Sampling Colorings with Fixed Color Class Sizes

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Abstract

In 1970, Hajnal and Szemerédi proved a conjecture of Erdős stating that any graph with maximum degree Δ admits an equitable $(\Delta + 1)$-coloring, that is, a coloring where color class sizes differ by at most 1. In 2007 Kierstead and Kostochka reproved their result and provided a polynomial-time algorithm which produces such a coloring. In this paper we study the problem of approximately sampling uniformly random equitable colorings. A series of works gives polynomial-time sampling algorithms for colorings without the color class constraint, the latest improvement being by Carlson and Vigoda for $q \geq 1.809\Delta$. In this paper we give a polynomial-time sampling algorithm for equitable colorings when $q > 2\Delta$. Moreover, our results extend to colorings with small deviations from equitable (and as a corollary, establishing their existence). The proof uses the framework of the geometry of polynomials for multivariate polynomials, and as a consequence establishes a multivariate local Central Limit Theorem for color class sizes of uniform random colorings.

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1 Introduction

1.1 Sampling with Constraints

Many interesting problems in combinatorics and its applications can be formulated as problems about independent sets, matchings, graph cuts, or colorings with global constraints. Thus, the ability to sample those fundamental objects with constraints solves a broad range of nuanced questions and strengthens current techniques and tools used for sampling. We begin by providing examples of some fundamental objects that we can sample and how introducing constraints alters the sampling problem.

One example comes from the ferromagnetic Ising model, a statistical physics model of graph cuts and magnets. It is possible to approximately count [20] and sample the states of the model [35] for all parameters and all graphs. However, when restricting the magnetization (number of vertices on one side of the cut/ number of particles with a positive spin), an algorithmic thresholds emerges. For the ferromagnetic Ising model with fixed magnetization, Carlson, Davies, Kolla, and Perkins established a computational threshold [4], meaning that there are regimes of temperature and magnetization for which efficient sampling algorithms exist and regimes in which no sampling is possible unless NP=RP (example of dynamical threshold also shown in [23]). Another example of interest comes from *independent sets*, often studied using the hard-core model. When introducing a size constraint on the independent sets, Davies and Perkins [9] showed the existence of a computational threshold; there exists α_c such that it is possible to sample independent sets of size αn for $\alpha < \alpha_c$ in polynomial time, but no sampling algorithm exists for $\alpha > \alpha_c$ unless NP=RP. Work of Jain, Michelen, Pham, and Vuong [17], establishes fast mixing of the down-up walk on independent sets of size $k \leq (1-\delta)\alpha_c$.

Placing constraints on the size of the object is natural for statistical physics models, since they differentiate between settings where the number of particles can vary, called *grand canonical ensembles*, versus settings where the number of particles is fixed, called *canonical ensembles*. In practice, there are a lot more results in sampling algorithms for the grand canonical ensemble, since placing an additional constraint adds a layer of complexity. Under additional constraints the behavior of the system can change completely; for example, a well behaved system (like the ferromagnetic Ising model on a random graph) can exhibit "glassy" behavior with the imposition of a constraint on the magnetization [28].

Observe that the examples of constraints we have mentioned so far are one-dimensional. In this paper we will explore the instance of placing multiple global constraints on the model. More precisely, we will show that given an integer vector (with some assumptions on it), we can sample colorings in which color class sizes form the requested vector. It is a delicate problem since each of the color class sizes is an independent set of restricted size, and moreover, the union of colors should cover all the vertices of the graph. We introduce the background about sampling colorings as well as known existence results in the next subsection.

1.2 Sampling Colorings with Multiple Constraints

Before looking at colorings with constraints, we give background for sampling colorings with no constraints, since this problem is also far from being well understood. The following is a classical conjecture about sampling colorings [19, 38]:

▶ **Conjecture 1.** *For the class of graphs of maximum degree Δ and $q \geq \Delta + 2$, there is a polynomial-time algorithm to approximately sample uniformly random q-colorings.*

Moreover, it is conjectured that a simple Markov chain, the Glauber dynamics, achieves this.

There has been a lot of progress in the past 30 years towards the resolution of the conjecture. In his seminal paper, Jerrum proved it for $q \geq 2\Delta + 1$ [19]. An exciting result by Vigoda improved that to sampling colorings with $q > \frac{11}{6}\Delta$ [38]. There were also a lot of substantial improvements on the bounds for special families of graphs [10, 30, 13, 27, 11, 31, 14, 37, 7]. For general graphs, q was further improved to $(11/6 - \varepsilon)\Delta$ by Chen, Delcourt, Moitra, Perarnau, and Postle [6], and recently to $q \geq 1.809\Delta$ by Carlson and Vigoda [5].

Despite the impressive progress in this direction, the problem presents a considerable challenge and might require many more ideas to decrease the necessary amount of colors down to $\Delta + 2$. Thus, it seems important to ask the following questions: How refined are our current tools and techniques? How well can we control the colorings that we can sample?

We now define the size constraints on color classes we have previously mentioned.

▶ **Definition 2** ($\vec{n}$-coloring). *Given a vector $\vec{n} = (n_1, \ldots, n_q)$, an $\vec{n}$-coloring is a proper coloring of the vertices of a graph such that there are n_i vertices of color i, $\forall i \in [q]$.*

Let us describe perhaps the simplest case of $\vec{n}$-colorings, equitable colorings.

▶ **Definition 3** (Equitable colorings). *An equitable q-coloring is a proper coloring with q colors such that the sizes of every pair of color classes differ by at most 1.*

Note that if q divides n, then all color class sizes must be equal.

Erdős conjectured in 1964 that any graph of maximum degree Δ has an equitable $(\Delta+1)$-coloring. In 1970, Hajnal and Szemerédi gave a proof of the conjecture [12]. Later, Kierstead and Kostochka provided a simpler proof, which also gives a polynomial-time algorithm for finding such a coloring. [22].

As a corollary of our main theorem, we establish a polynomial-time algorithm for approximately sampling uniform equitable colorings when $q \geq 2\Delta$.

▶ **Definition 4.** *An algorithm is said to ε-approximately sample from a target distribution μ if the distribution of its output ν satisfies $\|\nu - \mu\|_{TV} \leq \varepsilon$.*

▶ **Theorem 5** (Sampling Equitable Colorings). *There exists a sampling algorithm that, given $q \geq 2\Delta$ and any G on n vertices from the class of graphs with maximum degree Δ, ε-approximately samples equitable colorings on G with high probability with running time $O\big(n^{(q+1)/2} \log n \log(\frac{1}{\varepsilon})^2\big)$.*

In a more general version of the theorem, we show the existence of a sampling algorithm for colorings that we call "skewed", establishing their existence as a corollary.

▶ **Theorem 6** (Sampling Skewed Colorings). *Fix $q \geq 2\Delta + 1$. There exists a constant $c = c(\Delta) > 0$ small enough, such that given any G on n vertices from the class of graphs with maximum degree Δ and given $\vec{n}$ satisfying $\|\vec{n}\|_1 = n$ and $|n_i - \frac{n}{q}| < cn$ for all $i \in [q]$, there is an algorithm that ε-approximately samples $\vec{n}$-colorings on G with high probability with running time $O\big(n^q \log n \log(\frac{1}{\varepsilon})^2\big)$.*

As a simple corollary we obtain the existence of such colorings.

▶ **Corollary 7.** *Fix $q \geq 2\Delta + 1$. Then there exists constant $c > 0$ small enough such that for n large enough, and any $\vec{n}$ satisfying $\|\vec{n}\|_1 = n$ and $|n_i - \frac{n}{q}| \leq cn$ for all $i \in [q]$, every graph G of maximum degree Δ on n vertices has an $\vec{n}$-coloring.*

To the best of our knowledge, very little is known about the existence of skewed colorings beyond the equitable case (this can also be thought of as a way to generalize Hajnal and Szemerédi's theorem). One can apply a result by Hurley, Joos, and Lang [16] to guarantee that the complement of the graph can be decomposed into cliques of constant size, which is equivalent to finding a proper coloring with constant color class sizes, but the coloring requires linearly many colors.

We conjecture that the following bound on color class sizes (matching the size of an independent set guaranteed to exist in a graph of maximum degree Δ) suffices.

▶ **Conjecture 8.** *For any graph G with maximum degree Δ and $\vec{n}$ satisfying $\|\vec{n}\|_1 = n$ and $n_i \leq \left\lfloor \dfrac{n}{\Delta+1} \right\rfloor$ for all $i \in [q]$, there exists an $\vec{n}$-coloring of G.*

Note that the conditions on n_i are tight for a union of $(\Delta+1)$-cliques, and the floor can not be improved to a ceiling, since a union of $(\Delta+1)$-cliques together with one isolated vertex would serve as a counterexample.

Note as well that no assumption on q is made, though the assumptions on n_i and $\|\vec{n}\|_1$ already imply that $q \geq \Delta+1$. Since $q \geq 2\Delta$ constitutes a natural barrier for many approaches related to sampling colorings, a first step in proving Conjecture 8 might be to prove existence under the stronger condition that all $n_i \leq \left\lfloor \frac{n}{2\Delta} \right\rfloor$.

We also conjecture that it should be possible to sample colorings in that regime, although the proof would require new ideas.

▶ **Conjecture 9.** *There is a polynomial-time approximate sampling algorithm for uniform $\vec{n}$-colorings in graphs of maximum degree Δ, for $\vec{n}$ satisfying $n_i \leq \left\lfloor \frac{n}{2\Delta} \right\rfloor$ for all $i \in [q]$.*

1.3 Short Preliminaries: Potts and Coloring Models, Univariate Zero-Freeness

Let us give definitions of some statistical physics models that we use throughout the paper.

The following model is called anti-ferromagnetic Potts model, it is a generalization of another popular statistical physics model called anti-ferromagnetic Ising model. It differentiates between improper and proper colorings by introducing weights that penalize improper colorings.

▶ **Definition 10** (Anti-ferromagnetic Potts model)**.** *Let w be the penalty parameter. We define the partition function*

$$Z_G(w) := \sum_{\sigma \in [q]^V} w^{|m_G(\sigma)|},$$

where $m(\sigma)$ is the number of monochromatic edges in the coloring σ (not necessarily proper).

This also defines a natural Gibbs measure in which each coloring σ (not necessarily proper) appears with probability

$$\mathbb{P}_{G,w}[X = \sigma] = \frac{w^{m_G(\sigma)}}{Z_G(w)}.$$

Note that at $w = 0$ the partition function counts the number of proper colorings, and the Gibbs measure is simply a uniform distribution over proper colorings. If the number of colors satisfies $q \geq \Delta + 1$, a proper coloring exists, and thus the partition function at $w = 0$ is non-zero. Even though the probability distribution requires w to be non-negative and real, it is possible to work with the partition function for complex w. Liu, Sinclair, and Srivastava proved that in the complex plane around $w = 0$ there exists a zero-free region for $Z_G(w)$ [25].

▶ **Theorem 11** ([25])**.** *There exists a $\tau_\Delta > 0$ such that the following is true. Let D_Δ be a simply connected region in the complex plane obtained as the union of disks of radius τ_Δ centered at all points on the segment $[0, 1]$. For any graph G of maximum degree at most $\Delta \geq 3$ and integer $q \geq 2\Delta$, $Z_{G,q}(w) \neq 0$ when $w \in D_\Delta$.*

Using their zero-freeness result and a theorem by Barvinok [1], they get a deterministic algorithm for approximately counting proper colorings of a graph G with bounded maximum degree and a sampling algorithm for colorings of G. We will explain more on how zero-freeness results can be useful for combinatorial algorithms in the next section.

We will use a variant of the Potts model with external fields (weights) for each of the colors. Since the model is only defined over proper colorings, we will, for simplicity, refer to it as the coloring model (equivalently it can be defined over all colorings with monochormatic edges having penalty $w = 0$).

▶ **Definition 12** (Coloring model with external fields)**.** *The coloring model is the model defined through the partition function*

$$Z_G(\lambda_1, \ldots, \lambda_q) := \sum_{\substack{\sigma \in [q]^V \\ \sigma \text{ proper}}} \prod_{i=1}^{q} \lambda_i^{|\sigma^{-1}(i)|}.$$

The vector $\vec{\lambda} \in \mathbb{C}^q$ is called the fugacity, and represents the weight assigned to each of the colors. Observe that for $\vec{\lambda} = \vec{1}$, the partition function simply counts the number of proper colorings. For positive real $\vec{\lambda}$, the model induces a Gibbs measure on the space of proper colorings, given by

$$\mathbb{P}_{G,\vec{\lambda}}(\sigma) := \frac{\prod_{i=1}^{q} \lambda_i^{|\sigma^{-1}(i)|}}{Z_G(\lambda_1, \ldots, \lambda_q)}.$$

We will often drop the subscripts of $\mathbb{P}_{G,\vec{\lambda}}$ where there is no confusion.

As in the previous model, the probability distribution requires $\vec{\lambda}$ to be a non-negative real vector. However, we can still analyze the partition function for complex $\vec{\lambda}$, which is explained in the subsection below.

1.4 Multidimensional Zero-Freeness

An extremely powerful property of the model in a certain parameter regime is the absence of complex zeros (*zero-freeness*) of the partition function. It can imply properties of the model, like deterministic counting [40], Spectral Independence [8], Central Limit Theorems [29], Strong Spatial Mixing [36], and thus has applications in approximation and sampling algorithms.

There is a lot of existing work that establishes zero-freeness for different models and with different motivations. For example, Lee–Yang theorem shows that, for the ferromagnetic Ising model, the zeros of the partition function lie on the complex unit circle for any graph [39]. One of the theorem's implications is the absence of phase transition for non-zero external field. See also more work on the hard-core model [33], the Ising model [34], [32], and the monomer-dimer model [15]. It is important to note that in most cases one has to assume that a graph has bounded maximum degree Δ, since the zero-freeness radius often depends on Δ.

Our motivation for proving zero-freeness is getting very precise control over the cumulants (expectation, covariances and more) of the color class sizes. Derivatives of $\log Z$ are tightly connected with these cumulants, and thus showing zero-freeness of Z lets us control the cumulants via convergent power series.

For example, zero-freeness implies linear upper bounds on the cumulants of any order. As one of the consequences, we can compute the asymptotics of the determinant of the covariance matrix, which will be important for the runtime of our algorithm. Moreover, zero-freeness allows us to Taylor expand the characteristic function of the random vector of color class sizes and establish a Local Central Limit Theorem (LCLT). We use this for the rejection sampling part of our algorithm, see an overview in Subsection 1.5 and details in Section 3 of the extended version.

Let us now give an overview of the zero-freeness result for the coloring model (see Definition 12). The approach we are taking is similar to the blueprint developed by Liu, Sinclair, and Srivastava for the Potts model with penalty [25] (see Definition 10), however, we need to generalize their result to a multivariate partition function and handle q parameters (one for each color) instead of one. This is done so that each color receives its own variable, which lets us regulate each color class size. Generalizing existing tools to work in multivariate settings gives control over different parameters, which allows for better understanding and efficient algorithms for models with global constraints.

▶ **Definition 13** (Assumptions). *Let $\nu = 0.9$, $\varepsilon_I := 10^{-4}\Delta^{-4}$. From now on, we will assume that $\vec{\lambda} \in \mathbb{C}^q$ is chosen so that for any $i, j \in [q]$,*
1. $\lambda_i \notin (-\infty, 0]$*;*
2. $|\arg \lambda_i| \leq \nu\varepsilon_I/2$*;*
3. $|\operatorname{Re} \ln \frac{\lambda_i}{\lambda_j}| \leq \Delta\varepsilon_I$.

▶ **Theorem 14** (Zero-freeness around $\vec{1}$). *For $q \geq 2\Delta$, $\Delta \geq 1$ and graph G with maximum degree Δ, and $\vec{\lambda} \in \mathbb{C}^q$ such that the assumptions from Definition 13 hold, then $Z_G(\vec{\lambda}) \neq 0$.*

The following corollary of this result lets us rephrase it in terms of a zero-freeness ball with constant l_∞-radius:

▶ **Corollary 15.** *Let $R \in \mathbb{R}$ such that $0 < R \leq \nu\varepsilon_I/4 \approx 2.2 \times 10^{-5}\Delta^{-4}$. If $\vec{\lambda} \in \mathbb{C}^q$ is chosen so that $|\lambda_i - 1| \leq R$ for every $i \in [q]$, then the assumptions from Definition 13 are satisfied. Hence, for $q \geq 2\Delta$, $\Delta \geq 1$, and graph G with maximum degree Δ, $Z_G(\vec{\lambda}) \neq 0$ in a polydisc of radius R around $\vec{1}$.*

The proof can be found in Section 2. The main idea behind the proof is an inductive argument in terms of the number of unpinned (i.e. not yet colored) vertices. For each iteration of the induction, we show that the ratio of the probability of a vertex to be colored i vs be colored j is very close to 1.

We note that in the paper of Liu, Sinclair, and Srivastava [25] the theorem holds generally for colorings they call "admissible" and requires fewer colors for triangle-free graphs. It is also mentioned that their result should generalize to list colorings. We expect these generalizations to hold for our model as well. Note that we write theorems in a general enough way that an improvement to the zero-freeness region would lead to an immediate improvement on the sampling and existence result. Bencs, Berrekkal, and Regts [2] recently relaxed the condition on the number of colors for the Potts model (with penalty parameter for monochromatic edges) from $q \geq 2\Delta$ to $q \geq (2 - \eta)\Delta$, where $\eta \geq 0.002$. By adapting their ideas, we suspect a similar improvement is possible for our main results.

1.5 A Local Central Limit Theorem

As one of our main theorems, we prove a Local Central Limit Theorem (LCLT), which states that the distribution of color class sizes in the coloring model approaches a multivariate Gaussian distribution pointwise. We also obtain asymptotic control over the eigenvalues and determinant of the covariance matrix, and thus can lowerbound the probability of sampling an $\vec{n}$-coloring. This is the main ingredient that is needed for the rejection sampling procedure.

▶ **Theorem 16** (Local Central Limit Theorem). *Let $q \geq 2\Delta$, $\Delta \geq 1$, and $\vec{\lambda} \in \mathbb{R}^q$ such that $\|\vec{\lambda} - \vec{1}\|_\infty \leq R/2$, where R is a zero-freeness radius. Let $\vec{X} = (X_1, \ldots, X_{q-1})$ be the random vector of color class sizes with the last color dropped. We will denote its expected value and covariance matrix as $\mu := \mathbb{E}[\vec{X}]$ and $\Sigma := \operatorname{Cov}(\vec{X})$. Then, for $\vec{n} \in \mathbb{Z}_{\geq 0}^{q-1}$ such that $\sum n_i \leq n$,*

$$\mathbb{P}_{\vec{\lambda}}(\vec{X} = \vec{n}) = (1 + o(1)) \frac{1}{(2\pi)^{(q-1)/2}\sqrt{\det \Sigma}} \exp\left(-\tfrac{1}{2}(\vec{n} - \vec{\mu})^\top \Sigma^{-1} (\vec{n} - \vec{\mu})\right) + o\Big(n^{-(q-1)/2}\Big).$$

▶ **Lemma 17** (Covariance Determinant Asymptotics). *Let $q \geq \max\{2\Delta, 3\}$, $\Delta \geq 1$, and $\vec{\lambda} \in \mathbb{C}^q$ such that $|\vec{\lambda} - \vec{1}| \leq R/2$, where R is a zero-freeness radius. Then,*

$$\det(\Sigma) = \Theta(n^{q-1}).$$

Note that if we choose $\vec{\lambda} = \vec{1}$ (corresponding to uniform distribution), by symmetry $\mathbb{E}[X_i] = n/q$. We show that if $\|\vec{n} - \vec{\mu}\| < C\sqrt{n}$, $\exp\left(-\frac{1}{2}(\vec{n} - \vec{\mu})^\top \Sigma^{-1}(\vec{n} - \vec{\mu})\right) = \Omega(1)$, and so

$$\pi(\vec{X} = \vec{n}) = \Theta\left(n^{-(q-1)/2}\right).$$

This is an important component of the proof, since it provides a guarantee for the number of times a rejection sampling step has to be repeated to reach the desired coloring. We explain how that is done in the next subsection.

The proof of the LCLT uses two ingredients. The first one is the Taylor expansion of the logarithm of the characteristic function ($\varphi := \mathbb{E}[e^{i\langle t, X\rangle}]$) , which can be done in the region where the characteristic function is non-zero (this region is contained in the original zero-freeness region of $Z(\vec{\lambda})$). We show that for small t, φ is dominated by the terms corresponding to expectation and covariances (see Subsection 3.3 in the extended version). For larger t we utilize a lemma which shows an upper bound on the absolute value of the characteristic function, thus showing that the contribution of larger t is negligible (see Subsection 3.4 in the extended version).

1.6 Sampling Equitable Colorings

In this subsection we will explain how to sample equitable colorings and then in the next subsection we explain how to adjust the algorithm slightly to sample skewed colorings. We use the LCLT as the main ingredient to give us guarantees on the success of rejection sampling. This is a standard tool in sampling (see [21] for example).

By a classical result of Vigoda [38], there exists an algorithm that approximately samples colorings (uniformly) in $O(n \log n \log(1/\varepsilon))$ for $q \geq \frac{11}{6}\Delta$. One can use the Glauber dynamics as the chain in that algorithm (see [3], [26] for relevant details on its mixing time). We use this as the first step in the rejection sampling algorithm.

▶ **Definition 18** (Rejection Sampling Algorithm).

1. *Approximately sample a uniform coloring (for example by running Glauber dynamics for $T = O_\varepsilon(n \log n)$ steps).*
2. *Check if the coloring produced is an $\vec{n}$-coloring. If yes, output the coloring. If no, reject and repeat the first step.*

Since the LCLT together with bounds on the determinant of the covariance matrix imply that $\pi(\vec{X} = \vec{x}) = \Theta\left(n^{-(q-1)/2}\right)$, it is easy to show that after $O(n^{(q-1)/2}\log(1/\varepsilon))$ iterations of rejection sampling, the probability of the failure of the algorithm is at most $1-\varepsilon$. Hence, the total running time of the algorithm is $O\left(n^{(q+1)/2}\log(n)\log(\frac{1}{\varepsilon})^2\right)$. Note that this algorithm actually lets us sample more than just equitable colorings. For any constant $c > 0$ and $\vec{n}$ such that $\sum_i n_i = n$ and $|n_i - n/q| < c\sqrt{n}$ we know that the LCLT and bounds on eigenvalues of the covariance matrix imply that $\pi(\vec{X} = \vec{x}) = \Theta\left(n^{-(q-1)/2}\right)$, and hence the same rejection algorithm works.

One could also ask for deterministic algorithms to approximate the number of equitable colorings, just as the zero-freeness result of [25] gives a deterministic algorithm (FPTAS) for the total number of colorings. This should be possible using an algorithmic implementation of the LCLT, as in [18]. We leave this for future work.

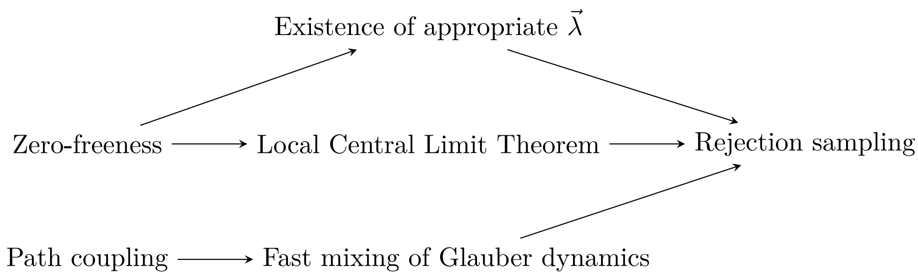

Figure 1 Proof outline.

1.7 Sampling Skewed Colorings

It is natural to predict that one could sample skewed colorings ($|n_i - n/q| < cn$ for small c) the same way that we sample equitable colorings. However, there are several challenges that appear. One of such challenges is showing that we can find (algorithmically) weights $\vec{\lambda}$ such that they give the right (or close to the right) color class sizes in expectation. For the equitable colorings (and colorings with $O(\sqrt{n})$ deviation from equitable) it was enough to take $\vec{\lambda} = \vec{1}$ (to sample from the uniform distribution). To be able to sample colorings with a linear deviation from equitable, we need to find $\vec{\lambda}$ such that $\|\mathbb{E}_{\vec{\lambda}}[\vec{X}] - \vec{n}\|_\infty \leq c\sqrt{n}$. This will allow us to sample colorings from the Gibbs distribution corresponding to $\vec{\lambda}$ and use LCLT to show that rejection sampling works. Note that the probability distribution we care about only makes sense in the zero-freeness region, thus it is not enough to only find appropriate choice of weights, but also show that they reside within the zero-freeness region.

To find $\vec{\lambda}$, we discretize our zero-freeness region such that $\lambda_i = 1 + k_i \frac{1}{\sqrt{n}}$ for $i \in [q-1], k_i \in \{-\lfloor R\sqrt{n} \rfloor, \ldots, \lfloor R\sqrt{n} \rfloor\}$ (we keep $\lambda_q = 1$ for simplicity). We show that the expectation map is $O(n)$-Lipschitz (Subsection 4.1 of the extended version) and since the distance between optimal $\vec{\lambda}$ and one of the candidate $\vec{\lambda}'$ is at most $\frac{1}{\sqrt{n}}$ (in l_2-norm), the difference between expectations is at most $\sqrt{n}$ for each color. Hence, we simply run the rejection sampling for each of the candidate $\vec{\lambda}$ until we obtain an $\vec{n}$-coloring. Since there are $\Theta(n^{(q-1)/2})$ candidates, the runtime is $O\big(n^q \log(n) \log(\frac{1}{\varepsilon})^2\big)$.

To understand what colorings we can sample with this technique, we show that the expectation map (restricted to a certain hyperplane) is surjective on the zero-freeness region (Subsection 4.2 of the extended version), which let's us show that this procedure works for any coloring such that $\sum n_i = n$ and $\|\vec{n} - \frac{n}{q}\vec{1}\|_\infty \leq cn$ for c small enough.

We also need to make sure that we can sample from the Gibbs distribution corresponding to $\vec{\lambda}$. This can be done by running the Glauber dynamics for $O(n \log n)$ steps. We prove the mixing time for $q \geq 2\Delta + 1$ using a standard path coupling technique in Subsection 4.3 of the extended version.

Figure 1 shows a diagram of the structure of the proof. Note that for sampling equitable colorings only the middle layer of the figure is necessary.

This extended abstract only contains some of the proofs; for full proofs, see the full version [24].

2 Zero-freeness

Note that the proofs of some claims in this section were removed or shortened, but the full proofs can be found in the extended version.

2.1 Overview

Recall the partition function and Gibbs measure of the coloring model:

$$Z_G(\lambda_1, \ldots, \lambda_q) := \sum_{\substack{\sigma\in[q]^V \\ \sigma \text{ proper}}} \prod_{i=1}^{q} \lambda_i^{|\sigma^{-1}(i)|}, \qquad \mathbb{P}_{G,\vec{\lambda}}(\sigma) := \frac{\prod_{i=1}^{q} \lambda_i^{|\sigma^{-1}(i)|}}{Z_G(\lambda_1, \ldots, \lambda_q)}.$$

For it to have a probabilistic interpretation, we would need $\lambda_i \in \mathbb{R}^+$, but we can still consider the partition function as a multivariate polynomial over the complex numbers.

▶ Remark 19. At $\vec{\lambda} = \vec{1} := (1, \ldots, 1)$, the partition function is counting the number of proper q-colorings. Since $q \geq 2\Delta$, a proper q-coloring always exists, and hence $Z_G(\vec{1}) \neq 0$.

The aim of this section is to show that in $\mathbb{C}^q$ there exists a ball B around $\vec{1}$ such that $\forall \vec{\lambda} \in B$, $Z_G(\vec{\lambda}) \neq 0$.

The proof here uses the framework of Liu, Sinclair and Srivastava [25] for the Potts model, in which the weight of a configuration is determined by the number of monochromatic edges in the coloring.

A variable ω specifies the "penalty" for a monochromatic edge. Let $m_G(\sigma)$ be the number of monochromatic edges introduced by the coloring σ. The partition function of the Potts model is a sum over all colorings (including improper ones), where the weight of a coloring is proportional to $\omega^{m_G(\sigma)}$. In other words, the partition function for the Potts model is

$$Z_G(\omega) := \sum_{\sigma\in[q]^V} \omega^{m_G(\sigma)}.$$

Note that this partition function is univariate. In contrast, our partition function is multivariate, since each color has its own weight; part of our work is therefore to lift several key ingredients of their proof to the multivariate case. Note also that in our model there are no monochromatic edges, so the penalty for such edges is simply $\omega = 0$.

Since all the steps had to be adapted for our model, we will provide detailed calculations, and we utilize the structure and definitions of [25] for the acquainted reader to track differences easily.

2.2 Main result

Throughout this section we fix the constants $\nu := 0.9$, $\varepsilon_R := 10^{-2}\Delta^{-2}$, and $\varepsilon_I := 10^{-4}\Delta^{-4}$. We will use ln to denote the principal branch of the logarithm, which is analytic on $\mathbb{C}\setminus(-\infty, 0]$ and satisfies $\arg(z) := \operatorname{Im} \ln z \in (-\pi, \pi)$.

We will commonly refer to the following assumptions on $\vec{\lambda} \in \mathbb{C}^q$.

▶ **Definition 20** (Assumptions on $\vec{\lambda}$). *We say that $\vec{\lambda} \in \mathbb{C}^q$ is valid if, for any $i, j \in [q]$,*

1. $\lambda_i \notin (-\infty, 0] \subset \mathbb{R}$
2. $|\arg(\lambda_i)| \leq \nu\varepsilon_I/2$
3. $|\operatorname{Re} \ln \frac{\lambda_i}{\lambda_j}| \leq \Delta\varepsilon_I$

We will denote $\lambda_{min} := \min\{1, \min_{i\in[q]} |\lambda_i|\}$ *and* $\lambda_{max} := \max\{1, \max_{i\in[q]} |\lambda_i|\}$.

▶ Remark 21. Note that we need assumption 1 for the quantity in assumption 2 to be well-defined, and then $|\arg(\lambda_i/\lambda_j)| \leq \nu\varepsilon_I < \pi$, so assumption 3 is also well-defined.

The main result we will prove is the following:

▶ **Theorem 22** (Zero-freeness around $\vec{1}$). *Let $\Delta \geq 1$, $q \geq 2\Delta$, and let G be a graph with maximum degree Δ. Assume that $\vec{\lambda} \in \mathbb{C}^q$ is valid, in the sense of Definition 20. Then, $Z_G(\vec{\lambda}) \neq 0$.*

The assumptions from Definition 20 are expressed in terms of the logarithm of the ratio between components of $\vec{\lambda}$. That may not be practical for some applications, but it can be shown that any $\vec{\lambda} \in \mathbb{C}^q$ sufficiently close to $\vec{1}$ (for example, in ℓ_∞-distance) satisfies those.

▶ **Lemma 23.** *Let $R := \nu\varepsilon_I/4$. Then, any $\vec{\lambda} \in \mathbb{C}^q$ such that $|\lambda_i - 1| \leq R$ for every $i \in [q]$ is valid, in the sense of Definition 20.*

Hence, we may derive the following corollary from Theorem 22:

▶ **Corollary 24** (Zero-freeness radius). *Let $\Delta \geq 1$, $q \geq 2\Delta$, and let G be a graph with maximum degree Δ. Then, for any $\vec{\lambda} \in B_{\ell_\infty}(\vec{1}, R)$, where $R = \nu\varepsilon_I/4$, we have that $Z_G(\vec{\lambda}) \neq 0$.*

▶ Remark 25. The constants used in the proof of Theorem 22 are not particularly optimized, so the zero-freeness radius is susceptible to improvement.

▶ Remark 26. According to Corollary 24, $Z_G(\vec{\lambda})$ is zero-free within the simply-connected region $B = B_{\ell_\infty}(\vec{1}, R) \subseteq \mathbb{C}^q$. Hence, there exists a branch of the logarithm such that $\log Z_G(\vec{\lambda})$ is well-defined and analytic for $\vec{\lambda} \in B$ (i.e. an analytic $f : B \longrightarrow \mathbb{C}$ satisfying $e^{f(\vec{\lambda})} = Z_G(\vec{\lambda})$). In the proofs in this section we are always able to take the principal branch of the logarithm, but that will not be the case for some results in the extended version of the paper.

The inductive proof not only shows that $Z_G(\vec{\lambda}) \neq 0$, but actually gives a lower bound on its norm (which, despite being exponentially decreasing in n, will be of use later in bounding $\left|\log Z_G(\vec{\lambda})\right|$).

▶ **Corollary 27.** *Let $\Delta \geq 1$, $q \geq 2\Delta$, and let G be a graph with n vertices and maximum degree Δ. Let $\vec{\lambda} \in \mathbb{C}^q$ be valid, according to Definition 20. Then,*

$$\left|Z_G(\vec{\lambda})\right| \geq 0.99^n \left(\frac{\lambda_{min}}{\lambda_{max}}\right)^{n\Delta},$$

where $\lambda_{min} := \min\{1, \min_{i\in[q]} |\lambda_i|\}$ and $\lambda_{max} := \max\{1, \max_{i\in[q]} |\lambda_i|\}$.

2.3 Partial colorings and recursive structure

We will prove Theorem 22 with a slightly stronger assumption; some vertices of the graph may already be "pinned" to a color. That will allow us to proceed by induction on the number of unpinned vertices of the graph. We will formalize this idea through the following definition:

▶ **Definition 28** (Partially-q-colored graph). *Let $q \geq 1$. A partially-q-colored graph (G, τ) is a graph G together with a partial q-coloring $\tau : V(G) \longrightarrow [q] \cup \{*\}$. The partial coloring needs to be proper, that is, for every edge $uv \in E(G)$ either $\tau(u) \neq \tau(v)$ or $\tau(u) = \tau(v) = *$ need to hold. Vertices $v \in V(G)$ such that $\tau(v) = *$ are said to be unpinned. Vertices such that $\tau(v) = c \in [q]$ are said to be pinned to color c. We additionally require that $\deg_G(v) = 1$ for all pinned v.*

▶ Remark 29. We may extend our model to a partially-colored graph (G, τ) by restricting the partition function to sum over the colorings that agree with τ on the pinned vertices. That is, we define

$$Z_{G,\tau}(\vec{\lambda}) := \sum_{\substack{\sigma \in [q]^{V(G)} \\ \sigma(v)=\tau(v) \text{ for pinned } v}} \prod_{v \in V(G)} \lambda_{\sigma(v)}.$$

We will use $\mathcal{C}_{G,\tau}$ to denote the set of proper q-colorings of G which agree with τ on the pinned vertices.

We will also introduce the following operation, which from a partially-q-colored graph (G, τ) generates another partially-q-colored graph $(\tilde{G}, \tilde{\tau})$ which has one unpinned vertex fewer:

▶ **Definition 30** (Pinning operation). *Let v be a vertex in (G, τ) with degree d. To pin vertex v to color c means that we have constructed a new graph $\tilde{G}$ in which we replaced the vertex v with d copies $v_1, \ldots, v_d$ (each of them joined to a different neighbor of v), and which is equipped with a partial coloring $\tilde{\tau}$ defined so that $\tilde{\tau}(v_i) = c$ for all $i \in [d]$ and $\tilde{\tau}(w) = \tau(w)$ for all other $w \in V(\tilde{G}) \setminus \{v_1, \ldots, v_d\} = V(G) \setminus \{v\}$.*

▶ Remark 31. Note that, after applying the previous operation, the newly-pinned vertices have all degree 1, as required in the definition of partially-colored graph.

Since we are working with a model that only allows proper colorings, we will never want to pin a vertex to a color which is already present in its neighborhood. Thus, following the notation from [25], we will classify the colors as either *good* or *bad* (for a given vertex):

▶ **Definition 32** (Good color, bad color). *A $c \in [q]$ is a good color for vertex v if it is different from all colors to which the neighbors of v are pinned to. Otherwise, it is called a bad color. The set of good colors of a vertex v will be denoted as Γ_v, while the set of bad colors will be denoted as B_v.*

We will denote (G, τ) as G, despite the abuse of notation, whenever it is clear from context that we are referring to a partially-colored graph and there is no ambiguity with respect to the underlying partial coloring τ. That extends as well to Z_G (in place of $Z_{G,\tau}$) and $\mathcal{C}_G$ (in place of $\mathcal{C}_{G,\tau}$).

Next, we develop the definitions needed to frame the zero-freeness problem recursively, in terms of graphs which have fewer unpinned vertices.

▶ **Definition 33** (Restricted partition function, marginal pseudo-probability, marginal ratio). *Let G be a partially-q-colored graph. Let $\vec{\lambda}$ be a vector of formal variables. For a given unpinned vertex $v \in V(G)$ and a color $i \in [q]$, we define*

$$Z^{(i)}_{G,v}(\vec{\lambda}) := \sum_{\substack{\sigma \in \mathcal{C}_G \\ \sigma(v)=i}} \prod_{v \in V(G)} \lambda_{\sigma(v)},$$

which corresponds to the restriction of the partition function to the colorings in which v has color i.

Let $\Delta \geq 1$ be the maximum degree of the graph G, and assume that $q \geq \Delta + 1$. Then, we define the marginal pseudo-probability

$$\mathcal{P}_{G,\vec{\lambda}}[\sigma(v) = i] := \frac{Z^{(i)}_{G,v}(\vec{\lambda})}{Z_G(\vec{\lambda})},$$

and, given another color $j \in \Gamma_v$, we define the marginal ratio

$$R_{G,v}^{(i,j)} := \frac{Z_{G,v}^{(i)}(\vec{\lambda})}{Z_{G,v}^{(j)}(\vec{\lambda})}.$$

▶ Remark 34. All the expressions above are to be interpreted as quotients of polynomials of the vector of formal variables $\vec{\lambda}$. Note that, under this interpretation, $Z_{G,v}^{(i)}(\vec{\lambda}) = 0$ for any $i \in B_v$. On the other hand, if $i \in \Gamma_v$ and $q \geq \Delta + 1$, we know that there is at least one proper coloring of G which is consistent with the current partial coloring, so $Z_{G,v}^{(i)}(\vec{\lambda}) \neq 0$ (and, by the same logic, $Z_G(\vec{\lambda}) \neq 0$). Hence, both the marginal pseudo-probability and the marginal ratios are well-defined. Nonetheless, it could still happen that $Z_{G,v}^{(i)}(\vec{\lambda}) = 0$ and/or $Z_G(\vec{\lambda}) = 0$ once we substitute the formal variables for a specific value $\vec{\lambda} \in \mathbb{C}^q$.

▶ Remark 35. The name and notation for the marginal pseudo-probability comes from the fact that, for real positive $\vec{\lambda}$, this is the marginal probability that v has color i under our model. Note, however, that for a general $\vec{\lambda} \in \mathbb{C}^q$, this quantity has no direct probabilistic meaning.

We would like to relate $R_{G,v}^{(i,j)}(\vec{\lambda})$ to the partition function of a graph with fewer unpinned vertices than G. The numerator $Z_{G,v}^{(i)}(\vec{\lambda})$ can be related to the graph in which we pin vertex v to color i, while the denominator $Z_{G,v}^{(j)}(\vec{\lambda})$ can be related to the graph in which we pin vertex v to color j. Next, we define a sequence of graphs that allows us to interpolate between these two cases.

▶ **Definition 36** (Graph $G_k^{(i,j)}$). *Given a partially-colored graph G, an unpinned vertex v with degree d, and an ordering of the neighbors of v (denoted as $w_1, \dots, w_d$), for every $k \in [d]$ and for every $i, j \in \Gamma_v$ we define the partially-colored graph $G_k^{(i,j)}$ obtained from G by replacing v with d copies $v_1, \dots, v_d$, attaching each v_ℓ to w_ℓ, pinning vertices $v_1, \dots, v_{k-1}$ to color i, pinning vertices $v_{k+1}, \dots, v_d$ to color j, and deleting v_k.*

One can then derive a recurrence relation that expresses the ratios in G in terms of the marginal pseudo-probabilities in each of the $G_k^{(i,j)}$.

▶ **Lemma 37** (Recurrence relation). *Let the vertices $w_1, \dots, w_{\deg_G(v)}$ be the neighbors of vertex v in the graph G. Then,*

$$R_{G,v}^{(i,j)}(\vec{\lambda}) = \frac{\lambda_i}{\lambda_j} \prod_{k=1}^{\deg_G(v)} \frac{1 - \mathcal{P}_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = i]}{1 - \mathcal{P}_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = j]}.$$

▶ Remark 38. The recurrence relation holds under the interpretation that $\vec{\lambda}$ is a vector of formal variables. It will also hold after plugging in a particular $\vec{\lambda} \in \mathbb{C}^q$, as long as all the terms are well-defined, that is,

- $Z_{G,v}^{(j)}(\vec{\lambda}) \neq 0$,
- $Z_{G_k^{(i,j)}}(\vec{\lambda}) \neq 0$, and
- $\mathcal{P}_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = j] \neq 1$ for all $k \in [\deg_G(v)]$.

Proof. Let us first define an auxiliary graph $H_k^{(i,j)}$, which is analogous to $G_k^{(i,j)}$, but in which v_k is pinned to color i, instead of being deleted.

▶ **Definition 39** (Graph $H_k^{(i,j)}$). *Given a partially-colored graph G, an unpinned vertex v with degree d, and an ordering $w_1, \dots, w_d$ of the neighbors of v, we define the graph $H_k^{(i,j)}$ by replacing vertex v with d copies $v_1, \dots, v_d$, attaching each v_ℓ to w_ℓ, pinning vertices $v_1, \dots, v_k$ to color i and pinning vertices $v_{k+1}, \dots, v_d$ to color j.*

Note that $H_0^{(i,j)}$ and $H_d^{(i,j)}$ correspond to the edge cases in which all copies of v are colored j, or all copies are colored i, respectively. Hence, $Z_{H_d^{(i,j)}}(\vec{\lambda}) = \lambda_i^{d-1} Z_{G,u}^{(i)}(\vec{\lambda})$ and $Z_{H_0^{(i,j)}}(\vec{\lambda}) = \lambda_j^{d-1} Z_{G,u}^{(j)}(\vec{\lambda})$. Therefore, we may write the marginal ratio as

$$R_{G,u}^{(i,j)}(\vec{\lambda}) = \frac{Z_{G,u}^{(i)}(\vec{\lambda})}{Z_{G,u}^{(j)}(\vec{\lambda})} = \frac{\lambda_j^{d-1}}{\lambda_i^{d-1}} \frac{Z_{H_d^{(i,j)}}(\vec{\lambda})}{Z_{H_0^{(i,j)}}(\vec{\lambda})} = \frac{\lambda_j^{d-1}}{\lambda_i^{d-1}} \prod_{k=1}^{d} \frac{Z_{H_k^{(i,j)}}(\vec{\lambda})}{Z_{H_{k-1}^{(i,j)}}(\vec{\lambda})}.$$

There is a correspondence between proper colorings of $H_k^{(i,j)}$ and proper colorings of $G_k^{(i,j)}$ in which $\sigma(w_k) \neq i$. Therefore,

$$Z_{H_k^{(i,j)}}(\vec{\lambda}) = \lambda_i (Z_{G_k^{(i,j)}}(\vec{\lambda}) - Z_{G_k^{(i,j)},w_k}^{(i)}(\vec{\lambda})),$$

and, similarly,

$$Z_{H_{k-1}^{(i,j)}}(\vec{\lambda}) = \lambda_j (Z_{G_k^{(i,j)}}(\vec{\lambda}) - Z_{G_k^{(i,j)},w_k}^{(j)}(\vec{\lambda})).$$

Hence,

$$R_{G,u}^{(i,j)}(\vec{\lambda}) = \frac{\lambda_j^{d-1}}{\lambda_i^{d-1}} \frac{\lambda_i^d}{\lambda_j^d} \prod_{k=1}^{d} \frac{Z_{G_k^{(i,j)}}(\vec{\lambda}) - Z_{G_k^{(i,j)},w_k}^{(i)}(\vec{\lambda})}{Z_{G_k^{(i,j)}}(\vec{\lambda}) - Z_{G_k^{(i,j)},w_k}^{(j)}(\vec{\lambda})} = \frac{\lambda_i}{\lambda_j} \prod_{k=1}^{d} \frac{1 - P_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = i]}{1 - P_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = j]}. \quad ◀$$

2.4 Complex analysis tools

In this subsection we introduce the ingredients from complex analysis we will need for the proof. The first three lemmas are from [25], while the last three are refinements of simple facts that were implicitly used in [25].

Let D be a domain in $\mathbb{C}$ with the following properties:

- For any $z \in D$, $\mathrm{Re}(z) \in D$.
- For any $z_1, z_2 \in D$, there exists a point $z_0 \in D$ such that one of the numbers $z_1 - z_0, z_2 - z_0$ has zero real part while the other has zero imaginary part.
- If $z_1, z_2 \in D$ are such that either $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$ or $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, then the segment $[z_1, z_2]$ lies in D.

As remarked by Liu, Sinclair and Srivastava [25], a rectangular region symmetric about the real axis will satisfy all of the above properties.

▶ **Lemma 40** (Mean value theorem for complex functions – Lemma 3.5 from [25]). *Let f be a holomorphic function on a domain D as above such that, for $z \in D$, $\mathrm{Im}(f(z))$ has the same sign as $\mathrm{Im}(z)$. Suppose further that there exist positive constants ρ_I and ρ_R such that*

- *for all $z \in D$, $|\mathrm{Im}(f'(z))| \leq \rho_I$;*
- *for all $z \in D$, $\mathrm{Re}(f'(z)) \in [0, \rho_R]$.*

Then for any $z_1, z_2 \in D$, there exists $C_{z_1,z_2} \in [0, \rho_R]$ such that

$$|\mathrm{Re}(f(z_1) - f(z_2)) - C_{z_1,z_2} \cdot \mathrm{Re}(z_1 - z_2)| \leq \rho_I \cdot |\mathrm{Im}(z_1 - z_2)|,$$

and furthermore,

$$|\mathrm{Im}(f(z_1) - f(z_2))| \leq \rho_R \cdot \begin{cases} |\mathrm{Im}(z_1 - z_2)|, & \text{when } \mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2) \leq 0; \\ \max\{|\mathrm{Im}(z_1)|, |\mathrm{Im}(z_2)|\}, & \text{otherwise.} \end{cases}$$

The first point of the next lemma is needed to show that the function

$$f(x) := -\ln(1 - e^x) \tag{1}$$

satisfies the conditions of the previous lemma.

▶ **Lemma 41** (Lemma 3.6 from [25]). *Consider the domain D given by*

$$D = \{z \mid \operatorname{Re}(z) \in (-\infty, -\zeta) \text{ and } |\operatorname{Im}(z)| < \tau\},$$

where $\tau < 1/2$ and ζ are positive real numbers such that $\tau^2 + e^{-\zeta} < 1$, and the function f defined in eq. (1)*. These satisfy the hypotheses of Lemma 40, if ρ_R and ρ_I in the statement of the theorem are taken to be $\frac{e^{-\zeta}}{1-e^{-\zeta}}$ and $\frac{\tau e^{-\zeta}}{(1-e^{-\zeta})^2}$, respectively.*

The next lemma gives us a way to lower-bound the sum of complex numbers, if we know that the angles between them are small enough.

▶ **Lemma 42** (Lemma 3.7 from [25]). *Let $z_1, z_2, \ldots, z_n$ be complex numbers such that the angle between any two non-zero z_i is at most $\alpha \in [0, \pi/2)$. Then,*

$$\left|\sum_{i=1}^{n} z_i\right| \geq \cos(\alpha/2) \sum_{i=1}^{n} |z_i| \,.$$

An important part of the proof will require going from bounds on z to bounds on e^z, and viceversa. That's what the following two lemmas provide.

▶ **Lemma 43.** *Let $r, \theta \in \mathbb{R}$ such that $|\theta| \leq \theta_0 < \pi$ and $|r| \leq r_0$ for certain $r_0, \theta_0 \in \mathbb{R}^{\geq 0}$. Let $z := r + i\theta$. Then, $|\arg(e^z)| \leq \theta_0$ and $e^{-r_0} - \theta_0^2/2 \leq \operatorname{Re}(e^z) \leq e^{r_0}$. Furthermore, if $\theta_0 \leq \ln 2 \cong 0.693$, then we also have that $e^{-r_0-\theta_0^2} \leq \operatorname{Re}(e^z)$.*

▶ **Lemma 44.** *Let $z \in \mathbb{C}$ with $\operatorname{Re}(z) \neq 0$ and $|\arg(z)| \leq \theta \leq 0.1$. Then, $\ln(\operatorname{Re}(z)) \leq \operatorname{Re}(\ln(z)) \leq \ln(\operatorname{Re}(z)) + \theta^2$ and $|\operatorname{Im}(\ln(z))| \leq \theta$.*

Finally, a result in the same vein as Lemma 42.

▶ **Lemma 45.** *Let $z_1, \ldots, z_k \neq 0$ be complex numbers such that $|\arg(z_i)| \leq \theta$ for all $i \in [k]$ and some $\theta < \pi/2$. Then, $\sum_{i=1}^{k} z_i \neq 0$ and $\left|\arg\left(\sum_{i=1}^{k} z_i\right)\right| \leq \theta$.*

2.5 Induction hypothesis

The main idea of the proof of the zero-freeness result of Theorem 22 is to pick a vertex u and rewrite the partition function as a sum of $Z^{(i)}_{G,u}(\vec{\lambda})$ for all $i \in \Gamma_u$. If we can prove that the angles between these terms are small, then it is enough for one of the terms to be non-zero for the whole sum to be non-zero (see Lemma 42).

Both the zero-freeness of $Z^{(i)}_{G,u}(\vec{\lambda})$ and the fact that the angles are small are shown by induction on the number of unpinned vertices of the graph. For the first, we simply use that $Z^{(i)}_{G,u}(\vec{\lambda}) = \lambda_i^{\deg(u)-1} Z_{\tilde{G}}(\vec{\lambda})$, where $\tilde{G}$ is the graph obtained by pinning vertex u to color i. For the second, we need to bound the argument of the ratios $R^{(i,j)}_{G,u}$. For that purpose, we use Lemma 37, in which we derived a recursive formula that expresses $R^{(i,j)}_{G,u}$ in terms of marginal pseudo-probabilities of certain graphs $G^{(i,j)}_k$, which all have one fewer unpinned vertex than G.

The following lemma summarizes the statements constituting our induction hypothesis.

▶ **Lemma 46.** *Let G be a partially-q-colored graph with maximum degree $\Delta \geq 1$ and $q \geq 2\Delta$. Let u be an unpinned vertex of G. Then, for a valid $\vec{\lambda} \in \mathbb{C}^q$, in the sense of Definition 20,*

1. *For all $i \in \Gamma_u$,*

$$\left|Z_{G,u}^{(i)}(\vec{\lambda})\right| \geq 0.99^{\ell-1} \frac{\lambda_{min}^{n+\ell(\Delta-1)}}{\lambda_{max}^{\ell\Delta}} > 0,$$

where ℓ the number of unpinned vertices and n the total number of vertices of G.

2. *For $i, j \in \Gamma_u$, if u has all of its neighbors pinned, then $R_{G,u}^{(i,j)}(\vec{\lambda}) = \lambda_i/\lambda_j$.*
3. *For $i, j \in \Gamma_u$, $R_{G,u}^{(i,j)}(\vec{\lambda}) \in \mathbb{C} \setminus (-\infty, 0]$.*
4. *For $i, j \in \Gamma_u$, if u has d_u unpinned neighbors, then*

$$\left|\mathrm{Re}(\ln R_{G,u}^{(i,j)}(\vec{\lambda})) - \ln R_{G,u}^{(i,j)}(\vec{1})\right| \leq d_u \varepsilon_R + \left|\mathrm{Re} \ln \frac{\lambda_i}{\lambda_j}\right|.$$

5. *For $i, j \in \Gamma_u$, if u has d_u unpinned neighbors, then*

$$\left|\mathrm{Im}(\ln R_{G,u}^{(i,j)}(\vec{\lambda}))\right| \leq d_u \varepsilon_I + \left|\mathrm{Im} \ln \frac{\lambda_i}{\lambda_j}\right|.$$

6. *For $i \notin \Gamma_u, j \in \Gamma_u$, we have that $R_{G,u}^{(i,j)}(\vec{\lambda}) = 0$.*

▶ Remark 47. The logarithm of λ_i/λ_j is well-defined due to the assumptions on $\vec{\lambda}$, while the ratios $R_{G,u}^{(i,j)}(\vec{\lambda})$ are well-defined due to item 1. We can take the logarithm of $R_{G,u}^{(i,j)}(\vec{\lambda})$ due to item 3.

▶ Remark 48. Note that $Z_{G,u}^{(i)}(\vec{1})$ counts the number of colorings of G which have $\sigma(u) = i$. For $i \in \Gamma_u$ and $q \geq \Delta + 1$, that number is non-zero, so $R_{G,u}^{(i,j)}(\vec{1})$ is a quotient of positive reals, and hence $\ln R_{G,u}^{(i,j)} \in \mathbb{R}$.

The zero-freeness of the full partition function follows easily once Lemma 46 is proved:

▶ **Corollary 49.** *Let G be a partially-q-colored graph with maximum degree $\Delta \geq 1$ and $q \geq 2\Delta$. Let u be an unpinned vertex of G. Then,*

$$|Z_G(\vec{\lambda})| \geq 0.99 \min_{i \in \Gamma_u} \left\{|Z_{G,u}^{(i)}(\vec{\lambda})|\right\} > 0.99^{\ell} \frac{\lambda_{min}^{n+\ell(\Delta-1)}}{\lambda_{max}^{\ell\Delta}} > 0,$$

where ℓ is the number of unpinned vertices and n the total number of vertices of G.

Proof. Observe that $Z_{G,u}^{(i)}(\vec{\lambda}) = 0$ for $i \in B_u$. Hence, $Z_G(\vec{\lambda}) = \sum_{i \in \Gamma_u} Z_{G,u}^{(i)}(\vec{\lambda})$. Since $\mathrm{Im}(\ln z) = \arg(z)$, from item 5 of Lemma 46 we know that the angle between the terms is at most

$$\left|\arg\left(R_{G,u}^{(i,j)}(\vec{\lambda})\right)\right| \leq \Delta\varepsilon_I + \left|\mathrm{Im} \ln \frac{\lambda_i}{\lambda_j}\right| \leq (\Delta + \nu)\varepsilon_I$$

where we also used assumption 2 from Definition 20. Using Lemma 42 together with the fact that $(\Delta + \nu)\varepsilon_I/2 \leq \Delta\varepsilon_I \leq 10^{-4} \leq \arccos 0.99$, we have that

$$\left|\sum_{i \in \Gamma_u} Z_{G,u}^{(i)}(\vec{\lambda})\right| \geq \cos\left((\Delta + \nu)\varepsilon_I/2\right) \sum_{i \in \Gamma_u} \left|Z_{G,u}^{(i)}(\vec{\lambda})\right| \geq 0.99 \min_{i \in \Gamma_u} \left|Z_{G,u}^{(i)}(\vec{\lambda})\right|$$

The result then follows by item 1 of Lemma 46. ◀

The previous corollary states that the partition function is zero-free for any partially-colored graph. Then, our main theorem follows as the particular case in which the partial coloring is "empty":

Proof of Theorem 22. Let G be a graph of maximum degree $\Delta \geq 1$, and let $q \geq 2\Delta$. By definition, $Z_G(\vec{\lambda}) = Z_{G,\tau}(\vec{\lambda})$, where (G, τ) is the partially-colored graph in which $\tau(v) = *$ for all $v \in V(G)$. Using Corollary 49, $Z_{G,\tau}(\vec{\lambda}) \neq 0$, from which the desired conclusion follows. ◀

Corollary 27 follows analogously by taking $\ell = n$ in Corollary 49, since all vertices are unpinned. Note that Corollary 49 is not only used now for the final zero-freeness conclusion, but also will be used at every step of the induction.

2.6 Consequences of the induction hypothesis

The remainder of the section deals with the inductive proof of Lemma 46. First, we will state a series of consequences of Lemma 46, which will be used to carry out the induction step. Their full proofs can be found in the extended version of the paper. We will assume throughout that G is a partially-colored graph of maximum degree $\Delta \geq 1$ for which Lemma 46 holds, that $q \geq 2\Delta$, that u is an unpinned vertex of G with d_u unpinned neighbors, and that $\vec{\lambda} \in \mathbb{C}^q$ is valid, in the sense of Definition 20.

▶ **Lemma 50.** *For any $i \in \Gamma_u$,*

- $\mathcal{P}_{G,\vec{1}}[\sigma(u) = i] \in \mathbb{R} \setminus \{0\}$, *and*
- $\mathcal{P}_{G,\vec{\lambda}}[\sigma(u) = i] \in \mathbb{C} \setminus (-\infty, 0]$.

▶ **Lemma 51.** *For any $i, j \in \Gamma_u$,*

$$e^{-(d_u+0.05)\varepsilon_R} \leq \frac{\operatorname{Re} R_{G,u}^{(i,j)}(\vec{\lambda})}{R_{G,u}^{(i,j)}(\vec{1})} \leq e^{(d_u+0.05)\varepsilon_R}.$$

▶ **Lemma 52** (Approximation of pseudo-probabilities by real probabilities)**.** *For any $i \in \Gamma_u$,*

$$\left| \operatorname{Im} \left(\ln \frac{\mathcal{P}_{G,\vec{\lambda}}[\sigma(u) = i]}{\mathcal{P}_{G,\vec{1}}[\sigma(u) = i]} \right) \right| \leq (d_u + \nu)\varepsilon_I,$$

and

$$\left| \operatorname{Re} \left(\ln \frac{\mathcal{P}_{G,\vec{\lambda}}[\sigma(u) = i]}{\mathcal{P}_{G,\vec{1}}[\sigma(u) = i]} \right) \right| \leq (d_u + 0.1)\varepsilon_R.$$

For the next result, we need that no color is "too probable" in vertex w_k of $G_k^{(i,j)}$. We quantify that through the following definition, analogous to the one from Liu, Sinclair and Srivastava [25]:

▶ **Definition 53** (Nice vertex)**.** *Let H be a partially-colored graph. We say vertex $v \in V(H)$ is nice if, for any color $c \in \Gamma_v$, we have that $\mathcal{P}_{H,\vec{1}}[\sigma(v) = c] \leq 1/(d_v + 2)$, where d_v is the number of unpinned neighbors of v.*

▶ **Lemma 54.** *Let $i, j \in \Gamma_u$ and $k \in [\deg_G(u)]$. Consider the graph $G_k^{(i,j)}$ from Definition 39, obtained from graph G and unpinned vertex u. Let $w_1, \ldots, w_k$ denote the neighbors of u in G. Then, the vertex w_k is nice in $G_k^{(i,j)}$.*

Proof. Let $d := \deg_{G_k^{(i,j)}}(w_k)$ and let $\tilde{d}$ be its number of unpinned neighbors. By definition of the graph, w_k has one neighbor less in $G_k^{(i,j)}$ than it had in G, so $\tilde{d} \le d \le \Delta - 1$. Since $q \ge 2\Delta \ge d + \tilde{d} + 2$, it suffices to show that $\mathcal{P}_{G_k^{(i,j)},\vec{1}}[\sigma(w_k) = c] \le 1/(q-d)$ for all $c \in \Gamma_{w_k}$.

Note that $Z^{(c)}_{G_k^{(i,j)},w_k}(\vec{1})$ counts the number of proper colorings which are consistent with the partial coloring of $G_k^{(i,j)}$ and that assign $\sigma(w_k) = c$. Hence, we need to prove that the number of colorings in which $\sigma(w_k) = c$ is at most a $1/(q-d)$-fraction of the total. We will do that by showing there exists an injective function $f : \mathcal{C}^{(c)} \times [q-d] \longrightarrow \mathcal{C}$, where $\mathcal{C}$ is the set of all possible colorings of $G_k^{(i,j)}$ that agree with its partial coloring, and $\mathcal{C}^{(c)} \subset \mathcal{C}$ is the subset of those which assign $\sigma(w_k) = c$.

Fix an order on the colors (for example, the natural one induced by $[q] \subset \mathbb{Z}$). For a given $\sigma \in \mathcal{C}^{(c)}$ and $\ell \in [q-d]$, we define $f(\sigma, \ell)$ as the coloring where $f(\sigma,\ell)(w_k)$ is the ℓ-th element of $[q] \setminus \sigma(N(w_k))$, while $f(\sigma,\ell)(z) = \sigma(z)$ for all $z \ne w_k$. Note that this is well-defined, as $|\sigma(N(w_k))| \le q - d$. It just remains to show that this function is injective.

Assume $\sigma, \tilde{\sigma} \in \mathcal{C}^{(c)}$ and $\ell, \tilde{\ell} \in [q-d]$ satisfy $f(\sigma,\ell) = f(\tilde{\sigma},\tilde{\ell})$. That means that $\sigma(z) = f(\sigma,\ell)(z) = f(\tilde{\sigma},\tilde{\ell})(z) = \tilde{\sigma}(z)$ for all $z \ne w_k$, while $\sigma(w_k) = c = \tilde{\sigma}(w_k)$, so $\sigma = \tilde{\sigma}$. Hence, we have the equality $\sigma(N(w_k)) = \tilde{\sigma}(N(w_k))$, so $f(\sigma,\ell)(w_k) = f(\tilde{\sigma},\tilde{\ell})(w_k)$ means that $\ell = \tilde{\ell}$. ◀

▶ Remark 55. Our niceness condition is equivalent to taking $w = 0$ in the one in [25]. Note that there is nothing special about the vertex w_k or the graph $G_k^{(i,j)}$: the same proof holds for any vertex with degree at most $\Delta - 1$.

For the next lemma, we adopt the notation $a^{(i)}_{G,u}(\vec{\lambda}) := \ln(P_{G,\vec{\lambda}}[\sigma(u) = i])$ for $i \in \Gamma_u$, and define $f(x) := -\ln(1 - e^x)$. Recall that this function is the same as f_1 from Lemma 41.

▶ **Lemma 56.** *Assume u is a nice vertex in G, and that $d_u \le \Delta - 1$. Then, for any colors $i, j \in \Gamma_u$, there exists a real constant $C_{G,u,i} = C \in [0, \frac{1}{d_u+\nu}]$ so that*

$$\left| \operatorname{Re}\left(f(a^{(i)}_{G,u}(\vec{\lambda})) - f(a^{(i)}_{G,u}(\vec{1})) \right) - C \cdot \operatorname{Re}\left(a^{(i)}_{G,u}(\vec{\lambda}) - a^{(i)}_{G,u}(\vec{1}) \right) \right| \le \varepsilon_I; \tag{2}$$

$$\left| \operatorname{Im} f(a^{(i)}_{G,u}(\vec{\lambda})) - \operatorname{Im} f(a^{(j)}_{G,u}(\vec{\lambda})) \right| \le \varepsilon_I; \tag{3}$$

$$\left| \operatorname{Im} f(a^{(i)}_{G,u}(\vec{\lambda})) \right| \le \varepsilon_I. \tag{4}$$

Proof. We will show that we can define parameters ζ and τ satisfying the hypothesis from Lemma 41, so that $a^{(i)}_{G,u}(\vec{\lambda}), a^{(i)}_{G,u}(\vec{1}), a^{(j)}_{G,u}(\vec{\lambda}) \in D_{\zeta,\tau}$, where D is the domain defined in Lemma 41. Then, we will be able to apply Lemma 40 to obtain the desired bounds.

▷ Claim 57. The parameters

$$\zeta := \ln(d_u + 2) - (d_u + 0.11)\varepsilon_R, \quad \text{and}$$
$$\tau := (d_u + 0.91)\varepsilon_I$$

satisfy the hypothesis from Lemma 41. ⌟

▷ Claim 58. Let $D = \{z \mid \operatorname{Re}(z) \in (-\infty, -\zeta) \text{ and } |\operatorname{Im}(z)| < \tau\}$, where ζ and τ are defined as in Claim 57. Then, for any $i \in \Gamma_u$, we have that $a^{(i)}_{G,u}(\vec{\lambda}) \in D$ and $a^{(i)}_{G,u}(\vec{1}) \in D$. ⌟

Next, we give some bounds for ρ_R and ρ_I, as defined in Lemma 41:

▷ Claim 59. Let $\rho_R := \frac{e^{-\zeta}}{1-e^{-\zeta}}$ and $\rho_I := \frac{\tau\rho_R}{1-e^{-\zeta}}$. Then, $\rho_R \le \frac{1}{d_u+0.91}$ and $\rho_I \le 3\varepsilon_I$. ⌟

From Lemma 41, we know that the function f and the domain D satisfy the hypothesis of Lemma 40. That lemma tells us that there exists a constant $C \in [0, \rho_R]$ (possibly depending on G, u, i), such that

$$\begin{aligned}&\left| \mathrm{Re}\left(f(a^{(i)}_{G,u}(\vec{\lambda})) - f(a^{(i)}_{G,u}(\vec{1}))\right) - C \cdot \mathrm{Re}\left(a^{(i)}_{G,u}(\vec{\lambda}) - a^{(i)}_{G,u}(\vec{1})\right) \right| \\ &\quad \leq \rho_I \cdot \left| \mathrm{Im}(a^{(i)}_{G,u}(\vec{\lambda}) - a^{(i)}_{G,u}(\vec{1}))\right| .\end{aligned} \tag{5}$$

Using Lemma 52 and Claim 59, we can bound the right-hand side of eq. (5) with

$$\rho_I \cdot \left| \mathrm{Im}(a^{(i)}_{G,u}(\vec{\lambda}) - a^{(i)}_{G,u}(\vec{1}))\right| \leq 3\varepsilon_I(d_u + \nu)\varepsilon_I < \varepsilon_I,$$

which proves eq. (2).

Note that $a^{(i)}_{G,u}(\vec{\lambda})$, $a^{(j)}_{G,u}(\vec{\lambda})$ and $a^{(i)}_{G,u}(\vec{\lambda}) - a^{(j)}_{G,u}(\vec{\lambda})$ all satisfy $|\mathrm{Im}\, z| \leq (d_u + \nu)\varepsilon_I$, in the first two cases due to Lemma 52, and in the third case due to item 5 of Lemma 46 and Definition 20. Therefore, applying the second part of Lemma 40 to the pair $a^{(i)}_{G,u}(\lambda)$, $a^{(j)}_{G,u}(\lambda)$, we get eq. (3):

$$|\,\mathrm{Im}(f(a^{i}_{G,u}(\lambda)) - f(a^{j}_{G,u}(\lambda)))| \leq \rho_R(d_u + \nu)\varepsilon_I \leq \varepsilon_I.$$

Finally, applying the second part of Lemma 40 to the pair $a^{(i)}_{G,u}(\vec{\lambda}), a^{(i)}_{G,u}(\vec{1})$ and using that $a^{(i)}_{G,u}(\vec{1}) \in \mathbb{R}$, we get:

$$|\,\mathrm{Im}(f(a^{(i)}_{G,u}(\lambda)) - f(a^{(i)}_{G,u}(\vec{1})))| \leq \rho_R \cdot |\,\mathrm{Im}(a^{(i)}_{G,u}(\lambda))| \leq \varepsilon_I.$$

which once again follows from Claim 59 and Lemma 52. This proves eq. (4) and finishes the proof of the lemma. ◀

▶ Remark 60. Note that these bounds are slightly tighter than the ones in the analogous lemma from [25], since we do not need to approximate f_κ with f.

2.7 Proof of Lemma 46

Note that the proof of Lemma 46 was shortened, but the full proof can be found in the extended version.

Proof. We will prove Lemma 46 by induction on the number of unpinned vertices. Let us first prove it for the base case in which u is the only unpinned vertex:

▷ Claim 61. If u is the only unpinned vertex of G, the induction hypothesis from Lemma 46 holds.

Proof.

(1) Note that $Z^{(i)}_{G,u}(\vec{\lambda})$ sums over the colorings which agree with the partial coloring on G and in which $\sigma(u) = i$. Since u is the only unpinned vertex, there is only one possible such coloring σ. Hence, we have that $\left|Z^{(i)}_{G,u}(\vec{\lambda})\right| = \prod_{v\in V} \left|\lambda_{\sigma(v)}\right| \geq \lambda^n_{\min}$. Note that this can not vanish, as $\vec{\lambda}$ satisfies assumption 1 from Definition 20.

(2) Since $Z^{i}_{G,u}(\vec{\lambda}) = \lambda_i \prod_{v\in V\setminus u} \lambda_{\sigma(v)}$, then $R^{(i,j)}_{G,u}(\vec{\lambda}) = \frac{\lambda_i \prod_{v\in V\setminus u} \lambda_{\sigma(v)}}{\lambda_j \prod_{v\in V\setminus u} \lambda_{\sigma(v)}} = \frac{\lambda_i}{\lambda_j}$.

(3) Using item 2 together with assumption 2 from Definition 20,

$$\left|\arg(R_{G,u}^{(i,j)}(\vec{\lambda}))\right| = |\arg(\lambda_i/\lambda_j)| \leq \nu\varepsilon_I,$$

which is much smaller than π, so $R_{G,u}^{(i,j)}(\vec{\lambda}) \notin (-\infty, 0)$.

(4), (5) Follows directly from item 2:

$$\left|\mathrm{Re}(\ln R_{G,u}^{(i,j)}(\vec{\lambda})) - \ln R_{G,u}^{(i,j)}(\vec{1})\right| = \left|\mathrm{Re}\ln\frac{\lambda_i}{\lambda_j}\right|, \quad \text{and}$$
$$\left|\mathrm{Im}\ln R_{G,u}^{(i,j)}(\vec{\lambda})\right| = \left|\mathrm{Im}\ln\frac{\lambda_i}{\lambda_j}\right|.$$

(6) If $i \notin \Gamma_u$, then by definition $Z_{G,u}^{(i)} = 0$, while if $j \in \Gamma_u$, then $Z_{G,u}^{(j)} \neq 0$ by item 1. Hence, $R_{G,u}^{(i,j)}(\vec{\lambda})$ is well-defined in this case and it takes value 0. $\triangleleft$

Let us now assume that G has $\ell \geq 2$ unpinned vertices (of which u is one) and that Lemma 46 holds for any partially-colored graph with at most $\ell - 1$ unpinned vertices. Items 1 and 2 are straightforward to prove. For item 1, observe that when we pin u to color i, we obtain a new graph G' with partition function $Z_{G'}(\vec{\lambda}) = \lambda_i^{\deg(u)-1} Z_{G,u}^{(i)}(\vec{\lambda})$. The graph G' has one fewer unpinned vertex than G, so we can assume Lemma 46 holds for it. On the other hand, note that the total number of vertices of G' is at most $\Delta - 1$ more than in G. Hence, Corollary 49 tells us that $\left|Z_{G'}(\vec{\lambda})\right| \geq 0.99^{\ell-1}\lambda_{\min}^{n+\ell(\Delta-1)}/\lambda_{\max}^{(\ell-1)\Delta}$. By definition, $|\lambda_i| \leq \lambda_{\max}$, so we conclude that $\left|Z_{G,u}^{(i)}(\vec{\lambda})\right| \geq 0.99^{\ell-1}\lambda_{\min}^{n+\ell(\Delta-1)}/\lambda_{\max}^{\ell\Delta} > 0$.

For item 2, note that if all neighbors of u are pinned, then $Z_{G,u}^{(k)}(\vec{\lambda}) = \lambda_k Z_{G-u}(\vec{\lambda})$ for any $k \in \{i, j\}$ (this is not true in general because an unpinned neighbor of u will have the restriction of not being colored i or j). Hence, $R_{G,u}^{(i,j)}(\vec{\lambda}) = \frac{\lambda_i Z_{G-u}(\vec{\lambda})}{\lambda_j Z_{G-u}(\vec{\lambda})} = \frac{\lambda_i}{\lambda_j}$.

For items 3 through 5 we will use the recurrence relation from Lemma 37:

$$R_{G,u}^{(i,j)}(\vec{\lambda}) = \frac{\lambda_i}{\lambda_j}\prod_{k=1}^{\deg_G(u)} \frac{1 - P_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = i]}{1 - P_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = j]}, \tag{6}$$

where $w_1, \dots, w_{\deg_G(u)}$ denote the neighbors of u in G.

We may restrict this product only to the vertices w_k that were unpinned in G. That's because if w_k was pinned in G to a certain color c, then $c \notin \{i, j\}$ (recall we are taking $i, j \in \Gamma_u$), so $P_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = i] = P_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = j] = 0$. That means that the factor corresponding to w_k does not alter the product. We will thus restrict this product to the unpinned neighbors of u, and rename them $w_1, \dots, w_{d_u}$.

For item 3, we will show a crude upper-bound on $\left|\arg R_{G,u}^{(i,j)}(\vec{\lambda})\right|$ (note that we already know that $R_{G,u}^{(i,j)}(\vec{\lambda}) \neq 0$ due to item 1). From eq. (6):

$$\left|\arg R_{G,u}^{(i,j)}(\vec{\lambda})\right| \leq \left|\arg\frac{\lambda_i}{\lambda_j}\right| + \sum_{k=1}^{d_u}\left|\arg(1 - \mathcal{P}_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = i])\right|$$
$$+ \sum_{k=1}^{d_u}\left|\arg(1 - \mathcal{P}_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = j])\right|.$$

The first term can be bounded through assumption 2 of Definition 20. For the rest, notice that the term involving $\mathcal{P}_{G_k^{(i,j)},\vec{\lambda}}[\sigma(w_k) = i]$ either vanishes (if $i \notin \Gamma_{w_k}$) or can be bounded with eq. (4) from Lemma 56 (if $i \in \Gamma_{w_k}$). The same logic applies to the terms involving color j. Thus,

$$\left|\arg R_{G,u}^{(i,j)}(\vec{\lambda})\right| \leq \nu\varepsilon_I + 2d_u\varepsilon_I \leq 2\Delta\varepsilon_I \leq 0.0002 < \pi,$$

which finalizes the proof of item 3. Note that we are able to apply Lemma 56 (and all the other lemmas which are a consequence of Lemma 46) due to the fact that $G_k^{(i,j)}$ has one unpinned vertex fewer than G, and hence is covered by the induction hypothesis.

For items 4 and 5, we take logarithms in equation (6) and split the terms depending on whether colors i and j are good/bad for each vertex w_k. Recall that $G_k^{(i,j)}$ has one pinned vertex fewer than G, so all the consequences of Lemma 46 hold for it. In particular, we know that w_k is nice in $G_k^{(i,j)}$, due to Lemma 54. Moreover, due to the definition of $G_k^{(i,j)}$, $\deg_{G_k}(w_k) = \deg_G(w_k) - 1 \leq \Delta - 1$. Therefore, we can apply Lemma 56, which combined with an elementary (yet somewhat involved) case analysis, yields the desired bounds. The full details can be consulted in the extended version of the paper.

Lastly, note that item 6 follows from item 1, as this implies that $Z_{G,u}^{(j)}(\vec{\lambda}) \neq 0$, while $Z_{G,u}^{(i)}(\vec{\lambda}) = 0$ by definition, since $i \notin \Gamma_u$. Hence, $R_{G,u}^{(i,j)}(\vec{\lambda}) = 0$ in that case. ◀

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