Pinning on Tight Cuts: Improved Algorithm and Bounds for Unsplittable Multicommodity Flows in Outerplanar Graphs

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Abstract

The multicommodity flow problem in an undirected capacitated graph G is specified by a set of source-sink pairs with nonnegative demands. A flow is *feasible* if it routes all demands without exceeding the edge capacities, and it is *unsplittable* if it routes each demand along a single path.

Let α be the *smallest value* such that the existence of a feasible flow implies the existence of an unsplittable flow that exceeds the edge capacities by at most $+\,\alpha\, d_{\max}$. Schrijver, Seymour, and Winkler showed that $\alpha \in [1.01,\, 1.5]$ if G is a cycle. These bounds were ultimately improved to $\alpha \in [1.1,\, 1.3]$ by Skutella and Däubel. Recently, Alemán Espinosa and Kumar extended this constant upper bound to the broader class of outerplanar graphs, and showed that if G is outerplanar then $\alpha \leq 3.6$.

We show that $\alpha \in \left[\frac{4}{3}, 2\right]$ if G is outerplanar. We introduce a novel technique that considers the global parameters of the instance, and that may be useful in other (more general) settings where the cut-condition is sufficient, or nearly sufficient, for the existence of a feasible flow.

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1 Introduction

Network flows are a fundamental and extensively studied class of problems in combinatorial optimization [1]. The *multicommodity flow* problem involves routing multiple distinct commodities through a shared network. An instance, which we denote by the tuple (G, u, H, d), is given by an undirected *supply graph* $G = (V, E(G))$ with edge capacities $u : E(G) \to \mathbb{R}_{\geq 0}$, and a collection of source-sink pairs $\{s_i, t_i\}$ in V with associated nonnegative demands $d(s_i, t_i)$. It will be convenient to think of the source-sink pairs as forming the edges of a *demand graph* $H = (V, E(H))$. The goal is to compute a flow in G that simultaneously routes all demands and respects the edge capacities, or to certify that no such flow exists. In the standard (*fractional*) version of the problem, the demand between a source-sink pair may be split across multiple paths, and this flexibility enables the precise formulation of the problem as a linear program that can be solved efficiently. This model has been studied for almost 70 years [10, 11].

Unsplittable multicommodity flows. Many practical applications often require the flow between each source-sink pair to be routed along a single path. Such a flow is called *unsplittable.*[1]

This additional requirement gives rise to a significantly harder and less well understood problem. Various fundamental NP-complete problems in combinatorial optimization such as bin packing, partitioning, and makespan minimization on identical and related machines, can be reduced to *unsplittable multicommodity flow* problems in networks consisting of only two nodes, which are the source-sink nodes of all demands, and parallel edges between them [12]. Moreover, already for very small feasible instances, any unsplittable flow violates some edge capacity by at least an additive amount of $d_{\max}$, where $d_{\max} = \max_i d(s_i, t_i)$ denotes the maximum demand value (see Figure 1). Furthermore, strong impossibility and/or hardness results imply that even with unit demands, and even when a feasible multicommodity flow exists, finding an *unsplittable flow* with only a small violation of edge capacities may be impossible or computationally intractable; see, e.g., the (updated) survey by Kolliopoulos [13].

Therefore, a natural question is to study the conditions under which the existence of a feasible flow implies the existence of an unsplittable flow that incurs only a small violation of edge capacities. Extensive work studies this question for special instances [8, 22, 25, 17, 26, 7, 18, 24, 28, 2, 16, 3, 27].

In the *single-source* setting (i.e., when the demand graph H is a star), the most prominent result of this kind is due to Dinitz, Garg, and Goemans [8], who proved that any feasible (fractional) flow can be converted into an unsplittable flow that violates the edge capacities by at most $d_{\max}$, the value of the maximum demand.[2]

However, in the multicommodity setting very few results of this kind are known for unsplittable flows [22, 24, 2, 16, 3]. A prominent example is the case where the supply graph G is a cycle and the demand graph H is arbitrary, which is known as the *ring-loading* problem, first studied by Cosares and Saniee [6]. In a classical result, Schrijver, Seymour, and Winkler [22] proved that any feasible (fractional) flow for a ring-loading instance can be converted into an unsplittable flow that exceeds the edge capacities by at most an additive amount of $1.5\, d_{\max}$; the best known bound, due to Däubel [7], is $1.3\, d_{\max}$.

Shapley and Shmoys [24] considered the much broader class of instances in which G is *outerplanar* and H is arbitrary. Recall that G is outerplanar if it admits a planar embedding in which all vertices lie on the unbounded face. They showed that any feasible flow can be converted into an unsplittable flow that exceeds capacities by at most $O(\log f)\, d_{\max}$, where f is the number of faces of G. This was recently improved by Alemán Espinosa and Kumar [2], who gave an additive $3.6\, d_{\max}$ congestion bound.

[1] In this work, whenever we refer to an unsplittable flow, we also assume that it routes *all demands.*
[2] This result also holds for *directed single-source* instances.

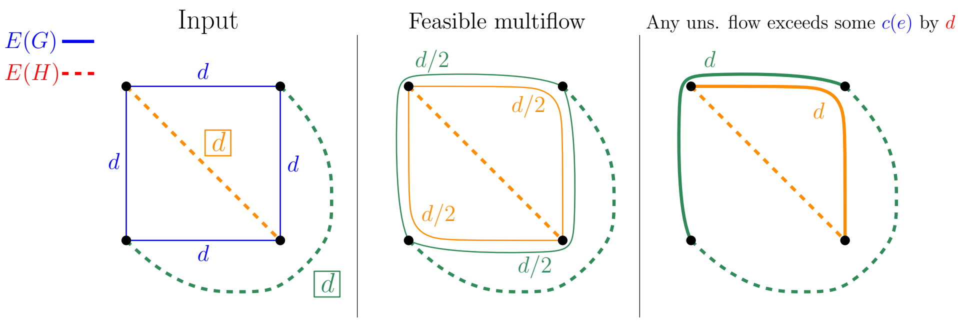

Figure 1 An example.

We show that an additive congestion of $2d_{\max}$ can be attained in outerplanar graphs. In terms of lower bounds on the additive violation needed, the only known result is a lower bound of $1.1\,d_{\max}$ by Skutella [26] for the ring-loading problem. We show that the additive congestion has to be at least $\frac{4}{3}\,d_{\max}$ for outerplanar graphs. We remark that outerplanar graphs are the only known nontrivial class of supply graphs for which the existence of a feasible flow guarantees the existence of an unsplittable flow exceeding edge capacities by at most $O(d_{\max})$, regardless of the demand graph H.

1.1 Our contributions

In this work, we significantly improve the state-of-the-art capacity-violation bounds for the existence of an unsplittable flow in outerplanar graphs.

Let $\alpha_O > 0$ be the smallest value such that every feasible instance with an outerplanar supply graph G admits an unsplittable flow that exceeds the edge capacities by at most an additive amount of $\alpha_O\, d_{\max}$. Similarly, let α_{RL} denote the smallest associated value for *ring-loading* instances.

Prior work showed that $1.01 \leq \alpha_{RL} \leq 1.5$ [22], which was ultimately improved to $\alpha_{RL} \leq 1.3$ [7], and the lower bound $\alpha_{RL} \geq 1.1$ [26]. Since every cycle is an outerplanar graph, we have $\alpha_O \geq \alpha_{RL}$, and prior to our work, it was not known that the parameters α_O and α_{RL} are necessarily different.

Our first main result establishes that $\alpha_O \leq 2$, improving on the bound of Alemán Espinosa and Kumar [2], who showed that $\alpha_O \leq 2\alpha_{RL} + 1 \in [3.2, 3.6]$. Our proof uses a more robust divide-and-conquer framework that treats the parameters of the instance more globally. We hope this approach to be useful for obtaining similar congestion bounds in other settings where the cut condition is sufficient, or approximately sufficient, for feasibility.

▶ **Theorem 1.** *Let G be an outerplanar graph. If an instance (G, u, H, d) is feasible, or equivalently, if it satisfies the cut-condition, then there exists an unsplittable flow y such that*

$$y(e) \leq u(e) + 2d_{\max} \quad \text{for all } e \in E(G).$$

Moreover, y can be computed in polynomial time.

We complement this result by showing that $\alpha_O \geq \frac{4}{3}$.

▶ **Theorem 2.** *For any $\varepsilon > 0$ there exists a feasible instance (G, u, H, d) on an outerplanar supply graph G, such that for any unsplittable flow y, there exists a supply edge $e \in E(G)$ with*

$$y(e) > u(e) + \left(\tfrac{4}{3} - \varepsilon\right) d_{\max}.$$

Recall that previously, the only lower bound known on α_O was that $\alpha_O \geq \alpha_{RL} \geq 1.1$. Our improved lower bound in Theorem 2, together with the fact that $\alpha_{RL} \leq 1.3$ [7], shows that the α-parameters for ring-loading and outerplanar graphs are different, and concretely shows that outerplanar graphs are more difficult settings than ring-loading instances for multicommodity unsplittable flows.

1.2 Related work

Very recently, Alemán Espinosa, Kumar, Poremba, and Shepherd [3] showed that in the *fully planar setting*, defined by instances in which $G + H$ is planar, any feasible flow can be converted into an unsplittable flow that violates the edge capacities by at most $2d_{\max}$. They also showed that the additive congestion has to be at least $1.5\, d_{\max}$ in this setting. The authors in [3] also showed that when G is *series-parallel* (i.e., G does not contain a K_4 minor), any feasible flow can be converted into an unsplittable flow such that the flow on any edge is at most twice its capacity plus $7.2\, d_{\max}$.

The directed single-source setting has been extensively studied; see, e.g., [25, 17, 18, 28, 27], including work on minimum-cost variants. Recently, Majthoub Almoghrabi, Skutella, and Warode [16] studied a multicommodity setting in a special class of directed series-parallel graphs and showed that any fractional flow can be converted into an unsplittable flow of no larger cost, such that the flow values on each arc differ by at most $d_{\max}$.[3]

1.3 The cut-condition

Interestingly, and perhaps not surprisingly, all of these settings have the common feature that one can identify a simple structural property, called the *cut-condition* that is sufficient, or nearly-sufficient, to guarantee the existence of a feasible flow. This condition requires that, for every cut, the total demand separated by the cut is at most the total capacity of the supply edges crossing it. The cut-condition is always necessary for feasibility of a multicommodity flow instance (see, e.g., [21]), but it is not always sufficient, even for small graphs such as $G = K_{2,3}$ [19]. It is well known that the cut-condition is sufficient for the following instances:

(i) (Single-source setting) G is arbitrary and H is a star [9].
(ii) (Okamura-Seymour setting) G is planar and $E(H)$ is incident to one face of G [19].
(iii) (Outerplanar setting) G is outerplanar and H is arbitrary (special case of (ii)).
(iv) (Fully planar setting) $G + H = (V, E(G) \cup E(H)\,)$ is planar [23].

Alemán Espinosa and Kumar [2] posed as an open problem whether every feasible Okamura–Seymour instance admits an unsplittable flow with additive congestion $O(d_{\max})$. Alemán Espinosa, Kumar, Poremba, and Shepherd [3] posed a more general question. Given a (supply graph, demand graph) pair (G, H), the *flow-cut gap* of (G, H) is the smallest value $\beta \geq 1$ such that, for all choices of capacities and demands, if every cut has capacity at least β times the total demand it separates, then a feasible flow exists. For example, $\beta = 1$ for the classes of instances (i)–(iv), it is known that $\beta = 2$ for instances in which G is series-parallel and H is arbitrary [5, 14]. For instances in which G is planar it is only known that $\beta \in [2, O(\sqrt{\log n}\,)]$ [20, 14]. In contrast, for general graphs the flow-cut gap is known to be $\beta = \Theta(\log k)$, where k is the number of source-sink pairs [15, 4]. Alemán Espinosa,

[3] We note that, as mentioned in [16] (see footnote on page 429 in [16]), that their results do not extend to the undirected (outerplanar) setting we consider here, nor the other way around.

Kumar, Poremba, and Shepherd [3] posed the natural question of whether a flow-cut gap of β for a class of instances implies the existence of an unsplittable flow that can be routed with capacities $O(\beta)\,u + O(d_{\max})$, or even $f(\beta)\,u + O(d_{\max})$ for some $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

The sufficiency of the cut-condition for the outerplanar setting is used crucially in [2] and also in this work. Its approximate sufficiency for series-parallel graphs is used in [3], where the framework in [2] is carefully adapted to handle this setting. We build on the ideas in [2] and develop a more robust divide-and-conquer framework that treats the instance parameters more globally. We hope that this approach can be extended beyond the outerplanar setting, for example, to the significantly more intricate Okamura–Seymour setting (*ii*), which is perhaps the most natural (intermediate) generalization.

1.4 Technical contributions and overview

Pinning a demand edge $h = (s, t) \in E(H)$ is a standard technique in (unsplittable) multicommodity flow problems. It subdivides h into a sequence $(s, v_1), (v_1, v_2), \ldots, (v_\ell, t)$ of demand edges, each with the same demand value, with the goal of transforming the instance into a more structured one. However, pinning can destroy feasibility: the resulting instance may not even admit a fractional flow. The algorithms of Dinitz et al. [8], Alemán Espinosa and Kumar [2], Majthoub Almoghrabi et al. [16], and Alemán Espinosa et al. [3] (for series-parallel graphs), perform pinnings in a controlled way so that an unsplittable flow exists, while incurring only an $O(d_{\max})$ additive capacity violation.

We first provide a high-level overview of the algorithm in [2] for outerplanar instances, which proceeds in two phases. In the first phase, it performs pinnings until every demand edge becomes parallel to a supply edge. They show that the cut-condition, and hence feasibility, can be preserved throughout this phase by increasing capacities by at most $2\alpha_{RL}\, d_{\max}$, where $\alpha_{RL} \leq 1.3$ [7] is the ring-loading parameter. In the second phase, they compute an unsplittable flow greedily, exceeding capacities by an additional amount of $d_{\max}$. We say that an ear of G is a path whose internal vertices have degree two, ignoring parallel edges. In each iteration of the first phase, their algorithm selects an ear $P = v_1, \ldots, v_\ell$ (with $\{v_1, v_\ell\} \in E(G)$), and considers the set $H_P \subseteq E(H)$ of demand edges with at least one endpoint in P. Each demand $(v_i, w) \in H_P$ is then pinned along a subpath of P. The key observation in [2] is that all demands in H_P can be pinned via a reduction to ring-loading, which preserves the cut-condition in the resulting instance, at the expense of increasing capacities on $E(P) + \{v_1, v_\ell\}$ by an additive amount of $\alpha_{RL}\, d_{\max}$. After the pinnings along P are made, a feasible flow is computed, and then each demand $(s, t) \in E(H) \cap E(G)$ and its corresponding s-t flow are removed from the first phase. After this removal, P can be treated as an edge that is parallel to $\{v_1, v_\ell\}$ for the remaining demands $E(H) \setminus H_P$. A similar ring-loading reduction approach was implemented by Alemán Espinosa et al. [3] in order to contract the ears of series-parallel graphs, where the goal of the first phase is to pin the demands so that the resulting instance becomes fully planar (i.e., $G + H$ is planar).

The main drawback of this procedure is that we can not remove the ear P after an iteration because the demands in $E(H) \setminus H_P$ may still require the residual capacity of P. This is not an issue in the outerplanar setting, but it becomes critical in, for example, an Okamura-Seymour instance (G, u, H, d). For example, suppose that both endpoints of every demand edge lie on the unbounded face of G. Not being able to remove P implies that one cannot pin the demands by using the inner nodes of G. If one makes such a pinning, then the resulting instance is no longer an Okamura-Seymour instance. Therefore, one can not get access to these inner nodes by relying on such a ring-loading reduction.

Our main technical contribution is to consider a tight cut $(S_1, S_2 := V(G) \setminus S_1)$, that is, a cut whose capacity equals the demand it separates, instead of an ear of G. We then solve a smaller instance that allows us not only to pin the demands in $\delta_H(S_1)$ on the cut, but also to fix their unsplittable routing on $\delta_G(S_1)$. This yields another smaller instance, which we solve recursively.

More concretely, tightness of the cut (S_1, S_2) implies that all demands in $H[S_i]$ are routed within $G[S_i]$. We construct a *cut instance* $\mathcal{I}'$ by picking $i \in \{1, 2\}$, deleting all demands in $H[S_i]$ and all inner (not outer) edges of G within S_i, and updating the capacities of the outer edges in S_i to guarantee feasibility of the cut instance.

We solve the cut instance recursively, i.e., we find an unsplittable flow y' for $\mathcal{I}'$ which violates edge capacities in $\mathcal{I}'$ by at most $2d_{\max}$. Actually, for the outer edges (edges on the boundary of the outer face) we will even get a stronger bound of $\frac{3}{2} \cdot d_{\max}$. The flow y' already determines our unsplittable flow for $\mathcal{I}$ on all edges except for $G[S_i]$.

To extend this partial unsplittable flow to all of G we construct a *split instance* by only considering $G[S_i]$ as supply graph and adding a demand edge between any vertices $s, t \in S$ that are connected within S_i in the unsplittable flow for some demand in y'. We also add the demands in $H[S_i]$ which were not considered in the cut instance at all. We show that since y' does not exceed capacities too much, this split instance becomes feasible after increasing the capacities of the edges $E^* \subseteq E(G[S_i])$ which are incident to $\delta_G(S_i)$, by $\frac{3}{2} \cdot d_{\max}$. This edge set contains at most one edge per block of $G[S_i]$.

We again use recursion to solve the split instance, which yields an unsplittable flow $\widehat{y}$ that can "fill the gaps" for y'. Our final unsplittable flow y will be equal to $\widehat{y}$ on $G[S_i]$ and to y' outside $G[S_i]$. For the edges E^* we will guarantee an even stronger bound of $\frac{1}{2} \cdot d_{\max}$ on the capacity violation of $\widehat{y}$, which will yield the capacity bounds needed for y.

The only case where our recursion step fails is if the tight cut $\delta_G(S_i)$ already contains all inner edges of G, because then the cut instance will not be smaller than the original instance. In this case, we consider a face F which is incident to at most one inner edge $\{v, w\}$ in G. W.l.o.g. both v and w have degree 3 in G, and we can find two cuts of size 2 among the edges incident to v and w. We will show that for one of these cuts we can perform the same procedure as above to reduce our instance to two smaller instances.

2 Preliminaries and Notation

All of the graphs that we consider in this work are undirected. Given a graph $G = (V, E)$, a function $c : E \to \mathbb{R}$, and an edge $\{v, w\} \in E$, we overload notation and use $c(v, w)$ to denote $c(\{v, w\})$. For any edge set $F \subseteq E$, we use $c(F)$ to denote $\sum_{e \in F} c(e)$. Given a vertex set $S \subseteq V$, we use $\delta_G(S)$ to denote the set of edges of G with precisely one endpoint in S. We sometimes use the term *path* interchangeably to mean either its edge set or its vertex sequence; the intended meaning will be clear from the context. For any universe U, any $S \subseteq U$, and any $e \in U$, we sometimes use $S + e$ and $S - e$ to denote $S \cup \{e\}$ and $S \setminus \{e\}$, respectively.

2.1 Multicommodity flows

An instance (G, u, H, d) of *multicommodity flow* is given by an undirected graph $G = (V, E(G))$ with edge capacities $u : E(G) \to \mathbb{R}_{\geq 0}$, and a graph $H = (V, E(H))$ with demand values $d : E(H) \to \mathbb{R}_{\geq 0}$. G and H may have parallel edges but no loops. We refer to G and $E(G)$ as the *supply graph* and *supply edges*, respectively. We refer to H and $E(H)$ as the *demand graph* and *demand edges*, respectively.

For a demand edge $h = \{s,t\} \in E(H)$, we use $\mathcal{P}_h$ to denote the set of all (simple) st-paths in G. A flow for h, or *h-flow*, is an assignment $x_h : \mathcal{P}_h \to \mathbb{R}_{\geq 0}$ of non-negative real numbers to paths in $\mathcal{P}_h$ such that $\sum_{P\in\mathcal{P}_h} x_h(P) = d(h)$. A collection of flows $x = (x_h)_{h\in E(H)}$ constitutes a *multicommodity flow* for the instance. For simplicity, we often refer to multicommodity flows simply as flows. For a supply edge $e \in E(G)$ and a demand edge $h \in E(H)$, we use $x_h(e) := \sum_{P:e\in P} x_h(P)$ to denote the total h-flow going through e. We use $x(e) := \sum_{h\in E(H)} x_h(e)$ to denote the total flow going through e. A flow x is *feasible* if it satisfies the edge-capacity constraints, that is, if $x(e) \leq u(e)$ for all $e \in E(G)$.

An h-flow x_h is called *unsplittable* if *exactly one* of the paths in $\mathcal{P}_h$ is assigned a non-zero value in x_h. We call a flow $x = (x_h)_{h\in E(H)}$ unsplittable if each x_h is unsplittable. In this work, whenever we refer to an unsplittable flow, we also assume that it routes all demands. We use $d_{\max} := \max_{h\in E(H)} d(h)$ to denote the maximum demand value. To simplify notation in the upcoming proofs, we adopt the notion of α-feasibility introduced in [2] and generalize it to the notion of $\vec{\beta}$-feasibility.[4]

▶ **Definition 3** ($\vec{\beta}$-feasibility). *For a vector $\vec{\beta} \in \mathbb{R}^{E(G)}_{\geq 0}$, we say that an (unsplittable) flow x is $\vec{\beta}$-*feasible *if*

$$x(e) \leq u(e) + \vec{\beta}(e) \cdot d_{\max} \text{ for each } e \in E(G);$$

*we overload notation, and say that x is α-*feasible *for a real number α, if $x(e) \leq u(e)+\alpha\cdot d_{\max}$ for all $e \in E(G)$ (i.e., x is $\vec{\beta}$-feasible, where $\vec{\beta} = \alpha \cdot \vec{1}$).*

We may assume w.l.o.g. that the supply graph G is 2-vertex-connected. Indeed, suppose that G has a cut vertex. Let $B_1 \subseteq V(G)$ denote the vertex set of a block[5] of G containing exactly one cut vertex $v \in B_1$ and define $B_2 := V(G) \setminus (B_1 \setminus \{v\})$. For any demand $h = \{a,b\} \in E(H)$ with $h \subseteq B_i$ for some $i \in \{1,2\}$ we have that any ab-path in G is contained inside B_i, so that demand does not interact with edges in the other component B_{3-i}. For a demand $h = \{a,b\} \in E(H)$ with $a \in B_1$, $b \in B_2$ we have that any ab-path in G contains v. Hence, we can replace h by two demands $\{a,v\}$ and $\{v,b\}$, each of value $d(h)$, obtaining an equivalent multicommodity flow instance. Repeating this transformation until no cut vertex exists reduces the problem to a collection of smaller instances, whose supply graphs are the blocks of G.

2.2 Outerplanar instances

A graph is outerplanar if it admits a planar embedding in which all vertices lie on the unbounded face. Throughout this work, the supply graph G of our instance (G,u,H,d) is outerplanar, and we fix an outerplanar embedding of G. We sometimes also call the unbounded face the *outer face*, and the other faces *inner faces*. Similarly, we call the edges of G incident with the outer face the *outer edges*, and the remaining edges the *inner edges*. We denote the sets of outer and inner edges by $E_{\text{outer}}(G)$ and $E_{\text{inner}}(G)$, respectively. Since we may assume that G is 2-vertex-connected, every face of G is bounded by a cycle.

[4] In [3] the authors defined (α,β)-feasibility, to indicate that a flow x satisfies $x(e) \leq \alpha\, u(e) + \beta\, d_{\max}$ for every $e \in E(G)$ (and some $\alpha,\beta \in \mathbb{R}_{\geq 0}$). In this work, we focus only on additive violations with respect to $d_{\max}$.

[5] A block of G is a maximal (inclusion-wise) vertex set $S \subseteq V(G)$ such that $G[S]$ is 2-vertex-connected, where we regard K_2 as 2-vertex-connected.

2.3 Cut-condition

Let (G, u, H, d) denote a multicommodity flow instance, and set $V := V(G)$. For any $S \subseteq V$, the *cut* $(S, V \setminus S)$ is a bipartition of the vertex set. The *cut-condition* requires that, for every cut, the total demand crossing the cut is at most the total capacity of the supply edges crossing it i.e.,

$$u(\delta_G(S)) \geq d(\delta_H(S)) \quad \text{for all } S \subseteq V.$$

A set $\emptyset \neq S \subsetneq V$ is called *central* if both of the induced graphs $G[S]$ and $G[V \setminus S]$ are connected. We say that $(S, V \setminus S)$ is a *central cut* if S is central. The following is a well known fact (see, e.g., [21]).

▶ **Lemma 4.** *An instance (G, u, H, d) satisfies the cut-condition if and only if $u(\delta_G(S)) \geq d(\delta_H(S))$ for all central sets $S \subseteq V$.*

Observe that in a 2-vertex-connected outerplanar graph G, the central sets are precisely the vertex sets of subpaths of the outer face.

A classical result of Okamura and Seymour states that the cut-condition is sufficient whenever G is planar and the endpoints of all demand edges lie on a common face of G.

▶ **Theorem 5** (Okamura–Seymour [19])**.** *Let G be a planar supply graph and H a demand graph. Suppose there exists a face F of G such that for every demand edge $\{s, t\} \in E(H)$, both s and t lie on F. Then, for any $u : E(G) \to \mathbb{R}_{\geq 0}$ and $d : E(H) \to \mathbb{R}_{\geq 0}$, the instance (G, u, H, d) admits a multicommodity flow if and only if it satisfies the cut-condition.*

Since every vertex of an outerplanar graph is incident to the outer face, an immediate consequence of the above theorem is that the cut-condition is sufficient for the feasibility of an instance if the supply graph G is outerplanar.

3 Improved Upper Bound for Outerplanar Instances

In this section, we give a proof of Theorem 1, which we restate for convenience.

▶ **Theorem 1.** *Let G be an outerplanar graph. If an instance (G, u, H, d) is feasible, or equivalently, if it satisfies the cut-condition, then there exists an unsplittable flow y such that*

$$y(e) \leq u(e) + 2d_{\max} \quad \textit{for all } e \in E(G).$$

Moreover, y can be computed in polynomial time.

Our main approach is to reduce our instance to two smaller instances, which can be solved recursively. Ultimately, we arrive at a ring-loading problem, which is better understood than the case where G is outerplanar. We introduce the results on the ring-loading problem that we use in Section 3.1. Afterwards, we prove Theorem 1.

3.1 Ring-loading

For the ring-loading problem, Däubel [7] showed how to compute a 1.3-feasible unsplittable flow in any feasible instance. For our purposes, however, it is essential to obtain a much stronger bound on the capacity violation for one special edge, while still keeping the capacity violations on all other edges small. Therefore, we make use of the algorithm of Schrijver, Seymour, and Winkler [22], which only gives a $\frac{3}{2}$-feasible unsplittable flow, but also directly implies a bound of $+\frac{1}{2}d_{\max}$ for the capacity violation on one edge that can be chosen arbitrarily in advance.

▶ **Theorem 6** (Schrijver, Seymour, Winkler [22]). *Let (G, u, H, d) be a feasible ring-loading instance and $e^* \in E(G)$. Then there is a $\vec{\beta}$-feasible unsplittable flow $y = (y_h)_{h \in E(H)}$, where $\vec{\beta}(e^*) = \frac{1}{2}$ and $\vec{\beta}(e) = \frac{3}{2}$ for all other edges $e \in E(G) \setminus \{e^*\}$.*

We also need an even stronger bound for the special case where all demands are incident to one of two specified vertices.

▶ **Lemma 7.** *Let $\mathcal{I} = (G, u, H, d)$ be a feasible ring-loading instance with two vertices $v, w \in V(G)$ such that any demand edge contains v or w. Let $e_1, e_2 \in E(G)$. Then there exists a $\vec{\beta}$-feasible unsplittable flow $y = (y_h)_{h \in E(H)}$ for $\mathcal{I}$, where $\vec{\beta}(e_1) = \vec{\beta}(e_2) = \frac{1}{2}$ and $\vec{\beta}(e) = \frac{3}{2}$ for $e \in E(G) \setminus \{e_1, e_2\}$.*

The proof of Theorem 6 follows by using the same algorithm and analysis as Schrijver, Seymour, and Winkler [22]. We include the details, together with the proof of Lemma 7, in the extended version of this paper.

3.2 Proof of Theorem 1

Recall that a nonempty vertex set $S \subsetneq V(G)$ is central if $G[S]$ and $G[V(G) \setminus S]$ are connected. A useful tool which we use in our proof is the notion of *x-nice central sets*:

▶ **Definition 8** (x-nice sets). *Let (G, u, H, d) be an outerplanar multicommodity flow instance, and let $x = (x_h)_{h \in E(H)}$ be a feasible flow. We call a central vertex set $S \subseteq V(G)$ x-nice if every $h \in E(H[S])$ is routed in $G[S]$, i.e., if $x_h(e) = 0$ for every edge $e \notin E(G[S])$.*

We make use of two types of x-nice sets, which are captured in the following two lemmata:

▶ **Lemma 9.** *Let (G, u, H, d) be a multicommodity flow instance and $x = (x_h)_{h \in E(H)}$ a feasible flow. Let $(S, V(G) \setminus S)$ be a central cut that is tight, i.e., $u(\delta_G(S)) = d(\delta_H(S))$. Then S is x-nice.*

Proof. Since x is feasible, we have

$$u(\delta_G(S)) \geq x(\delta_G(S)) = \sum_{h \in E(H)} x_h(\delta_G(S)) \geq \sum_{h \in \delta_H(S)} x_h(\delta_G(S)) \geq d(\delta_H(S)) = u(\delta_G(S)).$$

Therefore, equality holds throughout the above expression. Thus, $x_h(\delta_G(S)) = 0$ for every $h \in E(H[S])$. Since both endpoints of such an h lie in S, this implies that $x_h(e) = 0$ for all $e \notin E(G[S])$. Therefore, S is x-nice. ◀

▶ **Lemma 10.** *Let $\mathcal{I} = (G, u, H, d)$ be a multicommodity flow instance and $x = (x_h)_{h \in E(H)}$ a feasible flow that is minimal, i.e., for any feasible flow x' for $\mathcal{I}$ with $x'(e) \leq x(e)$ for all $e \in E(G)$ we actually have $x'(e) = x(e)$ for all $e \in E(G)$. Let $(S_1, S_2 = V(G) \setminus S_1)$ be a central cut with $|\delta_G(S_1)| = 2$. Then S_1 or S_2 is x-nice.*

Proof. Set $C := \delta_G(S_1) = \delta_G(S_2)$. Assume that neither S_1 nor S_2 is x-nice. Then, for each $i = 1, 2$, there is a demand $h_i = \{s_i, t_i\} \in E(H[S_i])$ such that $x_{h_i}(C) \neq 0$. Choose an $s_i t_i$-path P_i with $E(P_i) \cap C \neq \emptyset$ and $x_{h_i}(P_i) > 0$. Since both endpoints of h_i lie in S_i, the path P_i contains both edges of C. Let Q_i be the minimal subpath of P_i containing both edges of C. Then $Q_i - C$ is a path in $G[S_{3-i}]$. For each $i = 1, 2$, let P'_i be the $s_i t_i$-path obtained from P_i by replacing Q_i with $Q_{3-i} - C$ and possibly deleting cycles. Now, decrease $x_{h_i}(P_i)$ by a sufficiently small amount $\varepsilon > 0$ and increase $x_{h_i}(P'_i)$ by ε, for $i = 1, 2$. This yields another feasible flow x' with $x'(e) \leq x(e)$ for every $e \in E(G)$, and with $x'(e) = x(e) - 2\varepsilon$ for both edges $e \in C$, contradicting the minimality of x. ◀

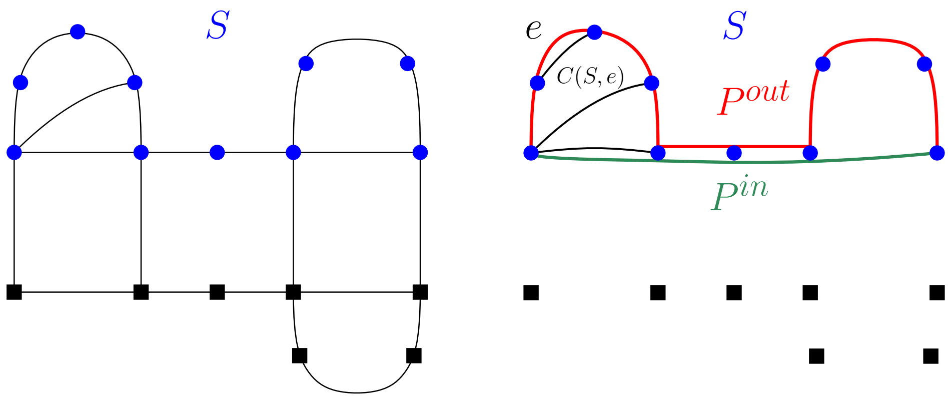

Figure 2 The left shows an outerplanar graph with a central set S (blue and circular vertices). The inner and outer paths are depicted on the right. Note that together they form the outer edges of $G[S]$. For the given edge e, the edge set $C(S, e)$ consists of e, the leftmost edge of P^{in}, and the inner black edge of $G[S]$ inbetween.

We can use an x-nice set in the following way to round the splittable flow x to an unsplittable one: First, we remove demands in $H[S]$, together with their flows, from our instance and replace $G[S]$ by a path (consisting of the outer edges of G within S) with adequate edge capacities. This yields a *cut instance* (cf. Definition 14). Due to the fact that S is x-nice, the cut instance differs from the original instance only within S, so after solving the cut instance recursively, we can fix the obtained unsplittable flow y' everywhere except within S. Afterwards, we need to route $H[S]$ and reconnect the segments of the unsplittable flows in y' that are routed in S. All of this needs to be routed through S, so we encode this task in a *split instance* (cf. Definition 16), which we also solve recursively.

▶ **Definition 11** (Outer and inner paths). *Let G be a 2-vertex-connected outerplanar graph and $S \subsetneq V(G)$ a central set.*

(i) *The* outer path *of S is the unique path P^{out} with $V(P^{\text{out}}) = S$ and $E(P^{\text{out}}) \subseteq E_{outer}(G)$.*

(ii) *For any $a, b \in S$ we define their outer path $P^{\text{out}}_{a,b}$ as the unique ab-subpath of P^{out}.*

(iii) *The* inner path *of S is the path P^{in} in $G[S]$ between the endpoints of P^{out} whose edge set is the union of $E_{outer}(G[S]) \cap E_{inner}(G)$ and the bridges[6] of $G[S]$. Equivalently, if G contains no parallel edges, P^{in} is the unique shortest path in $G[S]$ between the endpoints of P^{out}.*

See Figure 2 for an example.

▶ Remark 12. It is easy to verify that the definition of the inner path P^{in} is well-defined, i.e., the edge set of P^{in} actually forms a path. Also, the following useful observations are easy to see. We defer the proofs to the extended version of this paper.

- The vertex sequence of P^{in} is a subsequence of the vertex sequence of P^{out}.
- The edges in $P^{\text{in}} \cap P^{\text{out}}$ are precisely the bridges of $G[S]$.
- The edges in $P^{\text{out}} \cup P^{\text{in}}$ are precisely the outer edges of $G[S]$.
- The inner vertices of P^{in} are precisely the cut-vertices of $G[S]$.
- There is a one-to-one correspondence between the blocks of $G[S]$ and the edges of P^{in}. More precisely, each edge $\{v, w\} \in P^{\text{in}}$ corresponds to the block induced by the vertex set of $P^{\text{out}}_{v,w}$.

[6] A bridge is an edge whose removal disconnects the graph.

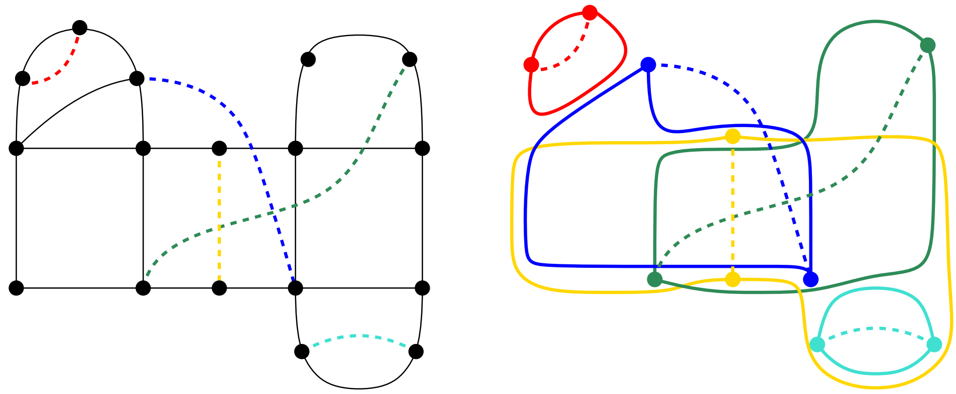

Figure 3 The left shows an outerplanar multicommodity flow instance (black edges) with 5 demands (colored dashed edges). The right image shows a feasible splittable flow, where the flow of each demand is drawn in the same color as the demand.

We now describe our procedure to compute an unsplittable routing by solving two smaller instances. We first formally define the cut instance.

▶ **Definition 13** (Semi-cut). *Let G be a 2-vertex-connected outerplanar graph, $S \subseteq V(G)$ a central set, and P^{out} the outer path of S. For any edge $e \in E(P^{\text{out}})$ we define the* semi-cut w.r.t. S and e *as the edge set $C(S,e) := \{\{v,w\} \in E(G[S]) : e \in E(P^{\text{out}}_{v,w})\}$. Equivalently, $C(S,e) := \delta_{G[S]}(X)$, where $X \subseteq S$ is the vertex set of a connected component of $P^{\text{out}} - e$. See Figure 2 for an example.*

▶ **Definition 14** (Cut instance). *Let (G,u,H,d) be an outerplanar multicommodity flow instance where G is 2-vertex-connected, and let $x = (x_h)_{h\in E(H)}$ be a feasible flow. Let $S \subseteq V(G)$ be an x-nice central set, and let P^{out} be the outer path of S. The* cut instance for x and S *is the instance $\mathcal{I}^{cut} = (G',u',H',d')$ defined as follows. See Figure 4 for an illustration. The supply graph G' is given by $V(G') = V(G)$ and $E(G') := (E(G) \setminus E(G[S])) \cup E(P^{\text{out}})$. The demand graph and demand values are given by $E(H') := E(H) - E(H[S])$ and $d'(h) := d(h)$ for all $h \in E(H')$. Edge capacities are defined as follows.*

$$u'(e) := \begin{cases} \sum_{e'\in C(S,e)} \sum_{h\in E(H')} x_h(e'), & \textit{if } \ e \in P^{\text{out}} \\ u(e), & \textit{otherwise.} \end{cases}$$

In other words, we first remove the flow of the demands that are routed completely inside S. We add the (residual) capacities of the inner edges of G inside S to the outer path of S, and then remove those inner edges from the instance. This makes sure that we maintain feasibility of the cut instance:

▶ **Lemma 15.** *In the situation of Definition 14, the cut instance $\mathcal{I}^{cut}$ is feasible.*

Proof. We construct a feasible flow $x' = (x'_h)_{h\in E(H')}$ for $\mathcal{I}^{\text{cut}}$ from the flow x. For any $h = \{s,t\} \in E(H')$ we construct x'_h as follows. For each st-path P in G that lies in the support of x_h, let P' be the st-path in G' obtained from P by replacing each edge $\{v,w\} \in E(P)$ with $v,w \in S$ by their outer path $P^{\text{out}}_{v,w}$, and then deleting cycles if necessary. We then increase $x'_h(P')$ by $x_h(P)$. Doing this for all st-paths in the support of x_h yields

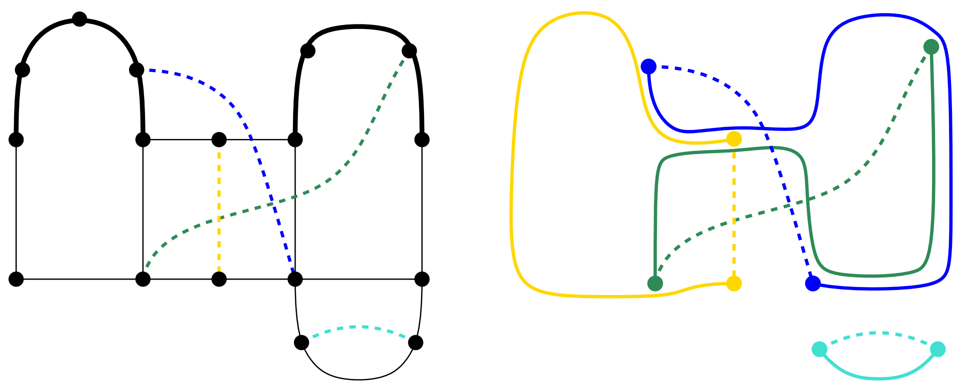

Figure 4 The left shows the cut instance for the instance from Figure 3, where S is chosen as in Figure 2. Note that the red demand is removed, and the capacities of inner edges in S are added along their outer paths. The right image shows a possible unsplittable flow for the cut instance.

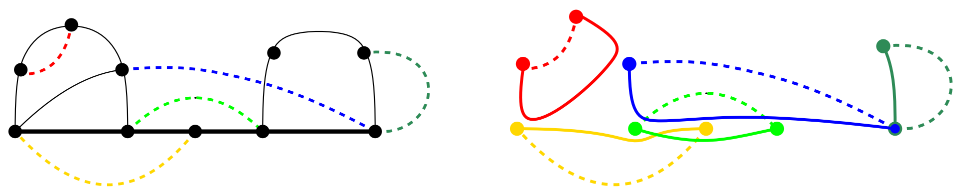

Figure 5 The left shows the split instance for the unsplittable flow from Figure 4. Note that the green demand from Figure 4 induces two demands here. The capacity on the inner path (straight horizontal edges) is increased to make it feasible. An unsplittable flow in this instance can be combined with the unsplittable flow from Figure 4 to a solution for the original instance.

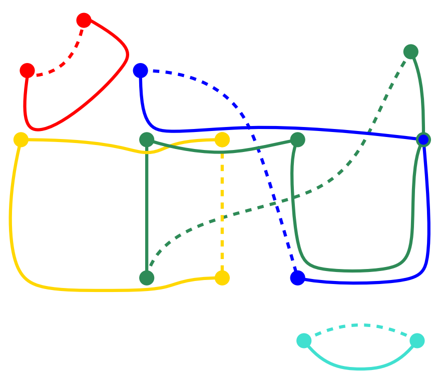

Figure 6 Combining the unsplittable flows from Figure 4 and Figure 5 yields an unsplittable flow for the original instance.

a flow x'_h of value $d(h) = d'(h)$ in G'. It remains to check that the capacities of $\mathcal{I}^{\text{cut}}$ are satisfied. If $e \in E(G') \setminus E(P^{\text{out}})$, the above procedure does not increase the flow on e, and hence $x'(e) \leq x(e) \leq u(e) = u'(e)$. Now let $e \in E(P^{\text{out}})$. By construction of x', for every $h \in E(H')$,

$$x'_h(e) \leq \sum_{\substack{\{v,w\} \in E(G[S]) \\ e \in E(P^{\text{out}}_{v,w})}} x_h(\{v,w\}) = \sum_{\{v,w\} \in C(S,e)} x_h(\{v,w\}).$$

Therefore,

$$x'(e) = \sum_{h \in E(H')} x'_h(e) \leq \sum_{h \in E(H')} \sum_{e' \in C(S,e)} x_h(e') = u'(e).$$

Thus, x' is feasible for $\mathcal{I}^{\text{cut}}$. ◀

Our rounding algorithm will find an adequate x-nice set such that the cut instance is simpler and can be solved recursively. Next, the obtained solution is be fixed on the edges not in $G[S]$ and the remaining task is to connect the endpoints of our partial unsplittable flow inside $G[S]$. This task is captured in the split instance, which we define now:

▶ **Definition 16** (Split instance). *Let $\mathcal{I} = (G, u, H, d)$ be an outerplanar multicommodity flow instance where G is 2-vertex-connected, and let $x = (x_h)_{h \in E(H)}$ be a feasible flow. Let $S \subseteq V(G)$ be an x-nice central set, and let P^{out} and P^{in} be the outer path and inner path of S, respectively. Let $\mathcal{I}^{cut} = (G', u', H', d')$ be the cut instance for x and S. Let $y' = (y'_h)_{h \in E(H')}$ be a $\vec{\beta}$-feasible unsplittable flow of $\mathcal{I}^{cut}$, for some $\vec{\beta} \in \mathbb{R}^{E(G')}_{\geq 0}$.*

The split instance for $\mathcal{I}, \mathcal{I}^{\text{cut}}$, y' and $\vec{\beta}$ *is the instance $\mathcal{I}^{split} = (\widehat{G}, \widehat{u}, \widehat{H}, \widehat{d})$ defined as follows. First, we set $\widehat{G} := G[S]$. Edge capacities are given by*

$$\widehat{u}(v,w) := \begin{cases} u(v,w) + \max\{\, \vec{\beta}(e') : e' \in P^{\text{out}}_{v,w} \,\} \cdot d_{\max}\,, & \textit{if } \{v,w\} \in E(P^{\text{in}}) \\ u(v,w)\,, & \textit{otherwise.} \end{cases}$$

The demands $E(\widehat{H})$ are constructed as follows. First, we add all the demands in $E(H[S]) = E(H) \setminus E(H')$ to $E(\widehat{H})$. Next, for each edge $h \in E(H')$, let P'_h denote the path on which y'_h routes h. Let $\mathcal{Q}_h$ denote the set of maximal subpaths of P'_h that are in $G'[S] = P^{\text{out}}$. For each $Q \in \mathcal{Q}_h$ with $E(Q) \neq \emptyset$ we add a demand h_Q between the endpoints of Q to $\widehat{H}$ and set $\widehat{d}(h_Q) := d(h)$. See Figure 5 for an example.

A key property of the split instance is that it is always feasible, although we only increased capacities along the inner path P^{in}.

▶ **Lemma 17.** *In the situation of Definition 16 the split instance $\mathcal{I}^{split}$ is feasible.*

Proof. We first observe that we may assume that $H' = H$. If this is not the case, we could first remove all demands in $E(H) \setminus E(H')$ from H and decrease all capacities $u(e)$ by $\sum_{h \in E(H) \setminus E(H')} x_h(e)$. This does not change the cut instance $\mathcal{I}^{\text{cut}}$. Thus, if the lemma holds for the resulting split instance, then restoring the removed demands and their flows yields the lemma for the original split instance.

Suppose that the outer face is bounded by the cycle $v_1, v_2, \ldots, v_n, v_1$. Assume w.l.o.g. that $S = \{v_1, v_2, \ldots, v_k\}$ (and thus $P^{\text{out}} = v_1, v_2, \ldots, v_k$). Let $X \subseteq S$ denote the set of vertices that are adjacent to $V(G) \setminus S$ in G (note that $X \subseteq V(P^{\text{in}})$). By Lemma 4 and Theorem 5, it suffices to show that the cut-condition holds for all central cuts of $\widehat{G}$. Let $U \subsetneq S$ be a central set of $\widehat{G} = G[S]$. W.l.o.g. $U = \{v_i, \ldots, v_j\}$ for some $1 \leq i \leq j \leq k$. We distinguish two cases:

Case 1: $i > 1$ and $j < k$.

Since $S \setminus U$ is also a central set of $G[S]$, it follows that U does not contain a cut-vertex of $G[S]$; otherwise, v_1 would be disconnected from v_k in $S \setminus U$. Since the cut-vertices of $G[S]$ are precisely the inner vertices of P^{in} (see Remark 12), and since $v_1, v_k \notin U$, it follows that $U \cap X \subseteq U \cap V(P^{\text{in}}) = \emptyset$. Therefore, $\delta_{\widehat{G}}(U) = \delta_G(U)$ and $u(\delta_G(U)) = \widehat{u}(\delta_{\widehat{G}}(U))$. Note that each demand $h \in E(H)$ induces at most one demand h_Q with an endpoint in $S \setminus X$. If such an h_Q exists, then this endpoint is the unique endpoint of h inside S. Hence, since $U \cap X = \emptyset$, we have $\delta_{\widehat{H}}(U) = \delta_H(U)$.

Thus, $\widehat{d}(\delta_{\widehat{H}}(U)) = d(\delta_H(U)) \leq u(\delta_G(U)) = \widehat{u}(\delta_{\widehat{G}}(U))$, where the inequality follows from the fact that the initial instance $\mathcal{I}$ satisfies the cut-condition.

Case 2: $i = 1$ or $j = k$. W.l.o.g. $i = 1$ and thus $j < k$. Define $e := \{v_j, v_{j+1}\}$, and observe that $\delta_{\widehat{G}}(U)$ is precisely the semi-cut $C(S, e)$ for S and e (see Definition 13). Since exactly one endpoint of P^{in} lies in U, there is some $e' \in C(S, e) \cap E(P^{\text{in}})$. We have $\widehat{u}(e') \geq u(e') + \vec{\beta}(e)$. Thus, $\widehat{u}(\delta_{\widehat{G}}(U)) \geq u(C(S, e)) + \vec{\beta}(e) \geq u'(e) + \vec{\beta}(e) \geq y'(e)$. Finally, observe that by definition of $\widehat{H}$ we have $\widehat{d}(\delta_{\widehat{H}}(U)) = y'(e)$. ◀

We now have all the ingredients to prove Theorem 1. We prove a slightly stronger statement that enables us to use an inductive argument. Rather than presenting the proof as an explicit induction, we use a minimal counterexample argument.

▶ **Theorem 18.** *Let $\mathcal{I} = (G, u, H, d)$ be an outerplanar multicommodity flow instance that is feasible, or equivalently, that satisfies the cut-condition. Let $E^* \subseteq E_{outer}(G)$ be a set of outer edges of G that contains at most one edge of each block of G. Then there exists a $\vec{\beta}$-feasible unsplittable flow $y = (y_h)_{h \in E(H)}$, where*

$$\vec{\beta}(e) := \begin{cases} \frac{1}{2}, & \text{if } e \in E^*, \\ \frac{3}{2}, & \text{if } e \in E_{outer}(G) \setminus E^*, \\ 2, & \text{if } e \in E_{inner}(G). \end{cases}$$

Proof. Assume that this statement is false, and let (G, u, H, d), together with $E^* \subseteq E(G)$, be a counterexample that lexicographically minimizes $(|E_{\text{inner}}(G)|, |E(H)|)$. As discussed in Section 2.2, the blocks of G are independent, so we may assume that G is 2-vertex-connected. In particular, $|E^*| \leq 1$ and we can assume w.l.o.g. that $E^* = \{e^*\}$ for some outer edge $e^* \in E_{\text{outer}}(G)$.

Furthermore, we may assume that for every edge $e \in E(G)$ there exists a central cut $(S_e, V(G) \setminus S_e)$ with $e \in \delta_G(S_e)$ that is tight, i.e., $u(\delta_G(S_e)) = d(\delta_H(S_e))$. Otherwise, we can decrease $u(e)$ while preserving the cut-condition, until either such a cut exists or $u(e) = 0$. The latter case can be ruled out by the minimality of the counterexample. Let $S := S_{e^*}$ and let $x = (x_h)_{h \in E(H)}$ be a feasible flow for $\mathcal{I}$. By Lemma 9, both S and $V(G) \setminus S$ are x-nice. Also, the fact that each edge belongs to a tight cut implies that x is minimal, i.e., if x' is feasible with $x'(e) \leq x(e)$ for all $e \in E(G)$ then also $x'(e) = x(e)$ for all $e \in E(G)$.

Case 1: $G[S]$ or $G[V(G) \setminus S]$ contains an inner edge of G.

W.l.o.g. $E(G[S]) \cap E_{\text{inner}}(G) \neq \emptyset$. Let P^{out} and P^{in} denote the outer path of S and inner path of S, respectively. Let $\mathcal{I}^{\text{cut}} = (G', u', H', d')$ be the cut instance for x and S.

Observe that $|E_{\text{inner}}(G')| < |E_{\text{inner}}(G)|$. Thus, by minimality of our counterexample, we can assume that there is a $\vec{\beta}'$-feasible unsplittable flow $y' = (y'_h)_{h \in E(H')}$ for $\mathcal{I}^{\text{cut}}$, where $\vec{\beta}'(e^*) = \frac{1}{2}$, $\vec{\beta}'(e) = \frac{3}{2}$ for $e \in E_{\text{outer}}(G') \setminus \{e^*\}$ and $\vec{\beta}'(e) = 2$ for $e \in E_{\text{inner}}(G')$.

Let $\mathcal{I}^{\text{split}} = (\widehat{G}, \widehat{u}, \widehat{H}, \widehat{d})$ be the split instance for $\mathcal{I}$, $\mathcal{I}^{\text{cut}}$, y' and $\vec{\beta}'$ according to Definition 16. By Lemma 17, $\mathcal{I}^{\text{split}}$ is feasible. Recall that $E_{\text{outer}}(\widehat{G}) = P^{\text{out}} \cup P^{\text{in}}$, and that for any edge $\{v, w\} \in P^{\text{in}}$, the vertex set $V(P^{\text{out}}_{v,w})$ is a block of $\widehat{G}$. Therefore, P^{in} contains precisely one edge from every block of $\widehat{G}$. Since $|E_{\text{inner}}(\widehat{G})| < |E_{\text{inner}}(G)|$, by minimality of our counterexample we can find a $\widehat{\vec{\beta}}$-feasible unsplittable flow $\widehat{y} = (\widehat{y}_h)_{h\in E(\widehat{H})}$ for $\mathcal{I}^{\text{split}}$, where $\widehat{\vec{\beta}}(e) = \frac{1}{2}$ for $e \in E(P^{\text{in}})$, $\widehat{\vec{\beta}}(e) = \frac{3}{2}$ for $e \in E_{\text{outer}}(\widehat{G}) \setminus E(P^{\text{in}})$ and $\widehat{\vec{\beta}}(e) = 2$ for $e \in E_{\text{inner}}(\widehat{G})$.

Now, we can define the unsplittable flow $y = (y_h)_{h\in E(H)}$ for the original instance (G, u, H, d): For $h \in E(H) \setminus E(H')$ we also have $h \in E(\widehat{H})$, and hence $y_h := \widehat{y}_h$ defines an unsplittable h-flow. For $h \in E(H')$ let P'_h denote the path on which y'_h routes h. As in Definition 16, let $\mathcal{Q}_h$ define the set of maximal subpaths of P'_h that are in $G'[S] = P^{\text{out}}$. For each $Q \in \mathcal{Q}_h$ we have added a demand h_Q to $E(\widehat{H})$; we construct the path P_h from P'_h by replacing each $Q \in \mathcal{Q}_h$ by the path that h_Q is routed on in the unsplittable flow $\widehat{y}$, and deleting possible cycles in the constructed walk. We define y_h to be the unsplittable h-flow that routes h along P_h (cf. Figure 6).

It is left to check the capacity constraints for y. For $e \in E(G) \setminus E(G[S])$ we have

$$\widehat{y}(e) \leq y'(e) \leq u'(e) + \vec{\beta}'(e) \cdot d_{\max} = u(e) + \vec{\beta}(e) \cdot d_{\max}.$$

Now consider an edge $e \in E(G[S])$. We have $y(e) \leq \widehat{y}(e) \leq \widehat{u}(e) + \widehat{\vec{\beta}}(e) \cdot d_{\max}$. If $e \notin E(P^{\text{in}})$ then $\widehat{u}(e) = u(e)$ and $\widehat{\vec{\beta}}(e) = \vec{\beta}(e)$, so $y(e) \leq u(e) + \vec{\beta}(e) \cdot d_{\max}$. Finally, consider the case $e \in E(P^{\text{in}})$. By our choice of $\vec{\beta}'$, Definition 16 implies $\widehat{u}(e) \leq u(e) + \frac{3}{2} \cdot d_{\max}$ in this case. If $e \in E(P^{\text{in}})$ is an outer edge of G then e must be a bridge in $\widehat{G}$; in particular we even have $\widehat{y}(e) \leq \widehat{u}(e) \leq u(e) + \frac{3}{2} \cdot d_{\max} = u(e) + \vec{\beta}(e) \cdot d_{\max}$. Otherwise, we have $\widehat{\vec{\beta}}(e) = \frac{1}{2} = \vec{\beta}(e) - \frac{3}{2}$, which concludes the case $e \in E(P^{\text{in}})$. This finishes the proof for case 1.

Case 2: $\delta_G(S)$ contains all inner edges of G.[7]

Let F be the (unique) inner face of G that is incident to e^*. If F is the only inner face of G then G is a cycle, and Theorem 6 yields a $\vec{\beta}$-feasible unsplittable flow. Otherwise, since $\delta_G(S)$ contains e^* and all inner edges of G, F is incident to exactly one inner edge, say $\{v, w\} \in E_{\text{inner}}(G)$. Let $v_1, v_2 \in V(G)$ such that $\{v_1, v\}$ and $\{v_2, v\}$ are the two outer edges of G incident to v, where $\{v_1\}$ lies on the boundary of F. Let $w_1, w_2 \in V(G)$ be defined analogously for w.

We can assume w.l.o.g. that $|\delta_G(v)| = 3$ and that no demand edges of H are incident to v; otherwise, replace v by two vertices v and v' that are connected by an edge $\{v, v'\}$ and connect all edges in $\delta_{G+H}(v)$ except $\{v_1, v\}$ and $\{v, w\}$ to v' instead of v. We can choose the capacity for the new edge $\{v, v'\}$ large enough such that our instance is feasible. Note that any $\vec{\beta}$-feasible unsplittable flow in the constructed instance directly induces a $\vec{\beta}$-feasible unsplittable flow in the original instance. Similarly, we can assume that $|\delta_G(w)| = 3$ and that no demand edges of H are incident to w. Now, we again distinguish two cases:

Case 2a: There exists a demand edge h between two vertices on the boundary of F.

Observe that $x_h(\{v_1, v\}) = x_h(\{w_1, w\})$. Note that if $x_h(\{v_1, v\}) = 0$ then x routes all the demand of h along a single path (on the boundary of F). Thus, removing h from H as well as removing x_h from x and u yields a counterexample with fewer demand edges, contradicting minimality of our counterexample. So w.l.o.g. assume $x_h(\{v_1, v\}) > 0$.

[7] In this case, deleting the vertex corresponding to the outer face from the planar dual results in a path.

By Lemma 10 one connected component $S \subseteq V(G)$ of $G - \{\{v_1, v\}, \{w_1, w\}\}$ is x-nice, and due to the flow x_h we know that $v, w \in S$. In particular, the supply graph of the cut instance for x and S has less inner edges than G, so we can proceed exactly as in Case 1 to finish the proof in this case.

Case 2b: Any demand edge contains at most one vertex on the boundary of F.
In this case we consider the set S of all vertices on the boundary of F. S is x-nice because $H[S] = \emptyset$. Let $\mathcal{I}^{\text{cut}} = (G', u', H', d')$ be the cut instance for x and S. We have $E(G') = E(G) \setminus \{\{v, w\}\}$, so by minimality of our counterexample there exists a $\vec{\beta}'$-feasible unsplittable flow $y' = (y'_h)_{h \in E(H')}$ for $\mathcal{I}^{\text{cut}}$, where $\vec{\beta}'(e) = \frac{3}{2}$ for $e \in E_{\text{outer}}(G')$ and $\vec{\beta}'(e) = 2$ for $e \in E_{\text{inner}}(G')$. Now let $\mathcal{I}^{\text{split}} = (\widehat{G}, \widehat{u}, \widehat{H}, \widehat{d})$ be the split instance for $\mathcal{I}$, $\mathcal{I}^{\text{cut}}$, y' and $\vec{\beta}'$. Clearly, $\widehat{G}$ is a cycle corresponding to the boundary edges of F. Furthermore, the assumption of case 2b implies that each edge of $\widehat{H}$ is incident to v or w. Therefore, we can apply Lemma 7 to find a $\widehat{\vec{\beta}}$-feasible unsplittable flow $\widehat{y}$ for $\mathcal{I}^{\text{split}}$, where $\widehat{\vec{\beta}}(e^*) = \widehat{\vec{\beta}}(\{v, w\}) = \frac{1}{2}$ and $\widehat{\vec{\beta}}(e) = \frac{3}{2}$ for all other $e \in E(\widehat{G})$.
Define the unsplittable flow y as in case 1. Analogously to case 1, it is straightforward to verify that the flow y fulfills the requirements of the lemma. ◀

Since $\vec{\beta}(e) \leq 2$ for all edges $e \in E(G)$ in Theorem 18, this directly implies the existence part of Theorem 1. Note that although the proof of Theorem 18 is existential, it still shows a clear way to obtain an unsplittable flow y as desired: In each iteration, after partitioning the instance into the blocks of G and removing demands which are already routed unsplittably, we construct two smaller instances, the cut instance $\mathcal{I}^{\text{cut}}$ and the split instance $\mathcal{I}^{\text{split}}$, and solve them recursively. Afterwards, we combine the obtained unsplittable flows to the flow y. Note that the combined number of inner faces of the supply graphs in $\mathcal{I}^{\text{cut}}$ and $\mathcal{I}^{\text{split}}$ equals the number of inner faces of G, so the total number of recursion steps is bounded linearly. Thus, we can also compute an unsplittable flow as guaranteed by Theorem 18 in polynomial time.

4 Improved Lower Bound for Outerplanar Instances

We prove Theorem 2 in this section, which we restate for convenience.

▶ **Theorem 2.** *For any $\varepsilon > 0$ there exists a feasible instance (G, u, H, d) on an outerplanar supply graph G, such that for any unsplittable flow y, there exists a supply edge $e \in E(G)$ with*

$$y(e) > u(e) + \left(\tfrac{4}{3} - \varepsilon\right) d_{\max}.$$

This lower bound improves over the previous best lower bound of $\frac{11}{10}\, d_{\max}$ in [26] for outerplanar graphs, which is achieved in a ring-loading instance.

We show that for any $n \in \mathbb{Z}_{>0}$, there is a feasible multicommodity flow instance supported on an outerplanar graph with $1 + 4n$ inner faces, such that any unsplittable flow must necessarily exceed the capacity of some edge by at least $\left(1 + \frac{n}{1+3n}\right) d_{\max}$. Since $1 + \frac{n}{1+3n} \to \frac{4}{3}$ as $n \to \infty$, Theorem 2 will follow.[8]

We start by considering a family of ring-loading instances on a common (supply graph, demand graph) pair (C, H). For any $n \in \mathbb{Z}_{\geq 0}$ and any $\ell \in [n]$ we define the following ring-loading instance $\mathcal{I}(n, \ell) := (C, u, H, d)$, which we illustrate in Figure 7.

[8] One can take $n = \lceil \frac{1}{9\varepsilon} \rceil$, where $\varepsilon > 0$ is as in the statement of Theorem 2.

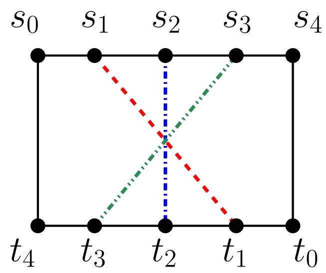

Figure 7 The ring-loading instance $\mathcal{I}(n,\ell)$.

Supply graph: $C = s_0, s_1, s_2, s_3, s_4, t_0, t_1, t_2, t_3, t_4, s_0$ is a 10-cycle,

Demand graph: $E(H) = \{\{s_1,t_1\}, \{s_2,t_2\}, \{s_3,t_3\}\}$,

Capacities: $u(s_1,s_2) = u(s_2,s_3) = 1 + \frac{n}{1+3n}$,
$u(s_3,s_4) = u(s_4,t_0) = u(t_0,t_1) = 1 - \frac{\ell}{1+3n}$,
$u(t_1,t_2) = u(t_2,t_3) = 1 + \frac{n}{1+3n}$,
$u(t_3,t_4) = u(t_4,s_0) = u(s_0,s_1) = 1 + \frac{\ell+2n}{1+3n}$,

Demands: $d(s_1,t_1) = 1, \quad d(s_2,t_2) = \frac{2n}{1+3n}, \quad d(s_3,t_3) = 1$.

▶ **Lemma 19.** *For any $n \in \mathbb{Z}_{\geq 0}$ and $\ell \in [n]$, $\mathcal{I}(n,\ell)$ is a feasible instance.*

Proof. For each $i \in \{1,2,3\}$, let r_i denote the amount of s_it_i-flow routed (clockwise) along the path $s_i, s_{i+1}, \ldots, s_4, t_0, \ldots, t_i$, and let $\overline{r_i}$ denote the amount of s_it_i-flow routed (counter-clockwise) along the path $s_i, s_{i-1}, \ldots, s_0, t_4, \ldots, t_i$. Take

$$r_1 = r_3 = \frac{\frac{1}{2}+n}{1+3n} - \frac{\ell}{2+6n}, \quad \overline{r_1} = \overline{r_3} = \frac{\frac{1}{2}+2n}{1+3n} + \frac{\ell}{2+6n}, \quad r_2 = \overline{r_2} = \frac{n}{1+3n}.$$

This flow routes the required demand, since $r_1 + \overline{r}_1 = r_3 + \overline{r}_3 = 1$ and $r_2 + \overline{r}_2 = \frac{2n}{1+3n}$. Observe that the flow on each edge of the paths s_1, s_2, s_3 and t_1, t_2, t_3 is equal to $1 + \frac{n}{1+3n}$. The flow on each edge of the path s_3, s_4, t_0, t_1 is equal to $1 - \frac{\ell}{1+3n}$. Finally, the flow on each edge of the path t_3, t_4, s_0, s_1 is equal to $1 + \frac{\ell+2n}{1+3n}$. Thus, on every edge the total flow equals its capacity, and the instance is feasible. ◀

Next, we construct an (intermediate) instance as follows. Consider the n vertex-disjoint instances $\{\mathcal{I}(n,\ell) = (C^{(\ell)}, u^{(\ell)}, H^{(\ell)}, d^{(\ell)})\}_{\ell\in[n]}$, where we assume that $\mathcal{I}(n,\ell)$ is supported on the cycle $C^{(\ell)} = s_0^\ell, \ldots, s_4^\ell, t_0^\ell, \ldots, t_4^\ell, s_0^\ell$ (and where we interpret $s_i^\ell \equiv s_i$ and $t_i^\ell \equiv t_i$). We concatenate these instances into a larger instance whose inner faces form a one-dimensional grid. Informally, for each $\ell \in [n-1]$, we identify $s_0^{\ell+1}$ with s_4^ℓ, identify $t_4^{\ell+1}$ with t_0^ℓ, and add the capacity of $\{t_4^{\ell+1}, s_0^{\ell+1}\}$ to that of $\{s_4^\ell, t_0^\ell\}$. The demands and the capacities of all other edges (i.e., the outer edges of the resulting instance) remain unchanged. This results in the following instance $\mathcal{J} = (G,u,H,d)$. See Figure 8 for an illustration.

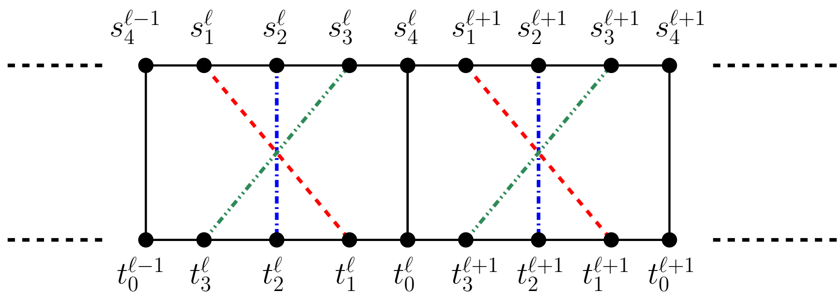

Figure 8 The ℓ-th and $\ell+1$-st inner face in the outerplanar instance $\mathcal{J}$.

Vertex set: $V = \bigcup_{\ell=1}^{n} V(C^{(\ell)}) \setminus \bigcup_{\ell=2}^{n} \{s_0^{\ell}, t_4^{\ell}\}$.

Supply edges: $E(G) = \bigcup_{\ell=1}^{n} E(C^{(\ell)}) \setminus \{\{t_4^{\ell}, s_0^{\ell}\} \,:\, 2 \le \ell \le n\,\}$.

Demand edges: $E(H) = \bigcup_{\ell=1}^{n} \{\{s_1^{\ell}, t_1^{\ell}\},\ \{s_2^{\ell}, t_2^{\ell}\},\ \{s_3^{\ell}, t_3^{\ell}\}\}$.

Capacities:
$$u(s_1^{\ell}, s_2^{\ell}) = u(s_2^{\ell}, s_3^{\ell}) = u(t_1^{\ell}, t_2^{\ell}) = u(t_2^{\ell}, t_3^{\ell}) = 1 + \tfrac{n}{1+3n} \quad \forall \ell \in [n]\ ,$$
$$u(s_3^{\ell}, s_4^{\ell}) = u(t_0^{\ell}, t_1^{\ell}) = 1 - \tfrac{\ell}{1+3n} \quad \forall \ell \in [n],$$
$$u(t_3^{\ell}, t_0^{\ell-1}) = u(s_4^{\ell-1}, s_1^{\ell}) = 1 + \tfrac{\ell+2n}{1+3n} \quad \forall 2 \le \ell \le n,$$
$$u(s_4^{\ell}, t_0^{\ell}) = u^{(\ell)}(s_4^{\ell}, t_0^{\ell}) + u^{(\ell+1)}(t_4^{\ell+1}, s_0^{\ell+1}) = 3 - \tfrac{n}{1+3n} \quad \forall \ell \in [n-1],$$
$$u(t_3^{1}, t_4^{1}) = u(t_4^{1}, s_0^{1}) = u(s_0^{1}, s_1^{1}) = 2 - \tfrac{n}{1+3n}, \quad u(s_4^{n}, t_0^{n}) = 1 - \tfrac{n}{1+3n}.$$

Demands: $d(s_1^{\ell}, t_1^{\ell}) = 1, \quad d(s_2^{\ell}, t_2^{\ell}) = \tfrac{2n}{1+3n}, \quad d(s_3^{\ell}, t_3^{\ell}) = 1 \quad \forall \ell \in [n]$.

Note that $d_{\max} = 1$ and that removing the outer face from the planar dual of G results in a path whose ℓ^{th} node corresponds to the inner face of G bounded by the cycle $C^{(\ell)}$.

▶ **Lemma 20.** *$\mathcal{J}$ is a feasible instance.*

Proof. For each $\ell \in [n]$ we can route the demands $\{s_1^{\ell}, t_1^{\ell}\}$, $\{s_2^{\ell}, t_2^{\ell}\}$, $\{s_3^{\ell}, t_3^{\ell}\}$ as in Lemma 19, around the boundary of the inner face that contains all s_i^{ℓ} and t_i^{ℓ} for $i = 1, 2, 3$. ◀

For simplicity, we introduce the following definition.

▶ **Definition 21.** *Consider an unsplittable flow y of $\mathcal{J}$, and let $\ell \in [n]$ and $i \in \{1, 2, 3\}$. We say that y routes $\{s_i^{\ell}, t_i^{\ell}\}$* to the right *if y routes $\{s_i^{\ell}, t_i^{\ell}\}$ along an $s_i^{\ell} t_i^{\ell}$-path containing the edge $\{s_i^{\ell}, s_{i+1}^{\ell}\}$; otherwise, we say that y routes $\{s_i^{\ell}, t_i^{\ell}\}$* to the left*, i.e., if y routes $\{s_i^{\ell}, t_i^{\ell}\}$ along an $s_i^{\ell} t_i^{\ell}$-path containing the edge $\{s_i^{\ell}, s_{i-1}^{\ell}\}$.*[9]

Later, we construct the instance satisfying the conditions of Theorem 2 by taking four vertex-disjoint copies of $\mathcal{J}$ and identifying the edge $\{t_4^1, s_0^1\}$ of each copy with an edge of a central cycle. The next lemma will allow us to assume later that, within every copy, all demands $\{s_1^{\ell}, t_1^{\ell}\}$ and $\{s_3^{\ell}, t_3^{\ell}\}$ must be routed to the left.

[9] We interpret $s_0^{\ell} \equiv s_4^{\ell-1}$ for any $2 \le \ell \le n$

▶ **Lemma 22.** *Let y be an unsplittable flow of $\mathcal{J} = (G, u, H, d)$. If, for some $\ell \in [n]$ and some $i \in \{1, 3\}$, y routes $\{s_i^\ell, t_i^\ell\}$ to the right, then there is an edge $e \in E(G) \setminus \{\{t_4^1, s_0^1\}\}$ with*

$$y(e) \geq u(e) + 1 + \tfrac{n}{1+3n}.$$

Proof. We start by observing that if the demands $\{s_1^\ell, t_1^\ell\}$ and $\{s_3^\ell, t_3^\ell\}$ are routed in opposite directions, then y exceeds the capacity of some edge of the ℓ-th face of G by at least $1 + \frac{n}{1+3n}$.

▷ Claim 23. If for some $\ell \in [n]$ and distinct $i, j \in \{1, 3\}$, y routes $\{s_i^\ell, t_i^\ell\}$ to the right and it routes $\{s_j^\ell, t_j^\ell\}$ to the left, then there is a supply edge $e \in E(G)$ with $y(e) \geq u(e) + 1 + \frac{n}{1+3n}$.

Proof. Suppose first that $i = 1$ and $j = 3$. Observe that in this case both $\{s_1^\ell, t_1^\ell\}$ and $\{s_3^\ell, t_3^\ell\}$ are routed in y by using the edges of the path $s_1^\ell, s_2^\ell, s_3^\ell$. Thus, if $\{s_2^\ell, t_2^\ell\}$ is routed to the left, then

$$y(s_1^\ell, s_2^\ell) \geq d(s_1^\ell, t_1^\ell) + d(s_2^\ell, t_2^\ell) + d(s_3^\ell, t_3^\ell) = 2 + \tfrac{2n}{1+3n} = 2\left(1 + \tfrac{n}{1+3n}\right) = u(s_1^\ell, s_2^\ell) + 1 + \tfrac{n}{1+3n}.$$

On the other hand, if $\{s_2^\ell, t_2^\ell\}$ is routed to the right, then

$$y(s_2^\ell, s_3^\ell) \geq d(s_1^\ell, t_1^\ell) + d(s_2^\ell, t_2^\ell) + d(s_3^\ell, t_3^\ell) = 2 + \tfrac{2n}{1+3n} = 2\left(1 + \tfrac{n}{1+3n}\right) = u(s_2^\ell, s_3^\ell) + 1 + \tfrac{n}{1+3n}.$$

The case in which $i = 3$ and $j = 1$ is analogous. In that case, the capacity of either $\{t_1^\ell, t_2^\ell\}$ or $\{t_2^\ell, t_3^\ell\}$ is exceeded by at least $1 + \frac{n}{1+3n}$ units of flow. ◁

By the previous claim, we can assume that for all $\ell \in [n]$ either both $\{s_1^\ell, t_1^\ell\}$ and $\{s_3^\ell, t_3^\ell\}$ are routed to the right, or that both are routed to the left. Take the largest ℓ such that both $\{s_1^\ell, t_1^\ell\}$ and $\{s_3^\ell, t_3^\ell\}$ are routed to the right. If $\ell = n$, then y routes at least $2 = d(s_1^n, t_1^n) + d(s_3^n, t_3^n)$ units of flow through the edge $\{s_4^n, t_0^n\}$. Thus,

$$y(s_4^n, t_0^n) \geq 2 = u(s_4^n, t_0^n) + 1 + \tfrac{n}{1+3n}.$$

Therefore, we can assume that $\ell < n$. For simplicity, let P_1^ℓ and P_3^ℓ denote the paths on which y routes $\{s_1^\ell, t_1^\ell\}$ and $\{s_3^\ell, t_3^\ell\}$, respectively. Similarly, let $P_1^{\ell+1}$ and $P_3^{\ell+1}$ denote the paths on which y routes $\{s_1^{\ell+1}, t_1^{\ell+1}\}$ and $\{s_3^{\ell+1}, t_3^{\ell+1}\}$, respectively. By our choice of ℓ (and the assumption that for each of these n indices, the two corresponding demands are routed in the same direction), we have that

$$\{s_3^\ell, s_4^\ell\} \in P_1^\ell \cap P_3^\ell \quad \text{and} \quad \{s_4^\ell, s_1^{\ell+1}\} \in P_1^{\ell+1} \cap P_3^{\ell+1}.$$

Observe that if these four (unit) demands are routed on the inner edge $\{s_4^\ell, t_0^\ell\}$ i.e., $\{s_4^\ell, t_0^\ell\} \in P_1^\ell \cap P_3^\ell \cap P_1^{\ell+1} \cap P_3^{\ell+1}$, then

$$y(s_4^\ell, t_0^\ell) - u(s_4^\ell, t_0^\ell) \geq 4 - \left(3 - \tfrac{n}{1+3n}\right) = 1 + \tfrac{n}{1+3n}.$$

Thus, we assume that either $\{s_3^\ell, s_4^\ell\} \in P_1^{\ell+1} \cup P_3^{\ell+1}$ or $\{s_4^\ell, s_1^{\ell+1}\} \in P_1^\ell \cup P_3^\ell$. If $\{s_3^\ell, s_4^\ell\} \in P_1^{\ell+1} \cup P_3^{\ell+1}$, then y would route (at least) 3 units of flow on $\{s_3^\ell, s_4^\ell\}$, implying that

$$y(s_3^\ell, s_4^\ell) - u(s_3^\ell, s_4^\ell) \geq 3 - \left(1 - \tfrac{\ell}{1+3n}\right) > 2.$$

On the other hand, if $\{s_4^\ell, s_1^{\ell+1}\} \in P_1^\ell \cup P_3^\ell$ then $\{s_1^{\ell+1}, s_2^{\ell+1}\}, \{s_2^{\ell+1}, s_3^{\ell+1}\} \in P_1^\ell \cup P_3^\ell$. Since $\{s_1^{\ell+1}, s_2^{\ell+1}\}, \{s_2^{\ell+1}, s_3^{\ell+1}\} \in P_3^{\ell+1}$ as well, this implies that y routes at least 2 units of flow on these two supply edges without counting the demand $\{s_2^{\ell+1}, t_2^{\ell+1}\}$. Thus, the same argument of Claim 23 implies that either $y(s_1^{\ell+1}, s_2^{\ell+1}) - u(s_1^{\ell+1}, s_2^{\ell+1}) \geq 1 + \frac{n}{1+3n}$ (if $\{s_2^{\ell+1}, t_2^{\ell+1}\}$ is routed to the left), or $y(s_2^{\ell+1}, s_3^{\ell+1}) - u(s_2^{\ell+1}, s_3^{\ell+1}) \geq 1 + \frac{n}{1+3n}$ (if $\{s_2^{\ell+1}, t_2^{\ell+1}\}$ is routed to the right). ◀

Now we describe the instance which will imply the proof of Theorem 2. The instance is illustrated in Figure 9. We create four disjoint copies of $\mathcal{J}$ and identify the edge $\{t_4^1, s_0^1\}$ of each of these copies with an edge of a central 12-cycle W containing four new vertices w_1, w_2, w_3, w_4 and two "crossing" unit demands $\{w_1, w_3\}$, $\{w_2, w_4\}$. We increase the capacity of the four copies of $\{t_4^1, s_0^1\}$ by one unit, and then assign one unit of capacity to the remaining edges of W. By Lemma 22, we can assume that the demands $\{s_1^1, t_1^1\}$ and $\{s_3^1, t_3^1\}$ of each of these copies are routed to the left; otherwise, the capacity of some edge within the corresponding copy is exceeded by at least $1 + \frac{n}{1+3n}$. We can then argue that $\{w_1, w_3\}$, $\{w_2, w_4\}$, together with the demands $\{s_1^1, t_1^1\}$, $\{s_3^1, t_3^1\}$ of one of the four copies of $\mathcal{J}$, must exceed the capacity of some edge by at least $1 + \frac{n}{1+3n}$. We provide a formal proof for completeness.

Consider four vertex-disjoint copies $\{\, \mathcal{J}^{(\theta)} = (G^\theta,\, u^\theta,\, H^\theta,\, d^\theta)\ \}_{\theta \in \{a,b,c,r\}}$ of $\mathcal{J}$. To simplify notation, we use t^θ and s^θ to denote the copies of nodes t_4^1 and s_0^1 in $\mathcal{J}^{(\theta)}$, respectively. Consider the 12-cycle

$$W = w_1, t^a, s^a, w_2, t^b, s^b, w_3, t^c, s^c, w_4, t^r, s^r, w_1.$$

Consider the following instance $\mathcal{I} = (G, u, H, d)$.

$$\textbf{Vertex set:} \quad V = \{w_1, w_2, w_3, w_4\} \cup \bigcup_{\theta \in \{a,b,c,r\}} V(G^\theta).$$

$$\textbf{Supply edges:} \quad E(G) = E(W) \cup \bigcup_{\theta \in \{a,b,c,r\}} \big(E(G^\theta) \setminus \{\{t^\theta, s^\theta\}\}\big).$$

$$\textbf{Demand edges:} \quad E(H) = \{\{w_1, w_3\},\ \{w_2, w_4\}\} \cup \bigcup_{\theta \in \{a,b,c,r\}} E(H^\theta).$$

$$\begin{aligned}
\textbf{Capacities:} \quad & u(e) = u^\theta(e), \qquad \forall \theta \in \{a,b,c,r\},\ \forall e \in E(G^\theta) \setminus \{\{t^\theta, s^\theta\}\}.\\
& u(t^\theta, s^\theta) = 1 + u^\theta(t^\theta, s^\theta) = 3 - \tfrac{n}{1+3n}, \qquad \forall \theta \in \{a,b,c,r\}.\\
& u(e) = 1, \qquad \forall e \in E(W) \setminus \{\{t^\theta, s^\theta\}\ :\ \theta = a,b,c,r\}.\\
\textbf{Demands:} \quad & d(h) = d^\theta(h), \qquad \forall \theta \in \{a,b,c,r\}, \quad \forall h \in E(H^\theta).\\
& d(w_1, w_3) = 1, \quad d(w_2, w_4) = 1.
\end{aligned}$$

▶ **Lemma 24.** *$\mathcal{I} = (G, u, H, d)$ is a feasible instance.*

Proof. First, route $\{w_1, w_3\}$ (resp. $\{w_2, w_4\}$) by sending $\frac{1}{2}$ units of flow along each of the two w_1w_3-paths (resp. w_2w_4-paths) in W. The residual capacity, with respect to u, on each edge $e \in E(G^\theta) \subseteq E(G)$ is precisely $u^\theta(e)$, for each $\theta \in \{a,b,c,r\}$. Since each instance $\mathcal{J}^\theta$ is feasible, the remaining demands can be routed within the corresponding copies. It follows that $\mathcal{I}$ is feasible. ◀

We can now prove Theorem 2.

Proof of Theorem 2. As mentioned at the start of the section, it suffices to show that for any unsplittable flow y of $\mathcal{I}$, there exists some $e \in E(G)$ with $y(e) \geq u(e) + 1 + \frac{n}{1+3n}$ (note that $d_{\max} = 1$). Let $P_{1,3}$ and $P_{2,4}$ denote the paths on which y routes $\{w_1, w_3\}$ and $\{w_2, w_4\}$, respectively. Then, there exists some $i \in [4]$ and $\theta \in \{a,b,c,r\}$ such that $\{w_i, t^\theta\}, \{s^\theta, w_{i+1}\} \in P_{1,3} \cap P_{2,4}$ (where we interpret $w_5 \equiv w_1$). Suppose without loss of generality (by the symmetry of the instance) that $\{w_1, t^a\}, \{s^a, w_2\} \in P_{1,3} \cap P_{2,4}$. Observe that if y routes any demand of $E(H^a)$ on some $e \in \{\{w_1, t^a\}, \{s^a, w_2\}\}$, then

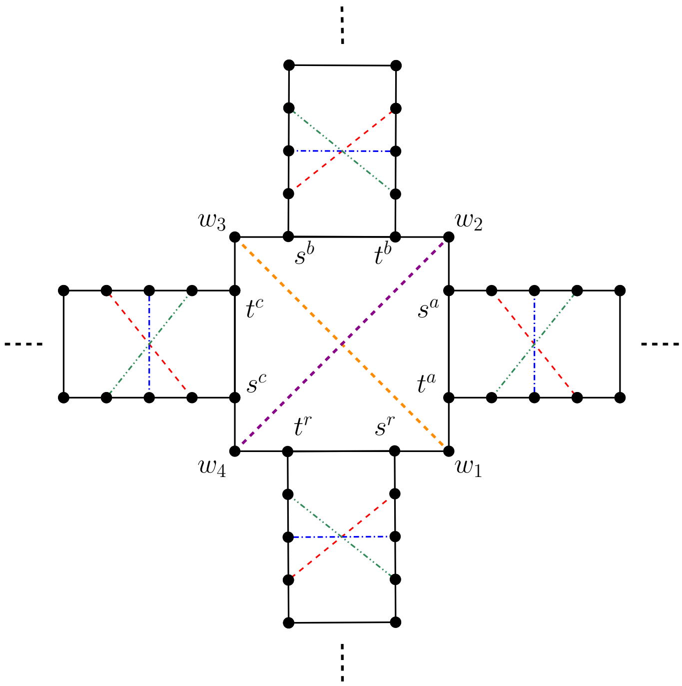

Figure 9 The instance $\mathcal{I}$ that consists of four copies of $\mathcal{J}$, joined at the cycle W.

$$y(e) \geq d(w_1, w_3) + d(w_2, w_4) + \tfrac{2n}{1+3n} = 2 + \tfrac{2n}{1+3n} = u(e) + 1 + \tfrac{2n}{1+3n},$$

where we use the fact that $\frac{2n}{1+3n}$ is the *minimum* demand value in $\mathcal{I}$. Thus, we can assume that y routes every demand in $E(H^a)$ using only the edges of $E(G^a)$. We overload notation and use s_i^1 and t_i^1 for each $i \in \{0, 1, 2, 3, 4\}$ to denote the corresponding copies of these nodes of $\mathcal{J}$ in $V(G^a)$ (note that $s_0^1 \equiv s^a$ and $t_4^1 \equiv t^a$). Since $u(e) = u^a(e)$ for every $e \in E(G^a) \setminus \{\{t^a, s^a\}\}$, Lemma 22 allows us to assume that, in $\mathcal{J}^a$, all demands $\{s_1^\ell, t_1^\ell\}$ and $\{s_3^\ell, t_3^\ell\}$ are routed to the left. In particular, this holds for $\{s_1^1, t_1^1\}$ and $\{s_3^1, t_3^1\}$. Let Q_1 and Q_3 denote the paths on which y routes $\{s_1^1, t_1^1\}$ and $\{s_3^1, t_3^1\}$, respectively. By the above, $\{t^a, s^a\} \equiv \{t_4^1, s_0^1\}$ is contained in $Q_1 \cap Q_3$. Observe that if $\{t^a, s^a\}$ is contained in $P_{1,3} \cap P_{2,4}$, then

$$y(t^a, s^a) \geq d(w_1, w_3) + d(w_2, w_4) + d(s_1^1, t_1^1) + d(s_3^1, t_3^1) = 4 = u(t^a, s^a) + 1 + \tfrac{n}{1+3n}.$$

Thus, we can w.l.o.g. assume that $\{s^a, s_1^1\} \equiv \{s_0^1, s_1^1\}$ is contained in $P_{1,3}$ (since s^a has degree 3). Since y routes $\{s_1^1, t_1^1\}$ and $\{s_3^1, t_3^1\}$ to the left, $\{s^a, s_1^1\} \in Q_1 \cap Q_3$. It follows that

$$y(s^a, s_1^1) \geq d(w_1, w_3) + d(s_1^1, t_1^1) + d(s_3^1, t_3^1) = 3 = u(s^a, s_1^1) + 1 + \tfrac{n}{1+3n}. \qquad \blacktriangleleft$$

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