Complexity of Firefighting on Graphs

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Abstract

We consider a pursuit-evasion game that describes the process of extinguishing a fire burning on the nodes of an undirected graph. We denote the minimum number of firefighters required by $\mathrm{ffn}(G)$ and provide almost sharp bounds to this graph parameter for complete binary trees. We show that deciding whether $\mathrm{ffn}(G) \leq m$ for given G and m is `NP-hard`. Furthermore, we show that shortest strategies can have superpolynomial length, leaving open whether the problem is in `NP`. We provide a construction that allows for transferring these results to a well-established Cops and Robbers variant called the "Hunter and Rabbit game".

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1 Introduction

We consider a game played on a simple undirected graph $G = (V, E)$. At the start of the game, we imagine all nodes of the graph to be on fire. A fixed number of firefighters are trying to extinguish the fire. Each round, every firefighter can extinguish one freely chosen node (without any restrictions like moving along edges), but must then leave to gather more water. In their absence, the fire spreads: Each node with a burning neighbour catches fire again. In particular, this can include nodes that have just been extinguished. We are interested in the smallest number of firefighters for which it is possible to extinguish the fire entirely, and call this number the *firefighter number* $\mathit{ffn}(G)$.

This problem was studied by Bernshteyn and Lee [7] in 2022.[1] They introduced a method of proving a lower bound on the firefighter number, which we improve, see Lemma 5.1. Furthermore, they showed that complete binary trees can have an arbitrarily high firefighter number. We extend this result by giving almost (up to an additive logarithmic term) tight bounds on the firefighter number of complete binary trees:

[1] They call it the "inspection number" of a graph.

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▶ **Theorem 7** (Bounds on the Firefighter Number of Binary Trees). *Let $\mathcal{B}_d$ be the complete binary tree of depth d. We have $ffn(\mathcal{B}_0) = 1$, $ffn(\mathcal{B}_1) = ffn(\mathcal{B}_2) = 2$, $ffn(\mathcal{B}_3) = ffn(\mathcal{B}_4) = 3$ and $ffn(\mathcal{B}_5), ffn(\mathcal{B}_6) \in \{3, 4\}$. For all $d \in \mathbb{N}_{\geq 7}$ it holds that*

$$\left\lfloor \frac{d-1}{2} \right\rfloor - \frac{1}{2} \log\left(\left\lfloor \frac{d-5}{2} \right\rfloor \right) - 2 < ffn(\mathcal{B}_d) \leq \left\lceil \frac{d}{2} \right\rceil + 1.$$

The game can also be interpreted as a pursuit-evasion game, specifically as a cops and robbers game with helicopter cops and an entirely invisible, omniscient robber with bounded speed. At any given time, the set of burning nodes corresponds to the set of still possible locations of the robber. The cops are not forced to move along edges, which was the case in the initial variant that was introduced by Tošić [22]. Pursuit-evasion games have been widely studied over the past few decades due to their broad range of applications. For example, the fire fighting version could be used to model the propagation of diseases or pests throughout certain ecosystems, i.e. how bark beetles spread between forests, and provide strategies to counter this spread even with limited resources.

A good overview of results regarding pursuit-evasion games can be found in the surveys by Alspach [2, 3], Bonato and Nowakowski [10], Bonato and Yang [11], Fomin and Thilikos [15], and Hahn [18]. For many variants, certain graph parameters, such as pathwidth or treewidth, can yield upper or lower bounds. In Table 1, we provide an overview of the relation between the firefighter number and some graph parameters.

A closely related variant of the problem discussed in this paper is the Hunter and Rabbit game, where the fugitive is forced to move to a neighbouring node in each time step (inspired by a rabbit moving when startled by a gunshot) while the hunters have no restrictions on which nodes to shoot at in every round. If the fugitive is also allowed to not move, this problem is equivalent to the firefighter variant. The Hunter and Rabbit game was studied by different researchers, including Abramovskaya, Fomin, Golovach and Pilipczuk [1], Bolkema and Groothuis [8], Britnell and Wildon [12], Gruslys and M'eroueh [17] and Haslegrave [19]. Among other results, they characterize the set of graphs with hunter number equal to one, and find the hunter number for certain graph classes. Similarly, we give a characterization of graphs G with $\text{ffn}(G) = 1$ and $\text{ffn}(G) = 2$ and determine the firefighter number for some graph classes in Section 3.

Apart from pursuit-evasion games, researchers are also studying some other problems related to a fire spreading on a graph: The graph burning problem (see e.g. [9] for a survey) concerns itself with the quickest way to burn an entire graph, where the player is allowed to set a new node on fire each turn, after which the fire spreads. In the firefighter problem (see e.g. [13] for a survey), a node of a graph is on fire and the fire spreads through the graph. The firefighters can (permanently) save a number of not yet burned nodes each turn, with the goal of containing the fire and saving the most nodes from being burned.

Although the concept of a fire spreading on a graph in these problems is similar to the problem we are concerned with, the underlying rules and objectives are fundamentally different, so to the best of our knowledge, there are no transferable results. In particular, if a node catches on fire, it is considered permanently burned in both of these problems, which gives them a monotonous nature. By contrast, a node may catch on fire and be extinguished an arbitrary amount of times in our problem.

The decision variants of many pursuit-evasion games are `NP-hard`, see [14] and [20]. In this paper, we analyze the following two decision problems.

> **(FireFighting): fire fighting**
> **Input:** A graph G and $m \in \mathbb{N}_{>0}$.
> **Output:** Is $\text{ffn}(G) \leq m$?

> **(FireFightingInTime): fire fighting within a given time horizon**
> **Input:** A graph G, $m \in \mathbb{N}_{>0}$ and $T \in \mathbb{N}_{>0}$.
> **Output:** Is $\text{ffn}(G) \leq m$ when restricted to at most T turns?

In 2025, Ben-Ameur and Maddaloni [6] proved that FireFightingInTime is `NP-hard` via a reduction from the 3-Partition problem. Moreover, they proved that the Hunter and Rabbit game is `NP-hard` on digraphs. In this variant, the movement of the rabbit is constrained to the direction of the edges. Together with Gahlawat, they extended their proof for undirected graphs [5]. We prove that FireFighting is `NP-hard`, which also implies that FireFighting on digraphs is `NP-hard`, since each undirected graph can be interpreted as a digraph where each edge has a reverse counterpart.

▶ **Theorem 11** (Hardness of FireFighting). *FireFighting is `NP-hard`.*

We provide a construction to show that Theorem 11 implies `NP-hardness` of the Hunter and Rabbit game, thus giving an alternative proof to the result by Ben-Ameur, Gahlawat and Maddaloni in [5]. Additionally, we show that FireFightingInTime is `NP-hard`, even on some heavily restricted graph classes.

▶ **Theorem 13** (Hardness of FireFightingInTime). *The problem FireFightingInTime is `NP-hard` even on trees. In particular, it is `NP-hard` even on trees with diameter at most 4 and on spiders (trees where at most one node has a degree greater than 2).*

Furthermore, we provide a class of graphs G_m whose shortest strategies can have superpolynomial length $T(G_m)$, leaving open whether FireFighting is in `NP`.

▶ **Theorem 18** (Shortest Strategies can have Superpolynomial Length). *There is an infinite class of graphs $(G_m)_{m \in \mathbb{N}_{\geq 2}}$ where the shortest strategies with $\text{ffn}(G_m)$ many firefighters have a length of at least $(m-1)!$, which is superpolynomial in $\text{size}(G_m) = \mathcal{O}(m^6)$.*

We can extend Theorem 18 to the Hunter and Rabbit game, answering one of the open questions in [5] regarding a bound on the length of shortest hunter strategies.

For all statements, the full rigorous proofs omitted in the paper can be found in the full version on arXiv [4].

2 Model

Any graphs mentioned in this paper are assumed to be simple and undirected. For a graph $G = (V, E)$ and a set $W \subseteq V$, we define the neighbourhood $N(W)$ as the set of nodes in $V \setminus W$ that are adjacent to at least one node in W. To keep the notation concise, we will sometimes refer to the node set of a graph G as G when it is clear from the context. For a number $n \in \mathbb{Z}$, we set $[n] = \{1, \ldots, n\}$ and $[n]_0 = \{0, 1, \ldots, n\}$.

Let us now introduce some basic notation for this game. A vector $S = (F_1, \ldots, F_T)$ with $F_i \subseteq V$ and $|F_i| \leq m$ for all $i \in [T]$ is called an *m-strategy* for G of length T. F_i is called the *firefighter set* at time i. Given an m-strategy $S = (F_1, \ldots, F_T)$, we define the set of *burning nodes* B_t at time t iteratively by setting $B_0 := V$ and $B_t := (B_{t-1} \setminus F_t) \cup N(B_{t-1} \setminus F_t)$ for

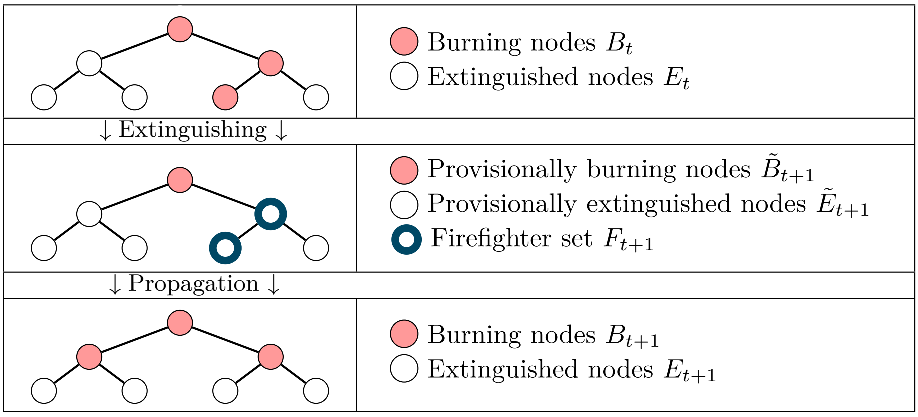


Figure 1 Two firefighters try to extinguish a partially burning tree. Visualization of the extinguishing and propagation process of the fire.

$t \geq 1$, where $F_t = \emptyset$ for $t > T$. Furthermore, let the set of *extinguished nodes* E_t at time $t \in \mathbb{N}_{\geq 0}$ be defined as $E_t \coloneqq V \setminus B_t$. For convenience, we write $\tilde{B}_t$ to denote $B_{t-1} \setminus F_t$, the *provisionally burning nodes*, as well as $\tilde{E}_t$ to denote $V \setminus \tilde{B}_t = E_t \cup F_t$, the *provisionally extinguished nodes*. A simple example of this extinguishing and fire propagation process is shown in Figure 1. An m-strategy for G is called a *T-winning m-strategy* if $B_T = \emptyset$ (or simply a *winning m-strategy* if T does not need to be specified). If there exists such a T-winning m-strategy for G, the graph G is called *m-winning* or, more precisely, *m-winning in time T*.

We denote the shortest possible length of a winning m-strategy for G by $T_m(G)$. If $\operatorname{ffn}(G) > m$, we set $T_m(G) = \infty$. We set $T(G) \coloneqq T_{\operatorname{ffn}(G)}(G)$ to be the length of the shortest possible winning strategy when using the smallest possible number of firefighters.

3 Basic Properties and Bounds

We characterize the classes of graphs with firefighter number one and two, and give some results for specific graph classes. Furthermore, Table 1 gives a short overview of some common graph parameters that can be used to find bounds on the firefighter number.

▶ **Corollary 1** (Characterization: $\operatorname{ffn}(G) = 1$ and $\operatorname{ffn}(G) = 2$). *For a graph $G = (V, E)$, it holds that $\operatorname{ffn}(G) = 1$ if and only if $|E| = 0$ and $\operatorname{ffn}(G) = 2$ if and only if $|E| > 0$ and any connected component of G is a caterpillar graph, i.e., a tree in which all the nodes are within distance 1 of a central path.*

▶ **Corollary 2** (Firefighter Number of K_n, C_n and $K_{n,m}$). *$\operatorname{ffn}(K_n) = n$, $\operatorname{ffn}(K_{n,m}) = \min\{n, m\} + 1$ and $\operatorname{ffn}(C_n) = 3$ for any $n, m \in \mathbb{N}_{>0}$, where K_n is a complete graph, $K_{n,m}$ is a complete bipartite graph and C_n is a circular graph.*

▶ **Corollary 3** (d-regular Graphs). *Let $d \in \mathbb{N}_{>0}$. Every d-regular graph G fulfills $\operatorname{ffn}(G) \geq d+1$. This bound is tight. For $d \in \{1, 2\}$ we have $\operatorname{ffn}(G) = d + 1$. For any $d \geq 3$, the firefighter number can reach arbitrarily high values.*

▶ **Corollary 4** (Order of a Forest). *For every forest $F = (V, E)$, we have $\operatorname{ffn}(F) \leq \log_3(2|V| + 1) + 2$.*

These results are direct consequences of the following useful lemmata providing bounds on the firefighter number. In particular, Lemma 5.1 will be an essential tool for proving theorems in later sections. It is a strictly stronger version of a criterion given in [7]. In their version, it is necessary to prove that for some $i \in [|V|]$, any $W \subseteq V$ with $i-m+1 < |W| < i$ fulfills $|N(W)| \geq m-1$, in order to prove $\text{ffn}(G) \geq m$. In our version, one only has to consider subsets W with $|W| = i$ instead of subsets W with an entire range of sizes, which makes applying the lemma significantly easier. As this Lemma is one of the only known techniques to show lower bounds, this is a valuable improvement. Note that Lemma 5.1 can give arbitrarily bad lower bounds, as shown in the full version [4].

▶ **Lemma 5** (Lower Bounds). *Let $G = (V, E)$ be a graph and $m \in \mathbb{N}_{>0}$. Then any of the following conditions imply that ffn$(G) \geq m$:*

1. *There exists an $i \in [|V|-m+1]$ such that any $W \subseteq V$ with $|W| = i$ fulfills $|N(W)| \geq m-1$.*
2. *$\delta_{\min}(G) \geq m-1$, i.e., each node in G has at least $m-1$ neighbours.*
3. *There exists a $G' \subseteq G$ with ffn$(G') \geq m$.*
4. *$|E| \geq m \cdot (|V| - \frac{m+1}{2})$.*

Proof of 1. Assume that there is a winning $(m-1)$-strategy $(F_1, \dots, F_T)$ for G, while an i as in the statement of the lemma exists. Then, there has to be a smallest t such that $|B_t| < i + (m-1)$. Note that $|B_0| = |V| \geq i + (m-1)$, hence $t \geq 1$. By the definition of t, we have $|B_{t-1}| \geq i + (m-1)$. Since $|F_t| \leq m-1$, it follows that $|\tilde{B}_t| \geq i$. In particular, this implies the existence of a set $\tilde{W} \subseteq \tilde{B}_t$ with $|\tilde{W}| = i$. Let $W := \tilde{W} \cup N(\tilde{W})$, i.e., the set of burning nodes after the fire spreads from the provisionally burning nodes $\tilde{W}$. Since $|N(\tilde{W})| \geq m-1$ by the definition of i, we have $|W| \geq |\tilde{W}| + (m-1) = i + (m-1)$. $\tilde{W}$ being a subset of $\tilde{B}_t$ implies $W \subseteq B_t$ and hence, we have $|B_t| \geq i + (m-1)$, a contradiction. ◀

▶ **Lemma 6** (Upper Bounds). *Let $G = (V, E)$ be a graph and $m \in \mathbb{N}_{>0}$. Then any of the following conditions imply that ffn$(G) \leq m$:*

1. *pw$(G) + 1 \leq m$, where pw(G) denotes the pathwidth of G.*[2]
2. *G is a tree with diam$(G) \leq 2m-2$.*
3. *There exists a graph $G_k = (V_k, E_k)$ for $k \in \mathbb{N}_{\geq 0}$ with ffn$(G_k) \leq m-k$, $G_k \subseteq G$ and $|V_k| \geq |V| - k$.*

Table 1 Tight upper and lower bounds on the firefighter number in terms of some graph parameters. For every missing entry except for the question mark, we prove that a bound based on this parameter is not possible. Proofs based on Lemma 5, Lemma 6, and [7] for the entries are given in the full version [4].

Graph parameter	Upper bound on ffn	Lower bound on ffn
Minimum degree $\delta_{\min}$	-	$\delta_{\min} + 1 \leq \text{ffn}(G)$
Maximum degree $\delta_{\max}$	-	-
Order $\|V\|$	$\text{ffn}(G) \leq \|V\|$	-
Depth d of a tree T	$\text{ffn}(T) \leq d + 1$	-
Beta index $\|E\|/\|V\|$	-	$\|E\| \leq (\text{ffn}(G) - 1) \cdot (\|V\| - \text{ffn}(G)/2)$
Pathwidth pw(G)	$\text{ffn}(G) \leq \text{pw}(G) + 1$	-
Treewidth tw(G)	-	?
Vertex cover number vcn(G)	$\text{ffn}(G) \leq \text{vcn}(G) + 1$	-

[2] Included just for the sake of completeness, has already been proven in [7].

4 Firefighting on Complete Binary Trees

A complete binary tree is a rooted tree in which the distance of any leaf node to the root is the same and every non-leaf node has exactly two children. In [7] it was shown that for every k there exists a complete binary tree $T = (V, E)$ such that $\mathrm{ffn}(T) > k$. We extend this result by giving almost tight bounds on the firefighter number of complete binary trees.[3]

▶ **Theorem 7** (Bounds on the Firefighter Number of Binary Trees)**.** *Let $\mathcal{B}_d$ be the complete binary tree of depth d. We have $\mathit{ffn}(\mathcal{B}_0) = 1$, $\mathit{ffn}(\mathcal{B}_1) = \mathit{ffn}(\mathcal{B}_2) = 2$, $\mathit{ffn}(\mathcal{B}_3) = \mathit{ffn}(\mathcal{B}_4) = 3$ and $\mathit{ffn}(\mathcal{B}_5), \mathit{ffn}(\mathcal{B}_6) \in \{3, 4\}$. For all $d \in \mathbb{N}_{\geq 7}$ it holds that*

$$\left\lfloor \frac{d-1}{2} \right\rfloor - \frac{1}{2} \log \left(\left\lfloor \frac{d-5}{2} \right\rfloor \right) - 2 < \mathit{ffn}(\mathcal{B}_d) \leq \left\lceil \frac{d}{2} \right\rceil + 1.$$

Proof sketch. To prove the upper bound, we provide a construction to extend a given winning k-strategy for $\mathcal{B}_{2d}$ to a winning $(k+1)$-strategy for $\mathcal{B}_{2d+1}$ that extinguishes the root node of $\mathcal{B}_{2d+1}$ in each step. Based on this specific strategy for $\mathcal{B}_{2d+1}$, we then construct a winning $(k+1)$-strategy for $\mathcal{B}_{2d+2}$.

We prove the lower bound by showing that we can apply Lemma 5.1 for some adequately chosen i, i.e., that any subset $W \subseteq \mathcal{B}_d$ with $|W| = i$ has at least $\lfloor \frac{d-1}{2} \rfloor - \frac{1}{2} \log_2 \left(\lfloor \frac{d-5}{2} \rfloor \right) - 2$ neighbours. To that end, we observe that any subset W of nodes in $\mathcal{B}_d$ can be constructed by iteratively adding or removing complete binary trees of decreasing size, as shown in Figure 2.

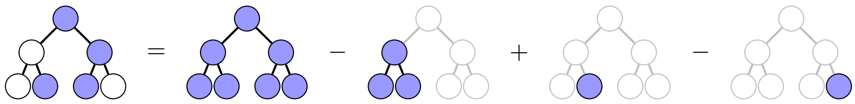


Figure 2 Decomposing a subset of a complete binary tree into multiple complete binary trees of varying depths.

The number of binary trees that is used in such a decomposition corresponds to the number of edges between W and the rest of the graph, which can be bounded in terms of the number of neighbours of W. Since the number of nodes in a complete binary tree equals $2^x - 1$ for some $x \in \mathbb{N}_{>0}$, we can use this decomposition to express the size of W as a sum of powers of two and their negatives, where the total number of summands is again bounded in terms of the number of neighbours of W. By applying some concepts from information theory (in particular the Hamming weight of the binary representation of a number), we show that representing i as such a sum requires a certain number of summands, which then implies that any $W \subseteq \mathcal{B}_d$ with $|W| = i$ has to have at least $\lfloor \frac{d-1}{2} \rfloor - \frac{1}{2} \log_2 \left(\lfloor \frac{d-5}{2} \rfloor \right) - 2$ neighbours, finishing the proof. ◀

5 NP-Hardness

Ben-Ameur and Maddaloni recently proved that FireFightingInTime is `NP-hard` (via a reduction from the 3-Partition problem) in [6]. Moreover, they proved that a directed variant of the Hunter and Rabbit game where the rabbit moves on a digraph is also `NP-hard`. However, their proof cannot be generalized to the undirected setting. Nevertheless, in this section, we use a novel construction to show that both FireFighting and FireFightingInTime are

[3] For a complete binary tree $\mathcal{B} = (V, E)$, the difference between our upper and lower bound is in $\mathcal{O}(\log(\log(|V|)))$.

`NP-hard` on undirected graphs as well. Proving that FireFighting is `NP-hard` also implies that FireFighting on digraphs is `NP-hard`, since each undirected graph can be interpreted as a digraph where each edge has a reverse counterpart. We also improve on their results by showing that FireFightingInTime is `NP-hard` even on trees with diameter at most 4 and on spiders (i.e. trees where at most one node has a degree greater than 2).

In order to show `NP-hardness` of FireFighting, we will build a gadget $H(G,T)$ for an arbitrary graph G such that there is a T-winning m-strategy for G if and only if $\text{ffn}(H(G,T)) \leq 4m$.[4] The main idea of this gadget is to attach the 2-blowup $\mathbb{G}$ of G (i.e. the graph that arises by replacing every node of G with a 2-clique and adding edges between all nodes of two cliques if the original two nodes of G were connected) to a circular structure consisting of a lower and an upper part, which serve as a timed fuse and as an interface between $\mathbb{G}$ and the fuse. As a result, the only way to extinguish the whole graph with $4m$ firefighters is to first extinguish the entire lower part, then move along the circle, extinguish $\mathbb{G}$ as fast as possible, and finally catch the fire that spreads in the lower part, right in time before it spreads to the upper part again. This is exactly possible if there is a T-winning m-strategy for G. To utilize this gadget we can use the `strongly NP-complete` BinPacking problem [16] to build a graph such that there is a T-winning m-strategy if and only if the BinPacking instance is a yes-instance, which proves that BinPacking can be reduced to FireFighting. Thus, we can conclude that FireFighting is `NP-hard`.

▶ **Definition 8** (Gadget $H(G,T)$)**.** *Let G be an arbitrary graph, $m \in \mathbb{N}_{>0}$ and $T \in \mathbb{N}_{\geq 2}$. The graph $H(G,T)$ is defined by a block structure, where the blocks X, Y, Z are $2m$-cliques, $\mathbb{G}$ is the 2-blowup of G, and all other blocks are m-cliques. Finally, we add an edge between every pair of nodes from two different blocks from the following list of block combinations: $(\mathbb{G}, Y), (A, X), (X, Y), (Y, Z), (Z, B)$, (A, P_i^1) for all $i \in [2T+2]$, (B, P_i^{T+1}) for all $i \in [2T+2]$ and (P_i^j, P_i^{j+1}) for all $i \in [2T+2]$ and $j \in [T]$.*

See Figure 3 for a visualization of this gadget. We further define the i-th path $P_i \coloneqq \bigcup_{j\in[T+1]} P_i^j$ and the set of all paths as $\mathcal{P} \coloneqq \bigcup_{i\in[2T+2]} P_i$.
Due to the following helpful lemma, we can restrict our analysis to strategies with firefighter sets such that each firefighter set either contains all nodes of a certain clique or none.

▶ **Lemma 9** (Cliques)**.** *Let G be an arbitrary graph containing a clique K with $N(v) \setminus K = N(w) \setminus K$ for all $v, w \in K$, and let S be a T-winning m-strategy. Then there exists a T-winning m-strategy $S' = (F_1', \ldots, F_T')$ such that for all $i \in [T]$ and $v \in K$, we have $v \in F_i'$ if and only if $K \subseteq F_i'$.*

▶ **Proposition 10** (Time Gadget)**.** *Let G be a graph. There is a T-winning m-strategy for G if and only if $\mathit{ffn}(H(G,T)) \leq 4m$.*

Proof sketch. If there is a T-winning m-strategy for G, we can explicitly state a winning $4m$-strategy for $H(G,T)$, which implies $\text{ffn}(H(G,T)) \leq 4m$.

Next, we assume that there is no T-winning m-strategy for G. Using Lemma 9, it is easy to see that this is equivalent to there not being a T-winning $2m$-strategy for $\mathbb{G}$. Before giving the technical details of the actual rigorous proof, let us first give some intuition on why this means that no $4m$-winning strategy for $H(G,T)$ exists. Every node in X, Y, Z, A or B has degree at least $4m$, so the first nodes to be extinguished have to be either in $\mathbb{G}$ or in $\mathcal{P}$.

[4] Note that this construction cannot serve as a polynomial reduction from FireFightingInTime to FireFighting, since the size of $H(G,T)$ will be quadratic in T and therefore not polynomial in the encoding size of the FireFightingInTime-instance (G,m,T).

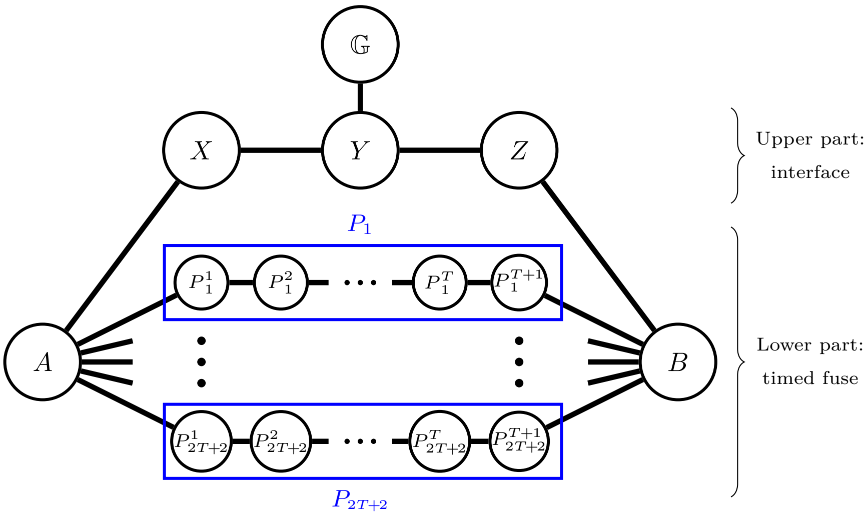


Figure 3 Gadget graph $H(G,T)$ with the property that there is a T-winning m-strategy for G if and only if $\text{ffn}(H(G,T)) \leq 4m$. The blocks X, Y, Z are $2m$-cliques and every P_i^j and the blocks A, B are m-cliques. An edge between two blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ in this image corresponds to connecting every node in $\mathcal{B}_1$ to every node in $\mathcal{B}_2$.

If we start by extinguishing nodes in $\mathbb{G}$, we either do not extinguish all nodes in Y afterwards, which lets all of the nodes in $\mathbb{G}$ reignite, or we keep extinguishing all nodes in Y. In the second case, we use $2m$ firefighters on Y, leaving only $2m$ firefighters for the rest of the graph. As each node in the graph outside of $\mathbb{G}$ has degree greater than $2m$, it is not possible to make any further progress.

Let us instead start by extinguishing nodes in $\mathcal{P}$. We can initially reach the point where $\mathcal{P}$ and A are completely extinguished. Trying to extinguish B next only mirrors the position, as we would lose A in turn. Extinguishing X requires all $4m$ firefighters (positioned in X and Y) and hence the last block of every path P_i catches on fire again. We can now stop the fire from spreading back to X and start working on extinguishing $\mathbb{G}$ by placing $2m$ firefighters in Y. If we try to extinguish $\mathbb{G}$ with the remaining $2m$ firefighters, it will take us at least $T+1$ steps. By Lemma 9, we can assume that we either use no firefighters or at least m firefighters in the path P_i for any $i \in [2T+2]$. Hence, with $2m$ firefighters in T steps, we can influence at most $2T$ of the $2T+2$ paths, meaning that at least 2 of the paths are fully burning again after T steps. Therefore, when we finish extinguishing $\mathbb{G}$, the block A will already be reignited. If we do not want the entire block $\mathbb{G}$ to reignite, we have to extinguish Y again. The only possible progress with the remaining $2m$ firefighters at this stage is to position them at Z, so that the nodes in Y actually stay extinguished for the first time. However, then only Y and $\mathbb{G}$ are extinguished, and one can see that it is not possible to extinguish any additional block without reverting to a (possibly mirrored) previous state of the game. As we have exhausted all reasonable solution approaches, there can be no $4m$-winning strategy for $H(G,T)$.

Let us now give a brief overview of the technicalities of our proof. In order to avoid nested case distinctions, we instead analyze the subsets between which the set of burning nodes in the graph can transition under a $4m$-strategy. In particular, we consider the following node subsets: $\Omega_1 = H(G,T) \setminus (A \cup \mathcal{P})$, $\Omega_2 = H(G,T) \setminus (\mathbb{G} \cup \mathcal{P})$, $\Omega_3 = \mathbb{G} \cup B \cup Y \cup Z \cup \bigcup_{i \in [2T+2]} P_i^{T+1}$, $\Omega_4 = B \cup Y \cup Z \cup P_k \cup P_\ell \cup \{v\}$, $\Omega_5 = A \cup B \cup Y \cup Z \cup P_k \cup P_\ell$, $\Omega_6 = A \cup B \cup X \cup Z \cup P_k \cup P_\ell$, and $\Omega_7 = A \cup B \cup Y \cup Z \cup P_k \cup (P_\ell \setminus P_\ell^1) \cup \{v\}$ where v is any node from $\mathbb{G}$ and $k, \ell \in [2T+2]$ with $k \neq \ell$. We call a subset of burning nodes Ω_ℓ*-blocked*, if it contains Ω_ℓ or one of its symmetric variants, regarding the following symmetries: Switching A and B, X and Z as well as P_i^j with P_i^{T+2-j} for all $i \in [2T+2], j \in [T+1]$ (i.e., mirroring the graph as shown in Figure 3 horizontally), switching the complete paths $\{P_1, \ldots, P_{2T+2}\}$ according to any permutation, or replacing v by any other node in $\mathbb{G}$.

In the full version [4] we prove that for any $4m$-strategy and any subset of burning nodes that is Ω_n-blocked for some $n \in [7]$, after finitely many steps, the subset of burning nodes will be $\Omega_{n'}$-blocked for some $n' \in [7]$. Since the initial state of a fully burning graph is Ω_1-blocked, this means that there is no winning $4m$-strategy for $H(G, T)$, as the empty set is not Ω_n-blocked for any $n \in [7]$. ◀

▶ **Theorem 11** (Hardness of FireFighting). *FireFighting is* `NP-hard`*.*

Proof. Let an instance of the following decision problem be given.

(BinPacking):
Input: Items $i_1, \ldots, i_n$, sizes $s_1, \ldots, s_n \in \mathbb{N}_{>0}$ and $b \in \mathbb{N}_{>0}$ bins with capacity $c \in \mathbb{N}_{>0}$ each, such that $\sum_{k \in [n]} s_k \leq b \cdot c$.
Output: Is there a packing of all items into the bins fulfilling the capacity constraints?

We construct a graph G as $\bigcup_{k \in [n]} K_{s_k}$, i.e., the graph G consists of n connected components, which are complete graphs of varying sizes. This graph G can be extinguished by c firefighters in at most b steps if and only if the BinPacking instance is a yes-instance, since we can assume that only full cliques are extinguished in each step due to Lemma 9. By using the construction from Proposition 10 with this G and $T = b$, we reduce this instance of BinPacking to determining if $\mathrm{ffn}(H(G, T)) \leq 4c$. Note that the size of this instance of FireFighting is in $\mathcal{O}(c^4 \cdot b^2)$ and therefore polynomial in the unary encoding size of the BinPacking problem. Since BinPacking is `strongly NP-complete` [16], it follows that in general, determining whether a given graph can be extinguished by m firefighters is `NP-hard`. ◀

Note that this Theorem implies that FireFighting is even `strongly NP-hard`, since for any non-trivial instance, m is bounded by $|V|$, so the input size of the problem remains in $\mathcal{O}(\mathrm{size}(G))$ even if m is encoded in unary.

While the proof strategy for Theorem 11 is based on first showing that an `NP-hard` problem can be reduced to solving an instance of FireFightingInTime and this specific instance is afterwards shown to be equivalent to an instance of FireFighting, we cannot generally reduce FireFightingInTime to FireFighting with our construction: Since the number of vertices of $H(G, T)$ is greater than T^2, the size of the graph $H(G, T)$ is in general not polynomial in the encoding size of G, T and m. However, a straightforward polynomial reduction for the converse direction exists:

▶ **Proposition 12** (Reducing FireFighting to FireFightingInTime). *FireFighting can be polynomially reduced to FireFightingInTime.*

Proof. Let G be the given graph and m the given number of firefighters. G can be solved by m firefighters if and only if it can be solved by m firefighters in at most $2^{|V(G)|}$ steps, since any shortest winning m-strategy will not reach any burning set more than once. As the encoding length of $2^{|V(G)|}$ is in $\mathcal{O}(|V(G)|)$, this reduction is polynomial in the encoding size of the initial question. ◀

In particular, this immediately implies that FireFightingInTime is NP-hard as well. However, we can go even further:

▶ **Theorem 13** (Hardness of FireFightingInTime). *The problem FireFightingInTime is NP-hard even on trees. Moreover, it is NP-hard even on trees with diameter at most 4 and on spiders (trees where at most one node has a degree greater than 2).*

Proof sketch. We prove this via a reduction from 3-Partition, which is strongly NP-hard, see [16]. For some $k \in \mathbb{N}_{>0}$, let $a_1, \ldots, a_{3k} \in \mathbb{N}_{>0}$ be the positive integer numbers in a given instance of 3-Partition, and set $s = \sum_{i=1}^{3k} a_i$. Without loss of generality, we may assume $\frac{s}{k} \in \mathbb{N}_{>0}$. Otherwise, a 3-partition of the numbers trivially cannot exist.

We now construct a graph G which we claim is $(\frac{s}{k} + 3s + 1)$-winning in time k if and only if there exists a 3-partition of $a_1, \ldots, a_{3k}$ (this claim is proven in the full version [4]). Let T_i be an arbitrary tree with $a_i + s$ nodes for each $i \in [3k]$. Then the graph G arises by adding a new node c and, for each $i \in [3k]$, adding an edge between c and an arbitrary node from T_i as visualized in Figure 4. Note that $|G| = s + 3sk + 1 = k \cdot (\frac{s}{k} + 3s + 1) - (k - 1)$, which implies that at most $k - 1$ nodes can be reignited during a successful strategy in time k with $\frac{s}{k} + 3s + 1$ firefighters. By choosing a star graph (resp. a path graph) for each T_i and attaching c to the internal node (resp. to an end of the path), we get the result for trees with diameter ≤ 4 (resp. for spiders). ◀

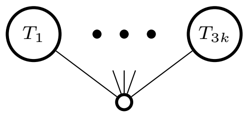


Figure 4 Construction of G. Every T_i is an arbitrary tree with $a_i + s$ nodes.

6 Graphs with Long Shortest Strategies

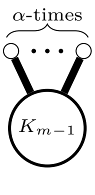
After proving that the problems FireFighting and FireFightingInTime are NP-hard, a naturally arising question is whether these problems are in NP, i.e., whether there exists a polynomial certificate for yes-instances. A natural candidate for such a certificate would be a winning m-strategy (resp. winning m-strategy in time T), since a strategy can be verified in polytime with respect to its size and the size of the graph. In this section, we will show that such a straightforward approach to attain a polynomial certificate does not work, by giving a class of graphs where the shortest possible winning m-strategy takes superpolynomially (in the size of the graph) many steps.

To this end, we first define the auxiliary graph H_m (see Figure 5 (a)), which has the useful property that any winning m-strategy on H_m has to use at least $m - 1$ firefighters for a certain number of consecutive steps. Here, α and β (which will be used in the upcoming definitions) denote some fixed values in $\mathbb{N}_{>0}$ that fulfill $2\beta + 2 \geq \alpha \geq \beta + 3$, e.g., $\beta = 1, \alpha = 4$. In particular, they do not depend on m or X.

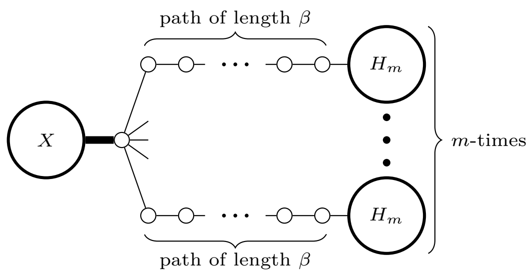
▶ **Definition 14** (Auxiliary Graph H_m). *For $m \geq 2$, we set $H_m = (V, E)$ with $V = \{v_1, \ldots, v_{m-1}\} \cup \{w_1, \ldots, w_\alpha\}$ and $E = \{\{v_i, v_j\} : i, j \in [m-1], i \neq j\} \cup \{\{v_i, w_j\} : i \in [m-1], j \in [\alpha]\}$.*

Next, we construct a graph $G(m, X)$ (see Figure 5 (b)) with $\text{ffn}(G(m, X)) = m$ that contains a subgraph X with $\text{ffn}(X) = m - 1$. We will prove that any m-winning strategy for $G(m, X)$ needs to fully extinguish the graph X with $m - 1$ firefighters at least $m - 1$ times.

▶ **Definition 15** ($G(m, X)$)**.** *For any $m \in \mathbb{N}_{\geq 2}$ and graph X with $\mathit{ffn}(X) = m - 1$, the graph $G(m, X)$ arises in the following way: First, add an additional node c to X, which shares an edge with every node in X. Next, add m paths with β nodes $(v_1^i, \ldots, v_\beta^i)$ for $i \in [m]$, and for each $i \in [m]$, connect c to v_1^i. Finally, add m auxiliary graphs H_m as defined previously, say $H_m^1, \ldots, H_m^m$, and for each $i \in [m]$, connect v_β^i to one arbitrary node u_i of the $(m-1)$-clique contained in H_m^i.*



(a) H_m.



(b) $G(m, X)$.

Figure 5 When a thick edge connects two subgraphs A and B, then every node in A is connected to every node in B. The rightmost node of each path of length β is connected to exactly one (arbitrary) node of K_{m-1} of the corresponding H_m.

▶ **Lemma 16** (Firefighter Number of $G(m, X)$)**.** *$\mathit{ffn}(G(m, X)) = m$.*

Proof sketch. By construction, the graph $G(m, X)$ contains the subgraph H_m, which in turn contains a m-clique. This shows $\text{ffn}(G(m, X)) \geq m$.

To give some intuition for why we have $\text{ffn}(G(m, X)) \leq m$, let us now sketch an outline of a winning m-strategy. Start by extinguishing $H_m^1 \setminus \{u_1\}$, which takes α steps. Next, extinguish the path from u_1 to c and position one firefighter in c while extinguishing X with the remaining $m - 1$ firefighters. Afterwards, it is possible to extinguish all other paths up to the u_i nodes without letting the fire spread back, so that only $H_m^2, \ldots, H_m^m$ still contain burning nodes. Then, continue by extinguishing the next auxiliary subgraph H_m^2. During this process, the fire barely does not reach H_m^1, which allows us to again extinguish the entire rest of the graph apart from $H_m^3, \ldots, H_m^m$. By repeating this process $m - 2$ more times, the entire graph is extinguished. ◀

After determining that $\text{ffn}(G(m, X)) = m$, we shall now find a lower bound to the length of a winning m-strategy for $G(m, X)$ by showing that such a strategy needs to be similar to the strategy described in the proof sketch of the previous lemma, and therefore needs to repeatedly extinguish X with $m - 1$ firefighters.

▶ **Lemma 17** (Lower Bound on $T(G(m, X))$)**.** *$T(G(m, X)) \geq (m - 1) \cdot T(X)$.*

By recursive applications of the construction $G(m, X)$, we can give a class of graphs such that the length of a shortest extinguishing strategy with the smallest possible number of firefighters is superpolynomial in the size of the graph.

▶ **Theorem 18** (Shortest Strategies can have Superpolynomial Length)**.** *Let $G_2 = G(2, (\{v\}, \emptyset))$ and $G_m = G(m, G_{m-1})$ for any $m \in \mathbb{N}_{\geq 3}$. For any $m \in \mathbb{N}_{\geq 2}$, we have $\mathit{ffn}(G_m) = m$ and $T(G_m) = T_m(G_m) \geq (m - 1)!$, which is superpolynomial in $\mathit{size}(G_m) = \mathcal{O}(m^6)$.*

Proof. By Lemmata 16 and 17, we have $\mathrm{ffn}(G_2) = 2$ and $T(G_2) \geq 1$. Using induction and the same two lemmata, we get $\mathrm{ffn}(G_m) = m$ and $T(G_m) \geq (m-1) \cdot T(G_{m-1}) \geq (m-1)!$ for any $m \geq 2$. Moreover, $G(m, X) \setminus X$ contains $1 + m \cdot (m - 1 + \alpha + \beta)$ nodes, which is in $\mathcal{O}(m^2)$. Since G_m is composed of $m - 1$ graphs that do not have a greater number of nodes than $G(m, X) \setminus X$, it follows that G_m has $\mathcal{O}(m^3)$ nodes and therefore has size in $\mathcal{O}(m^6)$. As $(m-1)!$ is superpolynomial in m^6, this finishes the proof. ◀

While the above theorem underlines the possibility that FireFightingInTime as well as FireFighting are not in NP, we can at least give an upper bound to their space complexity.

▶ **Proposition 19** (FireFighting is in PSPACE). *FireFightingInTime (and therefore also FireFighting) is in PSPACE.*

Proof. Given an instance $(G = (V, E), m, T)$ of FireFightingInTime, consider the following algorithm: Set $B = V$. Repeat the following steps T times: Pick a random $F \subseteq V$ with $|F| \leq m$. Set $\tilde{B} = B \setminus F$. Set $B = \tilde{B} \cup N(\tilde{B})$. If $B = \emptyset$ after T repetitions, return "yes", otherwise return "no". This non-deterministic algorithm has a probability strictly greater than 0 to return "yes" if (G, m, T) is a yes-instance of FireFightingInTime, and will always return "no" otherwise. Furthermore, the required space is in $\mathcal{O}(\mathrm{size}(G))$, so FireFightingInTime is in NPSPACE and, by the Theorem of Savitch [21], in PSPACE. Proposition 12 extends this result to FireFighting. ◀

7 Implications on the Rabbit and Hunter Game

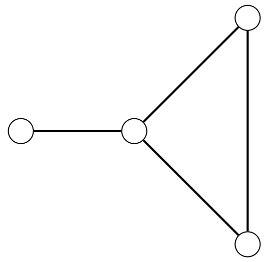
In this section, we show that the hardness and the existence of long shortest strategies transfer to the Rabbit and Hunter game. The only difference between FireFighting and the Rabbit and Hunter game is that the fugitive is not forced to move in each turn, i.e., we need to redefine $B_t := N(B_{t-1} \setminus F_t)$ to use our notation. To distinguish between settings, we will use $\mathrm{hn}(G)$ instead of $\mathrm{ffn}(G)$, m-hunter strategy instead of m-(firefighter-)strategy and $T^h(G)$ for the shortest m-hunter strategy instead of $T(G)$ for the shortest m-firefighter strategy.

(Hunting): Rabbit and Hunter game
Input: A graph G and $m \in \mathbb{N}_{>0}$.
Output: Is $\mathrm{hn}(G) \leq m$?

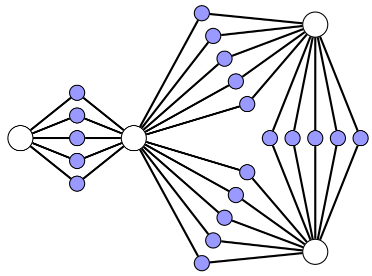
Let $G' = (V', E')$ be the graph that arises by replacing each edge $e \in E$ from a graph $G = (V, E)$ with $|V| + 1$ many disjoint paths, each with one intermediate node as visualized in Figure 6. Note that G' is bipartite with partition sets V and $V' \setminus V$.

▶ **Lemma 20.** *$\mathrm{ffn}(G) = \mathrm{hn}(G')$ and $2T(G) = T^h(G')$.*

Proof. Since G' is bipartite, the rabbit's position alternates between nodes in V and $V' \setminus V$. Let $F = (F_1, F_2, \ldots, F_n)$ be a winning $\mathrm{ffn}(G)$-firefighter strategy on G. If the rabbit starts in a node $v \in V$, it will be in $\{v\} \cup \{w \in V : \exists\{v, w\} \in E\}$ after two time steps. Thus, the strategy $(F_1, \emptyset, F_2, \emptyset, \ldots, F_{n-1}, \emptyset, F_n)$ catches the rabbit if it starts in V, and the strategy $(\emptyset, F_1, \emptyset, F_2, \emptyset, \ldots, F_{n-1}, \emptyset, F_n)$ catches the rabbit if it starts in $V' \setminus V$. Hence, $F' = (F_1, F_1, F_2, F_2, \ldots, F_n, F_n)$ is a winning $\mathrm{ffn}(G)$-hunter strategy. This implies $\mathrm{hn}(G') \leq \mathrm{ffn}(G)$ and $2T(G) \geq T^h(G')$.



(a) $G = (V, E)$.



(b) $G' = (V', E')$.

Figure 6 $G' = (V', E')$ arises from $G = (V, E)$ by replacing each edge $e \in E$ with $|V| + 1$ many disjoint paths, each with one intermediate node. Note that G' is bipartite.

Suppose $\text{hn}(G') < \text{ffn}(G)$. Let $F' = (F'_1, \ldots, F'_{n'})$ be a winning $\text{hn}(G')$-hunter strategy on G'. If $u \in \tilde{B}_t$ or $v \in \tilde{B}_t$ for some $\{u, v\} \in E$ and $t \in \mathbb{N}_{>0}$, then all intermediate nodes between u and v in G' are in B_t. Since there are $|V| + 1 > |V| \geq \text{ffn}(G) > \text{hn}(G')$ many of these nodes, at least one of them is in $\tilde{B}_{t+1}$. Therefore, we have $u, v \in B_{t+1}$, independent of the hunter strategy. This yields that the pruned hunter strategy $F'' = (F'_1 \cap V, \ldots, F'_{n'} \cap V)$ is also a winning $\text{hn}(G')$-hunter strategy on G'. The even and odd substrategies $(F''_{2i})_{i \in [\lfloor n'/2 \rfloor]}$ and $(F''_{2i-1})_{i \in [\lceil n'/2 \rceil]}$ are both winning $\text{hn}(G')$-firefighter strategies on G, since $B_{2t} \cap V$ in G' under hunter strategy F'' coincides with B_t in G under firefighter strategy $(F''_{2i})_{i \in [\lfloor n'/2 \rfloor]}$, and $B_{2t-1} \cap V$ in G' under hunter strategy F'' coincides with B_t in G under firefighter strategy $(F''_{2i-1})_{i \in [\lceil n'/2 \rceil]}$. This implies that both $(F''_{2i})_{i \in [\lfloor n'/2 \rfloor]}$ and $(F''_{2i-1})_{i \in [\lceil n'/2 \rceil]}$ are $hn(G')$-winning firefighter strategies for G. Hence, we have $\text{ffn}(G) \leq \text{hn}(G')$, which is a contradiction. Additionally, if F' was already a shortest winning $\text{hn}(G')$-hunter strategy, this implies that $2T(G) \leq T^h(G')$. We conclude $\text{ffn}(G) = \text{hn}(G')$ and $2T(G) = T^h(G')$. ◀

Observe that $\text{size}(G') \in \mathcal{O}(\text{size}(G)^2)$. Together with Theorem 11 and Theorem 18, this implies the following.

▶ **Corollary 21** (Hardness of Hunting and Long Shortest Strategies). *Hunting is* `NP-hard` *even on bipartite graphs and there exists an infinite family of graphs for which shortest hunter-strategies have superpolynomial length in their respective sizes.*

Corollary 21 answers one of the open questions in [5] regarding a bound on the length of shortest hunter strategies. Note that the `NP-hardness` of Hunting on bipartite graphs has also been shown in [5], using a completely different approach.

8 Open Problems

In Corollary 1, we gave a characterization of graphs with firefighter number one or two. A naturally arising question is whether it is possible to find a somewhat compact characterization for graphs with firefighter number three, or other bigger fixed numbers.

Table 1 covers some relations between the firefighter number and commonly used graph parameters. It is an interesting question whether one can relate the firefighter number to other graph parameters. In particular, we think that it would be worthwhile to investigate the existence of a lower bound based on the treewidth.

While we have proven that FireFighting is `NP-hard` and in `PSPACE`, one could investigate the placement of the problem within the polynomial hierarchy in more detail.

If applying Lemma 5.1 to all subgraphs suffices to characterize the firefighter number for some class of graphs, we can show that FireFighting restricted to this class is in `co-NP`, if either the number of firefighters m or the treewidth of the graphs is bounded by a constant. This would imply that these restricted variants are not `NP-hard`, unless `NP` equals `co-NP`. However, it is an open problem which graph classes satisfy this property, see Open Problem 70 in [7].

Finally, we are interested in algorithms to solve FireFighting or FireFightingInTime for medium-sized instances. The canonical IP formulation requires $\Omega(T \cdot |V|)$ integer variables for FireFightingInTime and $\Omega(2^{|V|} \cdot |V|)$ integer variables for FireFighting, which is already problematic for very small graphs.

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