Graph Classes Closed Under Self-Intersection

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Abstract

A graph class is monotone if it is closed under taking subgraphs. A monotone class defined by finitely many obstructions has bounded treewidth if and only if one of the obstructions is a tripod, i.e. a disjoint union of subdivided claws and paths. This dichotomy also characterizes exactly those monotone graph classes for which many NP-hard graph problems admit polynomial-time algorithms. These dichotomies do not extend to the universe of all hereditary classes. This leads to the question of whether we can extend known dichotomies for monotone classes to larger families of hereditary classes. We answer this question affirmatively by considering the family of hereditary graph classes closed under self-intersection. This family is known to be located strictly between the monotone and hereditary classes. We prove a new structural characterization of graphs in self-intersection-closed classes excluding a tripod. In contrast to monotone classes excluding a tripod, these classes do not necessarily have bounded treewidth; in fact, they do not even need to be sparse. We use our characterization to give a complete dichotomy for Maximum Independent Set, and its weighted variant, on self-intersection-closed classes defined by finitely many obstructions: these problems are in P if the class excludes a tripod and NP-hard otherwise. Our dichotomy generalizes several known results on Maximum Independent Set in the literature. We also apply our characterization to obtain a dichotomy for Maximum Induced Matching on self-intersection-closed classes of bipartite graphs defined by finitely many obstructions, and for Satisfiability and Counting Satisfiability on self-intersection-closed classes of (bipartite) incidence graphs defined by finitely many obstructions. Finally, we use our characterization to obtain a dichotomy for boundedness of clique-width for self-intersection-closed classes of bipartite graphs defined by finitely many obstructions.

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1 Introduction

The celebrated project of Robertson and Seymour on graph minors produced a variety of fundamental results of both structural and algorithmic nature, and culminated in the proof of Wagner's conjecture stating that the minor relation on graphs is a well-quasi-order. This implies that every minor-closed class can be characterized by a finite set of minimal forbidden minors, and any family of classes in this universe which is closed under taking subclasses, can be characterized by minimal classes not in the family. For instance, within the universe of minor-closed classes, the family of classes of bounded treewidth can be characterized by a unique "obstruction", i.e. a unique minimal minor-closed class of unbounded treewidth, namely the class of planar graphs. What is interesting is that the family of classes of bounded treewidth coincides with the family of minor-closed classes where many NP-hard algorithmic problems, such as Maximum Independent Set or Maximum Induced Matching, admit polynomial-time algorithms (unless $\mathsf{P} = \mathsf{NP}$). The same is true for Satisfiability represented by incidence graphs of CNF instances, i.e. the problem is NP-complete for planar incidence graphs [17] and can be solved in polynomial time for incidence graphs of bounded treewidth [13]. So, in the universe of minor-closed classes, polynomial-time solvability of many NP-hard algorithmic problems is equivalent to boundedness of treewidth.

These dichotomies do not extend to the more general family of hereditary classes, i.e. classes closed under taking induced subgraphs. The induced subgraph relation is not a well-quasi-order, as it contains infinite antichains, e.g. cycles. Hence, in this universe, there exist both classes characterized by infinitely many minimal forbidden induced subgraphs, and families of classes with no minimal classes outside of the families. Dichotomies are still possible for hereditary classes if we restrict ourselves to *finitely-defined* classes, i.e. classes defined by finitely many forbidden induced subgraphs. In this case, we relax the notion of minimal classes to the notion of boundary classes, i.e. minimal *limit* classes. Together, minimal and boundary classes are known as *critical classes*, or *critical properties*, and they enable a characterization of any family $\mathcal{F}$ of finitely-defined hereditary classes that is closed under taking subclasses, in terms of critical properties: a class X belongs to $\mathcal{F}$ if and only if it does not contain any critical class for $\mathcal{F}$. For instance, for the family of finitely-defined classes of bounded treewidth, there are precisely four critical classes [22]. Two classes, complete graphs and complete bipartite graphs, are minimal hereditary classes of unbounded treewidth, and the other two are boundary classes: the class $\mathcal{S}$ of forests in which every connected component is a tree with at most three leaves, and the class of line graphs of the graphs in $\mathcal{S}$.

Every connected graph in $\mathcal{S}$ with three leaves is a subdivision of the claw $K_{1,3}$ and is called, for example, a subdivided claw, long claw or tripod. With slight abuse of terminology, we will refer to any graph in $\mathcal{S}$ as a *tripod*. The class $\mathcal{S}$ is critical not only for graphs of bounded treewidth, but also for many algorithmic problems, including Maximum Independent Set [2], Maximum Induced Matching [3] and Satisfiability [19]. If a finitely-defined hereditary class X contains $\mathcal{S}$, then these problems are NP-hard on X.

It was conjectured in [18] that for Maximum Independent Set, $\mathcal{S}$ is the only critical class, i.e. by forbidding any graph from $\mathcal{S}$ as an induced subgraph we obtain a class where the problem can be solved in polynomial time. So far, this has only been confirmed for a few very special tripods with small connected components (see, e.g. [8, 14]). Recently, substantial progress has been made by proving weaker versions of the conjecture, such as in the setting of weakly sparse graph classes (i.e. hereditary classes excluding both a clique and a biclique) [1], and by establishing quasi-polynomial-time solvability of Maximum Independent Set on classes that exclude a tripod [12]. Despite this progress, the conjecture remains out of reach. Under these circumstances, it is natural to restrict the universe to a family of classes between minor-closed and hereditary. An important such family is that of monotone classes.

A class of graphs is *monotone* if it is closed under taking subgraphs, not necessarily induced. The subgraph relation is much richer than the induced subgraph relation, but it still contains infinite antichains, which is bad news. The good news is that it contains a *canonical* infinite antichain, i.e. an antichain that is unique in the sense that a monotone class is well-quasi-ordered under the subgraph relation if and only if it contains finitely many graphs from the canonical antichain [11]. This antichain consists of the minimal forbidden (induced) subgraphs for the class $\mathcal{S}$, which emphasizes the importance of $\mathcal{S}$ as a critical class.

For monotone classes and subject to $\mathsf{P} \neq \mathsf{NP}$, the class $\mathcal{S}$ is the unique critical class for many families, including classes of bounded treewidth and classes for which Maximum Independent Set, Maximum Induced Matching and Satisfiability are polynomial-time solvable (see also the survey [16]). Hence, we are back in the situation where polynomial-time solvability of the problems coincides with boundedness of treewidth, similarly to the case of minor-closed classes. This situation suggests that we step back and search for universes of classes that are strictly between monotone and hereditary. Is there life between these two worlds? We observe that any monotone class that excludes a graph G also excludes all graphs containing G as a subgraph. But what if we exclude only *some* of these graphs?

One specific family of such classes was proposed in [5] under the name of *self-intersection-closed* classes. Informally, we say that a graph H is a *self-intersection* of a graph G if H is the intersection of a collection of isomorphic copies of G (formal definitions will be given later). By definition, a self-intersection of G is isomorphic to a subgraph of G, but in general not every subgraph of G can be obtained in this way. To distinguish between the two notions, we will refer to a self-intersection of a graph G as an *si-subgraph* of G. A graph class $\mathcal{G}$ is closed under self-intersection if, for every $G \in \mathcal{G}$, the class $\mathcal{G}$ contains all si-subgraphs of G. As we shall see later, the family of classes closed under self-intersection is located strictly between monotone and hereditary classes. Determining a more precise location of this family between the two extremes is a challenging open question. We discuss this question at the end of Section 2, where we conclude that there is no obvious answer.

For our study on classes defined by forbidding graphs in $\mathcal{S}$ as si-subgraphs, we consider the following classical algorithmic problems as our testbed problems:

1. Maximum Weight Independent Set: Given a graph $G = (V, E)$ and a weight function $w\colon V \to \mathbb{Q}_{\geq 0}$, find an independent set $I \subseteq V$ maximizing the total weight $w(I) = \sum_{v \in I} w(v)$. The special case of Maximum Weight Independent Set when the function w is constantly equal to 1 is called Maximum Independent Set.
2. Maximum Induced Matching: Given a graph $G = (V, E)$, find an induced matching $M \subseteq E$ of maximum cardinality. An *induced matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges in M share an endpoint, and no edge in $E \setminus M$ connects two edges in M.
3. Satisfiability: Given a Boolean formula φ in conjunctive normal form (CNF), is there a truth assignment to the variables of φ that makes the formula evaluate to true?
4. Counting Satisfiability: Given a Boolean formula φ in CNF, determine the total number of satisfying truth assignments that make φ evaluate to true.

Alekseev and Sorochan [5] showed that Maximum Independent Set is polynomial-time solvable for every self-intersection-closed class excluding a path P_k. Sorochan [25] proved the same for self-intersection-closed graphs excluding a connected tripod as long as the input graph has a vertex with co-degree bounded above by the logarithm of the number of vertices.

Our Contributions

We extend the results of [5, 25] to a *full* dichotomy for both Maximum Independent Set and Maximum Weight Independent Set on classes defined by finitely many forbidden si-subgraphs. It is known [4] that a monotone class defined by finitely many forbidden subgraphs admits a polynomial-time algorithm for Maximum Independent Set and Maximum Weight Independent Set if and only if it excludes a graph from $\mathcal{S}$ (unless $\mathsf{P} = \mathsf{NP}$). We show exactly the same dichotomy for Maximum Independent Set and Maximum Weight Independent Set on self-intersection-closed classes defined by finitely many forbidden si-graphs. While for monotone classes, excluding a tripod is equivalent to boundedness of treewidth [23], this is not the case for self-intersection-closed classes excluding a tripod, as these may contain arbitrarily large complete graphs (see also Theorem 3).

The polynomial part of our dichotomy, shown in Section 3, is based on a structural characterization of self-intersection-closed classes excluding a tripod. We use this characterization in Section 4 to obtain the same dichotomy for Maximum Induced Matching on classes of bipartite graphs defined by finitely many forbidden si-subgraphs, and for Satisfiability and Counting Satisfiability on classes of (bipartite) incidence graphs defined by finitely many forbidden si-subgraphs. In Section 4, we also show a dichotomy for boundedness of clique-width for classes of bipartite graphs defined by finitely many forbidden si-subgraphs.

Finally, we discuss directions for future work in Section 5. Due to space restrictions, proofs of some results are omitted; they can be found in the full version of the paper [10].

2 Preliminaries

For a positive integer k, we let $[k]$ denote the set of the first k positive integers. Let $G = (V, E)$ be a graph. For a vertex $v \in V$, we let $N_G(v)$ denote the set of neighbours of v in G, and let $\deg_G(v)$ denote the degree of v in G, i.e. the cardinality of $N_G(v)$. When G is clear from the context, we simply write $N(v)$ and $\deg(v)$, respectively. For a set $U \subseteq V$, we let $G[U]$ denote the subgraph of G induced by U and let $G - U$ denote the subgraph of G induced by $V \setminus U$, i.e. $G - U$ is obtained from G by removing the vertices in U. If U consists of a single vertex u, we write $G - u$ instead of $G - U$ for brevity. A set of pairwise adjacent (respectively, non-adjacent) vertices in G is called a *clique* (respectively, an *independent set*) of G. A (possibly empty) set $U \subseteq V$ is a *cutset* in G if $G - U$ is disconnected and a *clique cutset* if U is both a clique and a cutset. A *cut-vertex* is a vertex v such that $\{v\}$ is a cutset. A graph is *biconnected* if it is connected and has no cut-vertices. A graph $H = (V, E')$ is a *spanning subgraph* of G if $E' \subseteq E$, and H is a *spanning supergraph* of G if $E' \supseteq E$. For two disjoint sets $A, B \subseteq V$, we say that A and B are *complete* to each other (in G) if every vertex in A is adjacent to every vertex in B. Similarly, A and B are *anticomplete* to each other if no vertex in A is adjacent to a vertex in B. A set $U \subseteq V$ is a *module* in G if every vertex in $V \setminus U$ is either complete or anticomplete to U. Two distinct vertices $x, y \in V$ are *twins* in G if $N(x) \setminus \{x, y\} = N(y) \setminus \{x, y\}$; equivalently, x, y are twins if $\{x, y\}$ is a module. Two non-adjacent twins are said to be *false* twins.

We denote the treewidth of a graph G by $\mathrm{tw}(G)$, and refer to [15] for the definitions of treewidth and clique-width, as we do not need these here.

We let $L(G)$ denote the *line graph* of G, i.e. the graph with vertex set $E(G)$, with distinct $e_1, e_2 \in E(G)$ being adjacent in $L(G)$ if and only if they share a vertex in G. We write $K_{p,q}$, K_s, P_s and C_s for the complete bipartite graph with p and q vertices in the parts, the complete graph, the path and the cycle on s vertices, respectively. For two vertex-disjoint graphs G and H, let $G + H$ denote the disjoint union of G and H, and let tG denote the disjoint union of $t \geq 1$ copies of G. For positive integers i, j, and k, we let $S_{i,j,k}$ denote the graph obtained from the claw $K_{1,3}$ by subdividing its three edges with $i-1$, $j-1$, and $k-1$ new vertices, respectively.

A graph G is H*-free* for some graph H if no induced subgraph of G is isomorphic to H. For a set M of graphs, Free(M) denotes the hereditary class of graphs containing no graph from M as an induced subgraph, and Subgraph-Free(M) denotes the monotone class of graphs containing no graph from M as a subgraph. If $M = \{F_1, \ldots, F_k\}$, we may write Free($F_1, \ldots, F_k$) and Subgraph-Free($F_1, \ldots, F_k$) instead of Free(M) and Subgraph-Free(M).

2.1 Self-intersection of Graphs

The *intersection* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$. We say that a graph H is a *self-intersection* of G, or that G *can be self-intersected* to H, denoted $G \xrightarrow{\cap} H$, if there exist graphs $G_1, G_2, \ldots, G_k$, each isomorphic to G, such that $H = G_1 \cap \cdots \cap G_k$; in other words, if H can be obtained as the intersection of finitely many graphs isomorphic to G (see Figure 1 for an example). Note that $\xrightarrow{\cap}$ is reflexive and transitive. Clearly, if H is a self-intersection of G, then H is isomorphic to a subgraph of G. However, in general, not every subgraph of G can be obtained by self-intersecting G. For example, a complete graph can only be self-intersected to a complete subgraph. To distinguish between the two notions, we refer to a self-intersection of G as an *si-subgraph* of G.

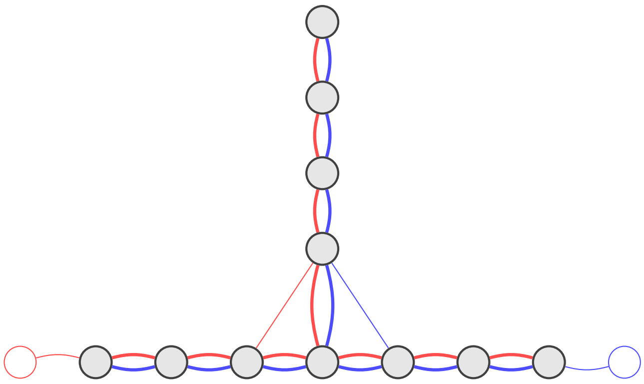

Figure 1 An illustration showing that $L(S_{4,4,4})$ can be self-intersected to $S_{3,3,4}$. The red and blue graphs are both isomorphic to $L(S_{4,4,4})$. Their intersection (highlighted vertices and thick double edges) is isomorphic to $S_{3,3,4}$.

The following lemma was proved by Alekseev and Sorochan [5], who introduced the notion of self-intersection.

▶ **Lemma 1** ([5]). *If H is an induced subgraph of a graph G, then $G \xrightarrow{\cap} H$.*

Below we prove a more general sufficient condition for a subgraph of G to be an si-subgraph. Let a graph H be isomorphic to a subgraph of a graph G and let $\varphi\colon V(H) \to V(G)$ be a *subgraph embedding* of H into G, that is, an injective mapping from $V(H)$ to $V(G)$ such that for every two adjacent vertices u and v of H, their images $\varphi(u)$ and $\varphi(v)$ are adjacent in G. Then, the inclusion $\varphi(N_H(v)) \subseteq N_G(\varphi(v)) \cap \varphi(V(H))$ holds for every $v \in V(H)$.

For a vertex $v \in V(H)$, the embedding φ is *induced with respect to* v if the neighbourhood of $\varphi(v)$ in $\varphi(V(H))$ in the graph G is exactly $\varphi(N_H(v))$, i.e. $\varphi(N_H(v)) = N_G(\varphi(v)) \cap \varphi(V(H))$.

Note that if φ is induced with respect to every vertex of H, then φ is an *induced* subgraph embedding of H into G, that is, two distinct vertices u and v of H are adjacent in H if and only if their images $\varphi(u)$ and $\varphi(v)$ are adjacent in G.

In our proofs we make use of the following lemma; its proof has been omitted but can be found in [10].

▶ **Lemma 2.** *Let G and H be graphs. If for every vertex $v \in V(H)$ there exists a subgraph embedding of H into G that is induced with respect to v, then $G \xrightarrow{\cap} H$.*

We say that a class $\mathcal{G}$ of graphs is *closed under self-intersection*, or *self-intersection-closed*, if $G \in \mathcal{G}$ and $G \xrightarrow{\cap} H$ imply $H \in \mathcal{G}$. From the above discussion, it follows that every monotone class is self-intersection-closed and every self-intersection-closed class is hereditary. Similarly to the forbidden subgraph characterization for monotone classes and the forbidden induced subgraph characterization for hereditary classes, every self-intersection-closed class admits a characterization in terms of forbidden si-subgraphs. For a set M of graphs, we let Si-Free(M) denote the class of graphs containing no graph in M as an si-subgraph. Clearly,

$$\text{Subgraph-Free}(M) \subseteq \text{Si-Free}(M) \subseteq \text{Free}(M).$$

Both inclusions can be strict even in the case when M consists of a single small graph in $\mathcal{S}$. For example, if $M = \{P_1 + P_3\}$, then the classes Subgraph-Free(M), Si-Free(M) and Free(M) are all pairwise distinct. Indeed, it is not difficult to see that Subgraph-Free$(P_1 + P_3) =$ Free$(C_3 + P_1, C_4, K_{1,3}, P_1 + P_3, P_4, \text{paw}, \text{diamond}, K_4)$ and Si-Free$(P_1 + P_3) =$ Free$(P_1 + P_3, P_4, \text{paw}, \text{diamond})$, where diamond is the K_4 minus an edge and paw is the graph formed by adding a pendant vertex to a K_3. In particular, note that if $F \in M$ is a graph on r vertices and M does not contain a complete graph, then $K_r \in$ Si-Free$(M) \setminus$ Subgraph-Free(M).

The above example of the forbidden graph $P_1 + P_3$ shows not only that the class Si-Free$(P_1 + P_3)$ is strictly between Subgraph-Free$(P_1 + P_3)$ and Free$(P_1 + P_3)$, but also that this class is almost "equidistant" from both extremes in terms of the number of minimal forbidden induced subgraphs. One more way of measuring the difference between the three types of classes is their *speed*, i.e. the number of n-vertex (unlabelled) graphs. Consider, for instance, the class of stars. For each $n \geq 3$, the hereditary closure of this class contains two n-vertex graphs (the n-vertex star and the n-vertex edgeless graph), the si-closure contains three n-vertex graphs (the n-vertex star, the n-vertex edgeless graph and the n-vertex graph with one edge), while the monotone closure contains n graphs with n vertices. In this case, the si-closure is much closer to the hereditary closure than to the monotone closure. The situation is similar in the case of complete bipartite graphs, because the si-closure of this class consists of graphs every connected component of which is complete bipartite, while the monotone closure consists of all bipartite graphs. However, the situation changes dramatically if we consider the class of complete bipartite graphs omitting one edge. In this case, the si-closure coincides with the monotone closure and is quite different from the hereditary closure. This discussion shows that the relationship between the three families of graph classes is a topic for a separate study.

3 Excluding Tripods

In this section we study graph classes excluding tripods as si-subgraphs.

Clearly, every tripod is an induced subgraph of $tS_{t,t,t}$ for some t, and hence, without loss of generality we will consider classes excluding $tS_{t,t,t}$ as an si-subgraph. For $t \in \mathbb{N}$, we let $\mathcal{X}_t =$ Si-Free$(tS_{t,t,t})$. We analyse structural properties of graphs in $\mathcal{X}_t$ that can be applied to develop efficient algorithms for the aforementioned algorithmic problems. Our main result is:

▶ **Theorem 3.** *Let $t \in \mathbb{N}$. Then, there exists $c \in \mathbb{N}$ such that if a graph $G \in \mathcal{X}_t$ contains neither false twins nor a clique cutset of size at most 2, then G is complete or $\mathrm{tw}(G) \leq c$.*

To prove this result we show that if a graph in $\mathcal{X}_t$ without small clique cutsets or twins contains a large induced complete bipartite graph or a large complete subgraph, then this subgraph is connected to the rest of the graph in a very structured way.

3.1 Handling Large Bicliques

We will show that if a graph G from $\mathcal{X}_t$ has no cut vertices, then any maximal induced complete bipartite graph in G of sufficiently large size forms a module in G.

▶ **Theorem 4.** *Let $t \in \mathbb{N}$ and G be a graph in $\mathcal{X}_t$ with no cut vertices. Then the vertex set of any maximal induced complete bipartite graph $K_{p,q}$ in G with $p, q \geq 3t^2 + t + 1$ is a module.*

To prove Theorem 4, we first develop several technical ingredients. We begin with the following lemma. Its proof can be found in [10] and uses Lemma 2.

▶ **Lemma 5.** *For an integer $t \in \mathbb{N}$, and a graph H, it holds that $H \xrightarrow{\cap} tS_{t,t,t}$ if $V(H)$ is the union of three pairwise disjoint sets A, B and C such that:*
1. *A and B are independent sets each of size $3t^2 + t$;*
2. *A and B are complete to each other;*
3. *$|C| \geq 2$ and $H[C]$ is a path with endpoints x and y;*
4. *both x and y have neighbours in $A \cup B$, and $C \setminus \{x, y\}$ is anticomplete to $A \cup B$;*
5. *$\deg_H(x) = 2$;*
6. *$N(x) \cap (A \cup B) \neq N(y) \cap (A \cup B)$.*

We now show the following lemma.

▶ **Lemma 6.** *Let H be a graph with vertex set $A \cup B \cup \{v\}$ such that:*
1. *A, B, $\{v\}$ are pairwise disjoint;*
2. *A is an independent set or a clique;*
3. *A and B are complete to each other;*
4. *v has a neighbour in A and a non-neighbour in A.*

Let F be a spanning subgraph of H obtained by removing some edges between v and vertices in A. Then $H \xrightarrow{\cap} F$.

Proof. Let $|A| = k$ and suppose without loss of generality that $A = [k]$, $N_H(v) \cap A = [s]$, and $N_F(v) \cap A = [t]$, for some $0 \leq t < s < k$.

For every vertex $x \in [s] \setminus [t]$, i.e. every vertex in A that is adjacent to v in H but non-adjacent to v in F, let Z_x be the graph obtained from H by swapping the adjacencies between v, x and v, k. More precisely, $E(Z_x) = (E(H) \setminus \{v, x\}) \cup \{v, k\}$. Note that x and k have the same neighbourhoods in $H - v$, and v is adjacent to exactly one of them in each H and Z_x, which implies that Z_x is isomorphic to H.

We claim that the intersection $(\bigcap_{x \in [s] \setminus [t]} Z_x) \cap H$ coincides with F. Indeed, for each $x \in [s] \setminus [t]$, the graph $Z_x - v$ is equal to $H - v$. Furthermore, v is adjacent to $[t] \cup (N_H(v) \cap B)$ in each of the graphs H, Z_x, $x \in [s] \setminus [t]$, and for every $y \in [k] \setminus [t]$, the vertex v is not adjacent to y in at least one of these graphs. Thus, in the intersection $(\bigcap_{x \in [s] \setminus [t]} Z_x) \cap H$, the neighbourhood of v is equal to $N_F(v) = [t] \cup (N_H(v) \cap B)$. Consequently, $F = (\bigcap_{x \in [s] \setminus [t]} Z_x) \cap H$, and therefore $H \xrightarrow{\cap} F$. ◀

In this section, we use Lemma 6 only in the case when A is an independent set; the case when A is a clique will be used in Section 3.2.

Let X_p denote the graph obtained from $K_{p,p}$ by adding a vertex that has exactly two neighbours, such that these neighbours belong to the same part of the $K_{p,p}$. Similarly, let Y_p denote the graph obtained from $K_{p,p}$ by adding a vertex that has exactly two neighbours, such that these neighbours belong to different parts of the $K_{p,p}$. See Figure 2 for an illustration.

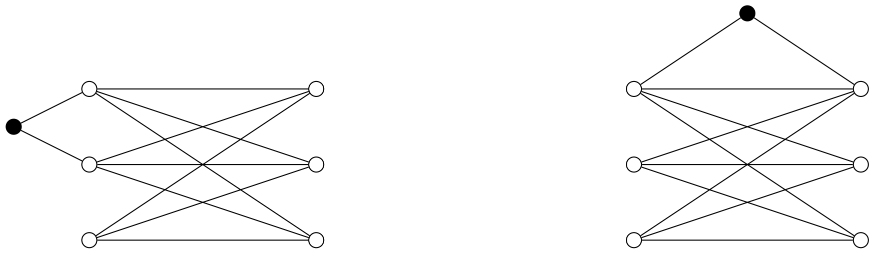

Figure 2 Left: the graph X_3: a vertex (black) is adjacent to two vertices in the same part of $K_{3,3}$. Right: the graph Y_3: a vertex (black) is adjacent to one vertex in each part of $K_{3,3}$.

Using the above notation, we prove the following lemma.

▶ **Lemma 7.** *Let $t, p \in \mathbb{N}$, where $p \geq 3t^2 + t + 1$. Let H' be a graph isomorphic to X_p or Y_p with vertex set $A' \cup B' \cup \{x\}$, where A' and B' are complete to each other and x has degree 2. Then $H' \xrightarrow{\cap} tS_{t,t,t}$.*

Proof. The lemma follows from Lemma 5 by noticing that both graphs X_p and Y_p contain an induced subgraph satisfying the conditions of Lemma 5. Indeed, let y, z be the neighbours of x in $A' \cup B'$. Then, by taking $C = \{x, y\}$, $A \subseteq A'$ and $B \subseteq B'$ such that $|A| = |B| = 3t^2 + t$, $y \notin A \cup B$ and $z \in A \cup B$, one can see that $H = H'[A \cup B \cup C]$ satisfies the conditions of Lemma 5. Thus, $H' \xrightarrow{\cap} H \xrightarrow{\cap} tS_{t,t,t}$. ◀

Lemmas 1, 6, and 7 imply the following sufficient condition for graphs that can be self-intersected to a tripod $tS_{t,t,t}$; see [10] for full proof details.

▶ **Lemma 8.** *Let $p \geq 3$ and let H be a graph with vertex set $A \cup B \cup \{v\}$ such that:*
1. *A, B, $\{v\}$ are pairwise disjoint;*
2. *A and B are independent sets that are complete to each other;*
3. *$|A| \geq p$ and $|B| \geq p$;*
4. *v has at least two neighbours in $A \cup B$;*
5. *v has a neighbour and a non-neighbour in at least one of the sets A, B.*

Then $H \xrightarrow{\cap} X_p$ or $H \xrightarrow{\cap} Y_p$. In particular, if $p \geq 3t^2 + t + 1$ for some $t \in \mathbb{N}$, then $H \xrightarrow{\cap} tS_{t,t,t}$.

We are now ready to prove the main result of this section, which we restate for convenience.

▶ **Theorem 4.** *Let $t \in \mathbb{N}$ and G be a graph in $\mathcal{X}_t$ with no cut vertices. Then the vertex set of any maximal induced complete bipartite graph $K_{p,q}$ in G with $p, q \geq 3t^2 + t + 1$ is a module.*

Proof. Let $s = 3t^2 + t + 1$, and let $A, B \subseteq V(G)$ be two disjoint independent sets in G such that $G[A \cup B]$ is a maximal induced complete bipartite graph in G and $|A| \geq s$ and $|B| \geq s$. It suffices to show that every vertex x outside $A \cup B$ is either complete or anticomplete to $A \cup B$. Suppose there exists a vertex $x \in V(G) \setminus (A \cup B)$ that is neither complete nor anticomplete to $A \cup B$. We will show that this implies $G \xrightarrow{\cap} tS_{t,t,t}$. We consider two cases.

Case 1: *x has at least two neighbours in $A \cup B$.*
Note that, due to the maximality of $G[A \cup B]$, the vertex x cannot be complete to one of the sets A and B, and anticomplete to the other. Furthermore, since x is not complete to $A \cup B$, it must have a neighbour and a non-neighbour in at least one of the parts A and B. Then $G \overset{\cap}{\to} tS_{t,t,t}$ by Lemma 8.

Case 2: *x has exactly one neighbour in $A \cup B$.*
Let y be the unique neighbour of x in $A \cup B$. Since G has no cut vertices, $G - y$ is connected. Let P be a shortest path from x to $(A \cup B) \setminus \{y\}$ in $G - y$. Now, $H = G[A \cup B \cup V(P)]$ satisfies the conditions of Lemma 5; thus, $G \overset{\cap}{\to} H \overset{\cap}{\to} tS_{t,t,t}$.

This completes the proof of the theorem. ◀

3.2 Handling Large Cliques

In this section we show that if a connected graph G from $\mathcal{X}_t$ has no clique cutsets of size 1 or 2, then G is either a complete graph or contains no large cliques. Before proving this result, we first establish some auxiliary technical results that will be used in the argument.

We start with the following lemma. Its proof can be found in [10] and uses Lemma 2.

▶ **Lemma 9.** *Let $t \in \mathbb{N}$ and let H be a graph with vertex set $A \cup T$ such that:*
1. *$|A| \geq 3t^2 + t + 1$;*
2. *A is a clique in H;*
3. *$T = V(Q_1) \cup V(Q_2) \cup V(Q_3)$, where Q_1, Q_2, Q_3 are paths in H from a vertex $w \notin A$ to A such that the subpaths $Q_1 - w$, $Q_2 - w$, $Q_3 - w$ are pairwise vertex disjoint;*
4. *$\deg_H(w) = 3$;*
5. *$|A \cap T| = 3$ and the three vertices in the intersection are the endpoints of Q_1, Q_2, Q_3 that are different from w; we denote these vertices by y_1, y_2, y_3, respectively;*
6. *$H[V(Q_1) \cup V(Q_2)]$ is a chordless cycle;*
7. *$A \setminus T$ and $T \setminus A$ are anticomplete to each other.*

Then $H \overset{\cap}{\to} tS_{t,t,t}$.

The following lemma is a corollary of Lemma 6 in the case when A is a clique and $B = \emptyset$.

▶ **Lemma 10.** *Let H be a graph with vertex set $A \cup \{v\}$ such that:*
1. *A and $\{v\}$ are disjoint;*
2. *A is a clique;*
3. *v has a neighbour and a non-neighbour in A.*

Let F be a spanning subgraph of H obtained by removing some edges incident to v. Then $H \overset{\cap}{\to} F$.

We are now ready to prove the main result of this section.

▶ **Theorem 11.** *Let $t \in \mathbb{N}$. Every connected graph in $\mathcal{X}_t$ with no clique cutsets of size at most 2 is complete or K_r-free, where $r = 3t^2 + t + 2$.*

Proof. For a contradiction, suppose there exists a non-complete graph $G \in \mathcal{X}_t$ with no clique cutsets of size at most 2 that contains a clique of size at least r. Let A' be a maximal clique in G of size at least r. Since G is non-complete and connected, there exists a vertex in $V(G) \setminus A'$ that has a neighbour in A'.

We consider three cases (see also Figure 3).

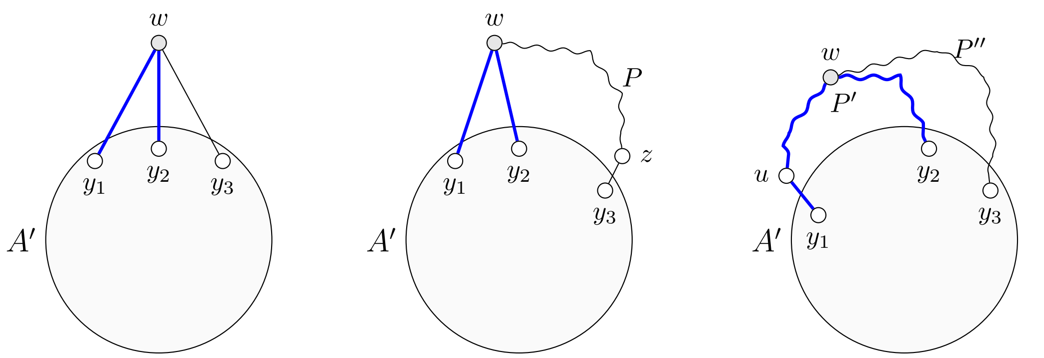

Figure 3 An illustration of the three cases in the proof of Theorem 11. Case 1 (left): w has at least three neighbours in A'. Case 2 (middle): w has exactly two neighbours in A'. Case 3 (right): three paths from w to A'. In all three figures, edges and paths highlighted in blue indicate that their vertices induce a chordless cycle, thereby satisfying Assumption 7 of Lemma 9.

Case 1: *There exists a vertex $w \in V(G) \setminus A'$ with at least three neighbours in A'.*
Since A' is maximal, w also has a non-neighbour in A'. Then, by Lemma 10, $G \xrightarrow{\cap} H$, where H is a spanning subgraph of $G[A' \cup \{w\}]$ obtained by removing all but three edges incident to w. Note that H satisfies the conditions of Lemma 9, and thus $G \xrightarrow{\cap} H \xrightarrow{\cap} tS_{t,t,t}$.

Case 2: *Every vertex outside A' has at most two neighbours in A', and there exists a vertex $w \in V(G) \setminus A'$ with exactly two neighbours in A'.*
Let y_1, y_2 be the neighbours of w in A'. Since G has no clique cutsets of size 2, the graph $G - \{y_1, y_2\}$ is connected. Let P be a shortest path from w to $A' \setminus \{y_1, y_2\}$ in this graph, let y_3 denote the endpoint of P in $A' \setminus \{y_1, y_2\}$ and let z denote the vertex on P adjacent to y_3. If z has a second neighbour y_3' in $A' \setminus \{y_1, y_2\}$, then we define $H = G[(A' \cup V(P)) \setminus \{y_3'\}]$; otherwise, we define $H = G[A' \cup V(P)]$. We observe that H satisfies the conditions of Lemma 9 with A being $A' \setminus \{y_3'\}$ or A', respectively, and $Q_1 = (w, y_1)$, $Q_2 = (w, y_2)$, and $Q_3 = P$. Therefore, $G \xrightarrow{\cap} H \xrightarrow{\cap} tS_{t,t,t}$.

Case 3: *Every vertex outside A' has at most one neighbour in A'.*
First, we argue that there exists a path with at least three vertices whose endpoints are in A' and all its internal vertices are outside A'. Indeed, let u be a vertex outside A' that has a unique neighbour in A', which we denote by y_1. Since G has no cut vertices, the graph $G - y_1$ is connected. By taking a shortest path from u to $A' \setminus \{y_1\}$ and extending it with the vertex y_1, we obtain a path P' with the desired properties. Without loss of generality, assume that among all such paths, the path P' has the least number of vertices. This, in particular, implies that $G[V(P')]$ is a chordless cycle. Let $y_2 \in A' \setminus \{y_1\}$ denote the second endpoint of P'.

Now, since G has no clique cutsets of size 2, the graph $G - \{y_1, y_2\}$ is connected. Let P'' be a shortest path between $V(P') \setminus \{y_1, y_2\}$ and $A' \setminus \{y_1, y_2\}$ in $G - \{y_1, y_2\}$. Let w be the end vertex of P'' in $V(P') \setminus \{y_1, y_2\}$ and y_3 be the end vertex of P'' in $A' \setminus \{y_1, y_2\}$.

Then, the graph $H = G[A' \cup V(P') \cup V(P'')]$ satisfies the conditions of Lemma 9 with A being A', and Q_1, Q_2 and Q_3 being the subpath of P' from w to y_1, the subpath of P' from w to y_2 and P'', respectively. Therefore, $G \xrightarrow{\cap} H \xrightarrow{\cap} tS_{t,t,t}$.

This completes the proof of Theorem 11. ◀

3.3 The Structure of Graphs Excluding a Tripod

In this section, we prove Theorem 3. In addition to Theorems 4 and 11, established in the previous sections, we require two more ingredients. One of them is the following implication of the characterization of bounded treewidth in finitely defined hereditary classes.

▶ **Theorem 12** (Lozin and Razgon [22]). *Let $r \in \mathbb{N}$, and let $\mathcal{X}$ be a hereditary graph class that excludes K_r, $K_{r,r}$, $rS_{r,r,r}$, and $L(rS_{r,r,r})$ as induced subgraphs. Then there exists a constant $c \in \mathbb{N}$ such that $\operatorname{tw}(G) \leq c$ for every $G \in \mathcal{X}$.*

The remaining ingredient is the following observation that the line graph of $tS_{q,q,q}$ can be self-intersected to $tS_{q-1,q-1,q}$ (see also Figure 1 and see [10] for a proof).

▶ **Lemma 13.** *Let $t \geq 1$ and $q \geq 2$. Then $L(tS_{q,q,q}) \xrightarrow{\cap} tS_{q-1,q-1,q}$.*

With these tools at hand we can now prove Theorem 3, which we restate for convenience.

▶ **Theorem 3.** *Let $t \in \mathbb{N}$. Then, there exists $c \in \mathbb{N}$ such that if a graph $G \in \mathcal{X}_t$ contains neither false twins nor a clique cutset of size at most 2, then G is complete or $\operatorname{tw}(G) \leq c$.*

Proof. Let $c \in \mathbb{N}$ be a constant given by Theorem 12 for $r = 3t^2 + t + 2$. Let G be a graph in $\mathcal{X}_t$ with at least two vertices, and assume that G contains neither false twins nor clique cutsets of size at most 2. If G is complete, the claim holds trivially. Thus, assume that G is not complete. We will show that $\operatorname{tw}(G) \leq c$.

First, by Theorem 11, the graph G is K_r-free. Second, we claim that G is $K_{r,r}$-free. Indeed, by Theorem 4, any maximal induced complete bipartite graph in G with parts of size $p, q \geq r$ would be a module in G, and thus any two vertices in the same part of such a complete bipartite graph would be false twins in G, which is not possible by our assumption. Finally, due to Lemma 13, G contains no $L(tS_{t+1,t+1,t+1})$, and in particular no $L(rS_{r,r,r})$, as an induced subgraph. Consequently, G contains no K_r, $K_{r,r}$, $L(rS_{r,r,r})$, or $rS_{r,r,r}$ as induced subgraphs, and therefore $\operatorname{tw}(G) \leq c$, by Theorem 12. ◀

4 Algorithmic and Structural Implications

In this section we provide several applications of our main structural result. In particular, we establish a number of dichotomies in finitely-defined self-intersection-closed graph classes.

To prove these dichotomies, we make use of the following hereditary graph classes. Recall that $\mathcal{S}$ denotes the class of tripods. It is known and easy to see that $\mathcal{S}$ is the limit (i.e. the intersection) of the infinite sequence of classes $\mathcal{S}_k \supset \mathcal{S}_{k+1} \supset \mathcal{S}_{k+2} \supset \cdots$, where $\mathcal{S}_k = \text{Subgraph-Free}(C_3, \ldots, C_k, H_0, H_1, \ldots, H_k)$ for $k \geq 3$, and H_k denotes the graph shown in Figure 4, where $H_0 = K_{1,4}$. Note that since $\mathcal{S}_k$ is monotone, it is also self-intersection-closed.

▶ **Observation 14.** *For every integer $k \geq 3$, there exists an integer $k' \geq k$ such that $\mathcal{S}_k$ contains $\text{Free}(C_3, \ldots, C_{k'}, H_0, H_1, \ldots, H_{k'})$.*

The importance of the classes $\mathcal{S}_k$ lies in the fact that various graph parameters (e.g. treewidth and clique-width) are unbounded and many algorithmic graph problems (e.g. Maximum Independent Set and Maximum Induced Matching) are NP-hard on these classes (see, e.g. [16]). In the following lemma, we show that if a finitely-defined, self-intersection-closed class does not exclude any tripod, then it contains one of the classes $\mathcal{S}_k$, and thus inherits their structural and algorithmic complexity (the proof of this lemma has been omitted but can be found in [10]).

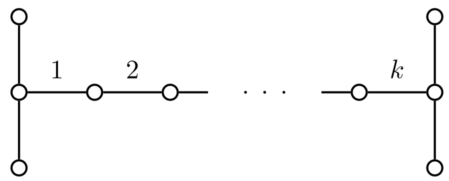

Figure 4 The graph H_k.

▶ **Lemma 15.** *Let M be a finite set of graphs none of which is a tripod, i.e. $M \cap \mathcal{S} = \emptyset$. Then there exists $r \in \mathbb{N}$ such that $\mathcal{S}_r \subseteq \text{Si-Free}(M)$.*

In Sections 4.1–4.4, we will use our structural results about the classes $\mathcal{X}_t = \text{Si-Free}(tS_{t,t,t})$, $t \geq 1$, to show that if M contains at least one tripod, the class Si-Free(M) becomes simple with respect to various structural parameters and algorithmic problems.

4.1 Dichotomy for Maximum (Weight) Independent Set

We prove the following dichotomy.

▶ **Theorem 16.** *Let M be a finite set of graphs and let $\mathcal{Y} = \text{Si-Free}(M)$. Then,* Maximum Independent Set *and* Maximum Weight Independent Set *are polynomial-time solvable on the class $\mathcal{Y}$ if $M \cap \mathcal{S} \neq \emptyset$, and* NP*-hard otherwise.*

Proof. Suppose first that $M \cap \mathcal{S} = \emptyset$. Then, by Lemma 15, $\mathcal{Y}$ contains $\mathcal{S}_r$ for some $r \geq 3$. This implies that Maximum Independent Set (and therefore Maximum Weight Independent Set) is NP-hard on $\mathcal{Y}$, because it is NP-hard on $\mathcal{S}_r$ for all $r \geq 3$ as a consequence of [18, Proposition 2] and Observation 14.

Now suppose that M contains a graph H from $\mathcal{S}$. Without loss of generality, we may assume that $H = tS_{t,t,t}$ for some $t \in \mathbb{N}$. Let G be an arbitrary graph in Si-Free($tS_{t,t,t}$). We may assume that G is not complete, as otherwise Maximum Weight Independent Set is trivial on G. Combining modular decomposition and decomposition by clique separators as in [9], we may further assume that G has no modules other than $V(G)$ and $\{v\}$, $v \in V(G)$, and has no clique cutsets. Then, from Theorems 4 and 11, we conclude that G contains neither large bicliques nor large cliques as induced subgraphs. It also does not contain the line graph of $tS_{t+1,t+1,t+1}$, because this graph can be self-intersected to $tS_{t,t,t}$ by Lemma 13. Therefore, due to Theorem 12, the treewidth of G is bounded. Consequently, by [6, 7], Maximum Weight Independent Set can be solved in polynomial time on $\mathcal{Y}$. ◀

In the rest of this section, we use $\mathcal{B}$ to denote the class of bipartite graphs.

4.2 Clique-width Dichotomy on Bipartite Graphs

We now prove our second dichotomy.

▶ **Theorem 17.** *Let M be a finite set of bipartite graphs and let $\mathcal{Y} = \text{Si-Free}(M) \cap \mathcal{B}$. Then $\mathcal{Y}$ has bounded clique-width if and only if $M \cap \mathcal{S} \neq \emptyset$.*

Proof. Suppose that $M \cap \mathcal{S} = \emptyset$. Then, by Lemma 15, $\mathcal{Y}$ contains $\mathcal{S}_k \cap \mathcal{B}$ for some $k \geq 3$. This implies that $\mathcal{Y}$ has unbounded clique-width, because bipartite graphs in $\mathcal{S}_k$ have unbounded clique-width for all $k \geq 3$ (see, e.g. [21]).

Now suppose that $M \cap \mathcal{S} \neq \emptyset$. Then, for some constant $t \in \mathbb{N}$, it follows that $\mathcal{Y} \subseteq \mathcal{X}_t \cap \mathcal{B}$. By Lemma 1, graphs in $\mathcal{Y}$ do not contain $tS_{t,t,t}$ as an induced subgraph. Furthermore, since graphs in $\mathcal{Y}$ are bipartite, they do not contain K_3, and therefore do not contain $L(S_{1,1,1})$, the line graph of $S_{1,1,1}$.

Now, if a biconnected graph G in $\mathcal{Y}$ contains an induced biclique $K_{p,q}$ with $p, q \geq 3t^2+t+1$, then, by Theorem 4, the vertex set of any maximal extension H of this biclique is a module in G. This implies that $G = H$ is a biclique, since otherwise any vertex in $V(G) \setminus V(H)$ that has a neighbour in H is complete to $V(H)$ and would hence lead to a K_3 in G, contradicting the fact that G is bipartite. Thus, any biconnected graph in $\mathcal{Y}$ is either a biclique or belongs to $\mathrm{Free}(tS_{t,t,t}, L(S_{1,1,1}), K_3, K_{p,p})$, where $p = 3t^2 + t + 1$. Since the latter class has bounded treewidth (by Theorem 12), and bicliques have clique-width 2, we conclude that the class of biconnected graphs in $\mathcal{Y}$ has bounded clique-width. The result now follows from the fact that, in any hereditary graph class, boundedness of clique-width in the subclass of biconnected graphs implies boundedness of clique-width of the entire class [20, Proposition 2]. ◀

4.3 Dichotomy for (Counting) Satisfiability

The *incidence graph of a CNF formula* φ is a bipartite graph G_φ, one part of which represents clauses and the other represents variables. The edges of G_φ connect variables to the clauses containing them (positively or negatively).

Our third dichotomy is as follows.

▶ **Theorem 18.** *Let M be a finite set of bipartite graphs and let $\mathcal{Y} = \text{Si-Free}(M) \cap \mathcal{B}$.*

(1) *If $M \cap \mathcal{S} = \emptyset$, then* Satisfiability *is* NP*-complete on instances with incidence graphs in $\mathcal{Y}$.*

(2) *If $M \cap \mathcal{S} \neq \emptyset$, then* Counting Satisfiability *is polynomial-time solvable on instances with incidence graphs in $\mathcal{Y}$.*

Proof. Suppose that $M \cap \mathcal{S} = \emptyset$. Then, by Lemma 15, $\mathcal{Y}$ contains $\mathcal{S}_k \cap \mathcal{B}$ for some $k \geq 3$. This immediately implies (1), as Satisfiability is NP-complete on instances with incidence graphs in the class $\mathcal{S}_k \cap \mathcal{B}$ for any $k \geq 3$, as a consequence of [19, Lemma 3] and Observation 14. Now suppose that $M \cap \mathcal{S} \neq \emptyset$. Then, by Theorem 17, the class $\mathcal{Y}$ has bounded clique-width, and therefore, according to [24], Counting Satisfiability can be solved in polynomial time on instances with incidence graphs in $\mathcal{Y}$. ◀

4.4 Dichotomy for Maximum Induced Matching on Bipartite Graphs

We now give our last dichotomy. Its proof has been omitted, but can be found in [10].

▶ **Theorem 19.** *Let M be a finite set of bipartite graphs and let $\mathcal{Y} = \text{Si-Free}(M) \cap \mathcal{B}$. Then* Maximum Induced Matching *is polynomial-time solvable on the class $\mathcal{Y}$ if $M \cap \mathcal{S} \neq \emptyset$, and* NP*-hard otherwise.*

5 Future Directions

Our paper not only extended previously known results, but also opens up a variety of new research directions. First, it would be natural to explore other algorithmic problems for which the tripods constitute the only critical property in the universe of monotone classes and determine whether the structural characterization of self-intersection-closed classes excluding a tripod can be applied to develop efficient algorithms for such problems.

Second, can we extend our dichotomy for classes of bipartite graphs of bounded clique-width defined by finitely many forbidden si-subgraphs to *all* classes defined by finitely many forbidden si-subgraphs? We conjecture that this is not possible, i.e. that there exist graph classes excluding a tripod as an si-subgraph of unbounded clique-width. On the other hand, we believe all graph classes excluding a tripod as an si-subgraph have bounded twin-width.

Third, can we extend the dichotomy for MAXIMUM INDUCED MATCHING on classes of bipartite graphs defined by finitely many forbidden bipartite si-subgraphs both to *all* classes defined by finitely many forbidden si-subgraphs and to the weighted case of the problem?

A fourth research direction is to consider the complexity of NP-hard algorithmic problems on classes defined by infinitely many forbidden si-subgraphs. For problems for which $\mathcal{S}$ is a critical class, polynomial-time solvability can only be obtained by forbidding a graph from $\mathcal{S}$ or by forbidding infinitely many graphs from the canonical antichain defining $\mathcal{S}$ (unless $\mathsf{P} = \mathsf{NP}$). This antichain consists of the cycles and the so-called H-graphs (see Figure 4). Therefore, one of the natural next steps is exploring the structure of self-intersection-closed classes excluding infinitely many cycles or H-graphs. Finally, do there exist other interesting families of classes intermediate between monotone and hereditary classes?

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