Hunting for Directed 2-Spiders

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Abstract

Hons, Klimošová, Mikšaník, Tkadlec, Tyomkyn and the second author proved that, for every integer $\ell \geqslant 1$, every directed graph with minimum out-degree at least $3.23 \cdot \ell$ contains a $(2, \ell)$-spider (a 1-subdivision of the in-star with ℓ leaves) as a subgraph. Hons et al. also conjectured that the bound on the minimum out-degree can be further improved to 2ℓ. In this note, we confirm this conjecture by showing that every directed graph with minimum out-degree at least 2ℓ contains a $(2, \ell)$-spider as a subgraph. This result is best possible, as the complete directed graph with 2ℓ vertices does not contain a $(2, \ell)$-spider.

2012 ACM Subject Classification Mathematics of computing → Extremal graph theory

Keywords and phrases Oriented and Directed Graphs, Extremal Graph Theory, Mathematics of Computing, Unavoidable Subgraphs

Digital Object Identifier 10.4230/LIPIcs.WG.2026.21

Related Version *Full Version*: https://arxiv.org/abs/2602.10340

Funding *Grzegorz Gutowski*: Partially supported by grant no. 2023/49/B/ST6/01738 from National Science Centre, Poland.
Gaurav Kucheriya: Partially supported by GAČR grant 25-17377S and GAUK project 92125.

Acknowledgements The authors would like to thank the organizers of the 10th Cracow Conference on Graph Theory (2025) which brought the authors together to work on this problem. The authors are also grateful to the organizers of Homonolo 2025, where the final stages of this research was carried out. The authors also thank Mykhaylo Tyomkyn for carefully reading an earlier draft of this paper and for helpful comments.

1 Introduction

The question of how local degree conditions force global structure has played a central role in the development of extremal graph theory. A classical result of Dirac [6] shows that every graph with $n \geqslant 3$ vertices and minimum degree at least $n/2$ contains a Hamiltonian cycle. Ore [12] strengthened Dirac's theorem by proving that a graph with n vertices is Hamiltonian whenever the sum of the degrees of any two non-adjacent vertices is at least n. In a different direction, Mader proved that for every integer k, there is an integer d_k such that every finite graph with sufficiently high average degree contains a k-connected subgraph [9] and a subdivision of the complete graph K_k [10].

In directed graphs, this line of research becomes more subtle. Already at the level of degree conditions, there is no longer a unique notion of vertex degree, and one may impose constraints either on both the in-degrees and out-degrees (so-called semidegree conditions) or on only one of these parameters. As a consequence, the extent to which local degree assumptions force global structural properties in digraphs is far less clear than in the undirected setting. For instance, in [11], Mader presents a construction which shows that even imposing large semidegree conditions does not always guarantee the existence of highly connected subgraphs in digraphs.

52nd International Workshop on Graph-Theoretic Concepts in Computer Science (WG 2026).
Editors: Jan Goedgebeur and Paweł Rzążewski; Article No. 21; pp. 21:1–21:6
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

When only minimum out-degree is considered, the directed setting presents additional challenges and several longstanding open problems. A prominent example is the Caccetta–Häggkvist conjecture [5], which asserts that every directed graph with n vertices with minimum out-degree at least n/k contains a directed cycle of length at most k. Partial progress has been made on the Caccetta–Häggkvist conjecture in special cases: Caccetta–Häggkvist [5] verified the case $k = 2$, whereas the case $k = 3$ was shown by Hamidoune [7].

Recently, Hons, Klimošová, Mikšaník, Tkadlec, Tyomkyn [8] and the second author initiated a systematic study of the following natural question arising from minimum out-degree conditions: given a directed graph G with sufficiently large minimum out-degree, which directed graphs H are *unavoidable*, i.e., which directed graphs H must be contained in G as a subgraph? Aboulker, Cohen, Havet, Lochet, Moura and Thomassé [1] proved that every in-arborescence T (i.e. a tree with all edges oriented towards a designated root vertex r) is contained as a subdivision in every directed graph G with minimum out-degree $d_T = d(T)$.

An oriented graph is a *(k, ℓ)-spider* if it is the $(k - 1)$-subdivision of the in-star with ℓ leaves. We say that a (k, ℓ)-spider is *rooted* at a vertex r when r is the central vertex of the subdivided in-star. See Figure 1 for an example.

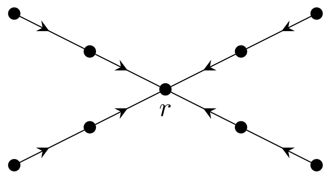

Figure 1 A $(2, 4)$-spider rooted at r.

As a (k, ℓ)-spider is an in-arborescence and any subdivision of a (k, ℓ)-spider contains a (k, ℓ)-spider, then by the result of Aboulker et al. [1], a (k, ℓ)-spider is unavoidable when the minimum out-degree is at least some large constant that depends only on k and ℓ. The next natural question is to determine the minimum out-degree d that any directed graph G must have so that a (k, ℓ)-spider is unavoidable. In this direction, generalizing a conjecture for oriented graphs of Thomassé [4, 13], Hons et al. proposed the following Giant Spider Conjecture:

▶ **Conjecture 1** (Giant Spider Conjecture [8]). *For all integers $k \geqslant 2$, $\ell \geqslant 1$, every directed graph with minimum out-degree at least $k\ell$ and every oriented graph with minimum out-degree at least $k\ell/2$ contains a (k, ℓ)-spider as a subgraph.*

The bounds conjectured above would be tight as witnessed by the examples of the complete directed graph and a regular tournament of order $k\ell$, respectively. In [8], a relaxed version of Conjecture 1 was proved directed graphs when $k = 2$ with a higher multiplicative constant. Hons et al. showed that every directed graph with minimum out-degree at least $3.23 \cdot \ell$ contains a $(2, \ell)$-spider as a subgraph. However, Conjecture 1 is still open. In this note, we confirm Conjecture 1 for $k = 2$, every $\ell \geqslant 1$ and every directed graph by proving the following theorem.

▶ **Theorem 2.** *For every integer $\ell \geqslant 1$, every directed graph with minimum out-degree at least 2ℓ contains a $(2, \ell)$-spider as a subgraph.*

For a directed graph G, let $X, Y, Z \subseteq V(G)$ be three subsets of vertices. We denote by $\overrightarrow{XY}$ the set of all edges oriented from X to Y. We denote by $\overrightarrow{XYZ}$ the set of all simple 2-paths oriented from X to Y to Z. For a vertex $v \in V(G)$, we denote by $N^+(v)$ the set of out-neighbors of v in G, and by $N^-(v)$ the set of in-neighbors of v in G. For a vertex v, we use $\deg^+(v) = |N^+(v)|$ and $\deg^-(v) = |N^-(v)|$ to denote the out-degree of v and the in-degree of v, respectively. We write $x \to y$ to denote that there is an edge $(x, y) \in E(G)$ and $x \not\to y$ to denote that (x, y) is not an edge in G. We write $x \to y \to z$ to denote $(x \to y) \wedge (y \to z) \wedge (x \neq z)$.

2 Main Result

Let G be a directed graph, and r be any vertex in $V(G)$. For any other vertex $x \in V(G) \setminus \{r\}$, let

$$O_{x,r} \coloneqq \{y \in V(G) : (x \to y \to r) \vee (y \to x \to r)\}$$

denote the set of vertices that together with x form a simple 2-path that ends in r. For any integer $i \geqslant 1$, we say that a vertex x is an *i-extender* for r if and only if $|O_{x,r}| \geqslant i$. Intuitively, when r is a fixed root of a spider and x is an i-extender for r, then there are at least i different vertices that can attach x to a spider rooted at r. Before we proceed to the proof of Theorem 2, we prove the following technical lemma that allows us to use extenders to extend the size of a constructed spider.

▶ **Lemma 3.** *For all integers $f \geqslant 0$, $s \geqslant 0$, a directed graph G, and a vertex $r \in V(G)$, if G contains a set F of f vertices, $F = \{x_1, x_2, \ldots, x_f\}$, such that for each $1 \leqslant i \leqslant f$, vertex x_i is an $(f + 2s + i - 1)$-extender for r, and a $(2, s)$-spider S rooted at vertex r with $V(S) \cap F = \emptyset$, then G contains a $(2, f + s)$-spider rooted at r.*

Proof. We prove this lemma by induction on f. For $f = 0$, the statement is clearly true. For $f > 0$, observe that the vertex x_1 is an $(f + 2s)$-extender for r, i.e. there are at least $f + 2s$ elements in $O_{x_1,r}$. As $x_1 \in F$, $x_1 \notin O_{x_1,r}$, $r \in V(S)$, and $r \notin O_{x_1,r}$, we get that there are at most $(f - 1 + 2s)$ elements in $(V(S) \cup F) \cap O_{x_1,r}$. Thus, there is an element $y \in O_{x_1,r}$ that is not in $V(S) \cup F$. We can construct a $(2, s + 1)$-spider S' by adding the path $x_1 \to y \to r$ or the path $y \to x_1 \to r$ to the spider S. Let $F' = F \setminus \{x_1\}$. The result follows by applying the inductive hypothesis to F' and S'. ◀

For every integer $i \geqslant 1$, note that an $(i + 1)$-extender for r is also an i-extender for r, and the requirement for x_f in Lemma 3 is the strongest among all vertices in F. We can simplify, and weaken, the statement of Lemma 3 so that all extenders in F are $(2f + 2s - 1)$-extenders.

▶ **Corollary 4.** *For all integers $f \geqslant 0$, $s \geqslant 0$, $f + s = \ell$, a directed graph G, and a vertex $r \in V(G)$, if G contains a set F of $(2\ell - 1)$-extenders for r with $|F| = f$, and a $(2, s)$-spider S rooted at vertex r with $V(S) \cap F = \emptyset$, then G contains a $(2, \ell)$-spider rooted at r.*

For a fixed integer ℓ, directed graph G, and vertex r, we call a $(2\ell - 1)$-extender for r to be a *strong extender* for r. Observe that for a vertex x with $x \to r$, and $\deg^-(x) \geqslant 2\ell$, there are at least $2\ell - 1$ in-neighbors of x that are different than r, and we get that x is a strong extender for r.

We note that Hons et al. [8] used a similar technique, extending a smaller spider using vertices of in-degree at least 2ℓ that are in-neighbors of the root vertex. The vertex set of the directed graph was partitioned into two sets: set A of vertices with in-degree at least 2ℓ, and $B = V \setminus A$. This approach was used to derive an upper bound on the number of edges in $\overrightarrow{AA}$. This allowed for a lower bound on the number of paths in $\overrightarrow{VBA}$ and the vertex r with the most $\overrightarrow{VBA}$ paths ending at r was shown to be a root of a $(2, \ell)$-spider. See the proof of Theorem 1.8 in [8].

Our approach is to use different kinds of extenders and apply Corollary 4 as the last step in the construction. We also use a similar partition of the vertex set of G, but we select the root of the spider a little more carefully. Then, we use edge coloring in an auxiliary graph to find a large spider that can be later extended to a $(2,\ell)$-spider.

Proof of Theorem 2. Fix $\ell \geqslant 1$ and $d = 2\ell$. Let G be a directed graph with $\deg^+(v) = d$ for every vertex $v \in V(G)$. We will show that G contains a $(2,\ell)$-spider. The theorem will follow, as every directed graph with minimum out-degree at least d contains a subgraph in which every vertex has out-degree exactly d.

Let $V = V(G)$ and $A \subseteq V$ denote the set of vertices of G with in-degree at least 2ℓ. Let $B = V \setminus A$ denote the set of vertices with in-degree at most $2\ell - 1$. As already observed in [8], we can easily get the following bound on $\left|\overrightarrow{VBA}\right|$:

$$\begin{aligned}\left|\overrightarrow{VBA}\right| &= \left|\overrightarrow{VBV}\right| - \left|\overrightarrow{VBB}\right| = \left|\overrightarrow{ABV}\right| + \left|\overrightarrow{BBV}\right| - \left|\overrightarrow{VBB}\right| \\ &= \left|\overrightarrow{AVV}\right| - \left|\overrightarrow{AAV}\right| + \left|\overrightarrow{BBV}\right| - \left|\overrightarrow{VBB}\right|\end{aligned}$$

Because $\deg^+(v) = d$ for all v and $\deg^-(v) \leqslant 2\ell - 1$ for every $v \in B$, we obtain:

$$\begin{aligned}\left|\overrightarrow{AVV}\right| &\geqslant d(d-1)|A|, \\ \left|\overrightarrow{AAV}\right| &\leqslant d\left|\overrightarrow{AA}\right|, \\ \left|\overrightarrow{BBV}\right| &\geqslant (d-1)\left|\overrightarrow{BB}\right|, \\ \left|\overrightarrow{VBB}\right| &\leqslant (2\ell-1)\left|\overrightarrow{BB}\right|.\end{aligned}$$

Hence, we have

$$\begin{aligned}\left|\overrightarrow{VBA}\right| &\geqslant \left|\overrightarrow{AVV}\right| - \left|\overrightarrow{AAV}\right| + \left|\overrightarrow{BB}\right| \cdot (d - 1 - (2\ell - 1)) \\ &\geqslant \left|\overrightarrow{AVV}\right| - \left|\overrightarrow{AAV}\right| \geqslant d(d-1)|A| - d\left|\overrightarrow{AA}\right|.\end{aligned}$$

For any vertex $x \in A$, let $A_x \subseteq A$ be the set of in-neighbors of x in A. Let $VB_x \subseteq \overrightarrow{VBA}$ be the set of all simple 2-paths with the second vertex in B and the third vertex x. Select vertex $r \in A$ to be a vertex that maximizes the value $d|A_r| + |VB_r|$. By averaging over all vertices in A, we get:

$$d|A_r| + |VB_r| \geqslant \frac{d\left|\overrightarrow{AA}\right| + \left|\overrightarrow{VBA}\right|}{|A|} \geqslant \frac{d\left|\overrightarrow{AA}\right| + d(d-1)|A| - d\left|\overrightarrow{AA}\right|}{|A|} \geqslant d^2 - d.$$

Observe that every vertex $x \in A_r$ has $x \to r$ and $\deg^-(x) \geqslant 2\ell$. Thus, every $x \in A_r$ is a strong extender for r. Let C_r be the set of all other strong extenders for r. Let $a = |A_r|$ and $c = |C_r|$. We have $A_r \cap C_r = \emptyset$, and $A_r \cup C_r$ is a set of $a + c$ strong extenders for r. In order to apply Corollary 4 it remains to find a $(2, \ell - a - c)$-spider rooted in r that is disjoint with $A_r \cup C_r$.

Let Q_r be the set of paths in VB_r that are disjoint with $A_r \cup C_r$. Each vertex x in A_r has $x \to r$ and $x \notin B$. Thus, there are no paths in VB_r that have x as the second vertex, and there are at most $d - 1$ paths in VB_r that have x as the first vertex. As $d = 2\ell$, we have that there are at most $2\ell - 1$ paths in VB_r that have $x \in A_r$ as a vertex. For each vertex x in C_r there are at most d paths in VB_r that have x as the first vertex. If there are paths in VB_r that have x as the second vertex, then $x \in B$, $\deg^-(x) < 2\ell$, and there are at most $2\ell - 1$ such paths. As $d = 2\ell$, we have that there are at most $d + 2\ell - 1 = 4\ell - 1$ paths in VB_r that have $x \in C_r$ as a vertex. Based on these observations, the previous bound on $|VB_r|$, and $|A_r| = a$ we get:

$$|Q_r| \geqslant |VB_r| - a(2\ell-1) - c(4\ell-1) \geqslant d^2 - d - d|A_r| - a(2\ell-1) - c(4\ell-1) \geqslant d^2 - d - (a+c)(4\ell-1).$$

Now, we define an undirected graph H with $V(H) = V(G) \setminus (A_r \cup C_r \cup \{r\})$. We add an edge $\{x, y\}$ to H if and only if there is a path $x \to y \to r$ or a path $y \to x \to r$ in Q_r. Observe that each undirected edge in H corresponds to at least one directed edge in G that can be extended to a simple 2-path that ends in r. Consider any vertex $x \in V(H)$. As vertices in $A_r \cup C_r$ are explicitly removed from H, we have that x is not a strong extender for r. As every neighbor of x in H is an element of $O_{x,r}$, we get that the degree of x in H is at most $2\ell - 2$. By Vizing's Theorem [14], there is a coloring of the edges of H using $2\ell - 1$ colors such that no two incident edges share a color. Let s be the largest size of a color class in the coloring, and T_r be some color class of size s. We have $s \geqslant |Q_r|/(2\ell - 1)$. As edges in T_r are pairwise disjoint, we can extend each edge in T_r to a simple 2-path that ends in r and get a $(2, s)$-spider S_r rooted at r. We easily get the following

$$s(2\ell - 1) \geqslant |Q_r| \geqslant d^2 - d - (a + c)(4\ell - 1),$$

and

$$(a + c + s)(4\ell - 1) \geqslant d^2 - d = 4\ell^2 - 2\ell,$$

and

$$(a + c + s) \geqslant \ell.$$

To finish the proof, we observe that if $s \geqslant \ell$, then G already contains a $(2, \ell)$-spider as a subgraph. Otherwise, we have $1 \leqslant \ell - s \leqslant a + c$, and we can select a set of strong extenders $F \subseteq A_r \cup C_r$ with exactly $\ell - s$ elements. Finally, by Corollary 4 applied to F and S_r, we get that G contains a $(2, \ell)$-spider as a subgraph. ◀

3 An Almost-Linear-Time Algorithm

It is worth mentioning that the presented proof of Theorem 2 yields a simple, almost-linear-time algorithm that finds a $(2, \ell)$-spider in any directed graph G satisfying the conditions of the theorem. Let $n = |V(G)|$, $m = |E(G)|$, and observe that selecting a subgraph of G with out-degree of every vertex equal to d is easily done in $O(n\ell)$ time. Further, calculating $d|A_x| + |VB_x|$ for every vertex $x \in A$ can be done in $O(n\ell)$ time. This allows to select the root r of the spider in linear time. The sets A_r and C_r are easily constructed in linear time. Recent breakthroughs in edge coloring algorithms [2, 3] allow us to construct the edge coloring of H in $m^{1+o(1)}$ time. The final application of Lemma 3 is also easily implemented in time $O(\ell^2)$. Thus, the total time used for the construction of a $(2, \ell)$-spider is almost-linear in the size of $E(G)$.

4 Future Directions

Although Theorem 2 establishes the optimal minimum out-degree threshold that forces the existence of a $(2, \ell)$-spider in directed graphs, several natural questions remain open. One direction is to classify the extremal examples that achieve this threshold while avoiding a

$(2,\ell)$-spider, and to understand the structural features that make such constructions possible. A second direction is to extend the result to general (k,ℓ)-spiders and prove the Giant Spider Conjecture for directed graphs. It seems plausible that Theorem 2 can be strengthened in the setting of oriented graphs by proving that any oriented graph with minimum out-degree at least $\sqrt{2}\cdot\ell$ contains a $(2,\ell)$-spider, through a more refined application of similar methods. However, achieving further improvements beyond this bound is likely to require fundamentally new techniques.

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