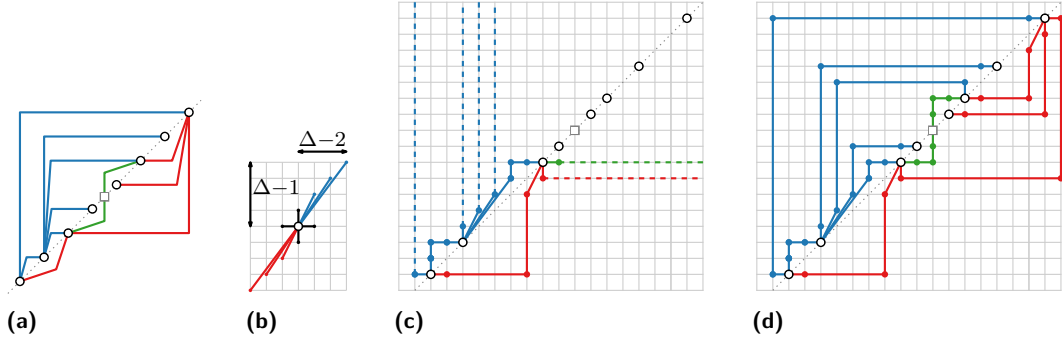
► **Corollary 5.** *Every planar graph G with maximum degree Δ admits a 2-bend planar drawing with at most $\lceil \Delta/2 \rceil + 1$ distinct slopes on a grid of size $O(n) \times O(\Delta n^2)$.*

5 4-bend planar drawings of degree- Δ graphs

In this section, we seek to prove that every planar graph of maximum degree Δ admits a 4-bend planar grid drawing with at most Δ slopes on an $O(n) \times O(n)$ grid. Our approach is based on an algorithm by Kaufmann and Wiese [23], which given a planar graph $G = (V, E)$ and a set of points P in the plane such that no two points have the same x -coordinate, it computes a 2-bend planar drawing of G that maps each vertex in V to a point in P . Let the *spine* be the x -monotone polyline whose bend-points are exactly the points in P . In the produced drawing, every edge is one of the following (see Figure 6a): (i) a *top edge*: it is drawn completely above the spine with one bend; (ii) a *bottom edge*: it is drawn completely below the spine with one bend; or (iii) a *spine-crossing edge*: it crosses the spine exactly once and has two bend points: one below and one above the spine. Note that the drawings produced by their algorithm require a linear number of slopes and, as stated in [23, Lemma 3.2], exponential area (in the number of vertices).

► **Theorem 6.** *Every planar graph of maximum degree Δ admits a 4-bend planar drawing with Δ slopes on a grid of size $O(n) \times O(n)$.*

Proof. We will create a drawing that is conceptually similar to the one by Kaufmann and Wiese. We first apply their algorithm on a set of points that lie on the diagonal $y = x$ to obtain a drawing Γ of G . Let Γ' be the drawing of the graph $G' = (V', E')$ obtained from Γ by subdividing every spine-crossing edge e by a dummy vertex placed on the crossing between e and the spine. We will show how to obtain a 3-bend planar drawing with Δ slopes for G' on a grid of size $O(n) \times O(n)$, in which the two edges incident to each dummy vertex use at most two bends and are drawn such that they use opposite (horizontal or vertical) ports at the incident dummy vertex. Thus, smoothing the dummy vertices yields a 4-bend drawing of G with Δ slopes on a grid of size $O(n) \times O(n)$.



■ **Figure 6** Illustration for our algorithm in Theorem 6 for the graph in Figure 6. (a) A 2-bend planar drawing obtained by the algorithm of Kaufmann and Wiese [23]. Top edges are drawn blue, bottom edges red, and spine-crossing edges green. (b) The vectors used around each vertex in Theorem 6. (c) The drawing after processing v_3 and (d) the final drawing.

Let $v_1, \dots, v_{n'}$ be the vertices of V' in the order that they appear along the spine in Γ' and assume that every edge (v_i, v_j) with $i < j$ is oriented from v_i to v_j . Let t_i and b_i be the number of top and bottom edges of v_i , respectively. We place v_1 at point $(0, 0)$ and every other vertex v_i with $1 < i \leq n'$ at point $p_{v_i} = p_{v_{i-1}} + (d_{i-1}, d_{i-1})$, where $d_{i-1} = \max\{1, t_{i-1}, b_{i-1}\}$. Furthermore, the j -th top (bottom, respective) edge incident to v_i in clockwise order around v_i starting from the spine will have a bend at point $p_{v_i} + \mathbf{v}_j$ ($p_{v_i} - \mathbf{v}_j$, respectively), where $\mathbf{v}_j = (j-2, j-1)$. This ensures that Δ distinct slopes are used by the edge segments incident to each vertex; all the edge segments not directly incident to a vertex are drawn either horizontally or vertically.

To achieve the above properties, we process the vertices in the order $v_1, \dots, v_{n'}$. Assume that we have already processed the vertices $v_1, \dots, v_{i-1}$, and consider the next vertex v_i . We call an edge (v_k, v_ℓ) with $k < \ell$ *open* if we have already processed v_k but not yet v_ℓ , that is, $k < i \leq \ell$. As an invariant of our algorithm, each open top edge is assigned to a grid column that will contain its vertical segment, and each open bottom edge is assigned to a grid row that will contain its horizontal segment; see Figure 6c.

Let $e_1^t, \dots, e_{t_i}^t$ denote the top edges of v_i in clockwise order around v_i , starting from the spine. As already mentioned, for each such edge e_j^t , we place a bend at point $p_{v_i} + \mathbf{v}_j$. If e_j^t is open, we assign it to the grid column $x(v_i) + j - 2$. Otherwise, we draw a horizontal segment to the left until reaching the grid column previously assigned to e_j^t , and then a vertical segment downward to the bend point located in the neighborhood of the other endpoint of e_j^t . In this way, each edge e_j^t is drawn using at most three bends. Symmetrically, let $e_1^b, \dots, e_{b_i}^b$ be the bottom edges of v_i in clockwise order around v_i , starting from the spine. For each such edge e_j^b , we place a bend at the point $p_{v_i} - \mathbf{v}_j$. If e_j^b is open, we assign it to the grid row $y(v_i) - j + 1$. Otherwise, we draw a vertical segment to the bottom and then a horizontal segment leftward to the bend point located in the neighborhood of the other endpoint of e_j^b . Hence, each edge e_j^b is also drawn using at most three bends and the invariant of our algorithm is satisfied.

Once all vertices have been processed, we have obtained a 3-bend drawing of G' ; see Figure 6d for an illustration. The obtained drawing is planar because (i) no two edge segments are assigned to the same horizontal or vertical grid column, and (ii) throughout the incremental drawing construction, the planar embedding of the algorithm by Kaufmann and Wiese is maintained due to the choice of the bend points around each vertex.

We now argue that after smoothing the dummy vertices, we obtain a 4-bend drawing of G . If v_i is a dummy vertex of G' , then it has exactly two edges: one incoming edge (v_h, v_i) and one outgoing edge (v_i, v_j) with $h < i < j$, one of which is a top edge, while the other one is a bottom edge. If (v_h, v_i) is a top edge, then both edges use a horizontal segment around v_i . Hence, after smoothing v_i , the spine-crossing edge (v_h, v_j) will be drawn with a short segment at v_h , followed by a long vertical segment, a horizontal segment that crosses the spine at the old position of v_i , a long vertical segment, and a short segment at v_j , so it has four bends in total. The case that (v_h, v_i) is a bottom edge is symmetric.

For the drawing area, recall that the difference in x -coordinates (and y -coordinates) between two consecutive vertices v_i and v_{i+1} is $d_i = \max\{1, t_i, b_{i+1}\} \leq 1 + t_i + b_{i+1}$, so we have $x(v_{n'}) = y(v_{n'}) \leq n' + \sum_{i=1}^{n'-1} t_i + \sum_{i=2}^{n'} b_i = n' + 2m - t_{n'} - b_1$. Some edges can extend to the bottom-left of v_1 and to the top-right of $v_{n'}$. Namely, one top edge might occupy one column to the left of v_1 , $b_1 - 2$ bottom edges might occupy one column to the left of v_1 each, and $b_1 - 1$ bottom edges might occupy one row to the bottom of v_1 each. Furthermore, to the right of $v_{n'}$, one column might be occupied by a bottom edge, and $t_{n'} - 2$ columns might be occupied by a top edge; above $v_{n'}$, $t_{n'} - 1$ rows might be occupied by top edges. Thus, the total width and height of the drawing is at most

$$\begin{aligned} x(v_{n'}) - x(v_1) + b_1 + 1 + t_{n'} + 1 &\leq n' + 2m - t_{n'} - b_1 - 0 + b_1 + t_{n'} + 2 \\ &= n' + 2m + 2 \leq (n + m) + 2m + 2 \leq 10n - 16 \in O(n). \quad \blacktriangleleft \end{aligned}$$

Since every *subhamiltonian* graph (i.e., a subgraph of a planar Hamiltonian graph) admits an embedding consisting exclusively of top and bottom edges, the following is a direct consequence of Theorem 6.

► **Corollary 7.** *Every subhamiltonian graph of maximum degree Δ admits a 3-bend planar drawing with Δ slopes on a grid of size $O(n) \times O(n)$.*

6 Conclusions and Open Problems

In this work, we studied trade-offs between the number of slopes, the number of bends per edge, and the area requirements in planar graph drawing. Our results show that allowing only a small number of bends per edge suffices to obtain polynomial-area drawings while maintaining a relatively small slope set. Our results narrow a gap between previous approaches that achieved few slopes at the expense of super-polynomial area and those that focused primarily on low-degree graphs. However, several questions remain open. Besides tightening the trade-offs between slopes and bends and extending the study to non-planar graph classes, we identify the following open problems.

- The algorithm presented in Section 3 yields drawings in which almost all edges have a bend. The only exception are the black edges and a red incoming edge per vertex. Following an argument of [27], we can choose a canonical order such that there are at most $(2n + 1)/3$ vertices without an incoming red edge. This gives us an upper bound of $(8n - 17)/3$ on the total number of bends for the constructed drawing. Adjusting the drawing algorithms, so as to reduce the total number of bends in the resulting 1-bend drawing is an interesting open problem for future consideration.
- We were unable to derive an extension of the algorithm presented in Section 3 to handle biconnected (and simply connected) planar graphs without increasing the number of slopes. Such an extension appears to be non-trivial and may require additional properties on the produced drawings in order to handle rigid 3-connected components of the graph.

- The drawings produced by our algorithms have low *angular resolution*, that is, the minimum angle between any two edge segments incident to the same vertex is small. Developing techniques that also account for angular resolution is an interesting direction for future work.
- It is known that the straight-line slope number of outerplanar graphs and partial 2-trees is $\Delta - 1$ and 2Δ , respectively [31, 33]. However, the existing algorithms for these results require superpolynomial area. A natural question is whether we can achieve polynomial area drawings for these graph classes by allowing a slight increase in the number of slopes.

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