On the Constructive Dimension Spectrum of Polynomials

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Abstract

Recently, Stull [18], [17] resolved a long-standing open problem posed by Lutz, on whether the set of effective Hausdorff dimensions of points on a straight line in $\mathbb{R}^2$ - the effective dimension spectrum of the line - contains a unit interval. This question is related to problems in classical fractal geometry like the Kakeya conjecture and Furstenberg sets. Stull posed an open question on the dimension spectra of polynomial curves.

For the first result, with new techniques which adapt the theory of classical real root-finding of polynomials to the current setting, we show that the dimension spectra of every polynomial curve contains at least two points. This answers an open question posed by Stull [18], [17]. We use the main result to construct a class of polynomials which have width strictly greater than 1, answering a second problem stated in [18],[17].

Stull [18] resolved the dimension spectrum conjecture for planar lines, showing that it contains a unit interval. For the second result, we resolve the conjecture for a subfamily of polynomials whose coefficients form a "low" dimension point in $\mathbb{R}^{d+1}$.

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1 Introduction

The theory of effective Hausdorff dimension was introduced by J. Lutz in 2000 [5], [7] initially over the Cantor Space of infinite binary sequences as a tool to study the relationships between complexity classes [6]. Subsequent works have adapted the theory to study the effective Hausdorff dimensions and points in Euclidean space and in general metric spaces ([10, 12, 21, 8, 14, 19, 17, 9, 15]). A major highlight of this theory is the "point-to-set principle", allowing classical Hausdorff dimensions to be derived using effective pointwise arguments. A central question arising in this setting concerns the effective dimension *spectrum* of sets in the Euclidean plane: what values are attained by the effective dimensions of points inside sets in $\mathbb{R}^2$?

In a recent work, Stull [17], extending an earlier work by N. Lutz and Stull [13], settled a long-standing conjecture of J. Lutz showing that the dimension spectrum of *every* line in $\mathbb{R}^2$ contains a unit interval. Stull [17] proposes extending the study to the dimensions of points along polynomial curves.

In this work, we resolve Stull's problem for every univariate polynomial with real coefficients. We show that its dimension spectrum of every polynomial contains at least two points. Further, we show that the dimension spectrum of every polynomial contains unit-length interval when the coefficients have low dimension. Our first main result is the following.

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▶ **Theorem 1.1.** *For* $(a_0, \ldots, a_d) \in \mathbb{R}^{d+1}$*,* $x \in \mathbb{R}$*, we have*

$$\dim \left(x, \sum_{i=0}^{d} a_i x^i \right) \geq \dim \left(x \mid a_0, \ldots, a_d \right) + \min \left\{ \dim \left(a_0, \ldots, a_d \right), \dim^{a_0, \ldots, a_d}(x) \right\}. \tag{1}$$

The second main theorem for this paper establishes that the dimension spectrum of polynomials with coefficients with effective dimension at most 1, contains a unit-length interval.

▶ **Theorem 1.2.** *Let* $\sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$ *be a degree* d *polynomial with* $\dim(\mathbf{a}) \leq 1$*. Then for every* $s \in [0, 1]$*, there is a point* $x \in \mathbb{R}$ *such that* $\dim \left(x, \sum_{i=0}^{d} a_i x^i \right) = s + \dim(\mathbf{a})$*.*

The proofs adapt the techniques in N. Lutz and Stull [13] to polynomials. We adapt the classical root-finding methods, namely, the bisection method, together with methods which count real roots from Sturm's theory [20] (see for example, von zur Gathen and Gerhard [22] and Yap [24]) to establish our results. Due to the fact that the coefficients can be arbitrary real numbers, we have to adapt these techniques to form short descriptions of the roots of these functions. These versions may be of broader interest.

The proof of the second result, building on the first, involves encoding the coefficients of a given polynomial in such a way so as to control the dimension of the corresponding point.

We conclude by discussing the implications of our result - in particular, that dimension level sets - sets consisting of points of the same effective dimension - cannot contain polynomials.

2 Prerequisites

We denote the binary alphabet by $\Sigma = \{0, 1\}$. The set of finite binary strings is denoted by Σ^* and the set of infinite binary sequences, by Σ^∞. Empty string is denoted by λ. The length of a finite string $w \in \Sigma^*$ is denoted by $\ell(w)$.

We denote the set of rational numbers by $\mathbb{Q}$ and the set of reals by $\mathbb{R}$. In this work, we assume a binary encoding $e : \mathbb{Q} \to \Sigma^*$ of the set of rationals.

We denote by $\mathbf{a}$ the tuple $a_0, a_1, \ldots, a_d$, denoting the coefficients of a polynomial $\sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$, such that $a_d \neq 0$.

We now introduce the basic notions in Kolmogorov complexity (see, for example, Downey and Hirschfeldt [3], Nies [16] or Li and Vitányi [4]).

▶ **Definition 1** (Kolmogorov Complexity of binary strings)**.** *For each pair of strings* $v, w \in \{0,1\}^*$*, the* Kolmogorov complexity *of* v *given* w *is defined as* $K(v|w) = \min_{\pi \in \{0,1\}^*} \{\ell(\pi) : U(\pi, w) = v\}$*, where* U *is a fixed universal (prefix) Turing machine. The* Kolmogorov complexity *of* v*, denoted by* $K(v)$*, is* $K(v \mid \lambda)$*.*

▶ **Definition 2** (Kolmogorov Complexity of binary strings relative to an oracle)**.** *For each pair of strings* $v, w \in \{0,1\}^*$*, the* Kolmogorov complexity *of* v *given* w *relative to an oracle* $A \subseteq \mathbb{N}$ *is defined as* $K^A(v|w) = \min_{\pi \in \{0,1\}^*} \left\{ \ell(\pi) : U^A(\pi, w) = v \right\}$*, where* U *is a fixed universal (prefix) Turing machine with oracle access to* A*.*

In recent works ([10, 12, 21, 8, 14, 19, 17, 9]) the theory of Kolmogorov complexity of binary strings has been adapted to the study of Kolmogorov complexity of reals. Instead of defining the complexity in terms of the truncated binary expansions of the real (which has known issues such as addition of reals being uncomputable - see Weihrauch [23] for a discussion), the approach defines the complexity of a real in terms of the complexities of its rational approximations. The intuition behind the following definition is this: any rational q within the open neighborhood of radius 2^{-r} around x is a valid description of x to within precision $r \in \mathbb{N}$. The shortest description of a real point x in $\mathbb{R}^n$ to within a precision $r \in \mathbb{N}$ is the shortest description of any rational point within the 2^{-r} neighborhood of x. Note that this may not necessarily be the rational obtained by truncating the binary expansion of x to r bits.

▶ **Definition 3** (Kolmogorov Complexity of Reals – Lutz and Mayordomo [10])**.** *The Kolmogorov complexity of a real number $x \in \mathbb{R}^n$ up to a precision $r \in \mathbb{N}$ is defined as*

$$K_r(x) = \min\left\{K(q) : q \in \mathbb{Q}^n \cap B\left(x, 2^{-r}\right)\right\},$$

where $B(x,d) = \{y \in \mathbb{R}^n : \|y - x\|_2 < d\}$ is the open ball of radius d centered at x.

The conditional Kolmogorov complexity of $x \in \mathbb{R}^m$ given $y \in \mathbb{R}^n$ is defined using the Kolmogorov complexity of rational approximations to x and y.

▶ **Definition 4** (Conditional Kolmogorov Complexity of Reals – Lutz and Mayordomo [10])**.** *The conditional Kolmogorov complexity of $x \in \mathbb{R}^m$ at precision r given $q \in \mathbb{Q}^n$ is*

$$\hat{K}_r\left(x|q\right) = \min\left\{K\left(p|q\right) : p \in B\left(x, 2^{-r}\right) \cap \mathbb{Q}^m\right\}.$$

The conditional Kolmogorov complexity of $x \in \mathbb{R}^m$ at precision $r \in \mathbb{N}$ given $y \in \mathbb{R}^n$ at precision $s \in \mathbb{N}$ is defined as

$$K_{r,s}\left(x|y\right) = \max\left\{\hat{K}_r(x|q) : q \in B\left(y, 2^{-s}\right) \cap \mathbb{Q}^n\right\}.$$

Notation. We use $K_r\left(x|y\right)$ to denote $K_{r,r}\left(x|y\right)$.

One of the characteristics of the theory of effective Hausdorff and packing dimension, in contrast to the classical theory, is that individual points can have strictly positive effective dimension. The effective Hausdorff dimension of a point is defined as follows.

▶ **Definition 5** (Effective Hausdorff dimension of a point – Lutz and Mayordomo [10])**.** *The* effective Hausdorff dimension *of a point $x \in \mathbb{R}^m$, denoted* $\dim(x)$*, is defined by* $\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r}$*. The* effective strong dimension *of $x \in \mathbb{R}^m$ is defined by*
$\mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}$.

Relativizing the above definitions with respect to an oracle $A \subseteq \mathbb{N}$, we can define the notions $K_r^A(x)$, $K_r^A(x|y)$, $\dim^A(x)$, and $\mathrm{Dim}^A(x)$, where the universal machine U has access to the oracle A. Since there is a bijective correspondence between subsets of natural numbers and reals, using any standard encoding of reals, we may also define, for any $y \in \mathbb{R}$, the notions $K_r^y(x)$, $\dim^y(x)$, and $\mathrm{Dim}^y(x)$.

▶ **Definition 6** (Dimension spectrum of a set – Lutz and Mayordomo [11])**.** *For any $S \subseteq \mathbb{R}^n$, the* dimension spectrum *of S, denoted by $sp(S)$, is defined by $sp(S) = \{\dim(x) : x \in S\}$.*

The following result by Lutz and Stull [13] gives a lower bound on the dimensions of points in the graph of a straight line $z \mapsto az + b$ in terms of the dimensions of a, b and the effective relative dimension of the co-ordinate $z \in \mathbb{R}$.

▶ **Theorem 7** (Lutz and Stull [13] – unrelativized)**.** *For every $a, b, x \in \mathbb{R}$, we have*

$$\dim\left(x, ax + b\right) \geq \dim^{a,b}\left(x\right) + \min\left\{dim(a, b), dim^{a,b}(x)\right\}. \quad (2)$$

3 Basic properties of Kolmogorov complexity of reals

In this section, we state a few basic properties of Kolmogorov complexity, conditional Kolmogorov complexity and relative Kolmogorov complexity of reals pertinent for the remainder of the work.

The following approximate symmetry of information holds for pairs of reals.

▶ **Lemma 8** (Lutz and Stull [19]). *For every $m, n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and precision parameters $r, s \in \mathbb{N}$ with $s \leq r$, we have*

1. $|K_{r,r}(x|y) + K_r(y) - K_{r,r}(x,y)| \leq O(\log r) + O(\log\log \|y\|)$.
2. $|K_{r,s}(x|x) + K_s(x) - K_r(x)| \leq O(\log r) + O(\log\log \|x\|)$.

The following lemma establishes a one-sided bound between the relativized Kolmogorov complexity and the conditional complexity, at a specific precision.

▶ **Lemma 9** (J. Lutz and N. Lutz [8]). *For all $m, n \in \mathbb{N}$, there is a constant $c \in \mathbb{N}$ such that for all reals $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and precision parameters $r, t \in \mathbb{N}$ we have*

1. $K_r^y(x) \leq K_{r,t}(x|y) + K(t) + c$.
2. $\dim^y(x) \leq \dim(x|y)$.

When the precision of either the conditioning variable or the conditioned variable changes, then the Kolmogorov complexity changes at most by an amount linear in the change in precision, uniformly in the points, as the following lemmas show.

▶ **Lemma 10** (Case and J. Lutz [1]). *There is a constant $c \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $x \in \mathbb{R}^n$ and precision parameters $r, s \in \mathbb{N}$, we have*

$$K_r(x) \leq K_{r+s}(x) \leq K_r(x) + K(r) + ns + a_s + c. \tag{3}$$

The following lemma states similar bounds for conditional Kolmogorov complexity.

▶ **Lemma 11** (J. Lutz and N. Lutz [8]). *For all $m, n \in \mathbb{N}$, there is a constant $c \in \mathbb{N}$ such that for all points $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $q \in \mathbb{Q}^n$ and precision parameters $r, s, t \in \mathbb{N}$, we have the following.*

1. $\hat{K}_r(x|q) \leq \hat{K}_{r+s}(x|q) \leq \hat{K}_r(x|q) + ms + 2\log(1+s) + K(r,s) + c$.
2. $K_{r,t}(x|y) \geq K_{r,t+s}(x|y) \geq K_{r,t}(x|y) - ns - 2\log(1+s) + K(t,s) + c$.

4 Outline of the Proof of Theorem 1.1

Let $A(x) = \sum_{i=0}^{d} a_i x^i$. It is easy to establish an upper bound on the dimension of a point $(x, A(x))$, given approximations to the coefficients of A and to x. We can give a lower bound on the dimension of the vector $(x, a_0, \ldots, a_d)$ using symmetry of information. However, the Kolmogorov complexity of the point $(x, A(x))$ may be lower than that of $(x, a_0, \ldots, a_d)$. The main task we accomplish is to establish lower bounds for the complexity of $(x, A(x))$ and consequently, its effective Hausdorff dimension.

The outline of the proof, adapting that of Lutz and Stull [19] is as follows. In Theorem 18, we show that if a different d-degree polynomial $B(x) = \sum_{i=0}^{d} b_i x^i$ intersects A at x, then either the coefficients of B are close in L_2 norm to A, or the coefficients of B must have very high Kolmogorov complexity. We introduce a technique of approximating roots when only approximations to the coefficients and rational approximations to points in the domain are available, adapting Sturm's theory to this new setting. This lower bounds the Kolmogorov complexity of $(x, A(x))$.

Lemma 21 (Lemma 3.3 from [13]) ensures an oracle D which limits the complexity of the coefficients of A. These results are then used to obtain the main technical lemma, Lemma 20, which yields the exact lower bound for the Kolmogorov complexity of $(x, A(x))$ at a specified precision r. The main lemma is then utilized to obtain a lower bound for the effective Hausdorff dimension of $(x, A(x))$.

5 Approximating intersecting polynomials

In this subsection, we prove a lower bound for the Kolmogorov complexity of *any* approximation to the coefficients of a polynomial in terms of Kolmogorov complexity of the exact coefficients and the roots. This is crucial bound which leads to the main result (Theorem 1.1). To this end, we provide an algorithm which lists approximations to all points where two given polynomials $B(x) = \sum_{i=0}^{d} b_i x^i$ and $A(x) = \sum_{i=0}^{d} a_i x^i$ intersect. This ensures that the complexity of any x such that $A(x) = B(x)$ can be lower bounded using the complexities of $\mathbf{a}$ and $\mathbf{b}$. Since the coefficients and x are real, this algorithm will take as input approximations to these values, and output approximations to the roots to any arbitrary precision.

The points of intersection of A and B are exactly the roots of the d-degree polynomial $C(x) = A(x) - B(x)$. Thus there are at most d points of intersection between A and B on $\mathbb{R}$. The problem at hand therefore reduces to computing all possible real roots, with multiplicities, of a polynomial with real coefficients, up to arbitrary precision. We start by outlining the bisection method, followed by the importance of Sturm's theorem, and then specify the algorithm.

In the special case of polynomials, the following method enumerates *all* the real roots of any univariate polynomial. Assume that we know the exact values of all the coefficients of a polynomial (in our case, we will have to work with approximations, and this makes the procedure much more technical). We start by getting a bound on the absolute value of any real root. This helps us to select an interval that is large enough to contain all the real roots of the polynomial. Recall that the roots of the polynomial are not guaranteed to be distinct - roots may have multiplicity greater than 1. In order to get all the real roots (with multiplicities), we use Sturm's theorem to get the number of distinct roots (with arbitrary multiplicities) of the polynomial in any given interval. This allows us to improve the bisection method whenever there are repeated roots (of even multiplicities) in any interval.

For a degree n polynomial P, we compute a sequence of polynomials, called the Sturm sequence of P as follows (see, for example, Gerhard, von zur Gathen [22, Ch 2], or Yap [24, Ch 7]). The first 2 polynomials in the sequence are the polynomial itself, followed by its derivative - *i.e* $P_0 = P$ and $P_1 = P'$. For $i \geq 2$, $P_i = P_{i-2} \mod P_{i-1}$, where $\mod$ is the remainder obtained when dividing the polynomial P_{i-2} by P_{i-1}. Any Sturm sequence has at most $d+1$ elements, for a degree d polynomial. Now, all the polynomials in the Sturm sequence obtained are evaluated at the end points of an interval, say $(a, b]$, which results in 2 sequences, each corresponding to an end point. We count the number of sign changes in each sequence, and denote them by $\sigma(a)$ and $\sigma(b)$ respectively.

▶ **Theorem 12** (Sturm, 1835 [20]). *For a square free polynomial P, the number of distinct roots of P in the interval $(a, b]$ is $\sigma(a) - \sigma(b)$. If the polynomial has repeated roots, and if neither a nor b is a root of P, then the number of distinct roots in $(a, b]$ is equal to $\sigma(a) - \sigma(b)$.*

If the end points happen to be the roots of the polynomial, we report them as is, and continue the search if required.

The following lemma bounds the roots of the polynomial P. We use this to determine the starting interval for our algorithm, ensuring that it is large enough to output all the real roots of P.

▶ **Lemma 13** (Cauchy Bound [2]). *Let $P(x) = \sum_{i=1}^{d} c_i x^i$ be a polynomial. If for any $x \in \mathbb{R}$, $P(x) = 0$, then $x \in (-\beta, \beta)$, where $\beta = 1 + \max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \dots, \left|\frac{a_0}{a_n}\right|\right\}$.*

Using the above classical results, we now introduce the algorithm that upper bounds the number of roots in an interval from rational approximations of the coefficients. We introduce the preliminary algorithms for the Sturm sequence and the approximate sign counting, concluding with the algorithm that enumerates the roots.

The main subtlety we deal with is this: since the algorithms work with approximations, the exact signs of the polynomial value cannot be determined. This is because a small negative value is considered an acceptable input approximation to a small positive value. This leads to a conservative estimate of the sign changes, leading to a slightly longer list of possible roots. No root will be omitted at the given precision. However, since it is impossible to algorithmically determine whether a function is exactly equal to 0 or is a small positive or small negative value, points besides the roots may be counted if the precision is not sufficiently high. The algorithm outputs a list sufficiently short so that the Kolmogorov complexity of describing any root from the output list remains acceptably small.

▶ **Lemma 14.** *There is an algorithm SturmSequence that, on input $(c_0, \dots, c_d)$ and $\alpha \in \mathbb{Q}$ outputs a list of rationals $\langle \hat{\alpha}_i \rangle_{i=1}^{d}$ of the evaluations of the Sturm Sequence corresponding to the polynomial $P_0(x) = \sum_{i=0}^{d} c_i x^i$ at α.*

The main step in our algorithm to find approximations to all real roots is the following sign computation. If the values of the polynomials are accurately known, then the sign determination is trivial. However, when the true value is nearly 0, it is difficult to determine its actual sign. We adopt a conservative upper bound for the actual number of sign changes.

▶ **Lemma 15.** *There are algorithms MaxSignChange and MinSignChange for any sequence $(q_0, \dots, q_d)$ of rational approximations to reals $(r_0, \dots, r_d)$ where for $0 \le i \le d$ we have $|q_i - r_i| < 2^{-r}$, MaxSignChange outputs an upper bound on the number of sign changes in $(r_0, \dots, r_d)$ and MinSignChange outputs a lower bound.*

Thus, we get the following upper bound on the output list of possible roots. This list is guaranteed to contain all the real roots of the polynomial to the given precision. However, since Lemma 15 is a conservative upper bound on the number of sign changes, there could be some values in the output list which are not the roots of the polynomial. This cannot be avoided in general. However, the output list of values is sufficiently small to control the number of bits used to describe *any* of its members. This upper bounds the Kolmogorov complexity of *all* the real roots of the polynomial.

▶ **Lemma 16.** *There is an algorithm RootEnum such that on input $(c_0, \dots, c_d) \in \mathbb{Q}^{d+1}$ and $r \in \mathbb{N}$, outputs a list of rationals $(q_1, \dots, q_\ell)$ of length at most $6d^2$ such that if $\hat{x}$ is a real root of $P(x) = \sum_{i=0}^{d} c_i x^i$, then there is an i, $1 \le i \le \ell$ such that $|q_i - \hat{x}| \le 2^{-r}$.*

▶ Remark 17. The root enumerating algorithm used in the proof of the above lemma is a standalone result about root-finding when coefficients and the domain is only available as an approximation, and is possibly of independent interest.

Now, in order to bound the dimension of a point on the graph of a polynomial $P(x) = \sum_{i=0}^{d} a_i x^i$, we try to bound the dimension of the coefficients of a polynomial of equal degree, denoted as $Q(x) = \sum_{i=0}^{d} b_i x^i$, *almost coinciding* with P, intersecting it at x. The coefficients of this polynomial will provide sufficient information about the original polynomial, which in turn will help estimating x.

▶ **Theorem 18.** *Let* $x \in \mathbb{R}, \mathbf{a} \in \mathbb{R}^{d+1}$. *For* $r \geq t = -\log \|\mathbf{a} - \mathbf{b}\|$, *and* $\mathbf{b} \in B(\mathbf{a}, 1)$ *where* $\sum_{i=0}^{d} a_i x^i = \sum_{i=0}^{d} b_i x^i$, *we have*

$$K_r(\mathbf{b}) \geq K_t(\mathbf{a}) + K_{r-t,r}(x|\mathbf{a}) + O(\log r).$$

6 Spectra of Polynomials of degree $d \geq 2$

Recall the statement of the main theorem.

▶ **Theorem 1.1.** *For* $(a_0, \ldots, a_d) \in \mathbb{R}^{d+1}$, $x \in \mathbb{R}$, *we have*

$$\dim\left(x, \sum_{i=0}^{d} a_i x^i\right) \geq \dim\left(x \mid a_0, \ldots, a_d\right) + \min\left\{\dim\left(a_0, \ldots, a_d\right), \dim^{a_0, \ldots, a_d}(x)\right\}. \quad (1)$$

Our proof follows the strategy of the work by Lutz and Stull [19]. However, since we have to work with degree-d polynomials, we deal with an increase in precision, as well as the presence of multiple roots. We indicate the strategy below, while simultaneously showing the similarity and emphasizing the differences from the work of Lutz and Stull [19]. Broadly, the steps involved in the proof are as follows.

First, note that any polynomial $P(x)$ as defined above is completely described by the list of its real coefficients $(a_0, \ldots, a_d)$. We show that any other polynomial as characterized by the list of coefficients $(b_0, \ldots, b_d)$ which coincides with $P(x)$ must either be very close to $(a_0, \ldots, a_d)$ in Euclidean distance, or must have very high Kolmogorov complexity. The technical steps in the proof, however, are radically different from the work of Lutz and Stull - since a degree-d polynomial can have d real roots, hence the intersection point of the polynomials is not uniquely specified by the lists of coefficients. Moreover, in order to compute the intersection points of the two polynomials, we employ a modification of the bisection method to find all real roots, where we have to manage the error introduced in this approximation, and possible multiplicities of real roots.

Second, we show a lower bound for $K_r(b_0, \ldots, b_d)$ in terms of $\|(a_0, \ldots, a_d) - (b_0, \ldots, b_d)\|$.

Then, as in Lutz and Stull [19], we use an oracle D such that the following sequences of inequalities hold.

$$\begin{aligned} K_r(x, P(x)) &\geq K_r^D(x, P(x)) \\ &\geq K_r^D(x, a_0, \ldots, a_d) - O(1) \\ &\geq K_r(x, a_0, \ldots, a_d) - O(\log\log\|x\|) - O_x(\log r), \end{aligned}$$

establishing that $K_r(x, P(x))$ is not much less than $K_r(x, a_0, \ldots, a_d)$.

We use these results to prove the required bound.

6.1 Error estimates for polynomial approximation

The following is a basic relation between $|a - b|$ and $|a^k - b^k|$, $k \geq 1$.

▶ **Lemma 19.** *If* $a, b \in \mathbb{R}$ *are such that* $|a - b| < 2^{-r}$, *then for any integer* $k \geq 1$, *we have*

$$\left|a^k - b^k\right| < 2^{-r} k \times \max\{|a|, |a|^k, |b|, |b|^k\}.$$

6.2 Lower bound for the complexity of $(x, P(x))$

▶ **Lemma 20.** *Let $\mathbf{a} = (a_0, \ldots, a_d) \in \mathbb{R}^{d+1}$ with $a_d \neq 0$, $x \in \mathbb{R}$, precision parameter $r \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and parameters $\epsilon, \eta \in \mathbb{Q}_+$ be such that $r \geq 1 + \log(2\max\{|a_i| \mid 0 \leq i \leq d\} + d(1 + \max\{|x|, |x|^{2d}\}))$ and the following conditions hold.*

(i) *$K_r(\mathbf{a}) \leq (\eta + \varepsilon)r$ and*

(ii) *For every $\mathbf{b} = (b_0, \ldots, b_d) \in B(\mathbf{a}, 1)$ such that x is a root of $\sum_{i=0}^{d}(b_i - a_i)x^i$, if $t = -\log||\mathbf{b} - \mathbf{a}||$ is at most r, then*

$$K_r(\mathbf{b}) \geq (\eta - \varepsilon)r + \delta(r - t).$$

Then,

$$K_r\,(x, P(x)) \geq K_r(\mathbf{a}, x) - \frac{4\varepsilon}{\delta}(d + 1)r - K(\varepsilon) - K(\eta) - O_{\mathbf{a}}(\log r). \tag{4}$$

This is the analogue of Lemma 3.1 in Lutz and Stull [19] generalized to degree d polynomials. The outline of the proof is along the lines in their work, but with the error analysis modified for degree d polynomials.

The following lemma from [13] is used to ensure one of the initial assumptions of Lemma 20.

▶ **Lemma 21** (Lutz and Stull [13]). *Let $r \in \mathbb{N}$, $z, \in \mathbb{R}^{d+1}$, and $\eta \in \mathbb{Q} \cap [0, \dim(z)]$. Then, there is an oracle $D = D(r, z, \eta)$ which satisfies*

- *$K_t^D(z) = \min\{\eta r, K_t(z)\} + O(\log r)$, $\forall t \leq r$.*
- *$K_{t,r}^D(y|z) = K_{t,r}(y|z) + O(\log r)$, and $K_t^{z,D}(y) = K_t^D(y) + O(\log r)$, for all $m, t \in \mathbb{N}$, and $y \in \mathbb{R}^m$.*

6.3 Proof of Theorem 1.1

Now, we proceed towards the proof of Theorem 1.1.

Proof Sketch of Theorem 1.1. The proof of this theorem follows the same steps as in the proof of the main theorem of Lutz and Stull [13], but with the bounds replaced appropriately. We summarize the argument below, highlighting the major steps. Let $\mathbf{a} \subseteq \mathbb{R}^{d+1}$ be the coefficients of the polynomial. Denote $\dim(\mathbf{a})$ by ρ. Let $\epsilon \in \mathbb{Q}_+$, $\eta \in [0, \min\{\dim(\mathbf{a}), \dim^{\mathbf{a}}(x)\}] \cap \mathbb{Q}$, and $\delta = \dim^{\mathbf{a}}(x) - \eta > 0$. For each $r \in \mathbb{N}$, let D_r be as defined in Lemma 21. We show that the conditions of Lemma 20 hold for the choices of $\mathbf{a}, x, r, \delta, \epsilon, \eta$ made, which would eventually yield the desired result.

By Lemma 21, the oracle D_r ensures that $K_r^{D_r}(\mathbf{a}) \geq \eta r + O(\log r)$. This satisfies condition (i) of Lemma 20.

Now we show that condition (ii) of Lemma 20 also holds, relative to D_r. Let $\mathbf{b} \in B(\mathbf{a}, 1)$ such that $\sum_{i=0}^{d} b_i x^i = \sum_{i=0}^{d} a_i x^i$, $t = -\log\|\mathbf{b} - \mathbf{a}\| \leq r$. Therefore,

$$\begin{aligned}
K_r(\mathbf{b}) &\geq K_t(\mathbf{a}) + K_{r-t,r}(x|\mathbf{a}) - O(\log r), && \textit{Theorem } 18\\
&\geq K_t^{D_r}(\mathbf{a}) + K_{r-t,r}(x|\mathbf{a}) - O(\log r), && \text{oracles never increase complexity}\\
&= \min\{\eta r, K_t(\mathbf{a})\} + K_{r-t,r}(x|\mathbf{a}) - O(\log r), && \textit{Lemma } 21.1\\
&\geq \rho t - o(t) + (\delta + \eta)(r - t) - o(\log r), && \textit{Lemma } 9\\
&\geq (\eta - \epsilon)r + \delta(r - t).
\end{aligned}$$

Since both the conditions of Lemma 20 have been met, towards the final argument, we have,

$$\begin{aligned}
K_r\left(x, \sum_{i=0}^{d} a_i x^i\right) &\geq K_r^{D_r}\left(x, \sum_{i=0}^{d} a_i x^i\right), && \text{oracles never increase complexity} \\
&\geq K_r^{D_r}(\mathbf{a}, x) - \frac{4\epsilon}{\delta}(d+1)r - K(\epsilon) - K(\eta) - O_{\mathbf{a}}(r), && \textit{Lemma } 20 \\
&\geq K_r^{D_r}(x|\mathbf{a}) + K_r^{D_r}(\mathbf{a}) - \frac{4\epsilon}{\delta}(d+1)r - O(r) \\
&\geq K_r^{D_r}(x|\mathbf{a}) + \eta r - \frac{4\epsilon}{\delta}(d+1)r - O(\log r), && \textit{Lemma } 21.1 \\
&\geq K_r(x|\mathbf{a}) + \eta r - \frac{4\epsilon}{\delta}(d+1)r - O(\log r). && \textit{Lemma } 21.2
\end{aligned}$$

Taking $\liminf$ as $r \to \infty$ on both sides on the modified inequality, we get,

$$\liminf_{r\to\infty} \frac{K_r\left(x, \sum_{i=0}^{d} a_i x^i\right)}{r} \geq \liminf_{r\to\infty} \frac{K_r(x|\mathbf{a}) + \eta r - \frac{4\epsilon}{\delta}(d+1)r - O(\log r)}{r}$$

$$\implies \dim\left(x, \sum_{i=0}^{d} a_i x^i\right) \geq \dim(x|\mathbf{a}) + \eta - \frac{4\epsilon}{\delta}(d+1).$$

Since η, γ, ϵ were chosen arbitrarily, we get,

$$\dim\left(x, \sum_{i=0}^{d} a_i x^i\right) \geq \dim(x|\mathbf{a}) + \min\{\dim(\mathbf{a}), \dim^{\mathbf{a}}(x)\}.$$ ◀

▶ **Corollary 22.** *For almost every real $x \in \mathbb{R}$, we have*

$$\dim\left(x, \sum_{i=0}^{d} a_i x^i\right) = 1 + \min\left\{\dim(\mathbf{a}), 1\right\}. \tag{5}$$

Proof. We know that the set $\{x \in \mathbb{R} : x \text{ is } \textit{not} \text{ Martin-Löf random given } \mathbf{a}\}$ has measure 0. Hence, for almost every $x \in \mathbb{R}$, $\dim^{\mathbf{a}}(x) = 1$. By Lemma 9, $\dim^{\mathbf{a}}(x) \leq \dim(x|\mathbf{a})$. Thus, for almost every $x \in \mathbb{R}$, equation 5 holds. ◀

7 Unit-length dimension spectrum for polynomials with $\dim(\mathrm{a}) \leq 1$

This section deals with the polynomials, wherein
$\rho = \dim(\mathbf{a}) \leq 1$. Following is the main theorem of the section.

▶ **Theorem 1.2.** *Let $\sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$ be a degree d polynomial with $\dim(\mathbf{a}) \leq 1$. Then for every $s \in [0, 1]$, there is a point $x \in \mathbb{R}$ such that $\dim\left(x, \sum_{i=0}^{d} a_i x^i\right) = s + \dim(\mathbf{a})$.*

The basic idea is to construct a real $x \in \mathbb{R}$ by alternating segments of a random point $y \in \mathbb{R}$ and approximations of the coefficients $\mathbf{a}$. This interleaving is done in a stage-by-stage manner, ensuring that $\dim(x) = s$. Simply adding random bits to x is not helpful, because the information content in the point won't have any control on the information content of the approximated function value. Instead, the constructed x contains information about the graph of the function $\sum_{i=0}^{d} a_i x^i$.

Construction of x. Let $y \in \mathbb{R}$ be Martin-Löf random relative to $(\mathbf{a})$. For the stage $j = 1$, let the *stage length* be $h_j = 2$. For any stage $j > 1$, we define a sufficiently large *stage length* h_j by

$$h_j = \min\left\{h \geq 2^{h_{j-1}} : K_h(\mathbf{a}) \leq \left(\rho + \frac{1}{j}\right) h\right\}. \tag{6}$$

Note that h_j is finite, since $\dim(\mathbf{a}) = \rho$ is finite.

Denote the block length $\frac{h_j}{d}(1-s)$ by i_j. At stage j, we define the stretch $x[h_{j-1}+1 \dots h_j]$ of the binary expansion of x by

$$x[r] = \begin{cases} y[r] & h_{j-1} \leq r < sh_j \\ a_{(d+r \mod i_j)}[\lfloor r/i_j \rfloor] & sh_j \leq r < h_j,\ j \in \mathbb{N} \end{cases} \tag{7}$$

In other words, the first segment of x, i.e. $x[h_{j-1} \dots sh_j]$, is the same as $y[h_{j-1} \dots sh_j]$, and the other segment $x[sh_j \dots h_j]$ follows the interleaved pattern

$$a_1[0]a_2[0] \dots a_d[0] \;\; a_1[1]a_2[1] \dots a_d[1] \;\; \dots \\ a_1[i_j-2]a_2[i_j-2] \dots a_d[i_j-2] \;\; a_1[i_j-1]a_2[i_j-1] \dots a_d[i_j-1]. \tag{8}$$

It should be noted that the entire information of $\mathbf{a}$ has not been encoded into x. For instance, a_0 has not been used during the encoding. Only the required amount of information such that the complexity of the point on the polynomial can be reduced has been encoded.

▶ **Lemma 23** (Local Lipschitz condition). *For a polynomial $P(x) \in \mathbb{R}[x]$, we have*

$$(\exists c \in \mathbb{R})\,(\forall x, y \in [0,1])\,(|P(y) - P(x)| \leq c|y - x|)\,,$$

where $c \in \mathbb{R}$ is a constant independent of x or y.

This lemma is especially useful to ensure minimal loss of precision, which will be clear in the subsequent lemmas.

The following two lemmas show the effect of encoding the information of $\mathbf{a}$ into x for every segment. These lemmas would be useful towards proving the main theorem.

▶ **Lemma 24.** *For every $j \in \mathbb{N}$, and for $sh_j < r \leq h_j$,*

$$K_{\frac{r-sh_j}{d},r}\left(\mathbf{a} \,\middle|\, x, \sum_{i=0}^{d} a_i x^i\right) \leq O(\log h_j).$$

Even though encoding $\mathbf{a}$ into x helps reduce complexity, to control the information content, random bits (from y) were added. The following lemma shows that access to $\mathbf{a}$ doesn't provide any more information than what is already encoded in x, thereby showing the utility of adding segments from y.

▶ **Lemma 25.** *For $j \in \mathbb{N}$, the following hold.*

- $K_t^{\mathbf{a}}(x) \geq t - O(\log h_j)$, $\forall t \leq sh_j$.
- $K_t^{\mathbf{a}}(x) \geq sh_j + t - O(\log h_j)$, $\forall h_j \leq t \leq sh_{j+1}$.

We now try to obtain a lower bound on the complexity of the constructed point. There are two cases to be looked at; one over the segment of x encoded by $\mathbf{a}$, and the other over the segment encoded by y. The following lemma talks about the segment of x encoded with $\mathbf{a}$.

▶ **Lemma 26.** *For every $\gamma > 0$ and for large enough $j \in \mathbb{N}$,*

$$K_r\left(x, \sum_{i=0}^{d} a_i x^i\right) \geq \left(s + \rho - \gamma(d+1)^2\right) r,$$

for every $r \in (sh_j, h_j]$.

The previous lemma dealt with the segment encoded with $\mathbf{a}$. The following lemma talks about the other case, i.e. the segment encoded with y.

▶ **Lemma 27.** *For every $\gamma > 0$ and for large enough $j \in \mathbb{N}$,*

$$K_r\left(x, \sum_{i=0}^{d} a_i x^i\right) \geq \left(s + \rho - \gamma(d+1)^2\right) r,$$

for every $r \in (h_j, sh_{j+1}]$.

Now that we have a lower bound on the information of the point, the next lemma helps get an upper bound on the complexity of the point on the polynomial.

▶ **Lemma 28.** *For sufficiently large $j \in \mathbb{N}$,*

$$K_{h_j}\left(x, \sum_{i=0}^{d} a_i x^i\right) \leq (s + \rho)\, h_j.$$

Following is the main theorem of this section.

▶ **Theorem 1.2.** *Let $\sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$ be a degree d polynomial with $\dim(\mathbf{a}) \leq 1$. Then for every $s \in [0,1]$, there is a point $x \in \mathbb{R}$ such that $\dim\left(x, \sum_{i=0}^{d} a_i x^i\right) = s + \dim(\mathbf{a})$.*

Proof. We have $\rho = \dim(\mathbf{a}) \leq 1$ and $s \in [0,1]$.

For $s = 0$, we have,

$$K_{r/d}\left(\mathbf{a} \setminus a_0, \sum_{i=0}^{d} a_i x^i\right) = K_{r/d}\left(\mathbf{a} \setminus a_0\right) + K_{r/d,r/d}\left(\sum_{i=0}^{d} a_i x^i \middle| \mathbf{a} \setminus a_0\right) + O(\log r).$$

We have,

$$\begin{aligned} K_{\frac{r}{d},\frac{r}{d}}\left(\sum_{i=0}^{d} a_i x^i \middle| \mathbf{a} \setminus a_0\right) &= K_{\frac{r}{d},\frac{r}{d}}\left(a_0 | \mathbf{a} \setminus a_0\right), && \text{by construction of } x \text{, since } s{=}0 \\ &\geq K_{\frac{r}{d}-\log c, r/d}\left(a_0 | \mathbf{a} \setminus a_0\right) + O(\log r) \\ &\geq K_{\frac{r}{d},\frac{r}{d}}\left(a_0 | \mathbf{a} \setminus a_0\right) - K\left(\frac{r}{d} - \log c\right) \\ &\qquad - \log c + O(\log r), && \textit{Lemma } 10 \\ &= K_{r/d,r/d}\left(a_0 | \mathbf{a} \setminus a_0\right) + O(\log r) \\ &= K_{r/d}(\mathbf{a}) + O(\log r). \end{aligned}$$

which gives the desired result.

For $s = 1$, by Theorem 1.1, for almost every point $x \in \mathbb{R}$ which is random relative to $(\mathbf{a})$, we have, $\dim\left(x, \sum_{i=0}^{d} a_i x^i\right) = 1 + \rho$.

For $s \in [\rho, 1)$, by Theorem 1.1, by selecting x to be a point which satisfies $\dim(x) = \dim^{\mathbf{a}}(x) = s$, we get that $\dim(x, \sum_{i=0}^{d} a_i x^i) = s + \rho$.

For $s \in (0, \rho)$, by Lemma 26 and Lemma 27, we get,

$$\begin{aligned} \dim\left(x, \sum_{i=0}^{d} a_i x^i\right) &= \liminf_{r\to\infty} \frac{K_r\left(x, \sum_{i=0}^{d} a_i x^i\right)}{r} \\ &\geq \liminf_{r\to\infty} \frac{\left(s + \rho - \gamma(d+1)^2\right) r}{r} \\ &= s + \rho - \gamma(d+1)^2. \end{aligned}$$

Since γ is arbitrarily chosen, we get $\dim\left(x, \sum_{i=0}^{d} a_i x^i\right) = s + \rho$. Combining this with Lemma 28, we get the desired result. ◀

8 Width of the dimension spectra

We provide some insights into the case where the dimension spectrum of points on a polynomial could have diameter strictly greater than 1. This leads to answer to another question posed by Stull [17].

▶ **Lemma 29.** *There is a class of polynomials of the form* $P(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{R}[x]$*, with dimension spectrum having diameter strictly greater than 1.*

Proof. Denote the vector $(a_d, a_{d-1}, \ldots, a_0)$ by $\mathbf{a}$. Consider the polynomial $P(x) = \sum_{i=0}^{d} a_i x^i$. Let $S = \{(x, P(x)) : x \in \mathbb{R}\}$ denote the graph of the polynomial. Let $\mathbf{a} \setminus \{a_0\}$ be Martin-Löf random, and let $a_0 \in \mathbb{R}$ be such that $\dim(a_0) = q$, where $q \in [0, 1]$. Note that $\dim(\mathbf{a}) > 1$.

Observe that $P(0) = a_0$, hence $\dim(0, P(0)) = q$. Hence, $q \in sp(S)$.

By [21], we know that $1 \in sp(S)$.

Next, let $x \in \mathbb{R}$ be such that it is Martin-Löf random relative to $\mathbf{a}$. Then by Theorem 1.1, we have $\dim(x, P(x)) \geq \dim(x|\mathbf{a}) + \min\{\dim(\mathbf{a}), \dim^{\mathbf{a}}(x)\} \geq 1 + \min\{1, 1\} = 2$. Hence diam $sp(S) \geq 2 - q$. Thus, the diameters of the dimension spectrum of polynomials in this class lie in the range $[1, 2]$. ◀

In the specific case of lines, we provide an answer to Stull's question - there are lines with computable intercepts with dimension spectrum greater than 1.

▶ **Corollary 30.** *For the class of polynomials* $\{P_i(x) \in \mathbb{R}[x] : \dim(P(0)) = 0\}_i$ *with computable intercepts,* diam $sp(S) = 2$*.*

Proof. Consider a_0 to be a computable point (dimension 0). Then, in Lemma 29, $q = 0$, and hence diam $sp = 2$. ◀

9 Open problems

A natural question to consider is whether the methods in this work extend to spectra of arbitrary continuous curves.

Building on our work, and Stull [18], we propose the dimension spectrum conjecture for high dimension polynomials. In other words, does the dimension spectrum of every high dimension polynomial also contain an interval of length 1?

It is also interesting to determine whether there is a trigonometric polynomial whose dimension spectrum is a singleton.

If every dimension level set in Euclidean space is path-connected, are such paths differentiable almost everywhere, or are there dimension level sets where every continuous path in them will be nowhere differentiable?

References

1 Adam Case and Jack H. Lutz. Mutual dimension. *ACM Trans. Comput. Theory*, 7(3):12:1–12:26, 2015. doi:10.1145/2786566.

2 Augustin-Louis Cauchy. Exercices de mathématique. *Œuvres 2*, 9:122, 1829.

3 Rodney Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*, 2008. In preparation.

4 Ming Li and Paul M. B. Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications, 4th Edition*. Texts in Computer Science. Springer, 2019. doi:10.1007/978-3-030-11298-1.

5 J. H. Lutz. Gales and the constructive dimension of individual sequences. In *Proceedings of the 27th International Colloquium on Automata, Languages, and Programming*, pages 902–913, 2000. Revised as [7].

6 J. H. Lutz. Dimension in complexity classes. *SIAM Journal on Computing*, 32:1236–1259, 2003. Preliminary version appeared in *Proceedings of the Fifteenth Annual IEEE Conference on Computational Complexity*, pages 158–169, 2000. doi:10.1137/S0097539701417723.

7 J. H. Lutz. Dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, 2003. doi:10.1016/S0890-5401(03)00187-1.

8 Jack H. Lutz and Neil Lutz. Algorithmic information, plane kakeya sets, and conditional dimension. *ACM Trans. Comput. Theory*, 10(2):7:1–7:22, 2018. doi:10.1145/3201783.

9 Jack H. Lutz, Neil Lutz, and Elvira Mayordomo. Extending the reach of the point-to-set principle. *Information and Computation*, 294:Paper No. 105078, 2023. doi:10.1016/j.ic.2023.105078.

10 Jack H. Lutz and Elvira Mayordomo. Dimensions of points in self-similar fractals. *SIAM Journal on Computing*, 38:1080–1112, 2008. doi:10.1137/070684689.

11 Jack H. Lutz and Elvira Mayordomo. Algorithmic fractal dimensions in geometric measure theory. *CoRR*, abs/2007.14346, 2020. arXiv:2007.14346.

12 Jack H. Lutz and Klaus Weihrauch. Connectivity properties of dimension level sets. *Math. Log. Q.*, 54(5):483–491, 2008. doi:10.1002/malq.200710060.

13 Neil Lutz and D. M. Stull. Bounding the dimension of points on a line. *Information and Computation*, 275:104601, 15, 2020. doi:10.1016/j.ic.2020.104601.

14 Neil Lutz and Donald M. Stull. Projection theorems using effective dimension. In Igor Potapov, Paul G. Spirakis, and James Worrell, editors, *43rd International Symposium on Mathematical Foundations of Computer Science, MFCS 2018, August 27-31, 2018, Liverpool, UK*, volume 117 of *LIPIcs*, pages 71:1–71:15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.MFCS.2018.71.

15 Elvira Mayordomo. Effective hausdorff dimension in general metric spaces. *Theory Comput. Syst.*, 62(7):1620–1636, 2018. doi:10.1007/s00224-018-9848-3.

16 André Nies. *Computability and randomness*, volume 51. OUP Oxford, 2009.

17 D. M. Stull. The dimension spectrum conjecture for planar lines. In *49th EATCS International Conference on Automata, Languages, and Programming*, volume 229 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 133, 20. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.ICALP.2022.133.

18 D. M. Stull. The dimension spectrum conjecture for planar lines. *Journal of the London Mathematical Society. Second Series*, 111(6):Paper No. e70216, 30, 2025. doi:10.1112/jlms.70216.

19 Donald M. Stull. Resource bounded randomness and its applications. *Algorithmic Randomness: Progress and Prospects*, 50:301, 2020.

20 C. Sturm. Mémoire sur la résolution des équations numériques. *Mémoires préséntés par divers savants à l'Acadèmie des Science de l'Institute de France*, 6:273–318., 1835.

21 Daniel Turetsky. Connectedness properties of dimension level sets. *Theoretical Computer Science*, 412(29):3598–3603, 2011. doi:10.1016/j.tcs.2011.03.006.

22 Joachim von zur Gathen and Jürgen Gerhard. *Modern computer algebra*. Cambridge University Press, Cambridge, third edition, 2013. doi:10.1017/CBO9781139856065.

23 Klaus Weihrauch. *Computable Analysis. An Introduction*. Springer-Verlag, 2000.

24 Chee Keng Yap. *Fundamental problems of algorithmic algebra*. Oxford University Press, New York, 2000.