

Colorful Minors

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Abstract

We introduce the notion of *colorful minors*, which generalizes the classical concept of rooted minors in graphs. A q -colorful graph is defined as a pair (G, χ) , where G is a graph and χ assigns to each vertex a (possibly empty) subset of at most q colors. The colorful minor relation enhances the classical minor relation by merging color sets at contracted edges and allowing the removal of colors from vertices. This framework naturally models algorithmic problems involving graphs with (possibly overlapping) annotated vertex sets. We develop a structural theory for colorful minors by establishing three core theorems characterizing $\mathcal{H}$ -colorful minor-free graphs, where $\mathcal{H}$ consists either of a clique or a grid with all vertices assigned all colors, or of grids with colors segregated and ordered on the outer face. Our results reveal that when exclusion is imposed not only on graphs but also to the way colors are distributed in them, a more refined structural landscape appears.

Leveraging our structural insights, we provide a *complete classification* – parameterized by the number q of colors – of all colorful graphs that exhibit the Erdős–Pósa property with respect to colorful minors. On the algorithmic side, we deduce that colorful minor testing is fixed-parameter tractable. Together with the fact that the colorful minor relation forms a well-quasi-order, this implies that every colorful minor-monotone parameter on colorful graphs admits a fixed-parameter algorithm. Furthermore, we derive two algorithmic meta-theorems (AMTs) whose structural conditions are linked to extensions of treewidth and Hadwiger number on colorful graphs. Our results suggest how known AMTs can be extended to incorporate not only the structure of the input graph but also the way the colored vertices are distributed in it.

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1 Introduction

Over the last decades, a fruitful interplay between structural graph theory and theoretical computer science has been the search for structural properties of graphs that facilitate the design of efficient algorithms for otherwise NP-hard problems (see for example [12]). One of the most studied such parameters is a measure for the “tree-likeness” known in its most popular form as *treewidth* [47, 2, 5, 3]. Treewidth plays an exceptional role not only in the world of (parameterized) algorithms, but also in the theory of graph minors by Robertson and Seymour [46, 48, 49, 50]. A central theorem of Robertson and Seymour is the *Grid Theorem* [48] that states that any minor-closed graph class $\mathcal{C}$ has bounded treewidth if and only if $\mathcal{C}$ does not contain all planar graphs. This reveals a fundamental limitation in the use of treewidth (or any minor-monotone parameter really) as a parametrization for an NP-hard problem: While there exists a wide range of problems that are tractable on classes of bounded treewidth, there exist many important problems that are NP-hard even on planar graphs [21, 11]. This means that, in the regime of minor-closed graph classes, such problems are tractable if and only if a planar graph is excluded, unless $P = NP$.

A particularly interesting class of problems tractable on graphs of bounded treewidth, but NP-hard on more general minor-closed graph classes, are certain problems whose instances come with specific vertex sets, typically called *roots*, *terminals*, or *annotated vertices*. A prototypical example of such a problem is STEINER TREE. The STEINER TREE problem receives a graph G accompanied with a *special* set $X \subseteq V(G)$ of annotated vertices as input and asks for the minimum number of edges in a connected subgraph of G containing all vertices of X . The problem is known to be NP-complete on planar graphs, however it can be solved in polynomial time with respect to structural parameterizations that instead of (or in addition to) restricting the *global structure* of the graph (like treewidth), focus on the structure *relative to the annotation* [14, 15, 32], thereby providing tractable algorithms *beyond* the regime of minor-closed classes of bounded treewidth.

This hints at the possibility of developing a general structural and algorithmic theory for annotated graphs, providing tools that supersede the limits imposed by classical graph minor theory.

Computational problems such as STEINER TREE can be naturally understood as problems on annotated graphs. Recently, several positive algorithmic results for STEINER TREE on annotated graphs excluding certain types of so-called *rooted minors*¹ have emerged [24, 27] alongside purely structural results on rooted minors [37, 26, 19] and found various applications [26, 1]. Moreover, STEINER TREE is just one example where a theory of rooted minors may open a new horizon of tractability. There exists a variety of other important problems exhibiting similar behavior, among them MULTIWAY CUT [36, 13, 40], DISJOINT PATHS [39, 55], #DEFECT MATCHING [11], and others [20, 11, 34]. Moreover, annotation has recently been shown to also play a key role in structural extensions of Courcelle’s Theorem [7, 8, 9] by restricting quantification to sets of low *bidimensionality*² [54].

Another area where annotation comes into play is the study of *Erdős-Pósa dualities*, which lies at the crossroads of structural and algorithmic graph theory. The Erdős-Pósa property for (rooted) minors asks, for a fixed (annotated) graph H , whether for any (annotated) graph

¹ An annotated graph (H, Y) is a *rooted minor* of an annotated graph (G, X) if there exists a collection $\{J_v\}_{v \in V(H)}$ of pairwise vertex-disjoint connected subgraphs of G such that for all $uv \in E(H)$ there is an edge between J_u and J_v , and for all $v \in Y$ we have $V(J_v) \cap X \neq \emptyset$.

² The *bidimensionality* of a vertex set X in a graph G , denoted by $\text{bidim}(G, X)$, is the largest k such that (G, X) contains the $(k \times k)$ -grid with all vertices annotated as a rooted minor [57].

G either G contains many vertex-disjoint occurrences of H as a (rooted) minor, or G has a small vertex set whose deletion removes all occurrences of H as a (rooted) minor. Note that not every (annotated) graph satisfies the Erdős-Pósa property. The Erdős-Pósa property has been investigated by many authors in a variety of settings [48, 41], and several key cases are also known for rooted minors [28, 44, 35, 26].

While the concept of rooted minors in annotated graphs appears to be relatively well explored, a general theory for deriving results such as those above still seems to be lacking. Beyond the fact that many of the results mentioned are inspired by the theory of graph minors, no systematic extension of its powerful toolkit has yet emerged. This is somewhat surprising given the significant structural depth and algorithmic potential of these concepts, as we elaborate below.

1.1 Our results

In this paper, we develop a comprehensive theory for a generalization of rooted minors in multi-annotated graphs, which we refer to as *colorful minors* of *colorful graphs*. Instead of a single annotated set of vertices X , we consider q sets $X_1, \dots, X_q$ of annotated vertices, which may be overlapping. We investigate both the algorithmic and structural properties of colorful graphs that exclude certain classes of *colorful minors* – an appropriate adaptation of the concept of rooted minors to our setting. Our results suggest that much of the combinatorial and algorithmic power of classical graph minor theory can be effectively extended to the colorful setting.

From structure... Our core contributions are a series of structural theorems for (H, ψ) -colorful minor-free q -colorful graphs, both for general (H, ψ) and for the particular case where H is planar – under two distinct assumptions on the distribution of colors. Our results use as a departure point the notion of the bidimensionality of a vertex set and related structural concepts [41, 42, 43]. Building on our structural theorems, we derive a complete classification of *all* q -colorful graphs that satisfy the Erdős-Pósa property, for every $q \in \mathbb{N}$ (under the colorful minor relation).

...to algorithms. Our next step is to apply our structural theorems to algorithmic settings, with the broader goal of developing an algorithmic graph structure theory applicable to problems involving vertex annotations. To this end, we first observe that colorful graphs are well quasi-ordered under the colorful minor relation. We also provide an algorithm that decides whether a q -colorful graph (H, ψ) is a colorful minor of a q -colorful graph (G, χ) in time³ $\mathcal{O}_{|H|+q}(\|G\|^{1+o(1)})$. These two facts imply that membership for any colorful minor-closed class of colorful graphs is decidable in polynomial time.

Next, we show how classical algorithmic meta-theorems (AMTs) – such as Courcelle’s Theorem and the recent algorithmic meta-theorem of Sau, Stamoulis, and Thilikos [54] – can be extended to colorful graphs, where the structural conditions are expressed via the exclusion of colorful minors. This framework enables the resolution of problems on graphs containing arbitrarily large grid or clique minors while, up to now, the exclusion of a minor appeared as a necessary (sparsity) condition. Our results support a key insight emerging from the body of work mentioned above:

³ By $g(n) \in \mathcal{O}_k(f(n))$ we mean that there is a function $h: \mathbb{N} \rightarrow \mathbb{N}$ such that $g(n) \in \mathcal{O}(h(k) \cdot f(n))$. Also, given a graph G , we use $|G|$ and $\|G\|$ for the number of its vertices and edges respectively.

Problem complexity does not only depend on the global structure of the input graph, but crucially on how the annotated sets are distributed within it.

The algorithmic results presented here are not intended to be exhaustive; rather, they serve to illustrate this principle and to lay the groundwork for future exploration of algorithmic problems involving vertex annotations. In particular, we provide algorithmic applications for two of our three main structural results – Theorems 2 and 3 – while leaving the investigation of algorithmic consequences of Theorem 4 to future work.

Colorful graphs. Let q be a non-negative integer. A q -colorful graph is a pair (G, χ) where G is a graph and⁴ $\chi: V(G) \rightarrow 2^{[q]}$. For a vertex $v \in V(G)$ we call $\chi(v)$ the *palette* of v . For illustrations, we use **magenta** for 1, **blue** for 2, **yellow** for 3, and **green** for 4.

Certainly, every vertex of G receives a (possibly empty) subset of colors. Given a vertex set $X \subseteq V(G)$, we define $\chi(X) := \bigcup_{v \in X} \chi(v)$. We also use notation $\chi(G) := \chi(V(G))$. Similarly, for $I \subseteq [q]$ we denote by $\chi^{-1}(I)$ the set $\{v \in V(G) : I \cap \chi(v) \neq \emptyset\}$. When $I = \{i\}$ is a singleton, we write $\chi^{-1}(i)$ instead of $\chi^{-1}(\{i\})$.

A q -colorful graph (G, χ) where $\chi(G) \subsetneq [q]$ is called *restricted* and a q -colorful graph (G, χ) with $\chi(G) = \emptyset$ is called *empty*. Note that any restricted q -colorful graph is also a $(q-1)$ -colorful graph, so the notion of being restricted is highly dependent on the context given by the total set of colors. Finally, a q -colorful graph (G, χ) is said to be *rainbow*⁵ if $\chi(v) = [q]$ for all $v \in V(G)$.

A colorful graph (H, ψ) is a (*colorful*) *subgraph* of a colorful graph (G, χ) , denoted by $(H, \psi) \subseteq (G, \chi)$, if H is a subgraph of G and $\psi(v) \subseteq \chi(v)$ for all $v \in V(H)$. If (G, χ) is a colorful graph and $H \subseteq G$ is a subgraph of G we write, in slight abuse of notation, (H, χ) for the q -colorful subgraph of (G, χ) where all vertices of H receive precisely the colors they received in (G, χ) .

A colorful graph (H, ψ) is a *colorful minor* of a colorful graph (G, χ) if (H, ψ) can be obtained from (G, χ) by means of the following operations:

- deleting an edge $e \in E(G)$,
- deleting a vertex $v \in V(G)$ and restricting the domain of χ to $V(G) \setminus \{v\}$,
- for some $v \in V(G)$ and $i \in \chi(v)$ overwrite $\chi(v) := \chi(v) \setminus \{i\}$ (we refer to this operation as *removing a color (from v)*), and
- contracting an edge $uv \in E(G)$, that is introducing a new vertex x_{uv} with neighborhood $N_G(u) \cup N_G(v)$, setting $\chi(x_{uv}) := \chi(u) \cup \chi(v)$, and then deleting both u and v from the resulting colorful graph.

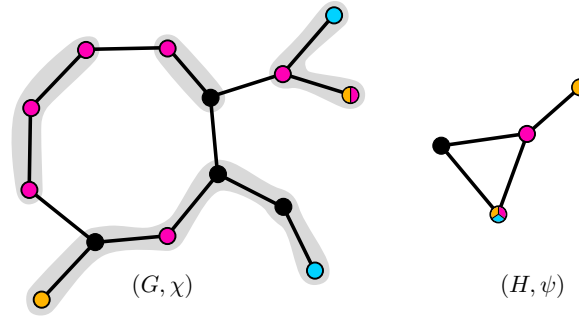
See Figure 1 for an example. Notice that 1-colorful graphs are exactly annotated graphs and the notions of colorful minors and rooted minors coincide on annotated graphs.

We derive the following, as a simple consequence of a deep theorem of Robertson and Seymour [53].

► **Theorem 1.** *For every non-negative integer q , the class of all q -colorful graphs is well-quasi-ordered by the colorful minor relation.*

⁴ For $n, m \in \mathbb{Z}$, we denote $\{x \mid 1 \leq x \leq n, x \in \mathbb{Z}\}$ by $[n]$ and $\{x \mid n \leq x \leq m, x \in \mathbb{Z}\}$ by $[n, m]$.

⁵ Notice that – in slight violation of the English language – we use the word “rainbow” as an adjective here.



■ **Figure 1** Two colorful graphs (G, χ) and (H, ψ) such that (H, ψ) is a colorful minor of (G, χ) . The gray subgraphs of (G, χ) indicate the connected vertex sets that have to be contracted in order to form (G, ψ) . Notice that it is necessary to remove some colors from some of the vertices of (G, χ) .

Structural results. We prove a total of three structure theorems as follows. The first two describe the structure of graphs excluding the q -colorful rainbow t -clique and the q -colorful rainbow $(k \times k)$ -grid respectively as a colorful minor. Moreover, we generalize a theorem of Marx, Seymour, and Wollan [37], that describes the structure of 1-colorful graphs excluding a $(k \times k)$ -grid where exactly the vertices of the first row receive color 1, to an arbitrary number of colors.

We begin with our simplest result – excluding a rainbow clique – which can be naturally phrased in terms of a modulator to restricted graphs. For a vertex set $X \subseteq V(G)$ in a graph G we define the *torso* of X in G , denoted by $\text{torso}(G, X)$, as the graph obtained from $G[X]$ by turning the neighborhood of J in X into a clique for every component J of $G - X$. In short, every q -colorful graph (G, χ) that excludes a rainbow clique as a colorful minor has a set X whose torso excludes a minor and where every component of $(G - X, \chi)$ is free of some (not necessarily the same) color.

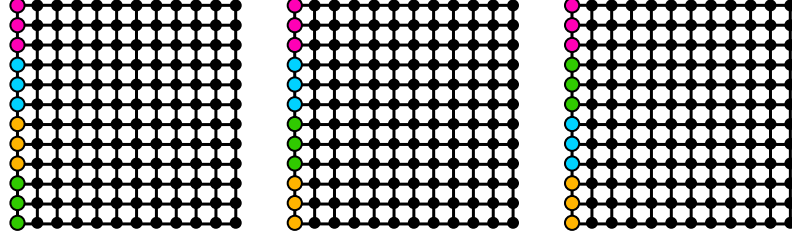
► **Theorem 2.** *There exists a function $\text{rc}: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for all non-negative integers q and t , and all q -colorful graphs (G, χ) one of the following holds:*

1. (G, χ) contains a rainbow t -clique as a colorful minor, or
2. *there exists a set $X \subseteq V(G)$ such that the torso of X in G is $K_{\text{rc}(q,t)}$ -minor-free and for all components J of $G - X$, the q -colorful graph (J, χ) is restricted.*

Moreover, $\text{rc}(q, t) \in \text{poly}(qt)$ and there exists an algorithm that finds one of the two outcomes above in time $2^{\text{poly}(qt)} \cdot |G|^3 \|G\| \log |G|$.

A *tree-decomposition* for a graph G is a pair $\mathcal{T} = (T, \beta)$ where T is a tree and $\beta: V(T) \rightarrow 2^{V(G)}$, called the *bags* of $\mathcal{T}$, assigns a set of vertices of G to every node of T such that $\bigcup_{t \in V(T)} \beta(t) = V(G)$, for every $e \in E(G)$ there is $t \in V(T)$ with $e \subseteq \beta(t)$, and for every $v \in V(G)$ the set $\{t \in V(T) \mid v \in \beta(t)\}$ is connected. The *adhesion* of $\mathcal{T}$ is $\max_{dt \in E(T)} |\beta(d) \cap \beta(t)|$ and the width of $\mathcal{T}$ is defined as $\max_{t \in V(T)} |\beta(t)| - 1$. The *treewidth* of a graph G , denoted by $\text{tw}(G)$, is the smallest integer k such that there is a tree-decomposition of width at most k for G .

Our next theorem generalizes a theorem of Marx, Seymour, and Wollan [37] (see also [26]). Let $q \geq 1$ be an integer. A (q, k) -*segregated grid* is a q -colorful graph (G, χ) where G is the $(qk \times qk)$ -grid where the vertices of the first column can be numbered as $v_1, \dots, v_{qk}$ in order of their appearance, and we have $\chi(u) = \emptyset$ for all $u \in V(G) \setminus \{v_1, \dots, v_{qk}\}$ and there exists a permutation π of $[q]$ such that for every $i \in [q]$, $\chi(\{v_j \mid j \in [(i-1)q+1, iq]\}) = \{\pi(i)\}$. In this situation, we say that (G, χ) *realizes* π . See Figure 2 for an example.



■ **Figure 2** Three non-isomorphic $(4, 3)$ -segregated grids.

Theorem 2 shows that every (H, ψ) -colorful minor-free colorful graph can be turned into a collection of restricted graphs by deleting an H' -minor-free subgraph. Segregated grids capture exactly the situation where we require H' to be a planar graph. We may express this in form of a parameter as follows. The *torso treewidth* of a q -colorful graph (G, χ) is the smallest integer k such that there exists a set $X \subseteq V(G)$ such that the treewidth of the torso of X in G is at most k and for every component J of $G - X$ the q -colorful graph (J, χ) is restricted.

► **Theorem 3.** *There exists a function $sg: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for all non-negative integers q and k , and all q -colorful graphs (G, χ) , $q \geq 1$, one of the following holds:*

1. *(G, χ) contains as a colorful minor some (q, k) -segregated grid and the torso treewidth of (G, χ) is more than $k - 1$, or*
2. *the torso treewidth of (G, χ) is at most $sg(q, k)$.*

Moreover, $sg(q, k) \in k^{2^{O(q)}}$ and there exists an algorithm that finds one of the two outcomes above in time $2^{k^{2^{O(q)}}} \cdot |G|^3 \|G\| \log |G|$.

The second part of the first outcome is technically not necessary for the structural statement but serves as a pointer to this min-max relation. Moreover, the min-max relation itself also supplies a guarantee for the approximation of torso treewidth via the algorithm from Theorem 3.

Notice that Theorem 3 is optimal in the sense that the torso treewidth of segregated grids is unbounded. In other words, Theorem 3 can be seen as a min-max duality theorem for segregated grids. We remark that for the case $q = 1$, the torso treewidth of (G, χ) has been independently defined by Jansen and Swennenhuis [27] and by Hodor, La, Micek, and Rambaud [26] under different names.

It is easy to see that Theorem 3 is far from a structure theorem for q -colorful graphs that exclude an arbitrary planar colorful minor. This is in stark contrast to the 0-colorful case provided by the Grid Theorem [48]. It turns out that, in order to fully describe the structure of q -colorful graphs that exclude a fixed but arbitrary planar colorful minor, we need the full power of Robertson and Seymour's *Graph Minor Structure Theorem* (GMST) [52, 30, 23]. The definitions necessary to state the GMST are rather involved and technical and are omitted in this extended abstract.

We say that a graph G has a k -near embedding in a surface Σ if there exists $A \subseteq V(G)$ with $|A| \leq k$ such that $G - A = G_0 \cup G_1 \cup \dots \cup G_\ell$, $\ell \leq k$, such that G_0 has an embedding into Σ with ℓ pairwise vertex-disjoint faces F_i such that for all $i \in [\ell]$, $V(G_0) \cap V(G_i) = V(F_i)$, the G_i 's are pairwise vertex-disjoint for positive i , and for each $i \in [\ell]$, G_i has a path-

decomposition (P_i, β_i) ⁶ of width at most k such that $V(P_i) = V(F_i)$, the vertices of P_i appear in agreement to their cyclic ordering on the boundary of F_i , and $v \in \beta_i(v)$, for all $v \in V(F_i)$. The set A is called the *apex set*, the graphs G_i , $i \in [\ell]$, are called the *vortices*, and the sets $V(G_i) \setminus V(F_i)$ are the *interiors* of the vortices.

In the context of q -colorful graphs, we need some additional information regarding the colors. Let (T, β) be a tree decomposition of a q -colorful graph (G, χ) . The *colorful torso* (G_t, χ_t) of (G, χ) at a node $t \in V(T)$ is the colorful graph obtained from $(G[\beta(t)], \chi)$ by turning, for every $dt \in E(T)$, the set $\beta(d) \cap \beta(t)$ into a clique and adding $\chi(\bigcup_{h \in V(T_d)} (\beta(h) \setminus \beta(t)))$ to $\chi(v)$ for all $v \in \beta(d) \cap \beta(t)$ where T_d is the unique component of $T - dt$ that contains d .

► **Theorem 4.** *There exists a function $\text{rg}: \mathbb{N}^2 \rightarrow \mathbb{N}$ such that for all non-negative integers q and k , and all q -colorful graphs (G, χ) one of the following holds:*

1. (G, χ) contains the q -colorful rainbow $(k \times k)$ -grid as a colorful minor, or
2. (G, χ) has a tree-decomposition (T, β) of adhesion at most $\text{rg}(q, k)$ such that for all $t \in V(T)$, either t is a leaf with unique neighbor d and $(G[\beta(t) \setminus \beta(d)], \chi)$ is restricted, or the colorful torso (G_t, χ_t) of G at t has an $\text{rg}(q, k)$ -near embedding and there exists $\emptyset \neq I_t \subseteq [q]$ such that all vertices $v \in \beta(t)$ with $I_t \cap \chi_t(v) \neq \emptyset$ belong to the apex set or the interior of a vortex.

Moreover, $\text{rg}(q, k) \in 2^{k^{\mathcal{O}(1)} 2^{2^{\mathcal{O}(q)}}}$ and there exists an algorithm that finds one of the two outcomes above in time $2^{2^{k^{\mathcal{O}(1)} 2^{2^{\mathcal{O}(q)}}}} \cdot |G|^3 \|G\| \log |G|$.

Notice that Theorem 4 can also be seen as a min-max duality as implied by results of Thilikos and Wiederrecht [57].

To rephrase Theorem 4, the absence of a rainbow grid as a colorful minor implies that, with respect to any large grid minor in G , and in the absence of a large clique minor, there is always at least one color of small *bidimensionality*; moreover, all vertices of such colors can be confined to the apex set and the vortices of a near-embedding.

In particular, for 1-colorful graphs, all colored vertices are confined entirely to the vortices and the apex set, while the surface part of the near-embedding contains no colored vertices. This appears to be a far-reaching generalization of the idea of covering all colored vertices in a planar graph by a bounded number of faces.

We believe that this provides a theoretical foundation for why many problems that are tractable on planar graphs when their annotated set is covered by few faces remain tractable on all colorful minor-closed classes of 1-colorful graphs that exclude some fixed planar 1-colorful graph. Moreover, it suggests that, in algorithm design, vortices in k -near embeddings may play a role analogous to that played by faces in planar graphs.

Delineating the Erdős-Pósa property for colorful minors. We say that a q -colorful graph (H, ψ) has the *Erdős-Pósa property* if there exists a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every k , every q -colorful graph (G, χ) either has k pairwise vertex-disjoint q -colorful subgraphs, each containing (H, ψ) as a colorful minor, or there is a set $S \subseteq V(G)$ with $|S| \leq f(k)$ such that $(G - S, \chi)$ does not contain (H, ψ) as a colorful minor.

It follows directly from a construction by Robertson and Seymour [48] that if a pair (H, ψ) has the Erdős-Pósa property, then the graph H must be planar. We prove that the boundary between instances that do and do not exhibit the Erdős-Pósa property shifts in subtle ways in the presence of colors – subtleties that are worth highlighting.

⁶ A *path-decomposition* of a graph is a tree-decomposition (T, β) where T is a path.

For every $q \in \mathbb{N}$, we provide the *full list* of graphs $\mathcal{O}_q$ such that a q -colorful graph has the Erdős-Pósa property if and only if it does not contain any of the graphs from $\mathcal{O}_q$ as a colorful minor. In other words, we provide, for every q , the set of *colorful minor-obstructions* for the Erdős-Pósa property of q -colorful graphs. We also stress that $|\mathcal{O}_q| \in \mathcal{O}(q^4)$, i.e., its size grows polynomially in the number of colors. Figure 3 gives a schematic definition of $\mathcal{O}_q$, for every q .

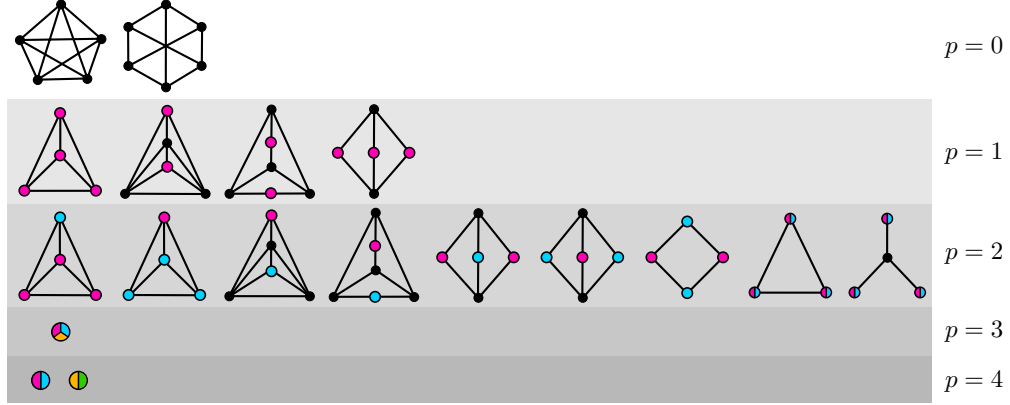


Figure 3 The obstructions to the Erdős-Pósa property: For $q \in \mathbb{N}$, the family $\mathcal{O}_q$ consists, for $p \in [0, 4]$, of the union of all p -colorful graphs displayed in the corresponding level above, where the levels for $p \in [4]$ are copied for each of the $\binom{q}{p}$ possible choices for p out of q colors.

► **Theorem 5.** *For every non-negative integer q , a q -colorful graph has the Erdős-Pósa property if and only if it does not contain any of the graphs in $\mathcal{O}_q$ as a colorful minor.*

Theorem 5 illustrates that the excluded minors for q -colorful graphs do not need to have all q colors. In fact $\mathcal{O}_q$ contains graphs without any colored vertices and no member of $\mathcal{O}_q$ has five colors no matter the choice of q . Our results on the Erdős-Pósa property indicate that colorful minors are a proper generalization of the concept of minors, able to express a variety of different properties.

Towards a general framework for colorful parameters. With Theorem 3 and Theorem 4 in hand, it becomes apparent that there are fundamentally distinct ways to extend a graph parameter to a colorful analogue, and that these choices can lead to remarkably different structural decompositions. Which extension should be regarded as “canonical” appears to depend strongly on the intended application and on the structural/algorithmic consequences one wishes to capture.

For instance, from the perspective of structural decompositions, torso treewidth emerges as a particularly natural analogue of treewidth, and this is further supported by the fact that it delineates the Erdős-Pósa property for colorful minors as Theorem 5 shows, mirroring the role played by treewidth in the uncolored setting. In contrast, when viewed through the lens of obstructions, bidimensionality provides the appropriate analogue, as it yields the correct structure theorem for colorful graphs excluding an arbitrary colorful planar graph as a colorful minor, and potentially, the correct structural parameterization to obtain precise dichotomy complexity results for problems on colorful graphs that become NP-hard in planar graphs, as previously discussed.

These observations point towards a broader, open-ended research direction, which is to better understand these phenomena by examining other classical graph parameters beyond treewidth, and by developing general principles for lifting such parameters to colorful analogues. We return to this perspective in Section 1.4, where we discuss this issue in more detail and propose a conjecture.

Algorithmic results. One of the ground-breaking results of Robertson and Seymour’s Graph Minors Series is the *Graph Minor Algorithm* that allows to check if a graph H is a minor of a graph G in time $\mathcal{O}_{|H|}(|G|^3)$ [51] (see [31, 33] for improvements). This algorithm also solves a rooted version of the MINOR CHECKING problem which, as a special case, contains the so-called k -DISJOINT PATHS problem.

The q -COLORFUL MINOR CHECKING problem has as input two q -colorful graphs (G, χ) and (H, ψ) , and the task is to determine whether (G, χ) contains (H, ψ) as a colorful minor.

We present a simple reduction to the rooted minor checking problem which allows us to leverage the algorithm of Korhonen, Pilipczuk, and Stamoulis [33] in order to deduce a similar result for q -COLORFUL MINOR CHECKING.

► **Theorem 6.** *There exist an algorithm that takes as input two q -colorful graphs (H, ψ) and (G, χ) and solves q -COLORFUL MINOR CHECKING in time $\mathcal{O}_{q+|H|}(\|G\|^{1+o(1)})$.*

Theorems 1 and 6 imply that the recognition problem for any proper colorful minor-closed class can be solved in polynomial time. We present below a parametric variant of this, based on the general framework introduced by Fellows and Langston [17, 18] for the existence of fixed-parameter tractable algorithms.

A (q -colorful) *graph parameter* is a map $\mathbf{p}$ assigning a non-negative integer to every (q -colorful) graph. We say that a q -colorful graph parameter $\mathbf{p}$ is *colorful minor-monotone* if for every q -colorful graph (G, χ) and every colorful minor (H, ψ) of (G, χ) we have $\mathbf{p}(H, \psi) \leq \mathbf{p}(G, \chi)$. Again, combining Theorems 1 and 6 we can deduce the following.

► **Theorem 7.** *For every colorful minor-monotone q -colorful graph parameter $\mathbf{p}$, there exist an algorithm that, given a q -colorful graph (G, χ) and a non-negative integer k , checks whether $\mathbf{p}(G, \chi) \leq k$ in $\mathcal{O}_{q+k}(\|G\|^{1+o(1)})$ time.*

We stress that the result of Theorem 7 is not constructive, i.e., it does not give any way to construct the claimed algorithm, as it is based on the finiteness of the obstruction set of the class $\{(G, \chi) \mid \mathbf{p}(G, \chi) \leq k\}$, which is only existentially provided by Theorem 1.

Meta-algorithmic Applications. Our two algorithmic results can be seen as extensions of Courcelle’s Theorem and of a recent result due to Sau, Stamoulis, and Thilikos [54] to colorful graphs. While the structural parameters in these theorems are treewidth, denoted by tw and Hadwiger number, denoted by hw (that is the maximum size of a clique minor) respectively, we now introduce two parameters on colorful graphs that serve as our new combinatorial restrictions.

A key condition for applying our algorithmic results is the notion of *folio-representability* of a class of colorful graphs $\mathcal{C}$ (i.e., the yes-instances of some algorithmic problem).

Informally, $\mathcal{C}$ is *folio-representable* if one can efficiently replace a subgraph P of a colorful graph (G, χ) with colored vertices only within a small subset B of the vertices of G – viewed as a “protuberance”⁷ interacting with the rest of the graph via the “boundary” B – by a bounded-size subgraph P' , such that the resulting graph (G', χ) belongs to $\mathcal{C}$ if and only if (G, χ) does.

This replacement is guided by a notion of folio-equivalence: P and P' must have the same set of rooted topological minors (up to a given size), with respect to the specified boundary B . Efficient computation of such replacements is enabled by results from [25], making folio-representability a natural combinatorial condition for the class $\mathcal{C}$.

In what follows, whenever we refer to 1-colorful graphs we may alternatively use the notation (G, X) instead of (G, χ) where $X := \chi^{-1}(1)$. Given a graph G and a set $X \subseteq V(G)$, we define the *monodimensionality* of X in G as the maximum order of a $(1, k)$ -segregated grid colorful minor of (G, X) . Given a q -colorful graph (G, χ) and a color $i \in [q]$ we define the *monodimensionality* of i in (G, χ) as the monodimensionality of $\chi^{-1}(i)$ in G .

We now give the logical conditions under which our framework operates. In our first result, we require the class $\mathcal{C}$ to be definable in *Counting Monadic Second-Order Logic*⁸ (CMSO). Courcelle’s Theorem says that, for every graph class $\mathcal{G}$ definable by some CMSO-formula ϕ , membership in $\mathcal{G}$ can be decided in time $\mathcal{O}_{|\phi|+\text{tw}(G)}(|G|)$. We prove the following analogue for colorful graphs.

► **Theorem 8.** *Let $q \in \mathbb{N}$, and let $\mathcal{C}$ be a q -colorful graph problem that is both folio-representable and definable by some CMSO-formula ϕ . Then, there exists an algorithm that, given a q -colorful graph (G, χ) , decides whether $(G, \chi) \in \mathcal{C}$ in time $\mathcal{O}_{q+|\phi|+k}(|G|^{\mathcal{O}(1)})$, where k is the maximum monodimensionality of a color in (G, χ) .*

From Theorem 3, the maximum monodimensionality of a color in a q -colorful graph (G, χ) is equivalent to the maximum torso treewidth of $(G, \chi^{-1}(i))$ over all colors $i \in [q]$. Notice that Theorem 8 can be restated by setting k to be this equivalent quantity, which plays the same role for colorful graphs as treewidth does for graphs in Courcelle’s Theorem.

For our second result, we consider a fragment of CMSO denoted as $\text{CMSO}/\text{tw} + \text{dp}$. In this logic, quantification is restricted to sets of bounded bidimensionality, and a special predicate $\text{dp}(t_1, \dots, t_r, s_1, \dots, s_r)$ is available, asserting the existence of r vertex-disjoint paths connecting the pairs $(t_1, s_1), \dots, (t_r, s_r)$. It was shown by Sau, Stamoulis, and Thilikos [54] that if a graph class $\mathcal{G}$ is definable in $\text{CMSO}/\text{tw} + \text{dp}$, then membership in $\mathcal{G}$ can be decided in time $\mathcal{O}_{|\phi|+\text{hw}(G)}(|G|^2)$.

The Hadwiger number of a color i in a q -colorful graph (G, χ) is the maximum k for which $(G, \chi^{-1}(i))$ contains the rainbow k -clique as a colorful minor. We prove the following.

► **Theorem 9.** *Let $q \in \mathbb{N}$, and let $\mathcal{C}$ be a q -colorful graph problem that is folio-representable and definable by some $\text{CMSO}/\text{tw} + \text{dp}$ -formula ϕ . Then, there exists an algorithm that, given a q -colorful graph (G, χ) , decides whether $(G, \chi) \in \mathcal{C}$ in time $\mathcal{O}_{q+|\phi|+k}(|G|^{\mathcal{O}(1)})$, where k is the maximum Hadwiger number of a color of (G, χ) .*

We emphasize that the applicability of Theorems 8 and 9, based on the structural theorems Theorems 2 and 3 respectively, exceeds the scope of Courcelle’s Theorem and the results of [54]. The generality of our extension is displayed by the fact that, for every $q \geq 1$, there

⁷ We use the term *protuberance* to distinguish it from the term “protrusion” introduced in [4], where similar replacement operations were applied. Unlike protrusions, which are required to have bounded treewidth, protuberances are characterized by being “internally colorless”.

⁸ There exists a formula $\phi \in \text{CMSO}$ such that $(G, \chi) \in \mathcal{C}$ if and only if (G, χ) models ϕ .

are classes of q -colorful graphs (G, χ) for which the maximum monodimensionality/Hadwiger number of a color in (G, χ) is bounded while the treewidth/Hadwiger number of G is unbounded.

This is a strong indication that, for problems on colorful graphs, the appropriate structural parameterizations toward deriving (meta-)algorithmic results should take into account the way the colors are distributed in the inputs.

We derive Theorems 8 and 9 as special cases of a more general result applicable to optimization problems.

1.2 Brief insight into our approach

The general approach to proving Robertson-Seymour-style structure theorems typically consists of establishing a local structure theorem with respect to a wall and then applying a well-known technique, originating from the work of Robertson and Seymour [50], that turns the local statement into a global one based on tree-decompositions. Our structure theorems 2, 3, and 4 follow this general paradigm.

In the uncolored setting, proving a local structure theorem with respect to a wall usually proceeds as follows: Starting from a large wall W in the graph G , one either finds

- a.i) a large clique-minor that is highly connected to W or
- a.ii) a structural decomposition of G relative to W , typically of a topological nature, such as a flat wall or, more generally, a near embedding of G in some surface with additional properties tailored to the theorem at hand.

For our results, handling outcome (a.ii) relies on sophisticated topological arguments stemming from the Graph Minor Structure Theorem [52] and its subsequent improvements [30, 23] as well as more recent refinements [56, 41, 43]. While technically demanding, this part largely builds on and adapts existing machinery to deal with annotation in near embeddings [43], except for Theorem 3 which uses different techniques which we comment on in the next paragraph.

Rainbow clique-minors and a multicolored Menger-type theorem. The genuinely new difficulties and contributions arise in outcome (a.i). In the colored settings, the mere presence of a large clique-minor provides little information on the structure of the colorful graph with respect to the colored vertices. For our purposes, it is therefore essential to understand how well the colored vertices are connected to this clique-minor. Concretely, given a q -colorful graph (G, χ) containing a large clique-minor, we aim to establish the following dichotomy:

- b.i) there exists a small set of vertices S and a color $i \in [q]$ such that the component of $(G - S, \chi)$ containing most of the clique-minor is free of vertices of color i , or
- b.ii) (G, χ) contains a large rainbow clique-minor that is highly connected to the original clique-minor.

This result constitutes a central new ingredient underpinning all our structural theorems.

In the context of the Disjoint Paths problem, a similar result was established by Robertson and Seymour [51] for 1-colorful graphs with a bounded number of colored vertices. Our setting is substantially more general: we allow q -colorful graphs with an arbitrary number of colored vertices. Our proof involves a *one-to-many* strengthening of Menger's Theorem which may be of independent interest. For all positive integers k, ℓ , given vertex sets $X_1, \dots, X_\ell$ and Y in a graph G , either

- c.i) there exist $k \cdot \ell$ disjoint paths in G , with exactly k paths linking X_i to Y , or
- c.ii) there exists an index $i \in [\ell]$ and a vertex set S of size less than $k \cdot \ell$ that separates X_i from Y .

Interestingly, our results on the Erdős-Pósa property for colorful minors imply that even a *two-to-two* strengthening of Menger’s theorem cannot hold in general, not even approximately. Consequently, our structural results are in a sense “tight”, as we cannot expect to replace condition (b.i) with a stronger alternative.

It is noteworthy that in the local version of Theorem 3, outcome (a.ii) also relies on the one-to-many strengthening of Menger above to control how well-connected the colored vertices are to a given flat wall in the graph.

Annotations and the Erdős-Pósa property. Erdős-Pósa-type dualities in graph minors and beyond are intrinsically related to structural characterization of restricted classes of graphs, often in terms of “global” decomposition theorems based on tree-decompositions. A prime example of this interaction is the celebrated characterization of the proper minor-closed classes which have the Erdős-Pósa property for minors, shown by Robertson and Seymour [48] to be precisely those of bounded treewidth.

Crucial to this duality is showing that graph classes of unbounded treewidth necessarily fail to satisfy the Erdős-Pósa property. By the grid theorem, such classes are precisely those containing all planar graphs. Consequently, this reduces to understanding why the two Kuratowski minors K_5 and $K_{3,3}$ themselves do not admit the Erdős-Pósa property. In [48], Robertson and Seymour address this by constructing families of graphs based on K_5 and $K_{3,3}$ that explicitly violate the Erdős-Pósa property. These constructions are inherently topological: The resulting graphs embed minimally in the torus or the projective plane, but not in the sphere. At a high level, the argument exploits the *additivity of Euler-genus* under disjoint union [38], which implies that repeatedly combining non-planar graphs forces unbounded Euler-genus.

In the colorful setting, we initially follow the same high-level strategy. We identify torso treewidth as the correct parameter governing the Erdős-Pósa property for colorful minors. Utilizing our grid theorem for torso treewidth (Theorem 3), we identify the set $\mathcal{O}_q$ of q -colorful graphs and for the reverse direction of Theorem 5, we have to show that each q -colorful graph in $\mathcal{O}_q$ violates the Erdős-Pósa property for colorful minors.

At this point, however, the classical approach breaks down. Unlike the uncolored case, most graphs in $\mathcal{O}_q$ are planar with particular assumptions on the topology induced by the colored vertices. As a consequence, the standard arguments are no longer applicable, as they fundamentally rely on non-planar embeddings.

Instead, our arguments crucially exploit the distribution of colored vertices within these graphs. We show that the topology induced by this distribution – rather than by the underlying graph alone – can still be leveraged to construct families that violate the Erdős-Pósa property. Once more, this highlights the importance of the arrangement of colored vertices within the underlying graph.

1.3 Related work

Various combinatorial problems concerning rooted minors with specific root configurations have been studied by Kawarabayashi [29], Wollan [58], Fabila-Monroy and Wood [16], and Moore [10]. For 1-colorful graphs, structural results have been obtained by Marx, Seymour, and Wollan [37]; Hodor, La, Micek, and Rambaud [26]; and Fiorini, Kober, Seweryn, Shantanam, and Yuditsky [19].

Building on the structural result of Marx, Seymour, and Wollan [37], Hodor, La, Micek, and Rambaud [26] established the Erdős-Pósa property for all connected planar 1-colorful graphs in which the colored vertices lie on the boundary of a single face. For q -colorful

graphs, the only known result concerning the Erdős–Pósa property involves cycles spanning a prescribed number of distinct colors, as shown by Kwon and Marx [35]. In the more intricate setting of labeled minors, Erdős–Pósa dualities have also been established by Bruhn, Joos, and Schaudt [6].

The most extensively studied problem on 1-colorful graphs is the STEINER TREE problem, which was shown to be solvable in polynomial time when the torso treewidth is bounded [27], and when a rainbow K_4 is excluded [24]. To the best of our knowledge, no combinatorial or algorithmic study has been undertaken on q -colorful graphs as a general and unifying combinatorial framework.

1.4 From faces to vortices

In this paper we presented some algorithmic consequences of two of our three structural theorems, i.e., Theorem 2 and Theorem 3. Clearly, the structure revealed by Theorem 4 is “intermediate” with respect to that of Theorem 2 and Theorem 3. It provides a structural decomposition of q -colorful graphs of small bidimensionality, i.e., q -colorful graphs excluding a large q -colorful rainbow grid as a minor, which we believe should be useful in algorithmic design. As we already mentioned, the algorithmic consequences of Theorem 4 go beyond the objectives of this paper. However, we outline below some directions on the STEINER TREE problem that are inspired by the current research.

In the introduction, we commented on STEINER TREE as one of the most studied problems on annotated graphs. Using the terminology of this paper, Jansen and Swennenhuis in [27] proved that STEINER TREE, with input a 1-colorful graph (G, χ) can be solved in $2^{\mathcal{O}(k)} \cdot |G|^{\mathcal{O}(1)}$ time where k is the torso treewidth of (G, χ) . Is torso treewidth the “best possible” structural parameter on (G, χ) that, when bounded, STEINER TREE can be solved in polynomial time? The answer is negative. This follows by the fact that a polynomial algorithm has been recently given by Groenland, Nederlof, and Koana in [24] for the case where (G, χ) excludes the 1-colorful rainbow K_4 as a colorful minor. As this 1-colorful graph belongs in the obstruction set $\mathcal{O}_1$, the results of [24] apply to all colorful minors of the $(1, k)$ -segregated grid on which torso treewidth may be arbitrarily large. We believe that the “limit” is given by the combinatorial bound of Theorem 4. We conjecture that the bidimensionality of $X = \chi^{-1}(1)$ in G is the structural frontier for the complexity of STEINER TREE problem in classes that are colorful minor-closed. In particular we conjecture the following complexity dichotomy.

► **Conjecture 10.** *Unless $P \neq NP$, STEINER TREE can be solved in polynomial time in a 1-colorful minor-closed class $\mathcal{C}$ if and only if $\mathcal{C}$ has bounded bidimensionality.*

Note that the negative part of the above conjecture already follows from the fact that the STEINER TREE problem is NP-complete in planar graphs [22]. We believe that the local structure provided by Theorem 4 should serve as a starting point for proving the positive part as well. In particular, we now have that all terminals in $X = \chi^{-1}(1)$ are confined to the apices and the “few” vortices of the near-embedding of G . Motivated by the fact that STEINER TREE can be solved in polynomial time when all terminals lie on “few” faces of a planar graph [14, 15, 32], the key challenge becomes how to generalize such face-based techniques to handle apices and vortices in near-embeddings of the kind provided by Theorem 4. We further anticipate that this methodological approach may lead to algorithmic dichotomies – such as the one conjectured above – for a broader class of problems on annotated graphs, where tractability arises from restricting terminals to few faces of the input graph.

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