Well-Quasi-Ordering Eulerian Digraphs: Bounded Carving Width

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Abstract

We prove that every class of Eulerian directed graphs of bounded carving width (equivalently, of bounded degree and treewidth) is well-quasi-ordered by strong immersion. In fact, we prove a stronger result, namely that every class of Eulerian directed graphs of bounded carving width, where every vertex is additionally labelled from a well-quasi-order, fixes a linear order on its incident edges, and may impose further restrictions on how the immersion is allowed to route paths through it, is well-quasi-ordered by an adequate notion of strong immersion. To this extent, we develop a framework seemingly suited to prove well-quasi-ordering for classes of Eulerian directed graphs by (strong) immersion and present a first meta theorem in that direction. We complement our results by observing that the class of Eulerian directed graphs of unbounded degree is *not* well-quasi-ordered by *strong* immersion, even if we assume the treewidth of the class to be at most two. We conclude with a dichotomy result, proving for a very restricted class of Eulerian directed graphs of unbounded degree that it is not well-quasi-ordered by strong immersion, but it is well-quasi-ordered by weak immersion.

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1 Introduction

Well-quasi-orderings of mathematical objects have been studied for many decades in various mathematical as well as computer science research areas, see e.g. [18, 8, 10, 19]. In general, given a set of objects V and a *binary relation* $\preceq$ on $V \times V$ that is reflexive and transitive, we call $(V, \preceq)$ a *quasi-order*. We call an infinite sequence $(e_i)_{i\in\mathbb{N}}$ of elements $e_i \in V$ a *chain* if $e_i \preceq e_j$ for all $i \leq j$. We call it an *antichain* if for all $i \neq j \in \mathbb{N}$ the elements e_i, e_j are incomparable with respect to $\preceq$. Finally, V is called *well-quasi-ordered by* $\preceq$ if for every infinite sequence $e_1, e_2, \ldots$ of objects $e_i \in V$, there exist $j_1, j_2 \geq 1$ with $j_1 < j_2$ such that $e_{j_1} \preceq e_{j_2}$. In this case, we refer to $(V, \preceq)$ as a *well-quasi-order*.

In 1960, Kruskal [17] proved one of the first impactful graph-theoretic results regarding well-quasi-ordering, namely that trees are well-quasi-ordered by topological containment[1]. Later, Nash-Williams [24] provided a simplified proof that finite trees are well-quasi-ordered by topological containment. Since then a whole field of research emerged, trying to identify classes of graphs (or related data structures) that are well-quasi-ordered with respect to some well-known "containment relation". In the late 20th century, Robertson and Seymour finally gave a proof of Wagner's Conjecture [29], stating that undirected graphs are well-quasi-ordered by the *minor* relation – H is a *minor* of G if it can be obtained from a subgraph of G by contracting edges – marking a major breakthrough in the area. The proof of this result culminated in the creation of a whole field of research known as graph minor theory, laying the groundwork for more general structural graph theory. A core idea of the proof is to decompose graphs into "structurally well-behaved" pieces forming a "tree-like" structure. Once given such a tree-like structure called *tree decomposition*, Robertson and Seymour showed that one may lift the simplified proof due to Nash-Williams for well-quasi-ordering trees, which in a nutshell shows that it suffices to prove that the "structurally well-behaved" pieces are well-quasi-ordered by the minor relation. Specifically, in [27] Robertson and Seymour prove a generalization of Nash-Williams' result for trees with bounded edge-weights satisfying a certain linkedness property, and whose vertices are labelled from a well-quasi-order, essentially representing the "structurally well-behaved" pieces. Intuitively speaking, they leverage Nash-Williams' proof to develop a framework that can be used to prove well-quasi-ordering for data structures encoded in trees in a fairly general way. In particular, graphs of bounded treewidth can be encoded using their framework, and they use it to prove that graphs of bounded treewidth are well-quasi-ordered by the minor relation. Similarly, as alluded to above, the graph minor structure theorem allows to decompose graphs via tree decompositions into pieces which are either of bounded treewidth or "structurally well-behaved." This can again be thought of as a labelled tree with bounded edge-weights (the adhesion) and labelled vertices. A lot of effort is then put into proving that their framework may again be applied, culminating in a proof of Wagner's Conjecture. This highlights that the fact that their framework works for graphs of bounded treewidth with vertices labelled from a well-quasi-order (and additional constraints) marks not only a base case, but a fundamental tool for the general proof of Wagner's Conjecture.

Besides the graph minor relation, another prominent relation between graphs is the *immersion* relation. We say that a graph H *immerses* in a graph G if there exists a map $\gamma : H \to \{P \mid P \subseteq G\}$ such that $\gamma|_{V(H)} : V(H) \to V(G)$ is injective and $\gamma|_{E(H)} : E(H) \to \{P \mid P \text{ is a trail}^2 \text{in } G\}$[3] guarantees that $P \coloneqq \gamma(\{u, v\})$ is a trail starting in $\gamma(u)$ and ending in $\gamma(v)$, and such that for two distinct edges $e_1, e_2 \in E(H)$, the trails $\gamma(e_1), \gamma(e_2)$ are edge-disjoint. We call γ an *immersion*[4].

The immersion γ is *strong* if no trail $\gamma(e)$ contains a vertex of $\gamma(V(H))$ as an internal vertex. We write $\gamma\colon H \hookrightarrow G$ to mean that γ strongly immerses H in G, and $\gamma : H \hookrightarrow^* G$ to mean that γ immerses H in G; we may call the latter *weak* immersion, to emphasise that it may not be strong. Immersions (weak and strong) are defined analogously for directed graphs, mapping directed edges to directed trails.

[1] The exact statement is stronger and more complex, and so is its proof.
[3] A trail is a walk in which no edge is traversed more than once.
[4] Note that immersion is often defined via mapping edges to paths; both notions are equivalent on (directed) graphs.

Later on, still part of their seminal work on graph minors, Robertson and Seymour derived Nash-William's Conjecture from their work. That is, they proved that undirected graphs are well-quasi-ordered by *weak* immersion [30], marking another breakthrough in the area. Their proof does not use *strong* immersions[5] and, as Robertson and Seymour discussed in [30], their framework cannot easily be adapted to prove well-quasi-ordering of undirected graphs by *strong* immersion. In fact, they claim that a proof of the strong immersion result is way more complicated: "*It seemed to us at one time that we had a proof of the stronger [immersion conjecture], but even if it was correct it was very much more complicated, and it is unlikely that we will write it down*" [30, p2)]. As far as we are aware, the strong immersion conjecture is still open. We want to emphasise that their proof for well-quasi-ordering graphs of bounded treewidth does imply well-quasi-ordering for bounded treewidth graphs by *weak* immersion, but it does not naturally lift to a proof for *strong* immersion. In fact, we are not aware of a result in the literature proving that graphs of bounded treewidth are well-quasi-ordered by strong immersion, highlighting once more the importance of results and new frameworks in the area.

Directed Graphs. After proving Wagner's and Nash-Williams' Conjecture, Robertson and Seymour turned their interest to directed graphs (also referred to as *digraphs*). For, it is natural to ask "What are good relations suited for well-quasi-ordering directed graphs?". Together with Johnson and Thomas, they introduced a notion of *directed treewidth* [13] seemingly suited to their needs – but unfortunately the natural extension of minors to digraphs, which is *butterfly minors*, does *not* yield a well-quasi-ordering for digraphs, not even on digraphs of low directed treewidth (we will elaborate on this below).

It quickly turned out that directed graphs behave very differently from undirected graphs: For most well-studied containment relations on digraphs, counterexamples to well-quasi-ordering have been found. Liu and Muzi [20] proved that digraphs are not well-quasi-ordered by immersion nor by strong immersion. Their counterexample is a growing sequence of slightly modified alternating paths (see the thick highlighted part of Figure 1 for an illustration), which is a counterexample to well-quasi-ordering with respect to many more minor relations on digraphs including the above mentioned butterfly minor relation. In the same paper they prove that alternating paths are, in a sense, the relevant obstruction to well-quasi-ordering with respect to strong immersion, for they prove that the class of digraphs *not* containing long alternating paths is indeed well-quasi-ordered by the strong immersion relation. It turns out that the same observation can be made for butterfly-minors [23], i.e., classes of digraphs that do not admit "long" alternating paths are well-quasi-ordered by butterfly-minors. Unfortunately, not containing long alternating paths is a very strict restriction for digraphs, in particular, such graphs must have low directed and even low undirected treewidth [20, 16].

Besides this, only a few positive results on well-quasi-ordering directed graphs seem to be known. One of the most prominent ones, due to Chudnovsky and Seymour [5], states that *tournaments* – graphs obtained from orienting edges of undirected cliques – are well-quasi-ordered by strong immersion; note that tournaments may contain arbitrarily long alternating paths. This result was later extended to semi-complete digraphs by Barbero, Paul, and Pilipczuk [2].

[5] Note that their results imply a proof for strong immersions on graph classes of bounded degree.

2 Well-Quasi-Ordering Eulerian Digraphs: A Counterexample

A class of directed graphs that lies somewhat between general directed and undirected graphs is the class of *Eulerian digraphs*. Given a digraph G we call a vertex $v \in V(G)$ *Eulerian*, if its in-degree equals its out-degree. Then, a digraph G is called *Eulerian* if every vertex in $V(G)$ is Eulerian[6]. Equivalently, G is the union of a set of pairwise edge-disjoint directed cycles. Eulerian digraphs have many nice properties, making them a particularly interesting class of digraphs to study. See [1, Chapter 4] for an introduction to Eulerian digraphs.

One prominent example relating Eulerian digraphs to undirected graphs is Menger's Theorem: For an undirected graph G and two disjoint vertex sets $A, B \subset V(G)$ there are either k edge-disjoint paths with one endpoint in A and one endpoint in B, or there is a set $F \subseteq E(G)$ of $(k-1)$ edges, whose deletion destroys all connectivity between A and B. For Eulerian digraphs, there is an equivalent result, stating that given two such sets A and B, either there is a set of $2k$ edge-disjoint paths (half of them going from A to B and the other half from B to A), or a set F of at most $2(k-1)$ edges, whose deletion destroys all connectivity between A and B. Although there is an analogue to Menger's Theorem for general digraphs, it unfortunately depends on a choice of whether we want to connect A to B or B to A. In particular, there may not be equally many paths in both directions, and any path from A to B may intersect every path from B to A. This is a crucial difference, and it may be the single most important reason why all of the results presented in this paper (and future work) hold true for Eulerian digraphs, while they fail for general digraphs.

The following is a more sophisticated example highlighting that Eulerian digraphs behave similarly to undirected graphs. Whereas it is known that the 2-Edge-Disjoint Paths problem is NP-complete on general digraphs, the k-Edge-Disjoint Paths problem was recently shown to be fixed-parameter tractable for Eulerian digraphs [4].

A further structural result tying Eulerian digraphs to undirected graphs is the following: when restricting to a class $\mathcal{C}$ of Eulerian digraphs of *bounded degree*, then the undirected and directed treewidth are qualitatively equivalent [13]. This is in stark contrast to general digraphs; consider an acyclic orientation of a large clique. This is another indication that, at least structurally speaking, Eulerian digraphs seem to be closely related to undirected digraphs.

It turns out that restricting to classes of bounded degree is crucial when working with strong immersion on Eulerian digraphs. On the one hand, note that when working with the general class of Eulerian digraphs, one must loosen the assumptions in the edge-disjoint paths problem to allow for the paths to pass through terminal vertices to get an FPT-algorithm[7]. This artifact disappears when working with classes of bounded degree. On the other hand, our first result is negative in the sense that it yields a counterexample to well-quasi-ordering Eulerian digraphs by strong immersion. The following construction relies on the aforementioned construction due to Liu and Muzi [20], leveraging that alternating paths are not well-quasi-ordered by strong immersion.

▶ **Observation 2.1.** *The class of Eulerian digraphs is not well-quasi-ordered by strong immersion.*

[6] It is common in the literature to assume Eulerian digraphs to be weakly connected. We do not impose this here, but mention that throughout most of the paper we assume our digraphs to be weakly connected, as the results easily transfer to connected components.

[7] This is a common assumption, even in the undirected setting.

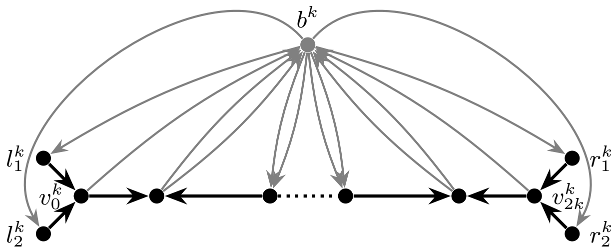

Figure 1 An infinite antichain of Eulerian digraphs G_k with respect to strong immersion. The thick highlighted subgraph forms an infinite antichain for digraphs with respect to strong immersions.

Proof. For every $k \geq 1$ let G_k be defined as follows; see the thick highlighted subgraph in Figure 1 for an illustration. We set $V(G_k) \coloneqq \{l_1^k, l_2^k, v_0^k, \ldots, v_{2k}^k, r_1^k, r_2^k\}$ and add edges from l_1^k and l_2^k to v_0^k, from r_1^k and r_2^k to v_{2k}^k, from v_0^k to v_1^k, and from v_{2k}^k to v_{2k-1}^k. Finally, for all $1 \leq i < k$ we add edges (v_{2i}^k, v_{2i-1}^k) and (v_{2i}^k, v_{2i+1}^k).

Thus, for every $k \geq 1$, the graph G_k consists of two special vertices, v_0^k and v_{2k}^k, which are the only vertices of degree 3 and between them a long "path" with edges in alternating directions.

We claim that if $i \neq j$ then $G_i \not\rightarrow G_j$. This is obvious if $i > j$, thus we assume $i < j$. But then, the two vertices v_0^i and v_{2i}^i of degree 3 in G_i must be mapped to the two vertices v_0^j and v_{2j}^j of degree 3 in G_j, as no other vertex in G_j has degree 3. This implies that the vertices $v_1^i, \ldots, v_{2i-1}^i$ on the alternating path in G_i must be mapped to vertices on the alternating path in G_j. As $j > i$ there must be two adjacent vertices v_t^i and v_{t+1}^i in G_i that are mapped to two vertices $v_{t_1}^j$ and $v_{t_2}^j$ which are not adjacent in G_j. But then there is no directed path in G_j between $v_{t_1}^j$ and $v_{t_2}^j$ and thus the edge between v_t^i and v_{t+1}^i cannot be immersed into G_j.

So far, the example only shows that the class of directed graphs is not well-quasi-ordered under the strong immersion relation. But the digraphs are not yet Eulerian. We can easily make them Eulerian using a standard construction from the theory of Eulerian digraphs: For $i > 1$ let G'_i be the digraph obtained from G_i by adding a fresh vertex b^i and for all $v \in V(G_i)$ that have more incoming than outgoing edges we add sufficiently many edges from v to b^i so that the in-degree of v equals its out-degree in G'_i and likewise we add edges from b^i to $v \in V(G)$ if v has more outgoing than incoming edges. See Figure 1 for an illustration of the construction. (To avoid parallel edges, one may subdivide parallel edges if need be.)

It is easily seen that the resulting graphs G'_i are Eulerian. But it is still the case that for $i < j$ there is no strong immersion of G'_i in G'_j. For, any such immersion must map the high degree vertex b^i of G'_i to the corresponding vertex b^j in G'_j and so the remaining vertices must be mapped to each other. Thus, the same argument as above shows that this is impossible. ◀

Note that, complementing the above discussion, the class $\mathcal{C} \coloneqq \{G'_k : k > 1\}$ has unbounded maximum degree and up to the vertices b^k for $k \geq 1$, every vertex is of degree at most four. Observe further that the undirected treewidth of the graphs G'_k in the proof of Observation 2.1 is ≤ 2 for every $k \geq 1$ and the graphs admit a planar embedding as depicted in Figure 1. This yields the following.

▶ **Corollary 2.2.** *The class of planar Eulerian digraphs of treewidth at most 2 is not well-quasi-ordered by strong immersion.*

Note here that one easily adapts the construction in Observation 2.1 to provide an antichain of unbounded *undirected* treewidth for strong immersion, by replacing the alternating paths with a version of "grids", where each path in the grid is alternating by making every vertex of the grid either a sink or a source. In particular, there are no "small" cuts that decompose these graphs into two "large" pieces.

Finally, we emphasise that the class $\mathcal{C}$ above is only an antichain for the *strong* immersion relation, but not for the weak immersion relation; we will get back to this observation below.

3 Conjectures and Implications

Observation 2.1 shows that to obtain a well-quasi-ordering of Eulerian digraphs by strong immersion we must restrict to classes of bounded maximum degree.

3.1 Bounded Degree

In contrast to and complementing Observation 2.1 we propose the following, which was already conjectured by Johnson [12].

▶ **Conjecture 3.1.** *For every $d \geq 1$, the class of Eulerian digraphs of maximum degree $2d$ is well-quasi-ordered by strong immersion.*

It should be noted that in his dissertation, Johnson [12] already discovered the strong structural similarities between Eulerian digraphs and undirected graphs. In fact, he proved a structure theorem for internally 6-connected 4-regular Eulerian digraphs, and mentions that he anticipated using it towards proving a well-quasi-order result for Eulerian digraphs by immersion[8] in the future. Unfortunately, to the best of our knowledge, he never published such results. Nevertheless, this shows that there has been a hunch that Eulerian digraphs may be a particularly interesting candidate for a directed graph class with tamable structural behavior.

Note further that, in contrast to the discussion about the results in [20, 23], if Conjecture 3.1 were true, then the obstructions to well-quasi-ordering Eulerian digraphs by strong immersion are not long alternating paths, but rather vertices of arbitrarily large degree.

Let us discuss a few immediate consequences and important applications that would follow from a proof of Conjecture 3.1.

Circle Graphs and Pivot-Minors. The most prominent one may be that a proof of Conjecture 3.1 for $d = 2$ would imply that *circle graphs* are well-quasi-ordered by the *pivot-minor* relation. This is a longstanding open problem, marking a crucial step towards the more general conjecture that undirected graphs are well-quasi-ordered by the pivot-minor relation (to the best of our knowledge, this conjecture is still open even in the weaker case of vertex-minors). We leave out the details and refer the interested reader to [14, 25, 22]. It turns out that every circle graph G naturally corresponds to a 4-regular Eulerian digraph $D(G)$ with a choice of Eulerian cycle, and every 4-regular Eulerian digraph naturally corresponds to a class of circle graphs (depending on a choice of Eulerian cycle) such that the following holds. Given two circle graphs H and G, H is a pivot-minor of G if and only if $D(H) \hookrightarrow D(G)$, i.e., strong immersion on the respective Eulerian digraphs corresponds to the pivot-minor relation on the circle graphs. It may be worth exploring whether the cases $d > 2$ may be of further

[8] His definition differs slightly from the standard definition.

help towards proving the general pivot-minor well-quasi-order conjecture, for example, by examining which classes of undirected graphs they correspond to when trying to translate strong immersions of Eulerian digraphs to pivot-minors of undirected graphs.

Membership Testing. Decision problems (in and outside of graph theory) can naturally be rephrased as *Membership Testing problems*. Let $\mathcal{S}$ be a class of structures and $\mathcal{P} \subseteq \mathcal{S}$, then we define the following:

$(\mathcal{P}, \mathcal{S})$-Membership Testing

Input: A structure $S \in \mathcal{S}$.
Question: Is S contained in $\mathcal{P}$?

A common example is Planarity Testing, where $\mathcal{S}$ is the class of undirected graphs and $\mathcal{P}$ is the class of planar graphs: Given a graph G one has to decide whether G is planar. By a well-known result due to Kuratowski and Wagner, this reduces to the question of whether G admits $K_{3,3}$ or K_5 as a minor. More generally, one may ask: "Given a list of graphs $\mathcal{F}$, does G admit a graph in $\mathcal{F}$ as a minor?". As we discuss next, this can be rephrased as a Membership Testing problem of particular importance.

Robertson and Seymour [26] proved that for any fixed graph H, the question of whether G admits H as a minor can be decided in time $O(f(H)n^3)$ for some function f. This was first improved to $O(f(H)n^2)$ by Kawarabayashi, Kobayashi, and Reed, [11], and has recently been brought down to almost linear time by Korhonen, Pilipczuk, and Stamoulis [15]. This result, combined with the fact that undirected graphs are well-quasi-ordered by the minor relation, implies the existence of a plethora of polynomial time algorithms for testing relevant graph properties by expressing them as Membership Testing problems. To make this more precise, let $\preceq$ induce a quasi-order on a class of structures $\mathcal{S}$; for example, $\mathcal{S}$ could be the class of undirected graphs and $\preceq$ the minor relation. A class $\mathcal{P} \subseteq \mathcal{S}$ of structures is called *$\preceq$-closed*, if $S \in \mathcal{P}$ implies that $S' \in \mathcal{P}$ for every $S' \preceq S$. Let now $\neg\mathcal{P} \subseteq \mathcal{S}$ be chosen so that $\neg\mathcal{P} \cap \mathcal{P} = \emptyset$ and such that $\neg\mathcal{P}$ is *$\preceq$-minimal* in the sense that for every $S \in \neg\mathcal{P}$ and every $S' \prec S$ (hence $S \neq S'$) it holds that $S' \in \mathcal{P}$. If the class $\mathcal{S}$ is well-quasi-ordered by $\preceq$, then $\neg\mathcal{P}$ is guaranteed to be finite. Thus, since $\mathcal{P}$ is $\preceq$-closed, to decide whether $S \in \mathcal{P}$ for some $S \in \mathcal{S}$, it suffices to check whether for all $S' \in \neg\mathcal{P}$ we have $S' \not\preceq S$. In particular, if $S' \preceq S$ can be efficiently decided, say in polynomial time for fixed S', then so can the $(\mathcal{P}, \mathcal{S})$-Membership Testing problem by iterating over the finite list $\neg\mathcal{P}$.

Robertson and Seymour [26] used this argument to prove that every minor-closed property of undirected graphs can be tested in $O(n^3)$-time (now improved to almost linear time by the above [15]). Letting $\preceq$ be the strong immersion relation and $\mathcal{S}$ be the class of Eulerian digraphs of bounded degree, a proof of Conjecture 3.1 together with the main result of [4] – The Edge-Disjoint Paths problem is fixed-parameter tractable for Eulerian digraphs – would imply that every property of Eulerian digraphs of bounded degree closed under strong immersion can be tested in polynomial time. Note that in [4], the authors do not solve the more general Folio problem (see [26]). Hence, their results imply that the Immersion Testing problem – given two Eulerian digraphs H and G, does $H \hookrightarrow G$? – is polynomial-time solvable for every fixed H by an algorithm running in time of $O(|V(G)|^{|V(H)|})$. However, this is not relevant for the mentioned application, as it is polynomial-time for fixed H. Furthermore, it seems reasonable that the results presented in [4] can be extended to the more general Folio problem, which would imply that the Immersion Testing problem could be solved in $O(f(|V(H)|)n^c)$ time for some function f and some constant c. This would likely make the polynomial-time algorithms discussed above far more practical.

Finally, combining this with the above discussion on pivot-minors and circle graphs, a proof of Conjecture 3.1 would further imply that $(\mathcal{P}, \mathcal{S})$-Membership Testing, for any fixed pivot-minor closed class $\mathcal{P}$, can be decided in polynomial time on the class of circle graphs $\mathcal{S}$. Pivot-Minor Testing as well as Vertex-Minor Testing are known to be NP-complete [6, 7]) in general, and there has been extensive research towards positive results ever since.

3.2 Unbounded Degree

Note that in the above discussion, we had to restrict to classes of Eulerian digraphs of *bounded degree*. Unfortunately, Observation 2.1 shows that there is no hope to extend Conjecture 3.1 to the class of *all* Eulerian digraphs. However, we believe that this is an artifact of the *strong* immersion relation. As the construction in Observation 2.1 shows, immersion forces us to map the high degree vertices to each other, and thus they become unusable for routing other paths when dealing with strong immersion. For weak immersion, this is not the case and the high degree vertex in the target digraph can still be used to route further paths. We believe that this is not just an artifact of our construction, but a fundamental difference; we conjecture the following.

▶ **Conjecture 3.2.** *The class of Eulerian digraphs is well-quasi-ordered by weak immersion.*

This marks a crucial distinction to undirected graphs: If the strong Nash-Williams Conjecture is true (which is commonly believed in the area), then undirected graphs are well-quasi-ordered by strong immersion, whereas Eulerian digraphs are not, but a proof of Conjecture 3.2 would imply that both are well-quasi-ordered by *weak* immersion. We want to emphasise that a proof of Conjecture 3.2 would immediately lift the above applications regarding Membership Testing to weak immersion and the class of all Eulerian digraphs. That is, any property closed under weak immersion could be tested in polynomial time on the class of Eulerian digraphs.

It may be fruitful to explore whether the above Conjectures 3.1 and 3.2 have implications for other subclasses of directed graphs via suited reductions or decompositions; we leave this for future research.

4 Main results

At the time of writing, we believe that we have a clear strategy to tackle both Conjectures 3.1 and 3.2. This paper is a crucial first step in that direction, and its goal is twofold.

As our first main contribution we establish a very general framework developed towards tackling Conjectures 3.1 and 3.2 and prove several important results to make the framework applicable. Our main result is a “meta theorem” (Theorem 4.5), opening the way to an inductive proof strategy for both conjectures, via decomposing our graphs into pieces of desirable structure. Such decomposition theorems lie at the core of well-quasi-ordering results exploiting structural graph theory as developed by Robertson and Seymour [27, 26, 28] as we discussed above. Note that the framework we develop is designed towards *strong* immersions. In light of the introductory discussion, we believe that the presented results may be of independent interest, and that our techniques and tools might generalise to undirected graphs and may help proving the strong Nash-Williams Conjecture and related results.

On the other hand, we prove the first important base case of Conjecture 3.1, handling the case where our graphs lack “structural richness.” The correctness of the base case discussed in this paper is a crucial first step towards a full proof of Conjectures 3.1 and 3.2, and the form in which we prove it can additionally function as another strong "meta theorem", suited for inductive arguments when proving well-quasi-ordering results by strong immersion, as we briefly elaborate on below. There are two main contributions to discuss.

4.1 Knitworks

First, we introduce the concept of Ω-*knitwork*, a data structure tailored towards strong immersion of (Eulerian directed) graphs, which seems well suited for proving Conjecture 3.1. In the following exposition, we have simplified the presentation somewhat and tried to focus on the main ideas and concepts. In particular, in several of the following definitions, e.g. in the definition of rooted Eulerian digraphs and of Ω-knitworks, we left out some extra requirements which are required for the proofs to work but which would only clutter this short version with technical details that we do not have space to explain properly. The full and correct, definitions and results can be found in the full version of this paper [3].

Before going into details, we motivate the concept of Ω-knitworks by the following example. Suppose we have two Eulerian digraphs H and G and want to immerse H into G. Suppose further that there is a set of edges $F_H \subseteq E(H)$ such that deleting F_H from H splits it into disjoint subgraphs H_1 and H_2; we call F_H a *cut*. Similarly, assume that there is a cut $F_G \subseteq E(G)$ splitting G into disjoint subgraphs G_1 and G_2. Now we could try to immerse H_1 into G_1 and H_2 into G_2 and from this obtain an immersion of H into G by somehow merging the two immersions of H_1 and H_2 into G_1 and G_2, respectively, along the edges in the cut F_G. If possible, this would allow for inductive proofs along cuts.

But surely simply recursing into the smaller subgraphs H_1, H_2, G_1, and G_2 will not be enough. For instance, H_1 may no longer be Eulerian as we removed the edges of the cut F_H. Furthermore, there is no guarantee that we can simply merge the immersion of the subgraphs together along the edges of F_G. It could be that the cut F_H in H allows for some connectivity between H_1 and H_2 that can not be replicated by the cut F_G in G. Thus, instead of simply recursing into the smaller subgraph H_1 along an edge cut we need to somehow remember the relevant properties of H_2 and the cut F_H in the smaller instance H_1. We refer to the data structure that will serve this purpose as an Ω-knitwork below.

We now present the framework of knitworks in more detail. Given a digraph G and a strict subset $X \subset V(G)$, we let $\bar{X} \coloneqq V(G) \setminus X$ and call

$$\rho(X) = \{e \in E(G) \mid e \text{ has one endpoint in } X \text{ and one in } \bar{X}\}$$

the *cut induced by* X. We denote by $\rho^-(X) \subseteq \rho(X)$ the set of edges with head in X, and by $\rho^+(X) \subseteq \rho(X)$ the set of edges with tail in X. If $X = \{v\}$ is a single vertex we write $\rho(v)$ as well as $\rho^-(v)$ and $\rho^+(v)$ for simplicity. A *rooted digraph* is a tuple (G, π, X) where G is a digraph, $X \subset V(G)$ is a strict subset and π is a map assigning to X a linear order $\pi(X)$ on $\rho(X)$.[9]

Given a well-quasi-order $\Omega = (V_\Omega, \preceq)$ we define Ω-knitworks as follows. (Again, we omit some details for the sake of simplicity.) Given two disjoint sets E_1, E_2 we write $\mathrm{Match}(E_1, E_2)$ for the set of all possible matchings on $E_1 \times E_2$, where a *matching* $M \subseteq E_1 \times E_2$ is a set of pairs such that no element of $E_1 \cup E_2$ is part of two distinct pairs in M.

[9] In the full paper we actually root digraphs in several cuts induced by disjoint sets $X_1, \ldots, X_\ell$. Furthermore, we allow some vertices of degree one in X. We call these graphs *quasi-Eulerian*, as they are not strictly Eulerian anymore. This seems necessary for our inductive arguments. We tacitly ignore these details in this exposition.

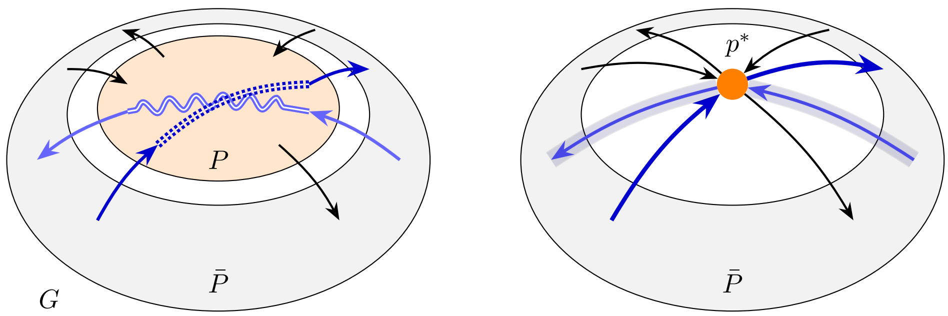

Figure 2 A "piece" P inducing a cut in G (left) is replaced with a placeholder vertex p^* by contracting it (right). Arrows represent cut-edges. Two edge-disjoint trails in $G[P]$ are highlighted on the left by a wiggly and dotted double-line. After contraction they are kept track of by $\mathfrak{m}_{\mathcal{G}}(p^*)$ on the right via the matching consisting of the pair of edges highlighted by a tube and the pair of thick edges.

▶ **Definition 4.1.** *An* Ω-knitwork *is a tuple* $\mathcal{G} = ((G, \pi, X), \mu_{\mathcal{G}}, \mathfrak{m}_{\mathcal{G}}, \Phi_{\mathcal{G}})$ *such that*

- (G, π, X) *is a rooted Eulerian digraph where* G *is loopless and connected*[10]*: We call it the* underlying rooted Eulerian digraph *of* $\mathcal{G}$*,*
- $\mu_{\mathcal{G}}$ *is a map with* $\mathrm{dom}(\mu_{\mathcal{G}}) \subseteq V(G) \setminus X$ *and* $\mu_{\mathcal{G}}(v)$ *fixes a linear order on* $\rho(v)$*,*
- $\mathfrak{m}_{\mathcal{G}}$ *is a map with* $\mathrm{dom}(\mathfrak{m}_{\mathcal{G}}) \subseteq \mathrm{dom}(\mu_{\mathcal{G}})$ *and* $\mathfrak{m}_{\mathcal{G}}(v) \subseteq \mathrm{Match}(\rho^-(v), \rho^+(v))$ *is a set of "feasible" matchings, such that* $M \in \mathfrak{m}_{\mathcal{G}}(v)$ *and* $M' \subseteq M$ *implies* $M' \in \mathfrak{m}_{\mathcal{G}}(v)$*,*
- $\Phi_{\mathcal{G}}$ *is a function with* $\mathrm{dom}(\Phi_{\mathcal{G}}) \subseteq V(G) \setminus X$ *and* $\Phi_{\mathcal{G}}(v) \in V_{\Omega}$*.*

The general idea of Ω-knitworks is that towards a proof of Conjecture 3.1 our aforementioned strategy would be to inductively decompose the graphs along "small cuts" into structurally rich "pieces" (we will not make this precise here) and contract these pieces into placeholder vertices at which one encodes the information of the pieces they replace. This information will essentially be kept track of by the maps $\mu_{\mathcal{G}}, \mathfrak{m}_{\mathcal{G}}$ and $\Phi_{\mathcal{G}}$. More precisely, the goal is to "cut out" pieces $P \subset V(G)$ of the graph that induce rooted Eulerian digraphs that fall into a class $\mathcal{C}$ for which we already proved that it is well-quasi-ordered by Ω-knitwork immersion. Let $\Omega = (\mathcal{C}, \hookrightarrow)$ be the well-quasi-order on $\mathcal{C}$. We then replace these pieces with new vertices p^*, say, and mark these with the respective "immersion type" $\Phi_{\mathcal{G}}(p^*) \coloneqq \sigma_P \in \mathcal{C}$ of the piece they replace. See Figure 2 for a schematic illustration, where P represents such a "piece", and p^* is the corresponding placeholder vertex. The map $\mu_{\mathcal{G}}$ then fixes a linear order on the respective cut $\rho(P) = \rho(p^*)$, so that the pieces we replaced can inductively be "glued back" together in the correct way. Finally, $\mathfrak{m}_{\mathcal{G}}$ keeps track of the "allowed edge-disjoint routings" inside the pieces we cut out, where each matching $M \in \mathfrak{m}_{\mathcal{G}}(p^*)$ of such a placeholder vertex p^* represents pairs of edges that can be simultaneously connected by edge-disjoint trails inside the piece P it replaces. This guarantees that we do not wrongly create new connectivity (recall that immersion is defined in terms of edge-disjoint trails, and thus several trails may possibly pass through the same vertex). See Figure 2 where a routing of two edge-disjoint trails through P is highlighted on the left side, and the respective matching consisting of two pairs of edges of the cut after replacing the piece by p^* is highlighted on the right side.

We lift the definition of immersion of digraphs to immersion of Ω-knitworks. To this extent, we define *sub-knitworks*.

[10] We show that it suffices to work with loopless and connected Eulerian digraphs.

▶ **Definition 4.2.** *Given an Ω-knitwork $\mathcal{G} = ((G, \pi_G, X), \mu_{\mathcal{G}}, \mathfrak{m}_{\mathcal{G}}, \Phi_{\mathcal{G}})$ a sub-knitwork $\mathcal{H} \subseteq \mathcal{G}$ is itself an Ω-knitwork $\mathcal{H} = ((H, \pi_H, X), \mu_{\mathcal{H}}, \mathfrak{m}_{\mathcal{H}}, \Phi_{\mathcal{H}})$ such that*

- *$H \subseteq G$ is a subdigraph satisfying $X \subseteq V(H)$ and $\rho_G(X) = \rho_H(X)$, and $\bar{H} = (H, \pi_H, X)$ shares its roots with $\bar{G}$, i.e., $\pi_H(X) = \pi_G(X)$,*
- *$\mathrm{dom}(\mu_{\mathcal{H}}) = \mathrm{dom}(\mu_{\mathcal{G}}) \cap V(H)$ and, for $\mu_{\mathcal{G}}(v) = (e_1, \ldots, e_k)$, we have $\mu_{\mathcal{H}}(v) = (e_{i_1}, \ldots, e_{i_t})$ for some $t \leq k$ and $1 \leq i_1 < \ldots < i_t \leq k$,*
- *$\mathrm{dom}(\mathfrak{m}_{\mathcal{H}}) = \mathrm{dom}(\mathfrak{m}_{\mathcal{G}}) \cap V(H)$ and $\mathfrak{m}_{\mathcal{H}}(v) \coloneqq \mathfrak{m}_{\mathcal{G}}(v) \cap \mathrm{Match}(\rho^-_H(v), \rho^+_H(v))$,*
- *$\Phi_{\mathcal{H}} \coloneqq \Phi_{\mathcal{G}}|_{V(H)}$.*

Note that by the third bullet point in Definition 4.1, $\mathfrak{m}_{\mathcal{H}}(v)$ is as expected. Given sub-knitworks, the following is a (slightly incomplete) definition of Ω-knitwork immersion.

▶ **Definition 4.3.** *Let $\Omega = (V_\Omega, \preceq)$ and $\mathcal{G}' = ((G', \pi_{G'}, X), \mu', \mathfrak{m}', \Phi')$ and $\mathcal{H} = ((H, \pi_H, A), \nu, \mathfrak{n}, \Psi)$ be Ω-knitworks. Then $\mathcal{H}$ weakly immerses in $\mathcal{G}'$ if $|\rho_{G'}(X)| = |\rho_H(A)|$ and there is a sub-knitwork $\mathcal{G} = ((G, \pi_G, X), \mu, \mathfrak{m}, \Phi) \subseteq \mathcal{G}'$ together with a weak immersion γ of H in G satisfying the following. Let $\pi_G(X) = (e_1, \ldots, e_\ell)$ and $\pi_H(A) = (f_1, \ldots, f_\ell)$ for some $\ell \geq 1$.*

- *γ maps A to X and $\bar{A}$ to $\bar{X}$ and f_i to a trail containing e_i for every $1 \leq i \leq \ell$.*
- *Let $v \in V(H)$. Then $\gamma(v) \in \mathrm{dom}(\mu)$ iff $v \in \mathrm{dom}(\nu)$ and, given the order $\nu(v) = (h_1, \ldots, h_r)$, we have $\mu(\gamma(v)) = (h'_1, \ldots, h'_r)$ and γ maps h_i to a trail containing h'_i.*
- *For $v \in \mathrm{dom}(\mathfrak{n})$ it holds $\gamma(v) \in \mathrm{dom}(\mathfrak{m})$ and for a set of matchings $M \in \mathfrak{n}(v)$ there is a set of matchings $M' \in \mathfrak{m}(\gamma(v))$ such that if $(h_i, h_j) \in M$ then $(h'_i, h'_j) \in M'$ where $(h_1, \ldots, h_r) = \nu(v)$ and $(h'_1, \ldots, h'_r) = \mu(\gamma(v))$.*
- *If $u \in \mathrm{dom}(\mathfrak{m}) \setminus \gamma(V(H))$, then the set of edge-disjoint trails $\mathcal{L} = \gamma(E(H))$ "respects" $\mathfrak{m}(u)$, in the sense that there is $M' \in \mathfrak{m}(u)$ such that for $\mu(u) = (h'_1, \ldots, h'_r)$ it holds that if (h'_i, h'_j) is a subpath of some trail in $\mathcal{L}$ then $(h'_i, h'_j) \in M'$.*
- *For every $v \in \mathrm{dom}(\Psi)$ it holds $\gamma(v) \in \mathrm{dom}(\Phi)$ and $\Psi(v) \preceq \Phi(\gamma(v))$.*

We say that $\mathcal{H}$ strongly *immerses in $\mathcal{G}$ if γ is additionally a strong immersion of H in G.*

It is crucial to note that, as opposed to standard immersion of digraphs, for Ω-knitwork immersion, it makes a difference whether we let edges be mapped to paths or trails. The main reason being that the fourth bullet point in the definition of Ω-knitwork immersion above may not allow to reroute trails to paths at vertices; we come back to this below.

As a first step we verify that the above defined structure and containment relation are compatible, i.e., strong and weak Ω-knitwork immersion induce quasi-orders on Ω-knitworks.

▶ **Lemma 4.4.** *Let Ω be a well-quasi-order. Then strong and weak Ω-knitwork immersion induce quasi-orders on the class of Ω-knitworks*[11]*.*

Although tedious and lengthy, a proof of Lemma 4.4 is straightforward. Note here that for weak immersion, one cannot add the assumption for edges to be mapped to paths instead of trails as the proof would fail, contrasting standard immersion of digraphs.

We then continue by introducing the main tools to prove decomposition results, namely *stitching* and *knitting*. In a nutshell, given a rooted Eulerian digraph (G, π, X) and a set $X \subseteq Y \subset V(G)$ together with a linear order $\pi(Y) = \pi(\bar{Y})$ on $\rho(Y)$, we define the *down-* and *up-stitches* of G at Y to be the rooted Eulerian digraphs obtained when contracting $\bar{Y}$ and Y into single vertices y_* and y^* respectively, keeping the roots $\pi(X)$ for the former,

[11] When allowing for several root sets, one needs to restrict to the class of Ω-knitworks admitting the same number of root sets.

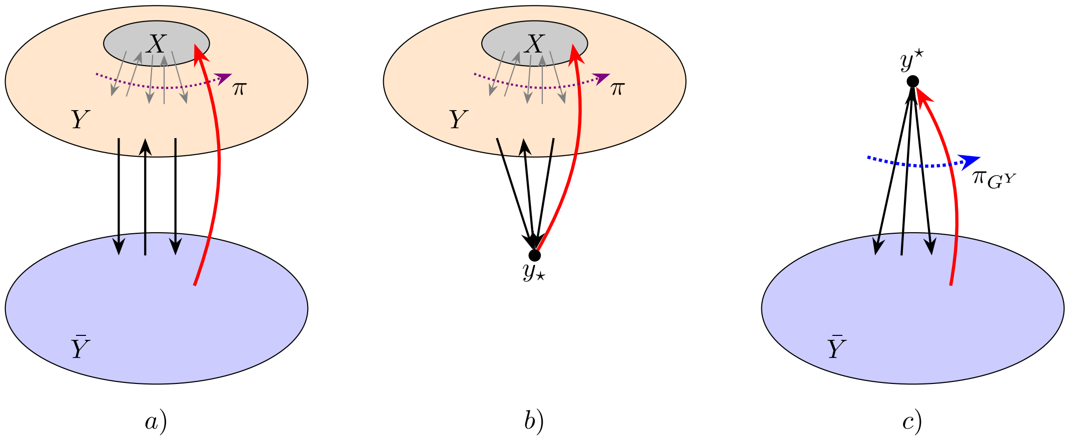

Figure 3 Down- and Up-Stitching. Figure $a)$ depicts a clamped graph $\bar{G} = (G, \pi, X)$ with a rooted cut induced by Y, a choice of $\pi(X)$ highlighted by a dotted oriented arrow, and a highlighted edge that is part of $\rho(X) \cap \rho(Y)$. Figure $b)$ depicts the down-stitch of $\bar{G}$ and Y with down-stitch vertex $y_\star$. Figure $c)$ depicts the up-stitch of $\bar{G}$ and Y with up-stitch vertex $y^\star$ fixing new roots $\pi_{G^Y}(y^*)$ on $\rho(y^*)$, highlighted by a dotted oriented arrow.

and fixing roots $\pi(y^*) := \pi(\bar{Y})$ for the latter. See Figure 3 for a schematic illustration. We say that Y induces a *rooted cut* in (G, π, X). Note here that $Y, \bar{Y}$ partition $V(G)$ and every incidence of an edge in G with a vertex in G is still part of one of the two stitches. In particular, it is straightforward to define stitches for Ω-knitworks since we do not alter any of the relevant incidences for $\mu_{\mathcal{G}}$, $\mathfrak{m}_{\mathcal{G}}$, or $\Phi_{\mathcal{G}}$. What we referred to as "cutting out pieces" above will essentially be handled by up- and down-stitching vertex sets inducing rooted cuts, where the down-stitches will essentially be the aforementioned "pieces"; we refer to the full paper for details.

Thus, stitching gives us a tool to decompose Ω-knitworks along partitions, where the newly introduced vertices obtained via stitching can be seen as the aforementioned placeholder vertices marking the cuts. We define *knitting* as a sort of inverse operation to stitching, where given two graphs G and H, both with partitions $X, \bar{X}$ and $A, \bar{A}$, and linear orders $\pi(X) = \pi(\bar{X})$ and $\pi(A) = \pi(\bar{A})$ on $\rho(X)$ and $\rho(A)$ respectively, we knit G and H along the cuts $\rho(X)$ and $\rho(A)$ if the cuts are of equal order, moving to a new graph on $X \cup \bar{A}$ identifying $\rho(X)$ with $\rho(A)$ – with respect to the linear orders $\pi(X)$ and $\pi(A)$ – and keeping all the incidences with X and those with $\bar{A}$. This will allow us to "knit back the pieces" we stitched off for inductive reasoning. Note that $\mu_{\mathcal{G}}$ keeps track of the respective linear orders when we cut out pieces inductively, and in future work we will essentially "knit back" the pieces with respect to $\mu_{\mathcal{G}}$ instead of $\pi(X)$.

We use both notions to prove two important results. First, the Stitch-and-Knit Lemma, which proves and makes precise what we mean by both operations being inverse to each other. It essentially states that, given an Ω-knitwork $\mathcal{G}$ and a rooted cut $Y \subset V(G)$ with a linear order $\pi(Y) = \pi(\bar{Y})$ on $\rho(Y)$, first taking the up- and down-stitches of $\mathcal{G}$ at Y, and then knitting them back together results in $\mathcal{G}$.

Second, applying the Stitch-and-Knit Lemma, we then prove a "decomposition theorem" marking a first main result. The exact formulation needs too many definitions hence we resort to the following intuitive description.

▶ **Theorem 4.5.** *Let Ω be a well-quasi-order. Let $\mathcal{H}$ and $\mathcal{G}$ be Ω-knitworks with underlying Eulerian digraphs H and G. Let $A \subset V(H)$ and $X \subset V(G)$ induce rooted cuts of equal order in the underlying rooted Eulerian digraphs. Then $\mathcal{H} \hookrightarrow \mathcal{G}$ if, and only if, the up- and down-stitch of $\mathcal{H}$ with respect to A strongly immerse in the up- and down-stitch of $\mathcal{G}$ with respect to X, respectively, fixing according linear orders on $\rho(A)$ and $\rho(B)$.*

Intuitively speaking, the theorem proves that stitching and knitting are the right tools to inductively decompose Ω-knitworks and then prove that the resulting pieces can be immersed in each other, complementing the introductory discussion of this section. Although this seems clear at first glance, the proof is rather technical.

We then use Theorem 4.5 to prove decomposition "meta theorems". Again fully stating the theorems would require a lot of machinery, but the main contribution can be intuitively understood as follows: "If we can decompose our graphs along small cuts in a tree-like fashion, then it suffices to prove that the resulting pieces of the trees are well-quasi-ordered". We want to emphasise that this is *not* the exact statement of the theorem and there is a little extra work needed to prove this exact claim (which we defer to another paper as it does not fit the scope of this exposition), but it portrays the main idea behind that theorem without the need to introduce further technical definitions. These kind of decomposition meta theorems mark a crucial first step towards an inductive proof of Conjecture 3.1, giving tools to reduce the conjecture on the more general graph class to that of "smaller well-behaved pieces".

Finally, we want to emphasise that the above results are given in the realm of *quasi-Eulerian* digraphs, where the non-Eulerian vertices – recall that there may be vertices of degree one which we tacitly ignored – are assumed to be covered by the root sets, but there is technically no immediate restriction on the number of such vertices (but rather on the size of the cut induced by the root sets). Every *digraph* D can easily be transformed into a quasi-Eulerian digraph as follows: Let $v \in V(D)$ with out-degree k and in-degree ℓ, say $k > \ell$, then add $t := k - \ell$ vertices $v_1^+, \ldots, v_t^+$ to D and add the edges (v_i^+, v) for every $1 \leq i \leq t$. Thus, it seems plausible that the results discussed above may be of independent interest for further non-Eulerian classes of digraphs that can be "decomposed" into quasi-Eulerian "pieces"; we leave this for future research.

4.2 Bounded Carving width

The second, and possibly most direct main contribution of this paper is a proof that the class of Eulerian digraphs lacking "structural richness" with respect to strong immersion is indeed well-quasi-ordered. Formally, we prove the following.

▶ **Theorem 4.6.** *Every class of Eulerian digraphs admitting bounded carving width is well-quasi-ordered by strong immersion.*

Before we give the definition of *carving width*, we mention that Theorem 4.6 is equivalent to the following.

▶ **Theorem 4.7.** *Every class of Eulerian digraphs admitting bounded treewidth and bounded degree is well-quasi-ordered by strong immersion.*

A *carving* of a digraph G is a tuple (T, ℓ) where T is a cubic tree and $\ell : V(G) \to \text{leaves}(T)$ is an injective map from $V(G)$ to the leaves of T. To every edge $e \in E(T)$ one can naturally associate an induced cut in G as follows: let T_1, T_2 be the two components of $T - e$, and let $V_i := \ell^{-1}(\text{leaves}(T_i))$ for $i = 1, 2$, then V_1, V_2 partition $V(G)$. In particular $\rho(V_1) = \rho(V_2)$ and we define the width of e to be $|\rho(V_1)|$. Finally, the width of a carving is the maximum width over all its edges, and the carving width of a digraph G is the minimum width over all

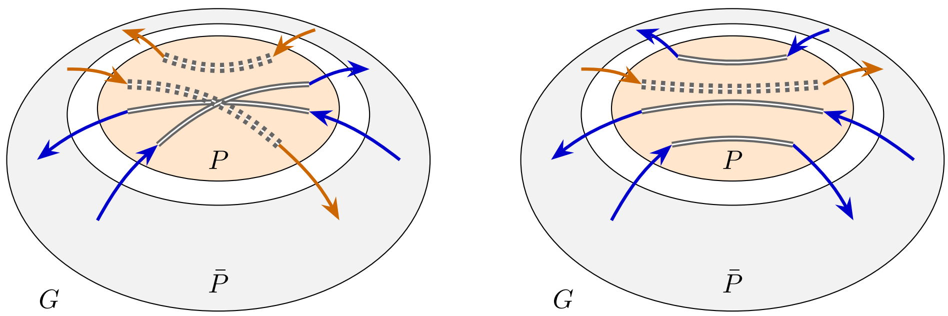

Figure 4 A schematic illustration of two matchings in $\mathfrak{m}_{\mathcal{G}}(p)$ of a well-linked Ω-knitwork corresponding to edge-disjoint trails in the "piece" P, highlighted in gray. E_1^-, E_1^+ are highlighted in dark orange, with the respective matching highlighted by a dotted gray double line, and E_2^-, E_2^+ in dark blue, with the respective matching highlighted by a gray double line.

possible carvings of G. Carvings for rooted digraphs and Ω-knitworks are defined analogously by defining it for the graph obtained from $\bar{G}$ after up-stitching X into a single vertex x^*; we omit the exact definitions here.

Theorem 4.6, and hence Theorem 4.7, is a corollary of a much stronger theorem, proving that *well-linked* Ω-knitworks of bounded *carving width* are well-quasi-ordered by strong Ω-knitwork immersion, marking a first and crucial class of Ω-knitworks for which we prove a well-quasi-order result. An Ω-knitwork $\mathcal{G}$ is *well-linked*, if for every vertex $p \in \mathrm{dom}(\mathfrak{m}_{\mathcal{G}})$ it holds that for every pair of partitions E_1^-, E_2^- of $\rho^-(p)$ and E_1^+, E_2^+ of $\rho^+(p)$ such that $|E_i^-| = |E_i^+|$ for $i = 1, 2$, there is a matching $M \in \mathfrak{m}_{\mathcal{G}}(p)$ such that $M = M_1 \cup M_2$ is the disjoint union of two perfect matchings $M_i \in \mathrm{Match}(E_i^-, E_i^+)$. Intuitively speaking, this means that for every such partition of $\rho(p)$ we can "link up" $E_1^- \cup E_2^+$ and $E_2^- \cup E_1^+$ "inside" the piece P represented by p via a respective set of edge-disjoint trails, i.e., every trail has as first edge one of E_i^- and as last edge one of E_i^+ for $i = 1, 2$. See Figure 4 for a schematic illustration of two choices of partitions E_1^-, E_1^+ highlighted in dark orange and E_2^-, E_2^+ highlighted in dark blue, and the respective matchings in $\mathfrak{m}_{\mathcal{G}}(p)$, or rather the sets of edge-disjoint trails in the associated piece P, highlighted in gray.

Recall that, as we briefly discussed above, it is crucial to define Ω-knitwork immersion using *trails* due to the fact that the immersion needs to respect the map $\mathfrak{m}_{\mathcal{G}}$, which in turn keeps track of the allowed edge-disjoint routings at vertices. In particular, edge-disjoint trails respecting $\mathfrak{m}_{\mathcal{G}}$ can in general not be rerouted to edge-disjoint paths respecting $\mathfrak{m}_{\mathcal{G}}$. However, we prove that, given a *well-linked* Ω-knitwork $\mathcal{G}$ and a set of edge-disjoint trails $\mathcal{L} = \{L_1, \ldots, L_k\}$ in the underlying directed graph G such that for ever $1 \leq i \leq k$, L_i has one endpoint in a $A \subseteq V(G)$ and the other in $B \subseteq V(G)$, say, with $A \cap B = \emptyset$, the trails in $\mathcal{L}$ can be rerouted to a set of edge-disjoint trails $\mathcal{L}' = \{L_1', \ldots, L_k'\}$ such that for every $1 \leq i \leq k$, L_i' still has one endpoint in A and the other in B – not necessarily the same as L_i – with the addition that $\mathcal{L}'$ now *respects* $\mathfrak{m}_{\mathcal{G}}$. More precisely, we prove the following.

▶ **Lemma 4.8.** *Let $\mathcal{G} = ((G, \pi, X), \mu, \mathfrak{m}, \Phi)$ be an Ω-knitwork, and let $A, B \subseteq V(G)$ be disjoint and non-empty inducing rooted cuts, in particular $Y := \bar{A} \cap \bar{B}$ is disjoint from X, such that $|\rho(A)| = |\rho(B)| = k$, say. Let $\mathcal{L}$ be a set of k edge-disjoint trails in G such that*

(i) *all the internal vertices of trails in $\mathcal{L}$ are in Y, and*

(ii) *every $L \in \mathcal{L}$ has one endpoint in A and the other in B.*

Then there exists a set of k edge-disjoint trails $\mathcal{L}'$ in G satisfying (i) and (ii) such that $\mathcal{L}'$ respects $\mathfrak{m}$.

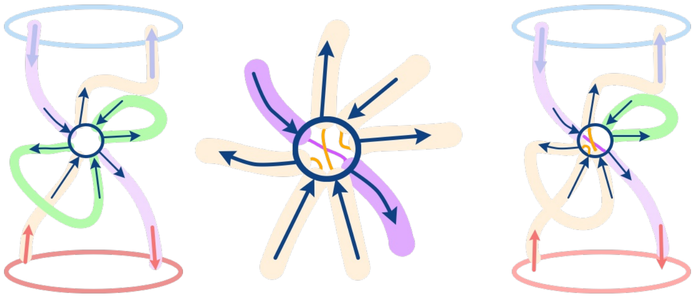

Figure 5 A schematic illustration of how to reroute trails at a single vertex v so that they respect $\mathfrak{m}_{\mathcal{G}}(v)$. The left hand figure shows $\mathcal{L}$ consisting of two trails highlighted in purple and orange, prior to rerouting. The right hand figure shows $\mathcal{L}'$ after rerouting at v respecting $\mathfrak{m}_{\mathcal{G}}(v)$, where a respective "feasible" matching of $\mathfrak{m}_{\mathcal{G}}(v)$ is highlighted in the middle figure.

This is a crucial result, allowing us to lift general "Menger-type" arguments on the underlying digraph G to the Ω-knitwork $\mathcal{G}$. See Figure 5 for an illustration of a single step in the proof of Lemma 4.8, where $\mathcal{L}$ – on the left hand side – and $\mathcal{L}'$ – on the right hand side – consist of two trails, one highlighted in orange and one in purple. $\mathcal{L}'$ is obtained from $\mathcal{L}$ by rerouting at a single vertex v, highlighted by a large cycle, where one "feasible" matching in $\mathfrak{m}_{\mathcal{G}}(v)$ is highlighted by a respective matching in the center of the cycle in the middle figure. The green highlighted trails "are remaining trails" – edge-disjoint from $\mathcal{L}$ and $\mathcal{L}'$ respectively, in the respective figure – in the graph with both endpoints v, that exist using the Eulerianness of G.

Finally, a weaker version of the aforementioned theorem implying Theorem 4.6 reads as follows.

▶ **Theorem 4.9.** *Let Ω be a well-quasi-order, and let $(\mathcal{G}_i)_{i\in I}$ be a sequence of well-linked Ω-knitworks*[12]*, each admitting carving width at most $k \in \mathbb{N}$. Then there exist $i < j$ such that $\mathcal{G}_i$ strongly immerses in $\mathcal{G}_j$.*

Outline of the Proof: The proof of the theorem builds on a well-known result due to Robertson and Seymour [27] – the so-called *Tree Lemma* – that has been refined by Geelen, Gerards, and Whittle [9] to a setting that fits our needs. Let us briefly elaborate on the overall proof technique. Forgetting about well-linked Ω-knitworks and focusing solely on Eulerian digraphs, the proof follows the same line of argumentation as those in [27] and [9]. That is, assume towards a contradiction that Theorem 4.6 is wrong, and let $(G_i)_{i\in\mathbb{N}}$ be a sequence of Eulerian digraphs of carving width $\leq k$ refuting the theorem. For every $i \in \mathbb{N}$ let (T_i, ℓ_i) be a respective carving of G_i witnessing its width. Next root the trees T_i at some vertex $x_i \in V(T_i)$, i.e., T_i is a directed graph where every vertex except for x_i has in-degree one, whereas x_i has in-degree 0. Let $e = (t_1, t_2) \in E(T_i)$ be a directed edge and define T_i^1, T_i^2 to be the components of $T_i - e$ containing t_1, respectively t_2; note that $x_i \in V(T_i^1)$. Then $\ell^{-1}(V(T_i^1)), \ell^{-1}(V(T_i^2))$ partition $V(G_i)$, and in particular $X(e) \coloneqq \ell^{-1}(V(T_i^1))$ induces a cut in G_i, uniquely defined by e.

[12] In general, when we allow for more than one root set, we need to further assume that the number of root sets for the sequence agree.

With these definitions at hand, intuitively speaking, the aforementioned *Tree Lemma* implies[13] the existence of an infinite index set $I \subseteq \mathbb{N}$ and for every $i \in I$ a directed edge $e_i \in E(T_i)$ such that the sequence of Eulerian digraphs $(G_i \mid e_i)_{i\in I}$ obtained from G_i by contracting $X(e_i)$ into a single vertex x_i^*, is an antichain with respect to strong immersion. Furthermore, for every choice $I' \subseteq I$ and every choice $e_i' \in E(T_i') \setminus \{e_i\}$ the respective sequence of Eulerian digraphs $(G_i \mid e_i')_{i\in I'}$ is a chain with respect to strong immersion. Let p_i be the head of e_i for every $i \in I$, and let $\mathcal{F}_i \subseteq E(T_i')$ be those edges that admit p_i as a tail. By a well-known theorem due to Higman [10] we derive the existence of $i < j$ with $i, j \in I$ and an injection $\alpha : \mathcal{F}_i \to \mathcal{F}_j$ such that for every $f \in \mathcal{F}_i$ it holds that $\eta_f : G_i \mid f \hookrightarrow G_j \mid \alpha(f)$. Collecting the maps η_f for $f \in \mathcal{F}_i$ one can then "combine" these strong immersions to construct a strong immersion $\eta : G_i \mid e_i \hookrightarrow G_j \mid e_j$ contradicting the fact that $(G_i)_{i\in\mathbb{N}}$ was an antichain.

We emphasise that the above sketch is far from complete. In particular, note that we are working with a far more complicated structure than Eulerian digraphs, and we need to carefully lift all of the mentioned results to Ω-knitworks: This is where the "well-linkedness" assumption will be crucial, and Lemma 4.8 will be needed.

Implications: The implications of Theorem 4.9 are three-fold. First, it immediately implies Theorem 4.6 by rooting the graphs in some arbitrary vertex and fixing the maps $\mu, \mathfrak{m}$ and Φ to be nowhere defined. Note that, as discussed above, this marks one of the first more versatile directed graph classes – besides the aforementioned results on tournaments and extensions [5, 2] – for which a positive well-quasi-ordering result is known. And, complementing the introductory discussion, a class $\mathcal{C}$ of Eulerian digraphs of bounded carving width may still admit arbitrarily long alternating paths.

Secondly, the theorem is an important step towards a proof of Conjecture 3.1, settling a crucial base case. Recall that our main strategy will be to decompose our graph into pieces that fall into classes for which we already proved well-quasi-ordering results. The inductive argument tries to cut out "structurally rich" pieces which are handled differently, where Theorem 4.6 essentially covers the case for pieces lacking "structural richness". It turns out that for many containment relations that do *not* yield well-quasi-orders on (directed) graphs, they already fail to do so on classes of "structurally non-rich" (directed) graphs, the exact definition of which depends on the respective relation.

Thirdly, it can be seen as another decomposition "meta theorem" towards a proof of Conjecture 3.1. To see this, note that Theorem 4.9 is much stronger than Theorem 4.6: Besides covering the base case, it proves that it is enough to decompose the graphs into possibly "structurally rich" pieces for which we already proved well-quasi-ordering, that are "loosely connected" to each other in a tree-like fashion. More precisely, if we loosen the definition of carving slightly by allowing the leaves of T to correspond to sets of vertices partitioning $V(G)$, and manage to carve the digraphs of a class $\mathcal{C}$ in a way that the width of this carving is bounded by k say, and such that the leaf-bags "induce" graphs (by taking up- and down-stitches for example) that fall into a class $\mathcal{C}^*$ for which we already proved well-quasi-ordering with respect to strong immersion, then we can contract the bags into single vertices which we then mark with the respective immersion-type of $(\mathcal{C}^*, \hookrightarrow)$ using the maps $\mathfrak{m}, \mu$ and Φ. This essentially defines for every $G \in \mathcal{C}$ an Ω-knitwork of carving width at most k, where $\Omega = (\mathcal{C}^*, \hookrightarrow)$. By Theorem 4.9 Ω-knitworks of bounded carving width are well-quasi-ordered by strong immersion, and thus, assuming that the carving defined above is *linked* (a notion to be made precise), we derive that $\mathcal{C}$ is well-quasi-ordered. We will not elaborate on this further and leave this application to future work.

[13] This relies on many further technicalities, such as assuming the carvings to be *linked*.

We want to mention here that very recently Lunel and Maria [21] independently derived a weaker variant of Theorem 4.6, i.e., they proved that "Eulerian embeddable"[14] 4-regular Eulerian digraphs of bounded carving width (without labels or roots) are well-quasi-ordered by strong immersion. They proved this as their main technical lemma to prove that graphs embedded in $\mathbb{R}^2$ are well-quasi-ordered by *embedded minors*, providing another nice use case and relation of Eulerian digraphs to undirected graphs.

Finally, we conclude the paper with a result that complements Corollary 2.2 and Conjecture 3.2; the following is a weaker version of the exact theorem.

▶ **Theorem 4.10.** *Let $\Delta \in 2\mathbb{N}$. The class of Eulerian digraphs of bounded treewidth admitting at most one vertex of degree larger than Δ is well-quasi-ordered by weak immersion.*

Thus, the class $\mathcal{C}(2,4)$ of planar Eulerian digraphs of treewidth at most 2, admitting at most one vertex of degree larger than 4 is *not* well-quasi-ordered by strong immersion following Corollary 2.2, but it *is* well-quasi-ordered by weak immersion. This is a dichotomy result in the following sense: If we additionally bound the maximum degree of $\mathcal{C}(2,4)$, then the class is well-quasi-ordered with respect to both weak and strong immersion by Theorem 4.6. If we restrict to treewidth at most 1, then it is not hard to see that this class is also well-quasi-ordered by both weak and strong immersion (it consists of bidirected trees, possibly with parallel edges).

We want to highlight that the proof of Theorem 4.10 uses a nice trick, linking the treewidth of the underlying undirected graph to the carving width of the directed graph (recall that this is, in general not possible if we do not impose restrictions on the degrees), which may be of independent interest. Theorem 4.10 further strengthens the evidence for Conjecture 3.2. We note that we believe that the arguments in the proof of Theorem 4.10 can be leveraged (by a careful and rather involved analysis) to prove the theorem for up to three vertices of unbounded degree, but the general case of bounded treewidth requires a substantial and non trivial amount of work. Since the bounded treewidth case follows from more "natural" results – the proofs of which we are currently writing up – we omit a direct proof here as it is more cumbersome than insightful.

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