# Determining the Outerthickness of Graphs Is NP-Hard

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## Abstract

We give a short, self-contained, and easily verifiable proof that determining the outerthickness of a general graph is NP-hard. This resolves a long-standing open problem on the computational complexity of outerthickness.

Moreover, our hardness result applies to a more general covering problem $P_{\mathcal{F},k}$, defined as follows. Let $\mathcal{F}$ be a proper graph class. Let $k \geq 1$ be an integer parameter. Given an undirected simple graph $G = (V, E)$, the task is to cover the edge set $E(G)$ by at most $k$ subsets $E_1, \ldots, E_k$ such that each subgraph $(V(G), E_i)$ for $i \in [k]$ belongs to $\mathcal{F}$. Note that if $\mathcal{F}$ is monotone (in particular, when $\mathcal{F}$ is the class of all outerplanar graphs), any such cover can be converted into an edge partition by deleting overlaps; hence, in this case, covering and partitioning are equivalent.

Our result shows that for every proper graph class $\mathcal{F}$ that satisfies all of the following conditions: (a) $\mathcal{F}$ is closed under topological minors, (b) $\mathcal{F}$ is closed under 1-sums, and (c) $\mathcal{F}$ contains a cycle of length 3, the problem $P_{\mathcal{F},k}$ is NP-hard for every integer $k \geq 3$. In particular:

- For $\mathcal{F}$ equal to the class of all outerplanar graphs, our result settles the long-standing open problem on the complexity of determining outerthickness.
- For $\mathcal{F}$ equal to the class of all planar graphs, our result complements Mansfield's NP-hardness result (1983) for the thickness, which applies only to the case $k = 2$.

It is also worth noting that each of the three conditions above is necessary. If $\mathcal{F}$ is the class of all eulerian graphs, then condition (a) fails. If $\mathcal{F}$ is the class of all pseudoforests, then condition (b) fails. If $\mathcal{F}$ is the class of all forests, then condition (c) fails. For each of these three classes $\mathcal{F}$, the problem $P_{\mathcal{F},k}$ is solvable in polynomial time for every integer $k \geq 3$, showing that none of the three conditions can be dropped unless P = NP.

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## 1 Introduction

Guy (1974) [16] defined the *outerthickness* of a graph $G$ as the minimum number of parts in an edge partition of $G$ such that each part induces an *outerplanar* subgraph; that is, a planar graph admitting an embedding in the plane in which every node lies on the outer


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face. Outerthickness is an analogue of thickness, except that each part is required to be outerplanar rather than merely planar. Although Mansfield (1983) proved that determining the thickness of a graph is NP-hard [24], the complexity of determining the outerthickness has remained unsettled [2, 3, 18, 23, 27–30].

The algorithmic study of outerthickness is motivated in part by a conjecture of Chartrand et al. (1971) [9], who proposed, among other things, that every planar graph admits an edge partition into two outerplanar graphs; equivalently, that every planar graph has outerthickness at most 2. Elmallah and Colbourn (1988) [13] cited this conjecture and addressed it for 4-node-connected planar graphs by presenting an algorithm that produces such a partition into two outerplanar subgraphs. Although they did not explicitly analyze the running time, using the state of the art of the required subroutines at the time [15, 20], their algorithm can be implemented in $O(n^3)$ time, where $n$ denotes the number of nodes in the input graph (and we use this convention throughout the paper).

Subsequently, Heath (1991) claimed a proof of Chartrand et al.'s conjecture and gave an $O(n^2 \log n)$-time algorithm to partition the edges of any planar graph into at most two outerplanar graphs [17]. Later, Gonçalves (2005) established the same statement with an optimal $O(n)$-time construction [14]. It is worth noting that Gonçalves remarked that some earlier claimed proofs were later found to be incorrect, and subsequent work attribute the resolution of Chartrand et al.'s conjecture to Gonçalves [4, 10, 11, 32].

Unlike the substantial progress on planar graphs, there has been little progress on determining outerthickness for general graphs. Mäkinen and Poranen (2004) conjectured in [27, 29] that deciding the outerthickness of a general graph is NP-hard. Since then, the open status of this problem has been repeatedly highlighted in subsequent work [3, 23, 28, 30], and was reiterated as recently as 2023 by Hlinený and Masarík [18] and 2024 by Balko et al. [2]. We further note that Balko et al. prove NP-hardness for computing the uncrossed number under the assumption that computing outerthickness is NP-hard [2].

In this paper, we settle the computational complexity of determining the outerthickness of general graphs, a question that has remained open since outerthickness was introduced in 1974. Formally, we consider the following decision problem.

▶ **Problem 1** (Outerthickness)**.**

- **Input:** An $n$-node undirected simple graph $G = (V, E)$.
- **Parameter:** An integer $k \geq 1$.
- **Question:** Does $G$ admit an edge partition $E = E_1 \cup E_2 \cup \cdots \cup E_k$ such that $(V, E_i)$ is outerplanar for every $i \in [k]$?

We use $k$-OuterThickness$(G)$ to denote Problem 1 with input graph $G$ and parameter $k$ for short. Our main result is the following.

▶ **Theorem 2.** *For every integer $k \geq 3$, $k$-OuterThickness$(G)$ for general graphs $G$ is NP-complete.*

Moreover, our hardness result applies to a more general covering problem $P_{\mathcal{F},k}$, defined as follows. Let $\mathcal{F}$ be a *proper* graph class, where a graph class is called proper if it does not contain all graphs. Let $k \geq 1$ be an integer parameter. Given an undirected simple graph $G = (V, E)$, the task is to cover the edge set $E(G)$ by at most $k$ subsets $E_1, \ldots, E_k$ such that each subgraph $(V(G), E_i)$ belongs to $\mathcal{F}$. Note that if $\mathcal{F}$ is *monotone*, meaning that whenever $G \in \mathcal{F}$ and $H$ is a subgraph of $G$, we have $H \in \mathcal{F}$, then any such cover can be converted into an edge partition by deleting overlaps. Hence, for monotone classes $\mathcal{F}$, covering and partitioning are equivalent. In particular, the class of all outerplanar graphs is monotone.

We generalize the NP-hardness of $k$-OUTERTHICKNESS$(G)$ to $P_{\mathcal{F},k}$ for every proper $\mathcal{F}$ that satisfies the three conditions stated below:

▶ **Theorem 3.** *For every proper graph class $\mathcal{F}$ that satisfies all of the following conditions:*
**(a)** *$\mathcal{F}$ is closed under topological minors*[1]*,*
**(b)** *$\mathcal{F}$ is closed under 1-sums*[2]*, and*
**(c)** *$\mathcal{F}$ contains a cycle of length 3,*
*the problem $P_{\mathcal{F},k}$ is NP-hard for every integer $k \geq 3$.*

In our NP-hardness reduction, we need an indicator array $A_{\mathcal{F},q}$ that records, for each graph with at most $q$ nodes, whether it belongs to $\mathcal{F}$, where $q$ is a function of $k$. Since there are $2^{O(q^2)}$ such graphs, $A_{\mathcal{F},q}$ is a bit string of length $2^{O(q^2)}$. Thus, the existence of the appropriate array is sufficient to establish, nonconstructively, the existence of a valid NP-hardness reduction, even if there is no uniform procedure for computing this array from $\mathcal{F}$ and $q$. However, if one requires the reduction to be constructible within some bounded amount of resources for algorithmic purposes, then one must impose an additional condition on $\mathcal{F}$ to ensure that $A_{\mathcal{F},q}$ can be computed within those resource bounds.

Moreover, if membership in $\mathcal{F}$ can be decided by an algorithm $M_{\mathcal{F}}$ in polynomial time, then $P_{\mathcal{F},k} \in$ NP. A certificate is an edge covering of $E(G)$ by $k$ subsets $E_i$ for $i \in [k]$, and we can verify in polynomial time that $(V(G), E_i) \in \mathcal{F}$ for every $i \in [k]$ by running $M_{\mathcal{F}}$ on each part. Hence the certificate certifies that the $\mathcal{F}$-thickness of $G$ is at most $k$. As a result, Theorem 3 also yields NP-completeness if membership in $\mathcal{F}$ can be decided in polynomial time.

As a corollary of Theorem 3, by setting $\mathcal{F}$ to be the class of all planar graphs, we obtain Corollary 4, which complements Mansfield's NP-hardness result (1983) for the thickness [24], established only for the case $k = 2$.

▶ **Corollary 4.** *For every integer $k \geq 3$, $k$-THICKNESS$(G)$ for general graphs is NP-complete.*

It may be worth noting that Mansfield's NP-hardness proof for the thickness proceeds via a reduction from Planar-3SAT [22] due to Lichtenstein (1982), which is technical. In contrast, our hardness proof uses other source problems: for $k = 3$, we reduce from the chromatic index problem for 3-regular graphs due to Holyer (1981) [19], and for every integer $k \geq 3$, we reduce from the chromatic index problem for $k$-regular graphs due to Leven and Galil (1983) [21].

As illustrated in Table 1, each of the three conditions mentioned in Theorem 3 is necessary. If $\mathcal{F}$ is the class of all eulerian[3] graphs (including disconnected ones), then condition (a) fails. If $\mathcal{F}$ is the class of all pseudoforests, that is, the class of all undirected graphs in which each connected component contains at most one cycle, then condition (b) fails. If $\mathcal{F}$ is the class of all forests, then condition (c) fails. For each of these three classes $\mathcal{F}$, the problem $P_{\mathcal{F},k}$ is solvable in polynomial time for every integer $k \geq 3$ (the case $P_{\text{eulerian},k}$ is due to Alon and Tarsi [1], and the cases $P_{\text{forests},k}$ and $P_{\text{pseudoforests},k}$ follow from Edmonds [12]),

[1] A graph class $\mathcal{F}$ is *closed under topological minors* if for every $G \in \mathcal{F}$ and every graph $H$ that is a topological minor of $G$, we have $H \in \mathcal{F}$. Here $H$ is a *topological minor* of $G$ if some subdivision of $H$ appears as a subgraph of $G$ (equivalently, $H$ can be obtained from a subgraph of $G$ by repeatedly *smoothing* degree-2 nodes). To smooth a degree-2 node means to remove the node and join its two neighbors by an edge.

[2] A graph class $\mathcal{F}$ is *closed under 1-sums* if for every $G_1, G_2 \in \mathcal{F}$ and every choice of nodes $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, the graph obtained from the disjoint union $G_1 \uplus G_2$ by identifying $v_1$ and $v_2$ into a single node also belongs to $\mathcal{F}$.

[3] We follow Catlin's convention [8] and use lowercase intentionally and consistently.

**Table 1** The complexity status of problem $P_{\mathcal{F},k}$ for some graph classes $\mathcal{F}$. Note that eulerian graphs here including disconnected ones, and $tw(G)$ denotes the treewidth of $G$. The results highlighted in blue are proved in Theorem 3, and NPC stands for NP-complete. The NP-hardness of $P_{\text{eulerian},2}$ follows by observing its equivalence to 3-edge-colorability for 3-regular graphs, as implicitly noted in the proof of Proposition 4 of [26].

| $\mathcal{F}$ | Cond. (a) | Cond. (b) | Cond. (c) | $k=1$ | $k=2$ | each $k \geq 3$ |
|---|---|---|---|---|---|---|
| forests | Y | Y | N | P | P [12] | P [12] |
| pseudo-forests | Y | N | Y | P | P [12] | P [12] |
| eulerian graphs | N | Y | Y | P | NPC [26] | P [1] |
| cacti | Y | Y | Y | P | | NPC |
| outerplanar graphs | Y | Y | Y | P [25] | | NPC |
| planar graphs | Y | Y | Y | P [20] | NPC [24] | NPC |
| graphs of $tw(G) \leq t$ | Y | Y | Y | P [5] | | NPC |

showing that none of the three conditions can be dropped unless P = NP. In addition to the three example problems above, we found that the class $\mathcal{F}$ of all graphs whose edge sets can be partitioned into at most a constant number of forests, that is, graphs with *arboricity* bounded by a constant, also serves as a useful counterexample when attempting to replace the three conditions with weaker ones; see Section 5 for details.

### 1.1 Paper Organization

In Section 2, we introduce our notation. In Section 3, we present a proof showing that outerthickness is NP-hard, thereby settling the long-standing open question on its computational complexity. Then, in Section 4, we prove our main theorem (Theorem 3), which in particular establishes Theorem 2 for every integer $k \geq 3$. We conclude with remarks in Section 5.

## 2 Preliminaries

All graphs and subgraphs considered in this paper are finite, undirected, and simple. For a graph $G$, we write $V(G)$ and $E(G)$ for its node set and edge set, respectively. For convenience, we sometimes use an edge subset $E' \subseteq E(G)$ to denote the subgraph of $G$ induced by $E'$, namely $(V(G), E')$.

Let $\mathcal{P}$ be a graph property. A graph $G$ is said to be *edge-maximal* with respect to $\mathcal{P}$ if $G$ satisfies $\mathcal{P}$, but for every pair of non-adjacent nodes $u, v \in V(G)$, the graph obtained by adding the edge $\{u, v\}$ to $G$ does not satisfy property $\mathcal{P}$.

For every integer $m \geq 1$, we use the notation $[m]$ to denote the set of integers $\{1, \ldots, m\}$. The reduction source of our problem is defined below.

▶ **Theorem 5** (Edge-Coloring [19,21]). *Let $k \geq 3$ be an integer. A proper $k$-edge-coloring of an undirected graph $G = (V, E)$ is a mapping $c : E \to \{1, 2, \ldots, k\}$ such that $c(e) \neq c(f)$ for every pair of edges $e, f \in E$ that share an endnode. For every $k \geq 3$, given a $k$-regular undirected simple graph $G$, deciding whether $G$ admits a proper $k$-edge-coloring is NP-complete.*

## 3 NP-Hardness of Outerthickness

Given an input instance $G$ of Edge-Coloring on 3-regular graphs, defined in Theorem 5, we compute a label function $\varphi$ defined below and an auxiliary graph $H$ in polynomial time. Then, we obtain a supergraph $G'$ from joining $G$ and $H$ using the label function $\varphi$ such that $G$ admits a 3-edge-coloring if and only if $G'$ has an edge-partition into 3 outerplanar subgraphs (Lemma 7). This proves the NP-hardness stated in Theorem 2 for $k = 3$. Moreover, 3-OuterThickness$(G) \in$ NP: a certificate is an edge partition of $E(G)$ into 3 subsets $E_i$ for $i \in [3]$, and we can verify in polynomial time that $(V(G), E_i)$ is outerplanar for every $i \in [3]$ by running the linear-time outerplanarity test of Mitchell [25] on each part. Hence the certificate certifies that the outerthickness of $G$ is at most 3.

- Let $\varphi : E(G) \to \mathcal{L}$ be a labeling function, where $\mathcal{L} = \{1, 2, \ldots, |\mathcal{L}|\}$ is a constant-size (not necessarily minimum-size) label set, such that for every path of length **at most** 3 in $G$, the edges on the path receive pairwise distinct labels under $\varphi$. The existence of $\varphi$, as well as a construction running in time polynomial in the input size of $G$, is guaranteed by Proposition 11.

- We pick a sufficiently large constant $C$ such that every $C$-node graph[4] with outerthickness 3 contains an (not necessarily maximum) independent set of $\alpha \geq |\mathcal{L}|$ nodes. Then we construct a $C$-node graph $H$ that is edge-maximal with outerthickness 3; that is, adding any edge not in $E(H)$ increases the outerthickness of $H$. Given the promise of $C$, let $w_1, w_2, \ldots, w_\alpha$ be the $\alpha$ nodes in an independent set in $H$. The existence of $H$, as well as a construction running in time constant in the input size of $G$, is guaranteed by Proposition 12.

We initialize $G'$ by taking a copy of $G$ and a copy of $H$ with disjoint node sets, and then identifying $G'$ with their union. Then, for each edge $e \coloneqq \{u, v\} \in E(G)$, we add the edges $\{u, w_{\varphi(e)}\}$ and $\{v, w_{\varphi(e)}\}$ to $G'$, as illustrated in Figure 1. Note that $G'$ remains a simple graph: since any two edges incident to the same node in $G$ receive distinct labels under $\varphi$, no parallel edges are created.

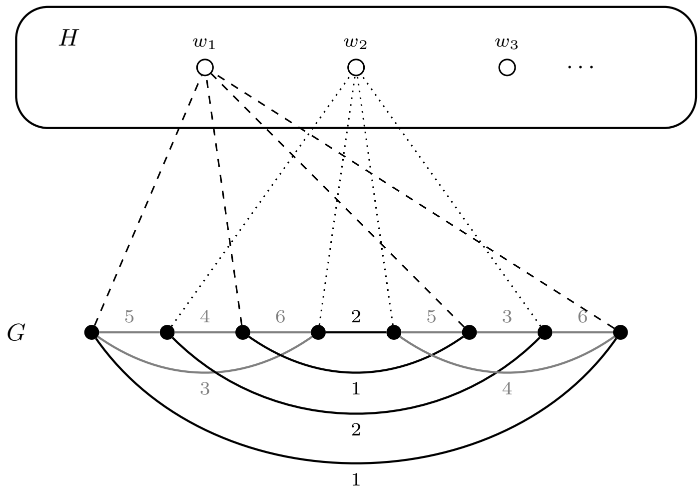



**Figure 1** An illustration of the construction of $G'$. For each edge $e \coloneqq \{u, v\} \in E(G)$, we add the edges $\{u, w_{\varphi(e)}\}$ and $\{v, w_{\varphi(e)}\}$ to $G'$. In this example, we have $\mathcal{L} = \{1, 2, \ldots, 6\}$, and we depict the added edges corresponding to $e$ with $\varphi(e) \in \{1, 2\}$.

We need the following observation for our main lemma.

[4] By a $C$-node graph, we mean a graph with $C$ nodes.

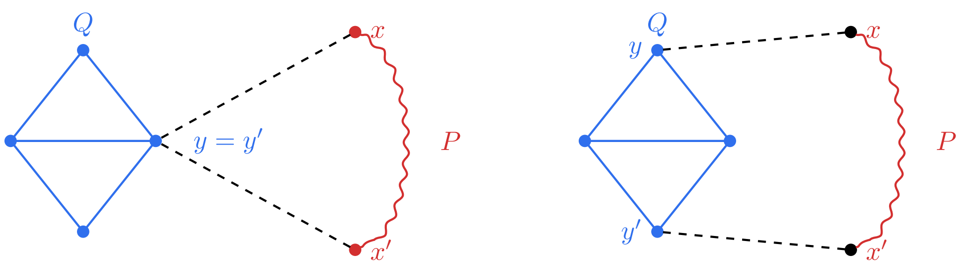
▶ **Observation 6.** *Let $z \geq 1$ be an integer. Let $Q$ be a graph that is edge-maximal with outerthickness $z$, and let $E_1, E_2, \ldots, E_z$ be an edge-partition of $E(Q)$ into $z$ outerplanar subgraphs. Let $P$ be a path with endpoints $x$ and $x'$ such that $V(P) \cap V(Q) = \varnothing$. For any nodes $y, y' \in V(Q)$, define*

$$S_{y,y'} \coloneqq E_1 \cup P \cup \big\{\{x, y\}, \{x', y'\}\big\}.$$

**(a)** *If $y = y'$, then $S_{y,y'}$ is outerplanar.*
**(b)** *If $y \neq y'$ and $\{y, y'\} \notin E(Q)$, then $S_{y,y'}$ is not outerplanar.*

**Proof.** (a) Fix an outerplanar embedding of $E_1$ in which all nodes, and in particular $y$, lie on the outer face. Draw the path $P$ in the outer face without crossings, and then connect both end-nodes of $P$ to $y$ by drawing the two edges within the outer face so that they intersect the embedding of $E_1$ only at $y$. This yields an outerplanar embedding of $S_{y,y'}$, as illustrated in Figure 2.

(b) If $y \neq y'$ and $\{y, y'\} \notin E(Q)$, suppose for a contradiction that $S_{y,y'}$ is outerplanar. Every node of $P$ has degree 2 in $S_{y,y'}$. Smoothing a degree-2 node in an outerplanar graph preserves outerplanarity: given an outerplanar embedding in which all nodes lie on the outer face, smoothing such a node yields an embedding in which all remaining nodes still lie on the outer face. Hence we may smooth all nodes of $P$ in $S_{y,y'}$ while preserving the outerplanarity of $S_{y,y'}$, thereby replacing the union of path $P$ and $\{\{x, y\}, \{x', y'\}\}$ in $S_{y,y'}$ by the single edge $\{y, y'\}$. It follows that $Q \cup \{\{y, y'\}\}$ has outerthickness $z$. Since $\{y, y'\} \notin E(Q)$, this adds an edge to $Q$, contradicting the assumption that $Q$ is edge-maximal with outerthickness $z$. Therefore, $S_{y,y'}$ is not outerplanar. ◀




(a) Case $y = y'$.

(b) Case $y \neq y'$ and $\{y, y'\} \notin E(Q)$.

**Figure 2** An illustration of $S_{y,y'}$ with $Q = K_4 - \{e\}$ and $z = 1$.

We are ready to prove the main lemma.

▶ **Lemma 7.** *$G$ admits a 3-edge-coloring if and only if $G'$ has outerthickness at most* 3.

**Proof.** If $G$ admits a 3-edge-coloring, then $E(G)$ can be partitioned into three matchings $M_1, M_2$, and $M_3$. Let $i \in [3]$ and $\ell \in \mathcal{L}$. Then the edges $e \coloneqq \{u, v\} \in M_i$ with $\varphi(e) = \ell$ give rise to a collection $T_{i,\ell}$ of triangles, each on nodes $\{u, v, w_\ell\}$, where the only node shared among these triangles is the common node $w_\ell$. Since $M_i$ is a matching, no triangle in $T_{i,\ell}$ shares a node with any triangle in $T_{i,\ell'}$ for $\ell \neq \ell' \in \mathcal{L}$. Let $H_1, H_2$, and $H_3$ be an edge partition of $H$ into three outerplanar subgraphs. By Observation 6(a), for each $i \in [3]$, the subgraph

$$H_i \cup \bigcup_{\ell \in \mathcal{L}} T_{i,\ell}$$

is outerplanar. Thus, $G'$ has outerthickness at most 3.

Otherwise, $G$ does not admit a 3-edge-coloring. Suppose $E(G')$ can be partitioned into three outerplanar subgraphs $E_i$ for $i \in [3]$, the following claims hold.

▷ Claim 8. Some path of length 2 in $G$ is contained entirely in an outerplanar subgraph $E_i$ for some $i \in [3]$.

Proof. Now $E(G')$ is partitioned into three outerplanar subgraphs $E_i$ for $i \in [3]$. Restricting this partition to $E(G) \subseteq E(G')$, the induced parts $E(G) \cap E_i$ for $i \in [3]$ cannot all be matchings; otherwise, they would yield a 3-edge-coloring of $G$. Therefore, for some $i \in [3]$, the subgraph $E_i$ contains a path of length 2 in $G$. ◁

▷ Claim 9. For each node $v \in V(G)$, let $w_{v,j}$ for $j \in [3]$ be neighbors of $v$ in $H$. Then, the edges $\{v, w_{v,j}\}$ for $j \in [3]$ have to be assigned to pairwise distinct outerplanar subgraphs $E_i$ for $i \in [3]$.

Proof. The nodes $w_{v,j}$ for $j \in [3]$ are pairwise distinct, since the three edges of $G$ incident to $v$ receive distinct labels in $\mathcal{L}$ by the construction of $\varphi$. Recall that $w_1, w_2, \ldots, w_\alpha$ (and hence $w_{v,j}$ for $j \in [3]$) form an independent set.

Suppose that, for some $i \in [3]$, the subgraph $(V(G'), E_i)$ contains at least two edges from $\{\{v, w_{v,j}\} : j \in [3]\}$, say without loss of generality $\{v, w_{v,1}\}$ and $\{v, w_{v,2}\}$. Let

$$E' \coloneqq (E(H) \cap E_i) \cup \{\{v, w_{v,j}\} : j \in [2]\}.$$

By Observation 6(b), the graph $(V(H) \cup \{v\}, E')$ is not outerplanar: apply the observation with $(Q, E_1, P, y, y')_{\text{Observation 6}} = (H, E_i, (v), w_{v,1}, w_{v,2})_{\text{here}}$. On the other hand, $E'$ is a subgraph of the outerplanar graph $E_i$, a contradiction.

Consequently, the three edges $\{v, w_{v,j}\}$ for $j \in [3]$ must be assigned to three distinct outerplanar subgraphs $E_i$ for $i \in [3]$. ◁

▷ Claim 10. For each edge $e \coloneqq \{x, y\} \in E(G)$, the three edges $\{x, w_{\varphi(e)}\}$, $\{y, w_{\varphi(e)}\}$, $\{x, y\}$ have to be assigned to the same outerplanar subgraph $E_i$ for some $i \in [3]$.

Proof. Since $E_1, E_2, E_3$ form a partition of $E(G)$, we may assume w.l.o.g. that $\{x, y\} \in E_1$. Let

$$W_x \coloneqq \{w_{x,1},\, w_{x,2},\, w_{\varphi(e)}\} \qquad \text{and} \qquad W_y \coloneqq \{w_{y,1},\, w_{y,2},\, w_{\varphi(e)}\}$$

denote the sets of nodes in $H$ that are adjacent to $x$ and $y$ in $G'$, respectively. By the construction of $\varphi$, each of $W_x$ and $W_y$ consists of three distinct nodes, and $W_x \cap W_y = \{w_{\varphi(e)}\}$. By Claim 9, there exist edges $\{x, w\}$ with $w \in W_x$ and $\{y, w'\}$ with $w' \in W_y$ that both lie in $E_1$.

If $w \neq w'$, then by Observation 6(b) the subgraph induced by the union of the path $P \coloneqq (w, x, y, w')$ and $E_1 \cap E(H)$ is not outerplanar: apply the observation with $(Q, E_1, P, y, y')_{\text{Observation 6}} = (H, E_1, (x, y), w, w')_{\text{here}}$.

Therefore $w = w' = w_{\varphi(e)}$, or equivalently, the three edges $\{x, w_{\varphi(e)}\}$, $\{y, w_{\varphi(e)}\}$, and $\{x, y\}$ are all assigned to $E_1$. ◁

By Claim 8, we assume w.l.o.g. that $E_1$ contains the path of length 2, denoted by $(a, b, c)$ with $a, b, c \in V(G)$ and $e_1 \coloneqq \{a, b\}, e_2 \coloneqq \{b, c\} \in E(G)$. By Claim 10, $E_1$ contains the four edges $\{a, w_{\varphi(e_1)}\}$, $\{b, w_{\varphi(e_1)}\}$, $\{b, w_{\varphi(e_2)}\}$, $\{c, w_{\varphi(e_2)}\}$ as well. However, by Claim 9, $\{b, w_{\varphi(e_1)}\}$ and $\{b, w_{\varphi(e_2)}\}$ have to be assigned to different $E_i$'s, a contradiction.

As a result, $E(G')$ cannot be partitioned into three outerplanar subgraphs $E_i$ for $i \in [3]$ or, equivalently, $G'$ has outerthickness greater than 3. This completes the proof. ◀

## 3.1 Proofs of Deferred Claims

▶ **Proposition 11.** *Let $k \geq 3$ be an integer. Let $G$ be a $k$-regular graph. There exists a labeling function $\varphi : E(G) \to \mathcal{L}$ with $|\mathcal{L}| \leq 2k(k-1)+1$, computable in polynomial time, such that the edges on any path in $G$ of length* ***at most*** *3 receive pairwise distinct labels.*

**Proof.** In a $k$-regular graph $G$, let $e = \{u, v\}$ be an edge. Any other edge $f \neq e$ that can appear together with $e$ on a path of length at most 3 must either be incident to $u$ or $v$ (at most $2(k-1)$ choices), or be incident to one of the $2(k-1)$ neighbors of $u$ and $v$ (at most $2(k-1)^2$ additional choices). Hence, for each edge $e$, the number of edges $f \neq e$ that can be contained with $e$ in some path of length at most 3 is at most $2k(k-1)$. Therefore, we can greedily assign to each edge $e$ a label from $\{1, 2, \ldots, 2k(k-1)+1\}$ so that no such edge $f$ receives the same label as $e$; since at most $2k(k-1)$ labels are forbidden at each step, at least one label remains available. The running time of the above greedy labeling is linear in the input size. ◀

▶ **Proposition 12.** *For each integer $t \geq 1$, there exists a constant $C$ such that every $C$-node graph $H$ with outerthickness 3 contains an independent set of size at least $\alpha \geq t$. Moreover, such a graph $H$ and an independent set of size $\alpha$ in $H$ can be computed in constant time.*

**Proof.** Consider an arbitrary constant $C$ such that

$$\binom{C}{2} > 3(2C-3). \tag{1}$$

Every $C$-node outerplanar graph has at most $2C-3$ edges, and hence any graph with outerthickness 3 has at most $3(2C-3)$ edges, implying that the complete graph $K_C$ has outerthickness greater than 3.

Let $H$ be any $C$-node graph with outerthickness 3. Such a subgraph exists: the empty graph has outerthickness 0, while $K_C$ has outerthickness greater than 3, and adding a single edge can increase the outerthickness by at most one.

By the Caro–Wei bound [7,31], every $C$-node $m$-edge graph has an independent set of size at least $\left\lceil \frac{C^2}{C+2m} \right\rceil$. Since $H$ has $m \leq 3(2C-3)$ edges, we obtain that $H$ has an independent set of size at least

$$\left\lceil \frac{C^2}{C+6(2C-3)} \right\rceil > t \text{ for some sufficiently large } C. \tag{2}$$

If we set $C > 13t$, then Equations (1) and (2) both hold.

Finally, since $C$ is a constant, we can compute such an $H$ and an independent set of size at least $t$ in constant time by brute force, using an outerplanarity testing algorithm [25]. ◀

It may be worth noting that a simpler proof of Proposition 12 can be obtained from the fact that every outerplanar graph is 3-colorable. However, we retain the current proof because it generalizes to the proof of Lemma 13.

# 4 NP-hardness of $\mathcal{F}$-Thickness

We generalize the NP-hardness proof for outerthickness in Section 3 to every proper graph class $\mathcal{F}$ that satisfies all of the following conditions:

**(a)** $\mathcal{F}$ is closed under topological minors,
**(b)** $\mathcal{F}$ is closed under 1-sums, and
**(c)** $\mathcal{F}$ contains a cycle of length 3.

By Condition (a), $\mathcal{F}$ is monotone. Thus, any covering of the edge set of an input graph for the problem $P_{\mathcal{F},k}$ can be converted into an edge partition by deleting overlaps; hence, in this setting, covering and partitioning are equivalent. **Thus, in this section, we treat $P_{\mathcal{F},k}$ as a partition problem:** Let $\mathcal{F}$ be a proper graph class. We define the *$\mathcal{F}$-thickness* of an undirected simple graph $G$, denoted by $\theta_{\mathcal{F}}(G)$, to be the minimum integer $k$ such that $E(G)$ admits a partition into at most $k$ subsets $E_1, E_2, \ldots, E_k$ with $(V(G), E_i) \in \mathcal{F}$ for all $i \in [k]$. Thus, the problem $P_{\mathcal{F},k}$ asks, given an undirected simple graph $G$, whether $\theta_{\mathcal{F}}(G) \leq k$.

Our NP-hardness reduction proceeds as follows. Let $k \geq 3$ be an integer. Given an input instance $G$ of Edge-Coloring on $k$-regular graphs, defined in Theorem 5, we compute a label function $\varphi$ defined below and an auxiliary graph $H$ in polynomial time. Then, we obtain a supergraph $G'$ from joining $G$ and $H$ using the label function $\varphi$ such that $G$ admits a $k$-edge-coloring if and only if $G'$ has $\theta_{\mathcal{F}}(G') \leq k$ (Lemma 16). This proves the NP-hardness claimed in Theorem 3 for every integer $k \geq 3$.

- Let $\varphi : E(G) \to \mathcal{L}$ be a labeling function, where $\mathcal{L} = \{1, 2, \ldots, |\mathcal{L}|\}$ is a constant-size (not necessarily minimum-size) label set, such that for every path in $G$ of length **at most** 3, the edges on the path receive pairwise distinct labels under $\varphi$. The existence of such a function $\varphi$, together with a construction running in time polynomial in the input size of $G$, is guaranteed by Proposition 11. The resulting label set $\mathcal{L}$ satisfies $|\mathcal{L}| = 2k(k-1) + 1 = O(1)$, as desired.
- We pick a sufficiently large constant $C$ such that every $C$-node graph with $\mathcal{F}$-thickness $k$ contains an (not necessarily maximum) independent set of $\alpha \geq |\mathcal{L}|$ nodes. Then we construct a $C$-node graph $H$ that is edge-maximal with $\mathcal{F}$-thickness $k$. Given the promise of $C$, let $w_1, w_2, \ldots, w_\alpha$ be the $\alpha$ nodes in an independent set in $H$. The existence of $H$, as well as a construction running in time constant in the input size of $G$, is shown in Lemma 13.

▶ **Lemma 13.** *For each integer $\ell \geq 1$, there exists a constant $C$ such that every $C$-node graph $H$ that is edge-maximal with $\mathcal{F}$-thickness $k$ contains an independent set of at least $\alpha \geq \ell$ nodes. Such an $H$ and an independent set in $H$ of $\alpha$ nodes can be computed in constant time.*

**Proof.** Since $\mathcal{F}$ is proper and closed under topological minors, there exists an integer $t$ such that the complete graph $K_t \notin \mathcal{F}$. By a theorem of Bollobás and Thomason [6], there is a constant $D_t$ such that every graph $F$ with more than $D_t|V(F)|$ edges contains a subdivision of $K_t$, and hence does not belong to $\mathcal{F}$. Therefore, every $F \in \mathcal{F}$ satisfies $|E(F)| \leq D_t|V(F)|$.

Choose an arbitrary constant $C$ such that

$$\binom{C}{2} > k(D_t C). \tag{3}$$

▷ Claim 14. For every $F \in \mathcal{F}$ and $e \notin E(F)$, $\theta_{\mathcal{F}}(F \cup \{e\}) \leq \theta_{\mathcal{F}}(F) + 1$.

Proof. Since $C_3 \in \mathcal{F}$ (Condition (c)) and $\mathcal{F}$ is closed under topological minors (Condition (a)), we have $P_2 \in \mathcal{F}$. Thus, we have $\theta_{\mathcal{F}}(F \cup \{e\}) \leq \theta_{\mathcal{F}}(F) + \theta_{\mathcal{F}}(\{e\}) \leq \theta_{\mathcal{F}}(F) + 1$. ◁

Since $\theta_{\mathcal{F}}(\emptyset) = 0$, the family

$$\mathcal{S} := \{S \subseteq K_C : \theta_{\mathcal{F}}(S) \leq k\}$$

is nonempty. Let $S^*$ be a member of $\mathcal{S}$ with maximum number of edges. Thus $S^*$ is edge-maximal with $\mathcal{F}$-thickness at most $k$. Here is why. For every edge $e \in E(K_C) \setminus E(S^*)$ we must have $\theta_{\mathcal{F}}(S^* \cup \{e\}) > k$, otherwise $S^* \cup \{e\} \in \mathcal{S}$ contradicts the choice of $S^*$. Moreover, by Claim 14, we have $\theta_{\mathcal{F}}(S^*) = k$.

By the Caro–Wei bound [7, 31], every $C$-node $m$-edge graph has an independent set of size at least $\left\lceil \frac{C^2}{C+2m} \right\rceil$. Since $H$ has $m \le k(D_tC)$ edges, we obtain that $H$ has an independent set of size at least

$$\left\lceil \frac{C^2}{C + 2k(D_tC)} \right\rceil > \ell \text{ for some sufficiently large } C. \tag{4}$$

If we set $C > (1 + 2kD_t)\ell$, then Equations (3) and (4) both hold.

Finally, since $C$ is a constant, we can compute such an $H$ and an independent set of size at least $\ell$ in constant time by brute force, using the indicator array $A_{\mathcal{F},q}$ for $q \le C$ mentioned in Section 1. ◀

▶ **Observation 15.** *Let $z \ge 1$ be an integer. Let $Q$ be a graph that is edge-maximal with $\mathcal{F}$-thickness $z$, and let $E_1, E_2, \dots, E_z$ be an edge-partition of $E(Q)$ into $z$ subgraphs in $\mathcal{F}$. Let $P$ be a path with endpoints $x$ and $x'$ such that $V(P) \cap V(Q) = \varnothing$. For any nodes $y, y' \in V(Q)$, define*

$$S_{y,y'} \coloneqq E_1 \cup P \cup \big\{\{x, y\}, \{x', y'\}\big\}.$$

*If $y \ne y'$ and $\{y, y'\} \notin E(Q)$, then $S_{y,y'}$ does not belong to $\mathcal{F}$.*

**Proof.** If $y \ne y'$ and $\{y, y'\} \notin E(Q)$, suppose for contradiction that $S_{y,y'}$ belongs to $\mathcal{F}$. Every node of $P$ has degree 2 in $S_{y,y'}$. Since $\mathcal{F}$ is closed under topological minors (Condition (a)), smoothing a degree-2 node preserves the membership in $\mathcal{F}$. Hence we may smooth all nodes of $P$ in $S_{y,y'}$ while preserving that $S_{y,y'} \in \mathcal{F}$, thereby replacing the union of path $P$ and $\{\{x, y\}, \{x', y'\}\}$ in $S_{y,y'}$ by the single edge $\{y, y'\}$. It follows that $Q \cup \{\{y, y'\}\}$ has $\mathcal{F}$-thickness $z$. Since $\{y, y'\} \notin E(Q)$, this adds an edge to $Q$, contradicting the assumption that $Q$ is edge-maximal with $\mathcal{F}$-thickness $z$. Therefore, $S_{y,y'}$ does not belong to $\mathcal{F}$. ◀

We initialize $G'$ by taking a copy of $G$ and a copy of $H$ with disjoint node sets, and then identifying $G'$ with their union. Then, for each edge $e \coloneqq \{u, v\} \in E(G)$, we add the edges $\{u, w_{\varphi(e)}\}$ and $\{v, w_{\varphi(e)}\}$ to $G'$. Note that $G'$ remains a simple graph: since any two edges incident to the same node in $G$ receive distinct labels under $\varphi$, no parallel edges are created.

We are ready to prove the key lemma.

▶ **Lemma 16.** *$G$ admits a $k$-edge-coloring if and only if $G'$ has $\theta_{\mathcal{F}}(G') \le k$.*

**Proof.** If $G$ admits a $k$-edge-coloring, then $E(G)$ can be partitioned into $k$ matchings $M_i$ for $i \in [k]$. Let $i^* \in [k]$ and $\ell^* \in \mathcal{L}$. The edges $e \coloneqq \{u, v\} \in M_{i^*}$ with $\varphi(e) = \ell^*$ give rise to a collection $T_{i^*,\ell^*}$ of triangles, each on nodes $\{u, v, w_{\ell^*}\}$, where the only node shared among these triangles is the common node $w_{\ell^*}$. Since $M_i$ is a matching, no triangle in $T_{i^*,\ell^*}$ shares a node with any triangle in $T_{i^*,\ell'}$ for $\ell^* \ne \ell' \in \mathcal{L}$. Since $H$ is edge-maximal with $\mathcal{F}$-thickness $k$, $H$ has an edge partition into $k$ subgraphs in $\mathcal{F}$, denoted by $H_i$ for $i \in [k]$. Since $\mathcal{F}$ is closed under 1-sums (Condition (b)) and $\mathcal{F}$ contains $C_3$ (Condition (c)), for each $i \in [k]$, the subgraph

$$H_i \cup \bigcup_{\ell \in \mathcal{L}} T_{i,\ell}$$

is contained in $\mathcal{F}$. To see why, for every $\ell \in \mathcal{L}$, we attach each triangle in $T_{i,\ell}$ to $H_i$ by taking a 1-sum that identifies the node $w_\ell$. Repeating this 1-sum operation for all triangles in $T_{i,\ell}$ for all $\ell \in \mathcal{L}$, yields exactly $H_i \cup \bigcup_{\ell \in \mathcal{L}} T_{i,\ell}$, which therefore lies in $\mathcal{F}$. This argument holds for every $i \in [k]$, so $G'$ has $\mathcal{F}$-thickness at most $k$.

Otherwise, $G$ does not admit a $k$-edge-coloring. Suppose $E(G')$ can be partitioned into $k$ subgraphs in $\mathcal{F}$, denoted by $E_i$ for $i \in [k]$, the following claims hold.

▷ Claim 17. Some path of length 2 in $G$ is contained entirely within a subgraph $E_i$ (in $\mathcal{F}$) for some $i \in [k]$.

Proof. Now $E(G')$ is partitioned into $k$ subgraphs $E_i$ in $\mathcal{F}$ for $i \in [k]$. Restricting this partition to $E(G) \subseteq E(G')$, the induced parts $E(G) \cap E_i$ for $i \in [k]$ cannot all be matchings; otherwise, they would yield a $k$-edge-coloring of $G$. Therefore, for some $i \in [k]$, the subgraph $E_i$ contains a path of length 2 in $G$. ◁

▷ Claim 18. For each node $v \in V(G)$, let $w_{v,j}$ for $j \in [k]$ be neighbors of $v$ in $H$. Then, the edges $\{v, w_{v,j}\}$ for $j \in [k]$ have to be assigned to pairwise distinct subgraphs $E_i$ (in $\mathcal{F}$) for $i \in [k]$.

Proof. The nodes $w_{v,j}$ for $j \in [k]$ are pairwise distinct, since the $k$ edges in $G$ incident to $v$ receive distinct labels in $\mathcal{L}$ by the construction of $\varphi$. Recall that $w_1, w_2, \ldots, w_\alpha$ (and hence $w_{v,j}$ for $j \in [k]$) form an independent set.

Suppose that, for some $i \in [k]$, the subgraph $E_i$ contains at least two edges from $\{\{v, w_{v,j}\} : j \in [k]\}$, say without loss of generality $\{v, w_{v,1}\}$ and $\{v, w_{v,2}\}$. Let

$$E' \coloneqq (E(H) \cap E_i) \cup \{\{v, w_{v,j}\} : j \in [2]\}.$$

By Observation 15, the graph $(V(H) \cup \{v\}, E')$ does not belong to $\mathcal{F}$: indeed, apply the observation with

$$(Q, E_1, P, y, y')_{\text{Observation 15}} = (H, E_i, (v), w_{v,1}, w_{v,2})_{\text{here}}.$$

On the other hand, $E'$ is a subgraph of $E_i$, which lies in $\mathcal{F}$. This contradicts Condition (a), since $\mathcal{F}$ is closed under topological minors (and hence under taking subgraphs).

Consequently, the $k$ edges $\{v, w_{v,j}\}$ for $j \in [k]$ must be assigned to $k$ distinct subgraphs $E_i$ for $i \in [k]$. ◁

▷ Claim 19. For each edge $e \coloneqq \{x, y\} \in E(G)$, the three edges $\{x, w_{\varphi(e)}\}$, $\{y, w_{\varphi(e)}\}$, $\{x, y\}$ have to be assigned to the same subgraph $E_i$ (in $\mathcal{F}$) for some $i \in [k]$.

Proof. Since $E_i$ for $i \in [k]$ form a partition of $E(G)$, we may assume w.l.o.g. that $\{x, y\} \in E_1$. Let

$$W_x \coloneqq \{w_{x,1}, \ldots, w_{x,k-1}, w_{\varphi(e)}\} \text{ and } W_y \coloneqq \{w_{y,1}, \ldots, w_{y,k-1}, w_{\varphi(e)}\}$$

denote the sets of nodes in $H$ that are adjacent to $x$ and $y$ in $G'$, respectively. By the construction of $\varphi$, each of $W_x$ and $W_y$ consists of $k$ distinct nodes, and $W_x \cap W_y = \{w_{\varphi(e)}\}$. By Claim 18, there exist edges $\{x, w\}$ with $w \in W_x$ and $\{y, w'\}$ with $w' \in W_y$ that both lie in $E_1$.

If $w \neq w'$, then by Observation 15 the subgraph induced by the union of the path $P \coloneqq (w, x, y, w')$ and $E_1 \cap E(H)$ does not belong to $\mathcal{F}$: apply the observation with

$$(Q, E_1, P, y, y')_{\text{Observation 15}} = (H, E_1, (x, y), w, w')_{\text{here}}.$$

Therefore $w = w' = w_{\varphi(e)}$, or equivalently, the three edges $\{x, w_{\varphi(e)}\}$, $\{y, w_{\varphi(e)}\}$, and $\{x, y\}$ are all assigned to $E_1$. ◁

By Claim 17, we assume w.l.o.g. that $E_1$ contains the path of length 2, denoted by $(a, b, c)$ with $a, b, c \in V(G)$ and $e_1 := \{a, b\}, e_2 := \{b, c\} \in E(G)$. By Claim 19, $E_1$ contains the four edges $\{a, w_{\varphi(e_1)}\}$, $\{b, w_{\varphi(e_1)}\}$, $\{b, w_{\varphi(e_2)}\}$, $\{c, w_{\varphi(e_2)}\}$ as well. However, by Claim 18, $\{b, w_{\varphi(e_1)}\}$ and $\{b, w_{\varphi(e_2)}\}$ have to be assigned to different $E_i$'s, a contradiction.

As a result, $E(G')$ cannot be partitioned into $k$ subgraphs $E_i$ in $\mathcal{F}$ for $i \in [k]$ or, equivalently, $G'$ has $\mathcal{F}$-thickness greater than $k$. This completes the proof. ◀

As a result, Theorem 3 follows.

## 5 Concluding Remarks

Finally, we remark on the case $k = 2$. It is known that every planar graph has outerthickness at most 2 due to Gonçalves (2005) [14], and hence 2-OuterThickness($G$) for planar graphs contains only Yes-instances. For general graphs, however, the complexity of 2-OuterThickness($G$) was, to the best of our knowledge, open at the time this paper was submitted for review. We later found that 2-OuterThickness($G$) is NP-hard as well; the proof is included in the arXiv version of this paper. More broadly, in the context of our hardness framework in Theorem 3, any proper graph class $\mathcal{F}$ satisfying Conditions (a)–(c) for which the corresponding problem $P_{\mathcal{F},k}$ with $k = 2$ is NP-hard.

It is also worth noting that Conditions (a)–(c) in Theorem 3 are necessary even for $k = 2$, in the following sense: relaxing any one of them allows graph classes $\mathcal{F}$ for which $P_{\mathcal{F},2}$ is solvable in polynomial time.

For instance, if $\mathcal{F}$ is the class of forests, then Condition (c) fails, and $P_{\mathcal{F},k}$ for $k = 2$ is in P by the same algorithm as in the case $k \geq 3$. Similarly, if $\mathcal{F}$ is the class of pseudoforests, then Condition (b) fails, and $P_{\mathcal{F},k}$ for $k = 2$ also is in P by the same algorithm as for $k \geq 3$. Finally, if $\mathcal{F}$ is the class of graphs of bounded arboricity, then Condition (a) is not satisfied[5], and the corresponding partitioning problem $P_{\mathcal{F},2}$ is in P by Edmonds' matroid-partitioning algorithm [12].

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