Odd-Cycle-Packing-Treewidth: On the Maximum Independent Set Problem in Odd-Minor-Free Graph Classes

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Abstract

We introduce the tree-decomposition-based graph parameter *Odd-Cycle-Packing-treewidth* (OCP-tw) as a width parameter that asks to decompose a given graph into pieces of bounded odd cycle packing number. The parameter OCP-tw is monotone under the odd-minor-relation and we provide an analogue to the celebrated Grid Theorem of Robertson and Seymour for OCP-tw. That is, we identify two infinite families of grid-like graphs whose presence as odd-minors implies large OCP-tw and prove that their absence implies bounded OCP-tw. This structural result is constructive and implies a $2^{\mathbf{poly}(k)}\mathbf{poly}(n)$-time parameterized $\mathbf{poly}(k)$-approximation algorithm for OCP-tw.

Moreover, we show that the (weighted) Maximum Independent Set problem (MIS) can be solved in polynomial time on graphs of bounded OCP-tw. Finally, we lift the concept of OCP-tw to a parameter for matrices of integer programs. To this end, we show that our strategy can be applied to efficiently solve integer programs whose matrices have entries in $\{-1, 0, 1\}$ and can be "tree-decomposed" into totally Δ-modular matrices with at most two non-zero entries per row.

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1 Introduction

The Maximum Independent Set *problem* (MIS) takes as input a graph G and asks for an *independent set* – a set of vertices that do not share any edges – of maximum size. MIS is known to be NP-hard in general and remains NP-hard on triangle-free graphs [44], graphs without induced $K_{1,4}$-subgraphs [41], and planar graphs of maximum degree at most three [22].

The field of parameterized algorithms aims to identify hierarchical graph classes $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \dots$ such that a given NP-hard problem can be solved in polynomial time in $\mathcal{C}_k$ for every fixed $k \in \mathbb{N}$. A simple example for a typical result in this area is the observation that, if we let $\mathcal{C}_k$ be the class of all graphs G with independent sets of order at most k, then for every k we can solve MIS on the graphs in $\mathcal{C}_k$ in time $n^{\mathcal{O}(k)}$ which is, under established complexity-theoretic assumptions, best possible (see [13] for an introduction to parameterized complexity). As the degree of this polynomial grows in the solution size, a need for different types of *parametrizations* for MIS becomes apparent.

Our contribution

In this paper we introduce the notion of *Odd-Cycle-Packing-treewidth* (OCP-tw) as a unifying way to understand both tree-like features and the behaviour of odd cycles in a graph. We leverage this definition to show that MIS can be solved in time $n^{f(k)}$ on graphs of OCP-tw at most k. This result can be understood as a major step forward in the understanding of the tractability horizon for MIS in *odd-minor-free* graph classes[1] (see Section 1.3 for a deeper discussion). Along the way we develop a list of technical tools we believe to be invaluable in the further advancement towards this goal. The full version of this paper [10] contains all details and proofs omitted from this extended abstract.

Structural parametrizations for MIS

A well-established strategy for the search for good parametrizations is the exclusion of certain substructures. An example of such a strategy is used for minor-closed graph classes: Via a result of Garey and Johnson [22], we know that MIS is NP-hard on every minor-closed graph class containing all planar graphs. However, the celebrated Grid Theorem of Robertson and Seymour [49] tells us that every such graph class excluding a planar graph H has *bounded treewidth*, i.e. treewidth at most $k = k(|V(H)|)$. As a consequence, there exists an algorithm for MIS on H-minor-free graphs running in $f(k) \cdot n$-time [13] if H is planar.

The underlying tree-like structure of graphs with bounded treewidth naturally lends itself to dynamic-programming and allows for the design of many parameterized algorithms [48, 3, 6, 7]. A downside of minor-closed graph classes – and therefore of treewidth as a parameter – is that H-minor-free graphs are necessarily *sparse*. Specifically, they always exclude many graphs on which MIS is naturally polynomial-time solvable such as *dense* bipartite graphs. In recent years, several research projects were aimed at overcoming this hurdle. On one side, the work of Fiorini, Joret, Weltge, and Yuditsky [18, 19] focussed on structural generalisations of bipartite graphs by proving that MIS can be solved in polynomial time on graphs with at most k pairwise disjoint odd cycles, i.e. graphs of bounded *odd cycle packing number*. On the other hand, based on an idea of Eiben, Ganian, Hamm, and Kwon [17], so-called *hybridisation*

[1] Such graph classes can be seen as generalisations of bipartite graphs. See Section 1.1 for a definition of odd-minors.

techniques have been used to combine the structural power of tree-decompositions – and related concepts – with the structural properties of bipartite graphs to create new width parameters [33, 34, 27, 31]. The goal here is to create parameters p such that MIS – or any other interesting problem – can be solved in polynomial time on all graphs G with $\mathsf{p}(G) \leq k$.

There are two slightly different genres of such hybrid parameters that can be found in the literature: One way – pioneered by Bulian and Dawar [8, 9] – fixes a minor-monotone[2] parameter q and a graph class $\mathcal{B}$ – for example bipartite graphs – and then fixes $\mathsf{p}(G)$ to be the smallest k for which there exists $X \subseteq V(G)$ whose *torso*[3] G_X satisfies $\mathsf{q}(G_X) \leq k$, and $G - X \in \mathcal{B}$. The case where q is treewidth is known as $\mathcal{B}$-treewidth.

One may also change the way a tree-decomposition is evaluated. This approach is central to our work. In simplified terms, the treewidth of a graph is the smallest k such that a graph G can be decomposed into a tree-like arrangement of vertex sets $X \subseteq V(G)$ of size at most $k+1$. Instead of focusing on the size of the sets, we could ask for any two such sets X and Y to satisfy $|X \cap Y| \leq k$ and for each torso G_X to satisfy a fixed property. A classic example of such a parameter is based on a theorem of Robertson and Seymour [47] and asks for every X to either be of size at most $k+1$ or G_X to be planar (see [36] for an application and [57] for a generalisation). Regarding MIS, the notions of *bipartite treewidth* [31] asking for every G_X to become bipartite by deleting at most k vertices, and *$\mathcal{B}$-blind-treewidth* [27], where $\mathcal{B}$ is the class of bipartite graphs, which only measures the treewidth of the non-bipartite blocks of G, fall into this category.

We further discuss some of the ideas above in relation to our results in Section 1.3.

1.1 Odd-Cycle-Packing-treewidth

Our results are centred around a novel extension of the ideas behind bipartite treewidth and odd cycle packing number. We combine both of these notions into a single parameter by choosing the parameter q in the construction above to be the *odd cycle packing number*. If we denote the class of bipartite graphs by $\mathcal{B}$, this results in a common generalisation of $\mathcal{B}$-treewidth, bipartite treewidth, and odd cycle packing number. To provide a full definition, we first define tree-decompositions.

A *tree-decomposition* of a graph G is a pair (T, β) where T is a tree and β assigns to every $t \in V(T)$ a set $\beta(t) \subseteq V(G)$ called the *bag* of t, such that $\bigcup_{t \in V(T)} \beta(t) = V(G)$, for each $xy \in E(G)$ there is $t \in V(T)$ with $x, y \in \beta(t)$, and for every $v \in V(G)$, the set $\{t \in V(T) \colon v \in \beta(t)\}$ is connected. The *adhesion* of (T, β) is the value $\mathsf{adhesion}(T, \beta) = \max_{dt \in E(T)} |\beta(d) \cap \beta(t)|$ and the *width* of (T, β) is the value $\max_{t \in V(T)} |\beta(t)| - 1$. In this language, the *treewidth* of a graph G, denoted by $\mathsf{tw}(G)$, is the minimum width over all tree-decompositions for G.

We now give a formal definition of Odd-Cycle-Packing-treewidth, which we mostly shorten to OCP-treewidth. This variant of treewidth measures how far each bag is from being bipartite.

In the following, for any graph G we denote by $\mathsf{OCP}(G)$ the largest integer k such that G contains k pairwise vertex-disjoint cycles of odd length.

[2] A graph parameter q is *minor-monotone* if for every graph G and every minor H of G it holds that $\mathsf{q}(H) \leq \mathsf{q}(G)$.

[3] The torso G_X is built from $G[X]$ by turning $N_G(V(H)) \subseteq X$ into a clique for every component H of $G - X$.

▶ **Definition 1** (OCP-treewidth). *An* OCP-tree-decomposition *of a graph G is a triple $\mathcal{T} = (T, \beta, \alpha)$, where T is tree and $\beta, \alpha \colon V(T) \to 2^{V(G)}$, satisfying the following conditions:*

(OCP1) *(T, β) is a tree-decomposition of G,*

(OCP2) *$\alpha(t) \subseteq \beta(t)$ for each $t \in V(T)$, and*

(OCP3) *for each $t \in V(T)$, if there is a path P in $G - \alpha(t)$ such that both endpoints of P are in $\beta(t)$, then $V(P) \subseteq \beta(t)$.*

The width of an OCP-tree-decomposition $\mathcal{T}$, denoted by $\mathsf{OCP\text{-}width}(\mathcal{T})$*, is the following value:*

$$\max\Big\{\mathsf{adhesion}(T, \beta),\ \max_{t \in V(T)} \big(|\alpha(t)| + \mathsf{OCP}(G[\beta(t) \setminus \alpha(t)])\big)\Big\}.$$

The OCP-treewidth *of G, denoted by* $\mathsf{OCP\text{-}tw}(G)$*, is the minimum width of an OCP-tree-decomposition of G. For $t \in V(T)$, we call $\beta(t)$ the* bag of t *and $\alpha(t)$ the* apex set of t*.*

A key feature of treewidth is that it is minor-monotone. A similar observation can be made for OCP-treewidth if we augment the minor relation to respect the parity of cycles.

For a set $X \subseteq V(G)$ we write $\partial_G(X) = \{xy \in E(G) \colon x \in X, y \notin X\}$. A *bond* in G is a set $\partial_G(X)$ such that there is no $Y \subseteq V(G)$ with $\partial_G(Y) \subsetneq \partial_G(X)$. We perform a *bond contraction* in a graph G by contracting all edges in a bond $\partial_G(X)$. A graph H is said to be an *odd-minor* of a graph G if it can be obtained from G by a sequence of edge deletions, vertex deletions, and bond contractions.

The notion of odd-minors has first been explicitly studied in the context of Hadwiger's Conjecture [24, 55, 42, 38]. It is a special case of the notion of minors in *signed graphs* and has strong relations with matroid minors. It is easy to see that OCP-treewidth is odd-minor-monotone which naturally raises the question of its structural behaviour under the odd-minor relation.

1.2 Our results

We split our overview into three categories: Algorithmic results, structural results, and building blocks for more structural inquiries concerning odd-minor-free graphs.

Algorithmic results

Our main result in this category is that MIS can be solved in polynomial time on graphs of bounded OCP-treewidth.

▶ **Theorem 2** (see Theorem 4.3 of the full version [10]). *For every fixed non-negative integer k, there exists a polynomial-time algorithm for (weighted) MIS on the class of all graphs of OCP-treewidth at most k.*

This algorithm consists of two parts. One that finds a OCP-tree-decomposition of small width and a second that is essentially a dynamic program along this decomposition which uses the algorithm of Fiorini et al. [18, 19] for weighted MIS in graphs of bounded odd cycle packing number as a blackbox. Due to the running time of the algorithm by Fiorini et al., the running time of our algorithm from Theorem 2 is of the form $n^{f(k)}$ where f is a computable function.

Tree-decomposing integer programs. The landscape of (tree-)decomposition based parameters measuring the matrices of integer programs and allow for parameterized algorithms is relatively sparse [21, 16]. Partially this lack of parameterized tools could be attributed to the fact that one of the most popular parameters – *treewidth* – fails in this regime [20]. Based on our structural insight, we lift Theorem 2 into a width parameter for integer programs whose coefficient matrices correspond to the signed incidence matrices of signed graphs. While our parameter imposes further restrictions on the *structure* of the input matrices, i.e. it requires them to be incidence matrices of signed graphs, it guarantees efficient algorithms way beyond the boundedness of typical parameters such as treewidth.

Fiorini et al. used their result to efficiently solve certain types of integer programs as follows. Let Δ be a positive integer. A matrix M is *totally Δ-modular* if the determinant of every square submatrix of M falls into the set $\{-\Delta, -\Delta+1, \dots, 0, \dots, \Delta-1, \Delta\}$. A major conjecture in the field of integer programming is that for every Δ there is a polynomial-time algorithm to solve all integer programs of the form $\max\{w^{\mathrm{T}}x \colon Ax \leq b, x \in \mathbb{Z}^n\}$ where A is totally Δ-modular. See [4, 53, 1] for further partial solutions and more background. In [18, 19], Fiorini et al. proved that for every Δ there exists a polynomial-time algorithm that solves all integer programs of the form $\max\{w^{\mathrm{T}}x \colon Ax \leq b, x \in \mathbb{Z}^n\}$ where A is totally Δ-modular and has at most two non-zero entries per row. Using the bridge between odd-minors, minors in signed graphs, and matroid minors, we are able to combine Theorem 2 with some of the methods from [19] to obtain algorithms for integer programs whose coefficient matrices correspond to signed graphs excluding certain minors. For the sake of brevity, we present here an abbreviated version of our result and refer the interested reader to Section 10 of the full version [10], where it appears in the form of Corollary 10.9.

To keep it short, a signed graph is a pair (G, γ) where G is a graph and $\gamma \colon E(G) \to \mathbb{Z}_2$ is a labelling of the edges of G with elements of $\mathbb{Z}_2$. A cycle C of G is *unbalanced* if $\sum_{e \in E(C)} \gamma(e) = 1$. We adapt our definition of OCP-treewidth to the world of signed graphs by replacing, for the evaluation of the width of a decomposition $\mathcal{T}$, the odd cycle packing number of $G[\beta(t) \setminus \alpha(t)]$ with the largest integer k such that $(G[\beta(t) \setminus \alpha(t)], \gamma)$ has k pairwise vertex-disjoint unbalanced cycles.

▶ **Theorem 3.** *There exist a computable function f and an algorithm that given a non-negative integer k and an integer program of the form $\max\{w^Tx \colon Ax \leq b, x \in \mathbb{Z}^n\}$ where $A \in \{-1, 0, 1\}^{m \times n}$ is the signed edge-vertex incidence matrix of a signed graph (G, γ), either decides correctly that OCP-tw$(G, \gamma) > k$ or solves the integer program in time $n^{f(k)}$.*

The incidence matrix of a signed graph (G, γ) is totally Δ-modular for some Δ if and only if any family of pairwise vertex-disjoint unbalanced cycles in (G, γ) has bounded size (see [19]). Hence, Theorem 3 contains the result from [19] for $\{-1, 0, 1\}$-matrices as a special case. Moreover, if (G_k, γ_k) is the disjoint union of $k \geq 1$ unbalanced cycles, then for every Δ there exists k such that the incidence matrix of (G_k, γ_k) is not totally Δ-modular, but OCP-tw$(G_k, \gamma_k) = 1$ for all k. Thus, Theorem 3 covers a larger class of $\{-1, 0, 1\}$-matrices than the work of Fiorini et al.

Finding tree-decompositions of small OCP-width. The core of our work is focussed on actually finding an OCP-tree-decomposition of bounded width or determining that the OCP-treewidth of the input graph is larger than k. A first roadblock in this endeavour is the observation that computing the OCP-treewidth of general graphs is as hard as computing the treewidth of bipartite graphs which is known to be equivalent to computing the treewidth of general graphs and therefore NP-complete [2].

▶ **Theorem 4.** *Computing OCP-treewidth is* NP*-complete.*

A major consequence of our structural results as explained below is the following parametrized approximation algorithm for OCP-treewidth.

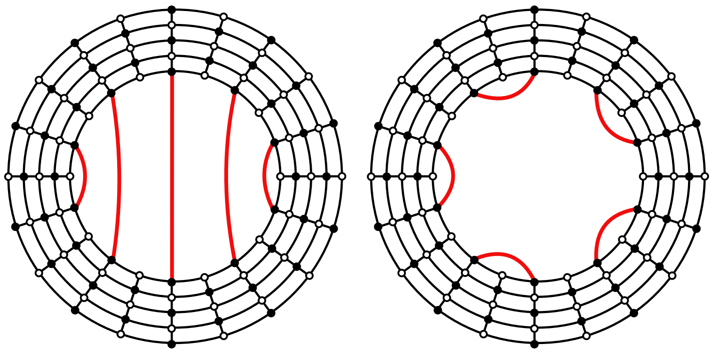

Figure 1 Representatives of our two obstructing families for OCP-treewidth: The parity handle of order 5 (left) and the parity vortex of order 5 (right). Notice that a cycle in either of these graphs is odd if and only if it contains an odd number of red edges.

▶ **Theorem 5.** *There exists an algorithm that takes as input a non-negative integer k and an n-vertex graph G and either finds a subgraph of G certifying that* OCP-tw$(G) > k$ *or computes an OCP-tree-decomposition of width in* $\mathbf{poly}(k)$ *for G in time* $2^{\mathbf{poly}(k)}n^{\mathcal{O}(1)}$.

Structural results

There are several equivalent ways to state the celebrated *Grid Theorem* of Robertson and Seymour [49]. One of them would be a twofold statement as follows:

1. For every $k \geq 1$, every graph containing the $(k \times k)$-grid as a minor has treewidth at least k.
2. There exists a function f such that for every $k \geq 1$, every graph with treewidth at least $f(k)$ contains the $(k \times k)$-grid as a minor.

A functionally equivalent statement would be:
A minor-closed graph class has bounded treewidth if and only if it excludes a planar graph.
Planarity in the statement above can be replaced by the phrase "*... $\mathcal{C}$ excludes a graph H such that there is a $k \geq 1$ for which H is a minor of the $(k \times k)$-grid.*" It just happens to be true that the class of all graphs that are a minor of some grid is exactly the class of planar graphs.

Our structural main result is an analogue of the Grid Theorem of Robertson and Seymour for OCP-treewidth under the odd-minor relation. This provides a satisfying answer to the question about structural properties of graphs of bounded OCP-treewidth as it classifies all odd-minor-closed graph classes where OCP-treewidth is bounded, and hence, where Theorem 2 can be applied. The role of the $(k \times k)$-grid is played by two infinite families of graphs – both consisting of planar graphs – to which we will refer to as the *parity grids* in the following. We present both types of parity grids in their *cylindrical* form here, but they could easily be expressed as square grids as well.

Let $k \geq 1$ be an integer and H_k be the cylindrical $(k \times 4k)$-grid.[4] Moreover, let $X = \{x_1, \dots, x_{2k}\}$ be a maximum subset of vertices of the kth copy of C_{4k} in H_k such that all vertices of X are pairwise at even distance. The *parity handle* $\mathcal{H}_k$ of order k is the graph

[4] The cylindrical $(n \times m)$-grid is the Cartesian (or box) product of the path P_n and the cycle C_m.

obtained from H_k by introducing the edges $x_i x_{2k-i+1}$ for all $i \in [k]$. The *parity vortex* $\mathcal{V}_k$ of order k is the graph obtained from H_k by introducing the edges $x_{2i-1}x_{2i}$ for all $i \in [k]$. See Figure 1 for examples.

Our structural main theorem reads as follows.

▶ **Theorem 6.** *There exists a polynomial $f \colon \mathbb{N} \to \mathbb{N}$ such that for every integer $k \geq 1$ and every graph G it holds that*

1. *if G contains $\mathcal{H}_k$ or $\mathcal{V}_k$ as an odd-minor, then* $\mathsf{OCP}\text{-}\mathsf{tw}(G) \geq \frac{k}{2}$, *and*
2. *if* $\mathsf{OCP}\text{-}\mathsf{tw}(G) \geq f(k)$ *then G contains $\mathcal{H}_k$ or $\mathcal{V}_k$ as an odd-minor.*

Moreover, $f(k) \in \mathbf{poly}(k)$ and there exists an algorithm that, given an integer k and a graph G as input, finds either a subgraph of G certifying that G contains $\mathcal{H}_k$ or $\mathcal{V}_k$ as an odd-minor, or produces an OCP-tree-decomposition of width at most $f(k)$ in time $2^{\mathbf{poly}(k)}|V(G)|^{\mathcal{O}(1)}$.

Since f is a polynomial, the algorithm guaranteed by Theorem 6 in fact yields Theorem 5.

Notably, every odd-minor of a parity handle can be embedded into the plane with at most two odd faces, while every odd-minor of the parity vortex can be embedded in the plane with a unique face that meets all odd cycles in the graph. These two properties illustrate that there exist planar graphs H such that H-odd-minor-free graphs have unbounded OCP-treewidth.

Maximum Independent Set in odd-minor-closed graph classes: remaining cases

As MIS is NP-hard on planar graphs, the core question underlying our research is the following:

▶ **Question 7.** *For which planar graphs H does there exist a polynomial-time algorithm for MIS on H-odd-minor-free graphs?*

To give a more concrete example of the gaps in our understanding of MIS in planar graphs, consider the following problem concerning a class of graphs with unbounded OCP.

▶ **Question 8.** *Let $k \in \mathbb{N}$. Does there exist a computable function $f \colon \mathbb{N} \to \mathbb{N}$ and an algorithm for MIS running in $f(k)n^{\mathcal{O}(1)}$-time on the class of planar graphs with at most k odd faces?*

In the above question k can clearly be taken to be even. The case $k \leq 2$ was solved by Gerards in [26]. All other cases are widely open. In Question 8 we ask explicitly for an FPT-algorithm, but even an XP-algorithm is unknown and would fully suffice to further our understanding of Question 7.

Beyond planar graphs the following question seems key to expanding our understanding of the tractability of MIS. A graph embedded in a surface Σ is said to have an *even-faced embedding* if all of its odd cycles describes a genus-reducing curve in Σ, equivalently no odd cycle bounds a disk in Σ.

▶ **Question 9.** *Does there exist a polynomial-time algorithm for MIS on graphs with a given even-faced embedding on the torus.*

Question 8 and Question 9 are not asked idly. We strongly believe that both of these questions must in some way be resolved to expand the structural understanding of odd-minor-free graph classes in which MIS is efficiently solvable *beyond* the results of this paper, that is the combination of Theorem 6 and Theorem 2. Notably, both questions diverge from the perspective of [12], [19], and [1] as the classes considered here all contain graphs of unbounded OCP and thus the corresponding matrices have *unbounded* subdeterminants. As such our more structural approach to studying odd-minor-closed graph classes seems more promising. Indeed, any positive progress on the questions above is likely to yield immediate progress towards Question 7 due to the robustness of our structural tools as laid out below.

Tools for the study of the structure of odd-minor-free graphs

As part of the more general project outlined above, we are working on understanding the structure of odd-minor-free graphs, with a particular focus on excluding planar graphs as odd-minors. In line with this, several of the theorems and tools in this article are geared not only towards proving the results advertised in this paper, but also for further use in future parts of this project. We highlight here the advantages of our particular approach and why it is necessary to proceed this way if we wish to prove theorems constructively and with explicit bounds.

The established approach to proving a structural result when excluding an odd-minor (or in a closely related fashion an *oriented or unoriented group-labelled-minor*[5]) is to take the existing results from the Graph Minor Series due to Robertson and Seymour (especially the Graph Minor Structure Theorem (GMST) from [52]) and then provide a tailor-made analysis of the structure provided by the GMST to solve the specific problem under consideration. Examples of this approach can be found in the work of Fiorini et al. [19] we already discussed, an approximate description of the structure of odd-minor-free graphs [15], a solution to the linkage problem for the oriented group-labelled setting [30], a structure theorem for oriented group-labelled graphs [23], and the solution to the linkage problem in the unoriented group-labelled setting (together with an associated structure theorem) [39]. We provide a rough description of what the GMST tells us about a graph excluding a given minor in Section 1.4.

This approach has two downsides. First, due to using older versions of the GMST and the associated machinery, such an approach often does not yield explicit bounds for the desired structural properties. More importantly, due to the ad-hoc nature of these individual solutions it is difficult to adapt them as tools for the further development of an efficient structure theory for odd-minors, where by efficient we mean theorems with explicit, hopefully low bounds, such as the polynomial bounds presented in this paper. This is particularly important as future progress towards resolving Question 7 will most likely heavily rely on such structural results.

We instead build robust tools that can be used in a more modular fashion. The key result here is a proof of an odd-minor version of the *Society Classification Theorem* (see Theorem 7.1 of the full version [10]), which is the engine behind a new constructive proof of the GMST with explicit bounds due to Kawarabayashi, Thomas, and Wollan [37, Theorem 10.1] (see also [28, Theorem 14.1]). This theorem is what allows Kawarabayashi et al. to prove the GMST constructively with explicit bounds via an induction. Our variant necessarily features more outcomes than the original Society Classification but importantly can be adapted to prove further results regarding excluded odd-minors and (in line with the variant in [28]) features polynomial bounds on the order of the structural elements involved.

As a second example, we provide a proof of an odd-minor variant of the *Flat Wall Theorem* (again see Section 1.4 for more details) with polynomial bounds, contrasting the worse than exponential bounds one would derive from the more general Flat Wall Theorem variant Thomas and Yoo prove in [58] for group-labelled graphs.

[5] We will not define the particulars of the group-labelled setting as they are not needed for our results.

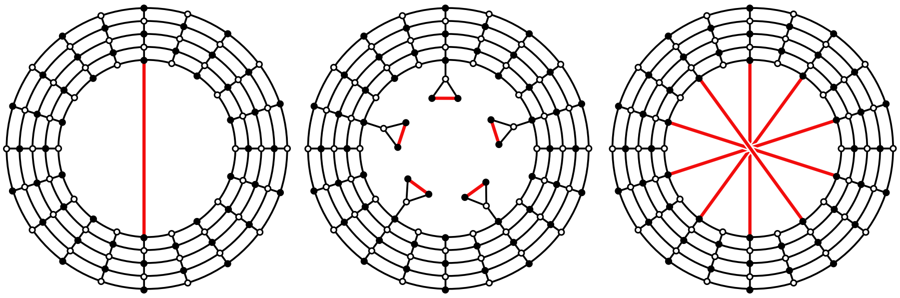

Figure 2 Representatives of three more types of parity grids: The single parity break of order 5 (left), a grid with odd cycle outgrows of order 5 (middle), and a parity crosscap, also known as an *Escher grid*, of order 5 (right). As before, a cycle in either of these graphs is odd if and only if it contains an odd number of red edges.

1.3 Related work

Let us now revisit related work on the topic of islands of tractability for the MIS problem. We partition this subsection into two parts: First we discuss how OCP-treewidth fits into the known landscape of odd-minor-free classes and odd-minor-related parameters that allow for efficient algorithms for MIS. In the second part, we briefly touch upon other structural strategies to approach MIS that have emerged in recent years.

Maximum Independent Set and odd-minors

It is easy to see that a graph is bipartite if and only if it does not contain K_3 as an odd-minor. So on K_3-odd-minor-free graphs MIS is famously tractable. Moreover, Gerards [26] has proven a min-max duality for the MIS-problem on K_4-odd-minor-free graphs that implies a polynomial-time algorithm. For any $t \geq 5$, the class of K_t-odd-minor-free graphs contains the class of all planar graphs and thus, MIS becomes NP-hard. However, on the positive side, Tazari [56] showed that MIS admits a PTAS on H-odd-minor-free graph classes for *every* graph H.

In the following we let $\mathcal{B}$ be the class of bipartite graphs. Recall that for a graph G we denote by $\mathsf{OCP}(G)$ the largest integer k such that G has k pairwise vertex-disjoint odd cycles. This is called the *odd cycle packing number*. The *odd cycle transversal number* of G, denoted by $\mathsf{OCT}(G)$, is the smallest k such that there is a set $S \subseteq V(G)$ of size at most k where $G - S$ is bipartite. By the definition, $\mathsf{OCP}(G) \leq \mathsf{OCT}(G)$ for all graphs G.

It is clear that $\mathsf{OCP}(G) \leq k$ if and only if G does not contain $k \cdot K_3$ as an odd-minor. For the odd cycle transversal number, Reed [46] proved that $\mathsf{OCT}(G)$ is bounded if and only if there exists a k such that G does not contain $k \cdot K_3$ nor the *Escher grid* of order k (see Figure 2) as odd-minors. The Escher grid acts as an excellent example that the parameters OCT and OCP are *comparable yet distinct* in the sense that for every positive integer k, OCT of the Escher grid of order k is k, while its OCP is equal to 1.

Since computing $\mathsf{OCT}(G)$ is *fixed-parameter tractable* (see for example [45]), MIS can be solved in polynomial-time on graphs of bounded OCT. For the odd cycle packing number, the situation is considerably more complicated. Fiorini et al. [19] proved that MIS is tractable on graphs of bounded OCP by using the graph minor structure theorem. Essentially, this

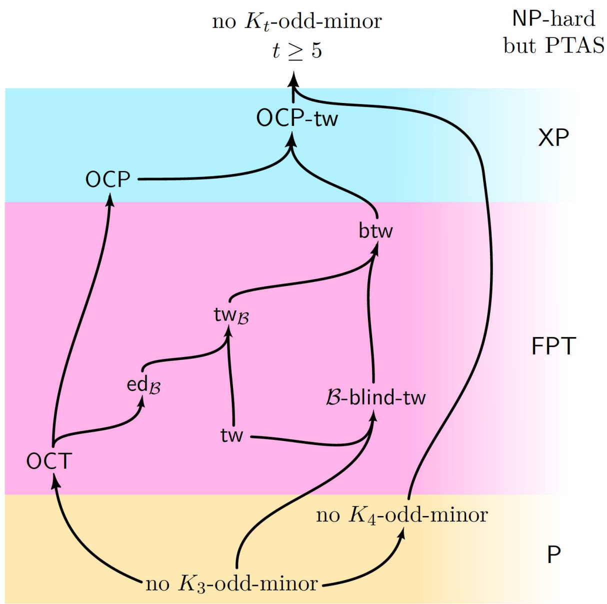

Figure 3 The complexity landscape of MIS in odd-minor-closed graph classes. The bottom shows the two instances of K_t-odd-minor-free classes where MIS is in P. The two middle areas depict odd-minor-monotone parameters and the corresponding (parameterized) complexity classes for MIS and finally, the very top indicates the general regime of K_t-odd-minor-free graphs for $t \geq 5$ where MIS is known to be NP-hard and to admit a PTAS. An arrow from x to y indicates that membership in the class x, or having the parameter x bounded also implies membership in the class y or that the parameter y is bounded.

algorithm is accomplished by exploiting how the graph with bounded OCP almost embeds on a surface with bounded genus. Note that Fiorini et al. converted the problem of finding an MIS to finding a minimum-cost circulation with certain property by using results of Conforti et al. [12]. Consequently, this approach alleviated the need to track the high connectivity within the near embedding that is required to find a specific minor model (see Section 7 of the full version [10]).

Two parameters that generalise the notion of OCT while maintaining both the comparability and the distinctness to OCP are the *elimination distance to* $\mathcal{B}$, denoted by $\mathsf{ed}_{\mathcal{B}}$, (see [8, 9]) and $\mathcal{B}$*-treewidth*, denoted by $\mathsf{tw}_{\mathcal{B}}$ (see [17, 34, 33]). Each of them is the minimum k such that a graph G has a set $X \subseteq V(G)$ such that $G - X \in \mathcal{B}$ and $\mathsf{p}(G_X) \leq k$ where G_X is the *torso* of X in G and p is the *treedepth*[6] in the case of elimination distance to $\mathcal{B}$ and the *treewidth* in the case of $\mathcal{B}$-treewidth. Moreover, it holds that $\mathsf{OCT}(G) \geq \mathsf{ed}_{\mathcal{B}}(G) \geq \mathsf{tw}_{\mathcal{B}}(G)$.

The $\mathcal{B}$*-blind-treewidth*, denoted by $\mathcal{B}$-blind-tw, of a graph G is the largest treewidth over all *non-bipartite* blocks of G. Gollin and Wiederrecht [27] proved an analogue of the Grid Theorem for $\mathcal{B}$-blind-treewidth where the grid is the *single parity break grid* (see Figure 2). Observe that the single parity break grid of order k has OCT equal to 1 for all $k \geq 1$. The notion of *bipartite treewidth* as given by Jaffke et al. [32] is relatively technical, instead we give a functionally equivalent definition in terms of OCP-tree-decompositions: The *bipartite treewidth*, denoted by btw, of a graph G is the minimum width of an OCP-tree-decomposition

[6] Since treedepth is not used anywhere in the paper, we omit its definition. See [8, 9, 34] for further information.

(T, β, α) where $G[\beta(t)] - \alpha(t)$ is bipartite for all $t \in V(T)$. Notice that $\mathcal{B}$-blind-tw$(G) \geq$ btw(G) and tw$_{\mathcal{B}}(G) \geq$ btw(G) for all graphs G. In particular, for every $k \geq 1$, the grid with odd cycle outgrowths of order k (see Figure 2) has bipartite treewidth 1, while its $\mathcal{B}$-treewidth is k. So with bipartite treewidth we now have a first parameter that properly subsumes both, the OCT-based parameters as well as $\mathcal{B}$-blind-treewidth.

The only parameter we have not discussed yet is OCP-treewidth itself. From the definition one can immediately deduce that OCP-tw$(G) \leq$ btw(G). Moreover, the Escher grid of order k can be seen to have bipartite treewidth k while, as established above, its OCP-treewidth is 1 for all $k \geq 1$. This makes OCP-treewidth the *most general* odd-minor-based parameter so far when it comes to parametrizations of MIS. In Figure 3 we give an overview of all classes and parameters discussed above and what they imply for the tractability of MIS.

With this it is now apparent that within the realm of known odd-minor-closed graph classes where MIS is tractable, there is a unique class, namely the K_4-odd-minor-free graphs, that does not fall under the regime of OCP-treewidth. Indeed, the parity handle grids are all K_4-odd-minor-free while K_4 is an odd-minor of $\mathcal{V}_3$. This raises the following question, related to Question 7.

▶ **Question 10.** *Let H be an odd-minor of the parity vortex of order k for some $k \geq 1$, is there a polynomial-time algorithm for MIS on H-odd-minor-free graphs?*

Non odd-minor-based approaches to MIS

To conclude this part, notice that we focussed here on the relevant literature for odd-minors. There are several other ongoing lines of research dealing with the computational complexity of MIS in settings of restricted structure. Among them is a recent interest in induced minors and the related *tree-independence number* [59, 14, 11]. Of particular interest is also the realm of H-vertex-minor-free graphs and classes of bounded *rankwidth* [43, 25]. In this area we have a particularly intriguing conjecture due to Geelen (see [40]) claiming that MIS should be tractable on *all* vertex-minor-closed graph classes. Finally, there exist many different kinds of "width" parameters that are fit to act as possible parametrizations for MIS (see [5] for some recent results and a good overview).

Existing tools for the study of odd-minor-free graphs

As mentioned earlier, there exist a few high-profile results on the structure of odd-minor-free and group-labelled-minor-free graphs [30, 23, 15, 58, 39], which all feature the downsides laid out earlier, namely non-constructive proofs, non-modular proof approaches, and non-explicit or huge bounds. Most other results concerning the structure of odd-minor-free graphs are more directly related to the odd-minor variant of Hadwiger's famous colouring conjecture [29, 35] (see also [54]). Sadly, with the exception with the work of Geelen et al. in [24], these results tend to not be very helpful in pursuing the kind of characterisations of odd-minor-free classes we are after.

1.4 An overview of the proof

We provide a brief overview of our proof for Theorem 6. Many of our techniques are based on the recent results in [28] on the so-called *Graph Minor Structure Theorem* (GMST). The GMST – originally due to Robertson and Seymour [52] – gives an approximate description of H-minor-free graphs, by stating that each such graph has a tree-decomposition where neighbouring bags intersect in a bounded number of vertices and the torso of every bag "*almost embeds*" into a surface where H does not embed. Here an "*almost embedding*" of H is roughly a drawing of H into a surface Σ after the removal of a bounded number of vertices, such that H is drawn into Σ "up to 3-separations" with a bounded number of special disks called "*vortices*" in which the drawing many include crosses but the connectivity of the part of H embedded in the disk is restricted severely. (We highly recommend [28] for its illustrations of these concepts.) We refine the techniques of [28] to be sensitive also to the parity of the embedded cycles and construct a more refined form of such an almost embedding, where the only cycles that may be of odd length are those that pass through the crosscaps of the surface an odd number of times and those that interact with vortices. Let us call this refined almost embedding an "*even-faced almost embedding*" for this overview.

Due to the high degree of technicality, we keep the details of the definitions involved purposefully vague and only provide a rough intuition. The relevant definitions can be found in Section 5 of the full version [10].

In the following we lay out five general steps. The first four of these steps have as their final goal to describe the structure of graphs without $\mathcal{H}_k$ and $\mathcal{V}_k$ as odd-minors "locally". Here the notion of locality is captured by being "highly connected" to a given grid minor (or *wall* or *mesh*). In the theory of Robertson and Seymour this connectivity is expressed through the notion of tangles (see [50]). The final step then puts these local pieces together to derive Theorem 6.

A bipartite Flat Wall

A central theorem of Robertson and Seymour is the *Flat Wall Theorem* [51]. It says that given a large wall[7] W in a graph G, there either exists a K_t-minor highly connected to W, or a small set $A \subseteq V(G)$ and a smaller but still big wall $W' \subseteq W$ such that the part of $G - A$ attached to the "inside"[8] of W' can be almost embedded into the plane without vortices. Such an embedding is called *flat*. We note that this is still an almost embedding and not a drawing, since we are still only embedding the graph up to 3-separations. Another key theorem for us is due to Geelen, Gerards, Reed, Seymour, and Vetta [24] stating that given a large enough complete graph as a minor in G, the graph G either contains every graph on t vertices as an odd-minor, or there is a small set $A \subseteq V(G)$ such that the part of $G - A$ that contains this complete minor is bipartite. Combining both results allows us to show that for any large enough wall W, we either find a set A and wall W' as above such that the inside of W' is also bipartite, or we find $\mathcal{H}_k$ as an odd-minor. Note that a similar theorem was proven by Thomas and Yoo [58]. In the interest of deriving polynomial bounds for our results, we provide a simple and independent proof.

A parity-sensitive society classification

A core technique called *society classification*, crucial for proving the GMST, pioneered by Kawarabayashi, Thomas, and Wollan [37], and refined in [28], is to fix our perspective onto a disk whose boundary is protected by many concentric cycles which appear together in a "flat" embedding on the outside of the disk, whilst on the inside of the disk we have an unembedded part of the graph. In [37] it is proved that in this setting we can either find a K_t-minor, a large linkage resembling a handle or crosscap that has its endpoints on the

[7] A *wall* is a subdivision of a hexagonal grid.
[8] The "inside" is everything except for the cycle bounding the outer face in a standard planar drawing of the wall.

boundary of the disk, or we can embed the entirety of the graph drawn on this disk in a flat way up to a small number of vortices whose internal connectivity is additionally restricted. In the last case, we in fact find an embedded, wall-like subgraph capturing each of these vortices in distinct walls that are all well-connected to a common wall living on the outside of our disk.

We refine this result by upgrading the embedding in each case to be "even-faced" in the sense described above and letting a variant of our obstructions be another outcome. In particular, we use the theorem from [24] discussed above to process the K_t-minor further.

Building a surface

Our society classification variant can be used to build a "local" form of the structure theorem based on a wall. First, the wall is made flat and bipartite via our modified flat wall theorem. Then, starting on the outside of the wall, we apply our refined society classification theorem to either find one of our obstructions, turn the block associated with our wall bipartite after the removal of a small number of vertices, embed what remains unembedded of the graph up to a handful of vortices and deleted vertices, or we find a large linkage resembling a crosscap or handle after removing a few vertices.

The last option allows us to iterate, as we can use this linkage and the structure on the boundary of our disk to cut out another disk that encompasses the previous one whilst preserving a large part of the handle or crosscap linkage. Thus we also have witnesses to the fact that we need to increase the genus of the surface we use in our even-faced almost embedding. The supporting infrastructure these linkages provide accumulates throughout the iterations, allowing us to find one of our obstructions if this process continues for too long. This procedure ends up either finding one of our obstructions or an even-faced almost embedding for the block associated with the remains of the initial wall.

Locally bounding OCP

Our cycles may be odd either when they are not entirely embedded, i.e. they go through a vortex, or they are embedded by going through an odd number of crosscaps. We first remove all odd cycles arising from vortices or obtain $\mathcal{V}_t$ as an odd-minor using a general technique exemplified in [57]. Consequently, all remaining odd cycles go through an odd number of crosscaps, allowing us to bound the odd-cycle-packing number by the maximum size of a disjoint set of such curves in our embedding via fairly simple counting arguments.

Deriving a decomposition

Being able to "locally" guarantee that our graph has bounded OCP, we modify a standard strategy stemming from one of the main proofs in [50] to derive the GMST from the local structure theorem to finally arrive at our desired OCP-tree-decomposition in the absence of any obstructions. This allows us to not only derive Theorem 6 at this point, but also prove Theorem 5 fairly directly.

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