Optimal b-Colourings and Fall Colourings in H-Free Graphs

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Abstract

In a colouring of a graph, a vertex is *b-chromatic* if it is adjacent to a vertex of every other colour. We consider four well-studied colouring problems: b-Chromatic Number, Tight b-Chromatic Number, Fall Chromatic Number and Fall Achromatic Number, which fit into a framework based on whether every colour class has (i) at least one b-chromatic vertex, (ii) exactly one b-chromatic vertex, or (iii) all of its vertices being b-chromatic. By combining known and new results, we fully classify the computational complexity of b-Chromatic Number, Fall Chromatic Number and Fall Achromatic Number in H-free graphs. For Tight b-Chromatic Number in H-free graphs, we develop a general technique to determine new graphs H, for which the problem is polynomial-time solvable, and we also determine new graphs H, for which the problem is still NP-complete. We show, for the first time, the existence of a graph H such that in H-free graphs, b-Chromatic Number is NP-hard, while Tight b-Chromatic Number is polynomial-time solvable.

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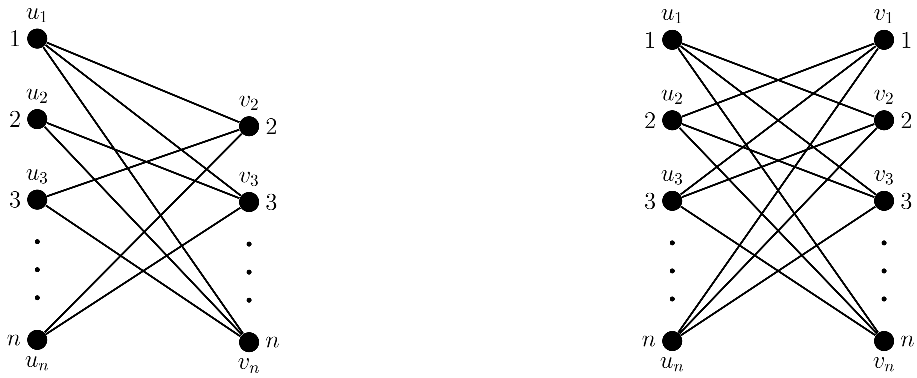

Figure 1 An example of a b-colouring of a $(2n-1)$-vertex complete bipartite graph minus a matching of size $n-1$ (left) with n colours that is not a fall colouring, and an example of a fall colouring of a $2n$-vertex complete bipartite graph minus a perfect matching (right) with n colours.

1 Introduction

Colouring is an extensively studied area of graph theory, mostly focussing on minimising the number of colours used [27, 35]. Namely, a *colouring* in a graph G is a mapping $c : V \to \mathbb{Z}^+$ such that $c(u) \neq c(v)$ for every edge $uv \in E$. If $|c(V)| = k$, then c is a *k-colouring*, and the *chromatic number* is the smallest k such that G has a k-colouring. However, there have also been well-studied graph colouring concepts that involve maximising the number of colours k subject to additional conditions, such as the achromatic number [29], b-chromatic number [22, 25], Grundy number [4] and fall achromatic number [12]. In particular, the *b-chromatic number* of a graph G is the maximum integer k for which G admits a k-colouring where every colour i has a vertex v_i of colour i that is adjacent to a vertex of every colour $j \neq i$. Such a vertex v_i is called *b-chromatic*, and a colouring where every colour class has a b-chromatic vertex is called a *b-colouring*. See Figure 1 for an example of a b-colouring.

The b-chromatic number was introduced by Irving and Manlove in 1999 [22] as the worst-case behaviour of a natural heuristic for approximating the chromatic number of a graph G. Starting with an arbitrary colouring, we iteratively try to improve it: at each step, pick a colour i and try to reduce the number of colours by recolouring each vertex v of colour i by a colour of G that is not in v's neighbourhood. If such a recolouring is not possible for colour i, try another colour. The process terminates when every colour has a b-chromatic vertex, and also shows that every graph has a b-colouring. Other applications of the b-chromatic number arise in graph clustering [14], postal mail sorting [17] and analysing hospital stays [15].

In this paper we study colourings with b-chromatic vertices: in particular, we consider the three natural cases where each colour class has (i) at least one b-chromatic vertex, (ii) exactly one b-chromatic vertex, and (iii) all of its vertices being b-chromatic. From Case (i) we derive the b-Chromatic Number problem, which is the problem of finding the b-chromatic number of a given graph. Case (ii) gives rise to the Tight b-Chromatic Number [20, 28, 37], which is the restriction of b-Chromatic Number to instances where the target number of colours is the maximum degree of the input graph and all maximum-degree vertices must become b-chromatic. Case (iii) is precisely the definition of a *fall colouring* of a graph [12].

A fall colouring is a b-colouring, but the converse is not necessarily true; see Figure 1 for an example. In fact, many graphs have no fall colouring; an example being the *paw*, the triangle with a pendant vertex. The problems Fall Chromatic Number and Fall Achromatic Number are to find the minimum and maximum integers k for which a given graph G admits a fall colouring with k colours (or 0 if G has no fall colouring).

Our focus. Forbidden subgraphs have been widely studied from a structural and algorithmic perspective [7], and in particular for graph colouring [18]. For a graph H, a graph G is H*-free* if G does not contain H as an *induced* subgraph. To increase our understanding of an NP-hard problem Π, we may aim to identify graphs H for which Π is polynomial-time solvable or NP-hard in H-free graphs, ideally to obtain a complete dichotomy for the computational complexity of Π in H-free graphs. This has been done for several well-known colouring problems, such as Chromatic Number [31], Precolouring Extension [19], ℓ-List Colouring [19], and apart from one missing case, for Acyclic Colouring, Star Colouring and Injective Colouring [3]. We continue this line of study:

Can we classify the complexity of b-Chromatic Number, Tight b-Chromatic Number, Fall Chromatic Number *and* Fall Achromatic Number *in* H*-free graphs?*

As we shall see, our systematic approach will allow us to unify and extend known results, and to identify general patterns of complexity behaviour and new interesting open cases.

Terminology and bounds. Before discussing known complexity results and our new results, we must first introduce some new terminology and basic observations. A fundamental upper bound for the b-chromatic number involves the *m-degree* of a graph G [22], denoted by $m(G)$, which is the largest integer k such that G has at least k vertices with degree at least $k-1$. Let $\chi(G)$ and $\varphi(G)$ denote the chromatic and b-chromatic number, respectively, of G. Then, for a graph G, it holds that $\chi(G) \leq \varphi(G) \leq m(G)$ [22].

A vertex of G with degree at least $m(G)-1$ is said to be *dense*, and G is *tight* [20, 37] (referred to as *m-tight* in [37]; see also [30, 36]) if G has exactly $m(G)$ dense vertices, each of which has degree (exactly) $m(G)-1$. For example, the 4-vertex cycle C_4 is not tight, but the triangle C_3 is tight (both graphs have m-degree 3). This leads to the Tight b-Chromatic Number problem, formally introduced by Havet, Sales and Sampaio [20], which is to decide if a tight graph G has a *tight b*-colouring, which is a b-colouring with $m(G)$ colours.

The *fall spectrum* $\mathcal{F}(G)$ of a graph G (also called the *fall set* of G) is the set of integers k for which G admits a *fall k-colouring*, which is a fall colouring with exactly k colours. The *fall chromatic number* $\chi_f(G)$ is the smallest integer in $\mathcal{F}(G)$. The *fall achromatic number* $\varphi_f(G)$ is the largest integer in $\mathcal{F}(G)$. We make the following known observation (see e.g. [43]):

▶ **Observation 1.** *For a graph G with $\mathcal{F}(G) \neq \emptyset$, it holds that $\chi(G) \leq \chi_f(G) \leq \varphi_f(G) \leq \delta(G)+1$, where $\delta(G)$ is the minimum degree of a vertex in G.*

1.1 Known Complexity Results

The problems b-Chromatic Number, Fall Chromatic Number and Fall Achromatic Number are all NP-hard. For b-Chromatic Number, this was originally shown in [22]. Later, it was proven that Tight b-Chromatic Number, and thus b-Chromatic Number, are NP-hard for line graphs [8][1], bipartite graphs [32, 39] and for chordal graphs, or more precisely, disjoint unions of a connected split graph and three stars [20]. In addition, b-Chromatic Number is also NP-hard for co-bipartite graphs [6]. In contrast, Tight b-Chromatic Number is polynomial-time solvable for co-bipartite graphs, block graphs and P_4-sparse graphs [20]. The last result was known for b-Chromatic Number [5], but tight P_4-sparse graphs allow for a faster algorithm [20]. Later, it was shown that b-Chromatic Number is polynomial-time solvable for graphs of girth at least 7 [9] and for a superclass of

[1] Campos et al. [8] state this result for b-Chromatic Number, but their proof holds in fact for Tight b-Chromatic Number.

P_4-sparse graphs: P_4-tidy graphs [46]. We note that P_4-tidy graphs have bounded clique-width [10]. More recently, Jaffke, Lima and Lokshtanov [24] proved that b-Chromatic Number is polynomial-time solvable for every graph class of bounded clique-width [24], by showing that this holds even for b-Colouring, which is the problem to decide if a graph has a b-colouring with exactly k colours for some given integer k. Jaffke and Lima [23] established a range of complexity results for b-Colouring.

Jaffke, Lima and Lokshtanov [24] also proved that Fall Colouring, which is to decide if a graph has a fall k-colouring for some given integer k, is polynomial-time solvable for graphs of bounded clique-width. Consequently, the same holds for Fall Chromatic Number and Fall Achromatic Number. Their result generalised results of Silva [43] for P_4-sparse graphs and Mitillos [40] for threshold graphs. The Fall Colouring problem, and therefore Fall Chromatic Number and Fall Achromatic Number, can also be solved in polynomial time for split graphs [40] and strongly chordal graphs [38]. It was shown that for a graph G that is threshold, split or strong chordal, G has a fall colouring if and only if $\chi(G) = \delta(G) + 1$, and thus by Observation 1, $\mathcal{F}(G) = \{\chi(G)\}$, or equivalently, $\chi_f(G) = \varphi_f(G) = \chi(G)$. We say that graphs whose fall spectrum consists of a singleton are *fall-unique*. Silva [43] proved that chordal graphs with a fall colouring are fall-unique, but that deciding non-emptiness of the fall spectrum is NP-complete for chordal graphs.

If the number of colours k is a fixed constant, then we obtain Fall k-Colouring. Already in 1998, Heggernes and Telle [21] proved that for every fixed $k \geq 3$, Fall k-Colouring is NP-complete. This was well before the notion of a fall colouring was introduced by Dunbar et al. [12]. The reason is that a graph $G = (V, E)$ has a fall k-colouring if and only if V can be partitioned into k independent dominating sets (or equivalently, k maximal independent sets). As such, Fall k-Colouring belongs to the widely studied framework of locally checkable vertex partitioning problems introduced by Telle [44, 45]. Subsequently, Laskar and Lyle [33] proved that Fall 3-Colouring is NP-complete for bipartite graphs, which was improved to planar bipartite graphs by Lauri and Mitillos [34]. The latter authors [34] also proved that for every $k \geq 3$, Fall k-Colouring is NP-complete for $(2k-2)$-regular line graphs.[2] As we will show later, all the above hardness results can be modified to hold for Fall Chromatic Number and Fall Achromatic Number as well by a simple trick.

1.2 Our Results

In our theorems, we let $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ denote the disjoint union of two vertex-disjoint graphs, and we let rG denote the disjoint union of r copies of G. We write $G_1 \subseteq_i G_2$ if G_1 is an induced subgraph of G_2, and let P_r be the path on r vertices.

We can now state a dichotomy for b-Chromatic Number in H-free graphs. We obtain this dichotomy using the aforementioned results from [5, 6, 32, 39] after observing that the hardness gadget of Bonomo et al. [6] is also $2P_2$-free; see Section 3 of the full version of this paper [1] for details.

▶ **Theorem 2.** *For a graph H,* b-Chromatic Number *in H-free graphs is polynomial-time solvable if $H \subseteq_i P_4$, and* NP*-hard otherwise.*

It is well known that for a graph H, the class of H-free graphs has bounded clique-width if and only if $H \subseteq_i P_4$ (see e.g. [11]). Hence, an alternative formulation of Theorem 2 is that subject to P $\neq$ NP, b-Chromatic Number is polynomial-time solvable in H-free graphs

[2] Theorem 15 in [34] is stated only for $(2k-2)$-regular graphs, but holds in fact for $(2k-2)$-regular line graphs, as explicitly stated in the first paragraph of the proof.

if and only if the class of H-free graphs has bounded clique-width. Our next result shows that for tight graphs this correspondence with clique-width no longer holds. We prove it in Section 3 by combining previous results from [5, 8, 20, 32, 39] with two new polynomial-time algorithms and two new NP-completeness results.

▶ **Theorem 3.** *For a graph H,* Tight b-Chromatic Number *in H-free graphs is*

- *polynomial-time solvable if $H \subseteq_i P_4$, $H \subseteq_i P_3 + P_1$ or $H \subseteq_i 2P_2 + P_1$ and*
- NP*-complete if H is not a linear forest, or $P_5 \subseteq_i H$, $2P_3 \subseteq_i H$ or $3P_2 \subseteq_i H$.*

Theorem 3 shows, together with Theorem 2, that there exist graphs H, namely $H = P_3 + P_1$, $H = 2P_2$, $H = 2P_2 + P_1$, $H = P_2 + 2P_1$ and $H = 3P_1$, for which b-Chromatic Number in H-free graphs is NP-hard, while Tight b-Chromatic Number is polynomial-time solvable. This difference in complexity for H-free graphs was not known before. All the new polynomial-time cases in Theorem 3 are for graph classes of unbounded clique-width. They generalise the aforementioned result for co-bipartite graphs [20], which are $3P_1$-free. See Section 5 for a discussion on the missing cases.

Our final result, Theorem 4, shown in Section 4, consists of two coinciding dichotomies for fall colourings, which equate to the dichotomy for Chromatic Number [31], but differ from the dichotomy in Theorem 2 and the partial dichotomy in Theorem 3. To prove it, we combine the results from [33, 34, 43] with a new hardness result for $(C_5, 2P_2, P_2 + 2P_1, 4P_1)$-free graphs and a new polynomial-time solvability result for $(P_3 + P_1)$-free graphs.

▶ **Theorem 4.** *For a graph H,* Fall Chromatic Number *and* Fall Achromatic Number *in H-free graphs are polynomial-time solvable if $H \subseteq_i P_4$ or $P_3 + P_1$, and* NP*-hard otherwise.*

Methodology. The new polynomial-time case in Theorem 4 follows by using a result of Olariu [41]. To prove the new polynomial-time results in Theorem 3, we could not make use of previous tools, such as the concepts of *b-closure* and *partial b-closure* as defined and used by Havet, Sales and Sampaio [20] to obtain polynomial-time algorithms for block graphs, co-bipartite graphs and P_4-sparse graphs. The b-closure and partial b-closure are obtained by adding edges within the set of dense vertices and/or within the set of their neighbours, in order to reduce to polynomial-time cases of Chromatic Number or Precolouring Extension. However, adding new edges to the input graph G may destroy the H-free property of G. Moreover, Chromatic Number and Precolouring Extension are NP-hard if H is an induced subgraph of neither $P_1 + P_3$ nor P_4 [31]. To circumvent this issue, we still exploit the idea behind precolouring extension and introduce the new technique of *tight b-precolouring extension.* We first construct a partial colouring c' of the input graph G that satisfies some additional properties. Afterwards, we will then try to extend c to a tight b-colouring of G.

2 Preliminaries

Let $G = (V, E)$ be a graph. For a set $S \subseteq V$, we let $G[S]$ denote the subgraph of G induced by S, and we write $G - S = G[V \setminus S]$. The *neighbourhood* of a vertex $u \in V$ is the set $N(u) = \{v \mid uv \in E\}$. The *degree* of u, denoted by $d(u)$, is equal to $|N(u)|$, and G is r-*regular* if every vertex of G has degree r; if $r = 3$, then G is said to be *cubic*. For a set $S \subseteq V$, the *outer boundary of* S [2] is defined as $\delta S = (\bigcup_{u \in S} N(u)) \setminus S$. Two disjoint vertex subsets S and T in G are *complete* to each other if every vertex of S is adjacent to every vertex of T. A set $D \subseteq V$ is *dominating* if every vertex in $V \setminus D$ has at least one neighbour in D. If $D = \{u\}$ for some $u \in V$, then u is a *dominating* vertex of G. For a colouring c of G, we say that $c(u)$

is the *colour* of u and that all vertices mapped by c to some specific colour i form a *colour class*. Note that the colour classes of c are independent sets that partition V. We say that G is $(H_1, \dots, H_r)$*-free* for some set of graphs $\{H_1, \dots, H_r\}$ if G is H_i-free for $i \in \{1, \dots, r\}$.

A *forest* is a graph with no cycles. A forest is *linear* if all its connected components are paths. We let C_r and K_r denote the cycle and complete graph, respectively, on r vertices. We also let $K_{1,r}$ denote the star on $r+1$ vertices; the graph $K_{1,3}$ is also known as the *claw*.

A *split graph* is a graph whose vertex set can be partitioned into a clique K and independent set I (where we allow K or I to be empty). It is well known that a graph is split if and only if it is $(C_4, C_5, 2P_2)$-free. A graph is *chordal* if it is $(C_4, C_5, \dots)$-free. A graph G is P_4*-sparse* if every set of five vertices in G induces at most one P_4. Note that P_4-free graphs are P_4-sparse. The *line graph* $L(G)$ of a graph $G = (V, E)$ has as vertices the edges from E, and there exists an edge between two distinct vertices e and f of $L(G)$ if and only if e and f share a common end-vertex in G. It is readily seen that every line graph is claw-free.

A graph $G = (V, E)$ is *complete multi-partite* if V can be partitioned into independent sets $V_1, \dots, V_r$ for some $r \geq 1$ such that for every i, j with $i \neq j$, there exists an edge between every vertex of V_i and every vertex of V_j. Recall that the *paw* is the graph on vertices u_1, u_2, u_3, v_1 and edges u_1u_2, u_2u_3, u_3u_1, u_1v_1. Olariu [41] proved that a graph G is paw-free if and only if each connected component of G is C_3-free or complete multi-partite. We let $\overline{G}$ denote the *complement* of a graph G, which is obtained from G by replacing each edge in G with a non-edge, and vice versa. Note that $\overline{C_3} = 3P_1$ and $\overline{\text{paw}} = P_3 + P_1$. A *co-component* of G is a (connected) component of $\overline{G}$, and a *co-bipartite* graph is the complement of a bipartite graph. We can now formulate the result of Olariu as:

▶ **Lemma 5** ([41]). *A graph G is $(P_3 + P_1)$-free if and only if each co-component of G is $3P_1$-free or a disjoint union of complete graphs.*

3 The Proof of Theorem 3

We start with a proposition (proved in [1]) that follows from the definitions of the concepts involved.

▶ **Proposition 6.** *Let G be a tight graph, and let T be the set of dense vertices of G. In every tight b-colouring c of G (if one exists), the following holds:*

(i) *the set T is exactly the set of b-chromatic vertices under c; and*

(ii) *each colour class of c has exactly one vertex of T, and this vertex has a unique neighbour in each of the other colour classes of c.*

(iii) *every vertex of $T_1 = \{v \in T : v \text{ is a dominating vertex of } G[T]\}$ belongs to a colour class of c that does not contain any vertex of δT.*

We now introduce two of our new key concepts, which are related to each other.

▶ **Definition 7.** *Let G be a tight graph with a set T of dense vertices. For a subset $S' \subseteq \delta T$, an S'-*partial b-colouring *of G is a colouring c' of $G[S' \cup T]$ if:*

1. *$c'(u) \neq c'(u')$ for every two distinct vertices $u, u' \in T$, and*
2. *for every $u \in T$ with $c'(u) = c'(s)$ for some $s \in S'$, it holds that every vertex of $T \setminus \{u\}$ has exactly one neighbour with colour $c'(u)$.*

▶ **Definition 8.** *Let G be a tight graph with a set T of dense vertices. A b-colouring c of G is a* b-precolouring extension *of an S'-partial b-colouring c' in G if:*

1. *$c(v) = c'(v)$ for every $v \in S' \cup T$ (that is, c is an extension of c'), and*
2. *every colour class of c with at least two vertices of δT contains no vertex of $\delta T \setminus S'$.*

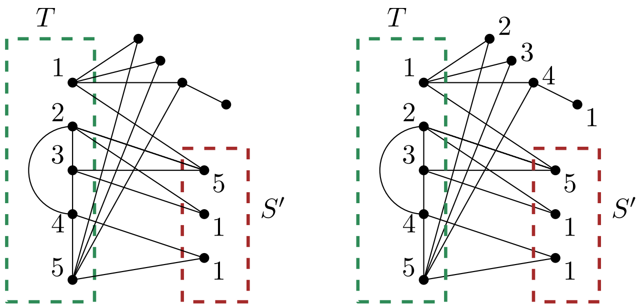

Figure 2 On the left: a tight graph G with a set T of dense vertices, where G is annotated with an S'-partial b-colouring c' (as per Definition 7) using colours $\{1,\ldots,5\}$. On the right: a b-precolouring extension of c' (as per Definition 8).

An S'-partial b-colouring c' and a b-precolouring extension of c' are illustrated for a tight graph in Figure 2. The following observation (proved in [1]) is a straightforward corollary of Definition 7.

▶ **Observation 9.** *Let G be a tight graph. If c is a b-precolouring extension of an S-partial b-colouring c' in G, then c is a tight b-colouring of G.*

We now present a polynomial-time algorithm for finding b-precolouring extensions.

▶ **Theorem 10.** *Let T be the set of dense vertices of a tight graph G, and let $S' \subseteq \delta T$, such that G has an S'-partial b-colouring c'. It is possible to determine in polynomial time if c' has a b-precolouring extension in G.*

Proof. Let $m = m(G)$, and let $T = \{u_1, \ldots, u_m\}$. As c' is an S'-partial b-colouring of G, we may assume that $c'(u_i) = i$ for every $i \in \{1, \ldots, m\}$. We let T' denote the set of vertices in T that have the same colour as a vertex of S'. We let $T_1 = \{v \in T : v \text{ is a dominating vertex of } G[T]\}$, so for all $u \in T_1$, every vertex of $T \setminus \{u\}$ is adjacent to u.

We claim that $T_1 \cap T' = \emptyset$. For a contradiction, suppose that there exists a vertex $u \in T_1 \cap T'$. Then, as $u \in T'$, there exists a vertex $s \in S'$ with colour $c'(u)$. Since u and s have the same colour, u and s are not adjacent. As $S' \subseteq \delta T$, this means that there exists a vertex $u' \in T \setminus \{u\}$ that is adjacent to s. However, now u' is adjacent to both u and s, contradicting the fact that c' is an S'-partial b-colouring of G.

We set $T_2 = T \setminus (T_1 \cup T')$. Note that $T = T_1 \cup T_2 \cup T'$, and that the sets T_1, T_2 and T' are pairwise disjoint. Moreover, we can determine T_1, T_2 and T' in polynomial time.

First, we assume that $T_2 = \emptyset$, so $T = T_1 \cup T'$. We check, in polynomial time, if $S' = \delta T$.

Case 1: $S' \neq \delta T$. We claim that c' has no b-precolouring extension in G. For a contradiction, suppose that c is a b-precolouring extension of c' in G. As $S' \subseteq \delta T$ and $S' \neq \delta T$, we find that $\delta T \setminus S'$ contains a vertex s. As c is a b-precolouring extension, c is a tight b-colouring of G by Observation 9. Hence, from Proposition 6 (iii) it follows that $c(s)$ is not a colour of a vertex in T_1. We recall that $1 \leq c(s) \leq m$ and that by definition, c' uses all m colours on the m vertices of T. Hence, $c(s) = c(u)$ for some vertex $u \in T'$. From the definition of T' it follows that $c'(u) = c'(s')$ for some $s' \in S'$. However, as c extends c', we also have that $c(s) = c(u) = c'(u) = c'(s') = c(s')$. In other words, c has a colour class with at least two vertices of δT, namely s and s', that contains a vertex, namely s, of $\delta T \setminus S'$. This contradicts our assumption that c is a b-precolouring extension of c' in G.

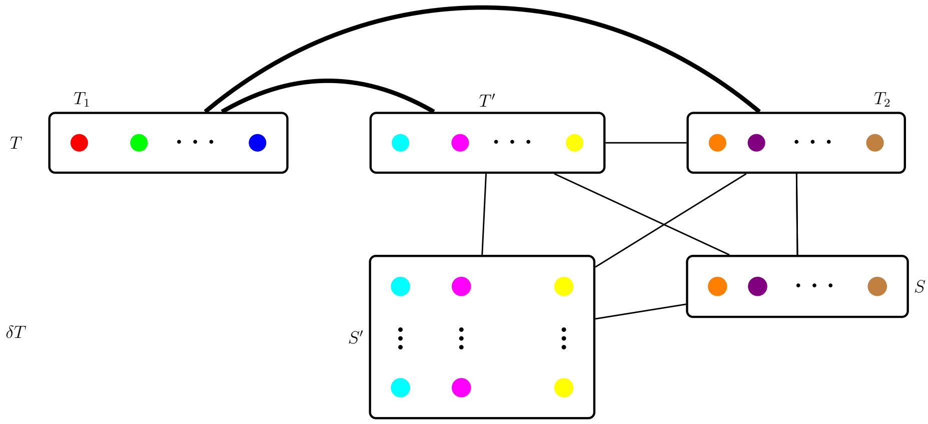

Figure 3 The sets T and δT from the proof of Theorem 10, where T is the disjoint union of T_1, T' and T_2, and δT is the disjoint union of S' and S. Thick edges indicate that the set T_1 is complete to both T' and T_2. Thin edges between sets indicate edges that may or may not exist between vertex pairs. Note that T_1 is a clique and that no edges exist between T_1 and δT. All "vertical layers" are monochromatic and given a unique colour. That is, the S'-partial b-colouring c' colours the vertices of $T \cup S'$, where each vertex in T' has at least one non-neighbour in S' of the same colour by definition. As displayed in the figure and shown in the proof, the vertices in T_2 and S must be in one-to-one correspondence in any b-colouring extension of G (if there exists one).

Case 2: $S' = \delta T$. Let $u \in T$. As c' is an S'-partial b-colouring, u has exactly one neighbour with colour $c'(u')$ for every $u' \in T' \setminus \{u\}$ by definition. As $T_1 \cap T' = \emptyset$ and every vertex of T_1 dominates $G[T]$, we also find that u has exactly one neighbour with colour $c'(u')$ for every $u' \in T_1 \setminus \{u\}$. As c' uses m colours on $G[T]$, we conclude that c' is a tight b-colouring of $G[T \cup S'] = G[T \cup \delta T]$.

Due to the above, we can extend c' to a b-precolouring extension c in G as follows. Set $c(u) = c'(u)$ for all $u \in T \cup \delta T$. As every vertex $w \in V(G) \setminus (T \cup \delta T)$ is not dense, w has degree at most $m-2$ in G. Hence, we can greedily colour (in c) the vertices of $V(G) \setminus (T \cup \delta T)$ one by one. Since $\delta T \setminus S' = \emptyset$, the second property of Definition 8 is now satisfied for c as well. Moreover, we found c in polynomial time.

From now on, suppose that $T_2 \neq \emptyset$. We let $S = \delta T \setminus S'$. At this point, we refer to Figure 3 for an illustration of the sets T and δT. From G we define a bipartite graph G^* as follows:

- let T_2 and S be the two partition classes of G^*, and
- add an edge between a vertex $u \in T_2$ and a vertex $s \in S$ if and only if $us \notin E(G)$ and moreover, u and s have no common neighbour in $(T_2 \cup T') \setminus \{u\}$ in G.

Note that constructing G^* takes polynomial time. We now prove that c' has a b-precolouring extension in G if and only if G^* has a perfect matching.

($\Rightarrow$) Suppose that c' has a b-precolouring extension c in G. By Observation 9, c is a tight b-colouring of G. By Proposition 6 (i), the set T is exactly the set of b-chromatic vertices under c. By Proposition 6 (ii), each colour class of c contains exactly one vertex of T. We say that a colour class of c is of *type 1* or *type 2* depending on whether its b-chromatic vertex from T belongs to $T_1 \cup T'$ or T_2, respectively.

By Proposition 6 (iii), no vertex of S belongs to the same colour class as a vertex of T_1. As c is a b-precolouring extension of c', no vertex of S belong to the same colour class as a vertex of T' either. Hence, all vertices of S are in colour classes of type 2. As $T_2 \subseteq T \setminus T_1$, every vertex $u \in T_2$ has at least one non-neighbour $u' \in T$. As every vertex of T_1 is a dominating vertex of $G[T]$, we find that $u' \notin T_1$. Since u' must still have a neighbour with colour $c(u)$, we find that u' must be adjacent to a vertex $s \in \delta T$ with colour $c(u)$. As $u \in T_2$ and $T_2 \cap T' = \emptyset$, we find that $s \in S$. Since c is a b-precolouring extension of c, every colour class of c with at least two vertices of δT contains no vertex of S. Hence, every colour class of type 2 contains exactly one vertex of T_2 and exactly one vertex of S.

The above implies that $|S| = |T_2|$, and we may write $S = \{s_1, \ldots, s_{|T_2|}\}$ and $T_2 = \{u_1, \ldots, u_{|T_2|}\}$. Let $V_1, \ldots, V_{|T_2|}$ be the colour classes of type 2 of G. By symmetry we may now assume that $V_i \cap T_2 = \{u_i\}$ and $V_i \cap S = \{s_i\}$ for every i ($1 \leq i \leq |T_2|$).

Let $1 \leq i \leq |T_2|$, and let $u_j \in (T_2 \cup T') \setminus \{u_i\}$. As u_j is b-chromatic, $d(u_j) = m - 1$ and $c(u_i) = c(s_i)$, it follows that u_j has exactly one neighbour in $\{u_i, s_i\}$. Hence, u_i and s_i have no common neighbour in $(T_2 \cup T') \setminus \{u_i\}$. As a colour class is an independent set, u_i and s_i are not adjacent. As a consequence, $u_i s_i$ is an edge of G^*. Moreover, as G^* has partition classes T_2 and S, the set $M = \{u_i s_i : 1 \leq i \leq |T_2|\}$ is a perfect matching of G^*.

($\Leftarrow$) Suppose that G^* has a perfect matching M. As G^* is bipartite with partition classes S and T_2, this means that $|S| = |T_2|$. Hence, we may write $S = \{s_1, \ldots, s_{|T_2|}\}$ and $T_2 = \{u_1, \ldots, u_{|T_2|}\}$ such that $M = \{u_i s_i : 1 \leq i \leq |T_2|\}$.

We assume without loss of generality that $c'(u_i) = i$ for every i ($1 \leq i \leq |T_2|$). We first colour the vertices of S by giving s_i colour i for every i ($1 \leq i \leq |T_2|$). As $T_2 \cap T' = \emptyset$, no vertex in S' has the same colour as a vertex in T_2. Hence, no two adjacent vertices in $G[T \cup \delta T]$ are coloured the same, and moreover, every vertex of S belongs to a colour class of size 2, together with a vertex of T_2. Since every vertex of $V(G) \setminus (T \cup \delta T)$ does not belong to T, it is not dense and thus has degree at most $m - 2$. So, just as before, we can safely colour the vertices of $V(G) \setminus (T \cup \delta T)$ one by one in a greedy way. Let c be the resulting colouring of G. We claim that c is a b-precolouring extension of c' in G.

By construction, c is an extension of c'. As each vertex of S belongs to a colour class of size 2, together with a vertex of T_2, every colour class of c with at least two vertices of δT contains no vertex of $S = \delta T \setminus S'$. This means that both properties of Definition 8 are satisfied. However, we must still show that c is not only a colouring but also a b-colouring. In order to do this, we first note that, since c' uses m colours and we did not introduce any new colours, c uses m colours as well. It remains to show that every vertex of T is b-chromatic.

First, consider a vertex $u \in T_1$. As $u \in T_1 \subseteq T$, it has degree $m - 1$. By the definition of c', every vertex of T belongs to a different colour class of c', and thus to a different colour class of c. Since T_1 dominates $G[T]$, this means that u has a neighbour in every colour class not its own. Hence, u is b-chromatic.

Now, consider a vertex $u \in T_2 \cup T'$. Since T_1 dominates $G[T]$ and $(T_2 \cup T') \subseteq T$, we find that u is adjacent to all vertices of T_1, each of which belongs to a different singleton colour class of c by construction. In other words, u has exactly one neighbour with colour $c(u')$ for every $u' \in T_1$, namely u'. Now suppose that $u' \in T' \setminus \{u\}$. By definition of T', $c'(u') = c'(s)$ for some $s \in S'$. As c' satisfies Property 2 of Definition 7, it follows that u has exactly one neighbour with colour $c'(u')$. As c is an extension of c', we conclude that u also has exactly one neighbour with colour $c(u')$ for every $u' \in T' \setminus \{u\}$.

It remains to show that u is also picking up every colour used on $T_2 \setminus \{u\}$ in c. For a contradiction, suppose that u is not adjacent to any vertex of $\{u_j, s_j\}$ for some $u_j \in T_2 \setminus \{u\}$. As c has m colour classes and u has degree $m-1$ in G, this means that u has two neighhbours with the same colour. Previously, we found that u has exactly one neighbour with colour $c(u')$ for every $u' \in (T_1 \cup T') \setminus \{u\}$. Thus, there must exist a pair $\{u_h, s_h\}$ with $u_h \in T_2 \setminus \{u, u_j\}$ such that u is adjacent to both u_h and s_h, by the pigeonhole principle. That is, $u \in (T_2 \cup T') \setminus \{u_h\}$

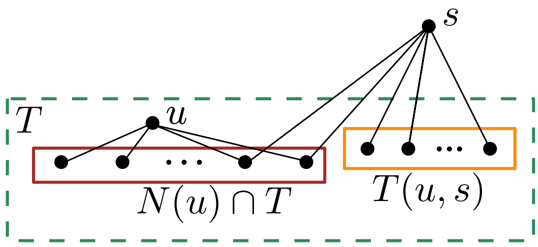

Figure 4 An illustration of a set $T(u, s)$ for two (non-adjacent) vertices $u \in T$ and $s \in S = \delta T$.

is a common neighbour of u_h and s_h. However, it now follows from the construction of G^* that $u_h s_h$ is not an edge in G^*, and thus $u_h s_h$ cannot be an edge of M, a contradiction. We conclude that u is b-chromatic.

It now suffices to check if G^* has a perfect matching, which takes polynomial time [13]. ◀

We will use our next result as a lemma in the proof of Theorem 3. In contrast, b-Chromatic Number is NP-hard even for the smaller class of co-bipartite graphs [6], which are $3P_1$-free.

▶ **Lemma 11.** Tight b-Chromatic Number *is polynomial-time solvable for* $(2P_2 + P_1)$*-free graphs.*

Proof. Let $G = (V, E)$ be a tight $(2P_2 + P_1)$-free graph. We write $m = m(G)$. Let $T = \{u_1, \dots, u_m\}$ be the set of dense vertices of G, so $|T| = m$ and every u_i $(1 \leq i \leq m)$ is a maximum-degree vertex of G with degree (exactly) $m - 1$.

We let $S = \delta T$ be the set of neighbours of vertices of T that do not belong to T. For two non-adjacent vertices $s \in S$ and $u \in T$, we let $T(u, s)$ consist of all neighbours of s in T that are not adjacent to u (see Figure 4); note that $T(u, s) = \emptyset$ is possible.

We first prove a claim. Assume that G admits a tight b-colouring c. By Proposition 6 (i), T is the set of b-chromatic vertices under c. Let $V_1, \dots, V_m$ be the m colour classes of c. By Proposition 6 (ii), we may assume without loss of generality that $V_i \cap T = \{u_i\}$.

▷ Claim 12. If $|V_i| \geq 3$ for some i $(1 \leq i \leq m)$ and $s \in S \cap V_i$, then $T(u_i, s)$ is complete to $T \setminus (T(u_i, s) \cup \{u_i\})$.

Proof. Let $|V_i| \geq 3$ for some i $(1 \leq i \leq m)$, say $i = 1$. Let $s \in S \cap V_1$. We note that s and u_1 are not adjacent, since V_1 is an independent set containing s and u_1, and thus, the set $T(u_1, s)$ is defined. For a contradiction, assume that $T(u_1, s)$ contains a vertex, say u_2, that is not adjacent to some vertex, say u_3, in $T \setminus (T(u_1, s) \cup \{u_1\})$. We distinguish two cases.

First suppose that u_3 is not adjacent to u_1. Since $u_3 \notin T(u_1, s)$ and u_3 is b-chromatic by Proposition 6 (i), this means that u_3 is adjacent to a vertex $s' \in S \cap V_1$. Since V_1 is an independent set, u_1, s, s' are pairwise non-adjacent. By Proposition 6 (ii), we find that u_2 and u_3 have a unique neighbour in V_1. As u_2 and u_3 are adjacent to $s \in V_1$ and $s' \in V_1$, respectively, this means that u_2 is adjacent neither to u_1 nor to s', and u_3 is adjacent neither to u_1 nor s. However, now $\{u_2, s, u_3, s', u_1\}$ induces a $2P_2 + P_1$ in G, a contradiction.

Now suppose that u_3 is adjacent to u_1. Since V_1 has size at least 3, there exists a vertex $s' \in V_1 \setminus \{u, s\}$. As u_3 has a unique neighbour in V_1 due to Proposition 6 (ii), and u_3 is adjacent to $u_1 \in V_1$, we find that u_3 is adjacent neither to s nor to s'. As u_2 has a unique neighbour in V_1 for the same reason, and u_2 is adjacent to $s \in V_1$, we find that u_2 is adjacent neither to u_1 nor to s'. As V_1 is an independent set, u_1, s, s' are pairwise non-adjacent. However, now $\{u_1, u_3, u_2, s, s'\}$ induces a $2P_2 + P_1$ in G, a contradiction. ◁

We now present our polynomial-time algorithm, which consists of four steps.

Step 1. We colour all the m vertices of T by giving them a unique colour from $\{1,\dots,m\}$.

Step 1 takes polynomial time and is correct due to Proposition 6 (ii).

Step 2. We check for each $u \in T$ and each $s \in S$ with $us \notin E(G)$, whether $T(u,s)$ is complete to $T \setminus (T(u,s) \cup \{u\})$. If so, we give s the same colour as u.

Step 2 takes polynomial time. It is correct for the following reasons. Suppose $T(u,s)$ is complete to $T \setminus (T(u,s) \cup \{u\})$ for some $u \in T$ and $s \in S$ with $us \notin E(G)$. We cannot give s the same colour as a vertex in $T(u,s)$, because s is adjacent to every vertex of $T(u,s)$ by definition. Moreover, we cannot give s the same colour as a vertex in $T \setminus (T(u,s) \cup \{u\})$. Otherwise, as $T(u,s)$ is complete to $(T \setminus (T(u,s) \cup \{u\})) \cup \{s\}$, the vertices from $T(u,s)$ would be adjacent to two vertices of the same colour. As $T(u,s) \subseteq T$, this is not possible due to Proposition 6 (ii). Hence, as s must receive a colour from $\{1,\dots,m\}$, we have no other choice than to give s the same colour as u. Also, by the same reasoning, if the same vertex s is coloured more than once during Step 2 then we return that G has no tight b-colouring.

Let S' be the subset of vertices of S already coloured. Let c' be the colouring constructed so far, so c' is a colouring of $G[S' \cup T]$. Note that we obtained S' and c' in polynomial time.

Step 3. We check if c' is an S'-partial b-colouring of G. If not, then we return that G has no tight b-colouring.

Step 3 takes polynomial time, as from Step 1, we know that $c'(u) \neq c'(u')$ for every two distinct vertices $u, u' \in T$, and it takes polynomial time to check if

- c' is a colouring of $G[S' \cup T]$,
- for every $u \in T$ with $c'(u) = c'(s)$ for some $s \in S'$, it holds that every vertex of $T \setminus \{u\}$ has exactly one neighbour with colour $c'(u)$.

Step 3 is correct for the following reasons. First, we have already shown above that the b-colouring that we constructed so far is unique (modulo permutation of colours). Hence, if two adjacent vertices in $G[S' \cup T]$ received the same colour, then we may safely output that G is a no-instance. By Proposition 6 (ii), we may do the same if there is a vertex in T that is adjacent to at least two vertices with the same colour.

Now suppose that there exists a vertex $u \in T$ with $c'(u) = c'(s)$ for some $s \in S'$, such that some vertex $u' \in T \setminus \{u\}$ is not adjacent to any vertex with colour $c'(u)$. In particular, this means that u and u' are not adjacent. If G still has a tight b-colouring, then u' must be adjacent to some vertex $s^* \in S \setminus S'$ that will be given colour $c'(u)$ later, due to Proposition 6 (i). Hence, every tight b-colouring of G (if there exists one) will have at least three vertices with colour $c'(u)$. However, by Claim 12, $T(u,s^*)$ must be complete to $T \setminus (T(u,s^*) \cup \{u\})$. This means that $s^* \in S'$, a contradiction.

Suppose we have not returned a no-answer yet.

Step 4. We check if G contains a b-precolouring extension c of c'. If so, then we return c, and otherwise we return that G has no tight b-colouring.

Step 4 takes polynomial time by Theorem 10 and is correct for the following reasons. For a contradiction, assume that G has a tight b-colouring c^* that is not a b-precolouring extension of c'. We already deduced that c' is unique (modulo permutation of colours). Hence, we may assume without loss of generality that c is an extension of c'. As c^* is not a b-precolouring extension, this means that c^* must have a colour class of size at least 3 that contains a vertex $s \in S \setminus S'$. Say the vertices in this colour class have colour i ($1 \leq i \leq m$). By Proposition 6 (ii), we find that T contains exactly one vertex u with $c^*(u) = i$. However, as there are at least three vertices of colour i, we find that $T(u, s)$ must be complete to $T \setminus (T(u, s) \cup \{u\})$ by Claim 12. This means that $s \in S'$, a contradiction.

As we showed that Steps 1–4 each take polynomial time, our algorithm runs in polynomial time. Correctness follows from the correctness proofs given for each of these four steps. ◀

We now give an algorithm for $(P_3 + P_1)$-free graphs (see [1] for the proof details). In it, we use our algorithm for $(2P_2 + P1)$-free graphs as a subroutine. As such, also this result relies (indirectly) on Theorem 10.

▶ **Lemma 13.** Tight b-Chromatic Number *is polynomial-time solvable for* $(P_3 + P_1)$*-free graphs.*

The next result is for line graphs and due to Campos et al. [8] who proved it for b-Chromatic Number (see [1] for more details).

▶ **Lemma 14** ([8]). Tight b-Chromatic Number *is* NP*-complete for line graphs.*

We now give four hardness results and one known polynomial-time result. Two hardness results are new and obtained these results by modifying the reduction from the proof of Lemma 15 (see [1] for the full details of the modifications required for Lemmas 16 and 17).

▶ **Lemma 15** ([20]). Tight b-Chromatic Number *is* NP*-complete for disjoint unions of a connected split graph and three stars, and thus for* (C_4, C_5, P_5)*-free graphs.*

▶ **Lemma 16.** Tight b-Chromatic Number *is* NP*-complete for* $3P_2$*-free graphs.*

▶ **Lemma 17.** Tight b-Chromatic Number *is* NP*-complete for* $2P_3$*-free graphs.*

▶ **Lemma 18** ([32, 39]). Tight b-Chromatic Number *is* NP*-complete for bipartite graphs.*

▶ **Lemma 19** ([5]). b-Chromatic Number *is polynomial-time solvable for* P_4*-sparse graphs.*

We are now ready to prove Theorem 3, which we restate below.

▶ **Theorem 3.** *For a graph* H*,* Tight b-Chromatic Number *in* H*-free graphs is*

- *polynomial-time solvable if* $H \subseteq_i P_4$, $H \subseteq_i P_3 + P_1$ *or* $H \subseteq_i 2P_2 + P_1$ *and*
- NP*-complete if* H *is not a linear forest, or* $P_5 \subseteq_i H$, $2P_3 \subseteq_i H$ *or* $3P_2 \subseteq_i H$*.*

Proof. If $H \subseteq_i 2P_2 + P_1$, $H \subseteq_i P_3 + P_1$ or $H \subseteq_i P_4$, we apply Lemma 11, 13 or 19, respectively. Now suppose H is not a linear forest. Either H has an induced C_3 or C_r for $r \geq 4$: apply Lemma 18 or 15, respectively, or H has an induced claw: apply Lemma 14. Finally, if $P_5 \subseteq_i H$, $3P_2 \subseteq_i H$ or $2P_3 \subseteq_i H$, then apply Lemma 15, 16 or 17, respectively. ◀

4 The Proof of Theorem 4

We start with two results of Silva [43]. Afterwards, we modify two known hardness results for Fall 3-Colouring from Laskar and Lyle [33] and Lauri and Mitillos [34], respectively, with a simple trick (see [1] for the details). We also show a new polynomial-time solvability result using Lemma 5 (again, see [1] for the proof details).

▶ **Lemma 20** ([43]). Fall Chromatic Number *and* Fall Achromatic Number *are* NP*-hard for chordal graphs, that is,* $(C_4, C_5, \ldots)$*-free graphs.*

▶ **Lemma 21** ([43]). Fall Colouring, *and thus* Fall Chromatic Number *and* Fall Achromatic Number, *are polynomial-time solvable for* P_4*-sparse graphs, and thus for* P_4*-free graphs.*

▶ **Lemma 22** ([33]). Fall Chromatic Number *and* Fall Achromatic Number *are* NP*-hard for* C_3*-free graphs.*

▶ **Lemma 23** ([34]). Fall Chromatic Number *and* Fall Achromatic Number *are* NP*-hard for line graphs, and thus for claw-free graphs.*

▶ **Lemma 24.** Fall Colouring, *and thus* Fall Chromatic Number *and* Fall Achromatic Number, *is polynomial-time solvable for* $(P_3 + P_1)$*-free graphs.*

We need one final lemma. Its proof (contained in [1]) is based on a construction from [42] for testing if a graph has a critical vertex, which in turn was based on a construction of Král et al. [31] for proving that Chromatic Number is NP-hard for $(C_5, 2P_2, P_2 + 2P_1, 4P_1)$-free graphs.

▶ **Lemma 25.** Fall Chromatic Number *and* Fall Achromatic Number *are* NP*-hard for* $(C_5, 2P_2, P_2 + 2P_1, 4P_1)$*-free graphs.*

▶ **Theorem 4.** *For a graph* H*,* Fall Chromatic Number *and* Fall Achromatic Number *in* H*-free graphs are polynomial-time solvable if* $H \subseteq_i P_4$ *or* $P_3 + P_1$*, and* NP*-hard otherwise.*

Proof. If H has an induced C_3, we apply Lemma 22. If H has an induced C_r for some $r \geq 4$, we apply Lemma 20. From now on, assume H is a forest. If H has an induced claw, we apply Lemma 23. Now also assume H is claw-free so H is a linear forest. If H has at least four connected components, then H has an induced $4P_1$: apply Lemma 25. Suppose H has exactly three connected components. If H has an edge, then H has an induced $P_2 + 2P_1$: apply Lemma 25 again. Else $H = 3P_1 \subseteq_i P_3 + P_1$: apply Lemma 24. Suppose H has two connected components. Hence, as H is a linear forest, $H = P_s + P_r$ for some r, s with $r \leq s$. If $r \geq 2$, then H has an induced $2P_2$: apply Lemma 25. Now assume $r = 1$. If $s \geq 4$, then $P_2 + 2P_1 \subseteq_i P_4 + P_1 \subseteq_i H$: apply Lemma 25. If $s \leq 3$, then $H \subseteq_i P_3 + P_1$: apply Lemma 24. Finally, if H is connected, then, as H is a linear forest, $H = P_r$ for some $r \geq 1$. If $r \geq 5$, then H has an induced $2P_2$: apply Lemma 25. If $r \leq 4$, then $H \subseteq_i P_4$: apply Lemma 21. ◀

5 Conclusions

We extended the known results for b-Chromatic Number, Tight Chromatic Number, Fall Chromatic Number and Fall Achromatic Number with several new polynomial-time algorithms and hardness results for H-free graphs to obtain three complexity dichotomies and one partial complexity classification. Our results provided the first evidence that the computational complexities of b-Chromatic Number and Tight b-Chromatic Number may differ in H-free graphs, namely if $3P_1 \subseteq_i H \subseteq_i 2P_2 + P_1$ or $H = P_3 + P_1$. To obtain these new polynomial-time cases, we introduced the technique of tight b-precolouring extension.

As our main open problem, we ask about the computational complexity of Tight b-Chromatic Number for H-free graphs for the remaining cases not covered by Theorem 3:

▶ **Open Problem 26.** *Determine the complexity of* Tight b-Chromatic Number *for* H*-free graphs if* $H = P_4 + P_2 + sP_1$ *for* $s \geq 0$*,* $H = P_4 + sP_1$ *for* $s \geq 1$*,* $H = P_3 + P_2 + sP_1$ *for* $s \geq 0$*,* $H = P_3 + sP_1$ *or* $H = 2P_2 + sP_1$ *for* $s \geq 2$*,* $H = P_2 + sP_1$ *for* $s \geq 3$*, or* $H = sP_1$ *for* $s \geq 4$.

In particular: does there exist an integer $s \geq 4$, such that Tight b-Chromatic Number is NP-complete on sP_1-free graphs, or equivalently, graphs of independence number at most $s-1$? Graph classes of bounded independence number are well-studied and new complexity results continue to be found (see e.g. [16, 26]). All known hardness gadgets for Tight b-Chromatic Number have arbitrarily large independent sets. It also does not seem straightforward to adjust the hardness gadget for b-Chromatic Number in co-bipartite graphs from [6] or the hardness gadget from Lemma 25. Hence, new ideas are needed.

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