The S-Hamiltonian Cycle Problem

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Abstract

Determining if an input undirected graph is Hamiltonian, i.e., if it has a cycle that visits every vertex exactly once, is one of the most famous NP-complete problems. We consider the following generalization of Hamiltonian cycles: for a fixed set S of natural numbers, we want to visit each vertex of a graph G exactly once and ensure that any two consecutive vertices can be joined in k hops for some choice of $k \in S$. Formally, an *S-Hamiltonian cycle* is a permutation $(v_0, \ldots, v_{n-1})$ of the vertices of G such that, for $0 \leq i \leq n-1$, there exists a walk between v_i and $v_{i+1 \bmod n}$ whose length is in S. (We do not impose any constraints on how many times vertices can be visited as intermediate vertices of walks.) Of course Hamiltonian cycles in the standard sense correspond to $S = \{1\}$. We study the *S-Hamiltonian cycle problem* of deciding whether an input graph G has an S-Hamiltonian cycle. Our goal is to determine the complexity of this problem depending on the fixed set S. It is already known that the problem remains NP-complete for $S = \{1, 2\}$, whereas it is trivial for $S = \{1, 2, 3\}$ because any connected graph contains a $\{1, 2, 3\}$-Hamiltonian cycle.

Our work classifies the complexity of this problem for most kinds of sets S, with the key new results being the following: we have NP-completeness for $S = \{2\}$ and for $S = \{2, 4\}$, but tractability for $S = \{1, 2, 4\}$, for $S = \{2, 4, 6\}$, for any superset of these two tractable cases, and for S the infinite set of all odd integers. The remaining open cases are the non-singleton finite sets of odd integers, in particular $S = \{1, 3\}$. Beyond cycles, we also discuss the complexity of finding S-Hamiltonian paths, and show that our problems are all tractable on graphs of bounded cliquewidth.

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1 Introduction

The Hamiltonian cycle problem is a well-known problem in graph theory and complexity theory. It asks, given an undirected graph, whether it contains a *Hamiltonian cycle*, i.e., a cycle that visits every vertex exactly once. This problem is well-known to be NP-complete, and the same is true if we want to find a *Hamiltonian path*, i.e., a path that visits every vertex exactly once.

Faced with the hardness of this problem, one natural relaxation is to allow greater distance bounds between pairs of consecutive vertices instead of requiring them to be adjacent. The simplest case is to allow hops of length 1 or 2 (instead of just 1). Equivalently, we consider the *square* G^2 of the input graph G, in which we connect any two vertices that can be joined by a path of length at most 2, and we ask whether G^2 is Hamiltonian. It is still NP-complete to determine whether an input graph admits such a relaxed cycle [25], even though many graph families are known whose square is always Hamiltonian: this is the case of two-connected graphs [13, 20] and other families [15, 1, 9, 10, 11]. (We note that there are also other different (and seemingly unrelated) ways to define generalizations of Hamiltonicity, e.g., taking the iterated line graph as in [8, 23].)

The natural next question is to allow hops of 1, 2, or 3. However, it is then known that the cube G^3 of every connected graph G is Hamiltonian [18, 24], and this result is a commonly used trick for efficient enumeration algorithms (the so-called "even-odd trick" [18, 24] (see also [26]). The case of hops of length at most 3 immediately implies that relaxed Hamiltonian cycles always exist in connected graphs for any other set of the form $S = \{1, \ldots, k\}$ with $k > 3$.

However, this does not address the question of other fixed sets of allowed hop lengths. For instance, we can consider $S = \{2\}$: this asks whether we can find a permutation $(v_0, \ldots, v_{n-1})$ of the vertices of the input graph G in which any two consecutive vertices (including the endpoints v_{n-1} and v_0) can be joined by a walk of length *exactly* two. We call such a cycle a *{2}-Hamiltonian cycle*. This question relates to the Hamiltonicity of the *common neighborhood graph* of G [16], i.e., the graph where we connect pairs of vertices that are joined by such walks. Hossein et al. [16, Theorem 4.1] in particular give a sufficient condition for graphs to have a $\{2\}$-Hamiltonian cycle, but they know that the condition is not necessary – so to our knowledge there is no characterization of the graphs admitting a $\{2\}$-Hamiltonian cycle and no complexity bounds on the problem of recognizing them. Alternatively, one can consider, say, $S = \{1, 2, 4\}$: do all graphs admit a $\{1, 2, 4\}$-*Hamiltonian cycle*, i.e., a cycle where any two consecutive vertices are joined by a walk with some length in S?

Contributions. In this paper we study which graphs admit S-Hamiltonian cycles, and the complexity of recognizing them, depending on the fixed set S. We show that, depending on the choice of S, the problem behaves very differently: there are sets S for which every (connected) graph admits an S-Hamiltonian cycle, other cases where the decision problem is NP-hard, and last some cases where the decision problem is non-trivial but efficiently solvable. We entirely classify these behaviors, except for one remaining family of open cases.

The main results come from the sets $S = \{2\}$, $S = \{2, 4\}$, $S = \{2, 4, 6\}$ and $S = \{1, 2, 4\}$. For $S = \{2\}$, we prove that the $\{2\}$-Hamiltonian cycle problem is NP-complete. This proceeds by a reduction from the classical $\{1\}$-Hamiltonian path problem: we reduce it to the $\{2\}$-Hamiltonian path problem with specified endpoints by building the incidence graph of the input graph and attaching a triangle, and we then reduce that problem to the $\{2\}$-Hamiltonian cycle problem by attaching a specific gadget. For $S = \{2, 4\}$, we show that the {2,4}-Hamiltonian cycle problem is NP-hard under Cook reductions. Specifically, we reduce from the {1,2}-Hamiltonian cycle problem (which is known to be NP-hard [25]) and build graphs by attaching a specific gadget on every pair of sufficiently close vertices. As for $S = \{2, 4, 6\}$, we show that a connected graph admits an {2,4,6}-Hamiltonian cycle if and only if it is non-bipartite: this gives a linear-time recognition algorithm, and we show that a witnessing cycle can also be built in linear time when one exists. Finally, for $S = \{1, 2, 4\}$, we prove that every connected graph admits a {1,2,4}-Hamiltonian cycle by first showing the same result on trees. The proof is by induction on the number of vertices, and it also yields a linear-time algorithm to build a witnessing cycle.

The remaining family of unclassified cases are the non-singleton finite sets of odd integers, i.e., the sets $S = \{1, \ldots, 2k + 1\}$ for $k > 1$, for which the complexity of the S-Hamiltonian cycle problem remains open.

We also discuss three variants of the S-Hamiltonian cycle problem, the first being the variants when the set S is infinite, which are all tractable. The second variant is the study of the complexity of S-Hamiltonian path, i.e., we study the complexity of deciding whether an input graph contains an S-Hamiltonian path (possibly with specified endpoints); we show that in most cases this has the same complexity as the S-Hamiltonian cycle problem. Finally, we also study the tractability of the S-Hamiltonian cycle problem when restricting the graphs that are allowed as inputs: we show that, for any fixed finite set S, the problem is tractable when the input graphs are required to have bounded cliquewidth.

Paper outline. In Section 2, we introduce the definitions and notations that are used in this paper, and give more formal details about the S-Hamiltonian cycle problem and its variants. In Section 3, we present our main result and a decision tree that delineates the tractable, intractable, and open cases. We then give the detailed proofs for each case. First, Section 4 is dedicated to the proofs of the NP-completeness of the $\{2\}$-Hamiltonian cycle and $\{2,4\}$-Hamiltonian cycle problems. Then, Section 5 deals with the $\{1,2,4\}$-Hamiltonian cycle problem, and Section 6 with the $\{2,4,6\}$-Hamiltonian cycle problem. We then discuss problem variants in Section 7. More specifically, in Section 7.1, we study the complexity of the S-Hamiltonian cycle problem when S is an infinite set. We discuss the complexity for the S-Hamiltonian path variants of our problem in Sections 7.2 and 7.3. Finally, in Section 7.4, we study the case of graphs of bounded cliquewidth. We conclude in Section 8. For lack of space, most detailed proofs are deferred to the appendix of the arXiv version of our paper [2].

2 Preliminaries and Problem Statement

Standard graph notions. All graphs in this paper are undirected, do not feature self-loops, are non-empty (i.e., have at least one vertex), and are simple graphs (i.e., without parallel edges). Furthermore, we always assume that graphs are connected. Formally then, a *graph* $G = (V, E)$ consists of a finite set of vertices V (also denoted $V(G)$) and a set of edges E (also denoted $E(G)$) which is a set of pairs of vertices, i.e., the edge $\{u, v\}$ connects the distinct vertices u and v, and we also say the edge is *incident* to u and to v, and that u and v are *adjacent*. A *spanning subgraph* of G is a graph (V, E') with $E' \subseteq E$ that is connected. A *tree* is an acyclic graph, and it is a *rooted tree* if a vertex has been designated as the root. A *spanning tree* T of G is a spanning subgraph of G which is a tree.

A *walk* in a graph G is a sequence of vertices $(v_0, v_1, \ldots, v_k)$ such that for each $i \in \{0, \ldots, k-1\}$ the vertices v_i and v_{i+1} are adjacent. The *length* of the walk is defined as the number of edges traversed, which is k. A *simple path* in a graph G is a walk where all vertices are distinct, and a *simple cycle* is a simple path where the first and the last vertices are adjacent. The length of a cycle is then defined as one more than the length of its defining walk. We require graphs to be *connected*, i.e., for any choice of two vertices there exists a walk that connects them.

The *incidence graph* of $G = (V, E)$ is the graph $G' = (V', E')$ whose vertices are the vertices and edges of G and where each edge of G is connected in G' to its incident vertices; formally $V' \coloneqq V \uplus E$ and $E' \coloneqq \{\{u, e\} \mid e \in E, u \in e\}$. The *line graph* of G, denoted $L(G) = (V', E')$, is the graph on the edges of G where two edges of G are connected in G' if they are incident in G to a common vertex; formally $V' \coloneqq E$ and $E' \coloneqq \{\{e, e'\} \mid e, e' \in E, e \cap e' \neq \emptyset\}$. We call G *bipartite* if its vertex set can be partitioned into two non-empty sets X and Y, called the *parts*, such that each edge of G connects a vertex in X to a vertex in Y. We recall that a graph is bipartite if and only if it does not admit an odd-length simple cycle, and that checking whether an input graph is bipartite can be done in linear time.

Problem definition. We now formally define the main notions studied in this paper. In the following, we assume S is a non-empty finite subset of $\mathbb{N}^+$, the positive integers (we will study the case of infinite S in Section 7.1).

▶ **Definition 2.1** (S-path)**.** *Let $G = (V, E)$ be a graph, and let $S \subseteq \mathbb{N}^+$. A sequence of vertices $P = (v_1, v_2, \ldots, v_n)$ is called an* S-path *of G if, for each $i \in \{1, \ldots, n-1\}$, the vertices v_i and v_{i+1} are connected by a walk of length $\ell \in S$, i.e., there exists some length $\ell \in S$ and some walk in G of length ℓ that starts at v_i and ends at v_{i+1}. Furthermore, P is called a* simple S-path *if all its vertices are distinct.*

▶ **Definition 2.2** (S-Hamiltonian path and S-Hamiltonian cycle)**.** *Let $G = (V, E)$ be a graph, and let $S \subseteq \mathbb{N}^+$. An* S-Hamiltonian path *of G is a simple S-path that contains all the vertices of G. An* S-Hamiltonian cycle *of G is an S-Hamiltonian path whose first and last vertices are connected by a walk of a length $\ell \in S$ in G.*

For example, the classical Hamiltonian cycle problem corresponds to the case $S = \{1\}$, as two vertices are connected by a walk of length 1 precisely when they are adjacent.

Alternative definitions. Note that our definitions above require the existence of a *walk* of a length in S connecting any two consecutive vertices. The rest of the paper only considers this choice of definition, but we briefly discuss here some alternative possible choices, in order to dispel potential confusion.

First, we do *not* require that the witnessing walk is a *shortest path*. In other words, we do not require that the *distance* between two consecutive vertices u and v belongs to S, only that *some possible walk* has length in S: there may be shorter walks connecting u and v with a length not in S. Requiring shortest paths as connecting walks would be a more stringent requirement in general, even though there are some sets S for which both definitions coincide (e.g., sets of the form $S = \{1, \ldots, k\}$ for some $k > 0$). We note that, already for the set $S = \{2\}$, requiring shortest paths between consecutive vertices would be different, and it would relate to the Hamiltonicity of the *exact-distance square* of the input graph [3]. We do not study this question in the present work.

Second, we do *not* require that the witnessing walks between consecutive vertices are *simple paths*. Requiring simple paths would again be a more stringent requirement in general – though again it would make no difference for sets of the form $S = \{1, \ldots, k\}$, and it would also make no difference for $S = \{2\}$. Again, in this work we do not study this alternative definition requiring simple paths as connecting walks.

Third, we reiterate that our definitions do not pose any restriction on the number of times that a vertex may occur in the connecting walks. In particular, note that vertices that have already been visited earlier in the permutation can still be traversed as intermediate vertices of later connecting walks.

Last, we point out one important consequence of our choice to allow connecting walks that may feature repeated vertices: when two vertices are connected by a walk of length $\ell > 0$, then they are also connected by a walk of length $\ell + 2$, simply by going back-and-forth on the last edge. Thus, in our problem, we can always assume without loss of generality that the set S is *closed under subtraction of* 2, i.e., whenever $\ell \in S$ is an allowed distance and $\ell - 2 > 0$ then the distance $\ell - 2$ is also allowed. In other words, whenever S contains a number ℓ, then we assume that it also contains all smaller numbers of the same parity – for clarity we will always write down these numbers explicitly.

Computational complexity. The main problem we study is the S-Hamiltonian cycle problem for fixed sets S, which asks, given a graph G, whether G admits an S-Hamiltonian cycle. We also study in Section 7 the complexity of two variants the *S-Hamiltonian path problem*, for which we need to decide whether G has an S-Hamiltonian path, and the *S-Hamiltonian path problem with specified endpoints*, which asks whether G admits an S-Hamiltonian path that starts and ends at some specified endpoints given as input. For all these three problems, the complexity is always measured as usual as a function of the input graph. We note that these problems are always in NP for any choice of fixed S: the certificate is the permutation of vertices $s = (v_1, \ldots, v_n)$, and it is easy to check in polynomial time that the certificate is correct. We will show cases when the problem is NP-hard, and other cases when the problem can be efficiently solved: either because it is trivial (i.e., all graphs have an S-Hamiltonian cycle), or because there is an efficient algorithm to decide it. In the efficient cases, we will also study the complexity of the problem of efficiently finding a witness (e.g., an S-Hamiltonian cycle) when one exists, and give tractable algorithms for this task.

For some problems, we will only be able to show NP-hardness under *Cook reductions* (i.e., polynomial-time Turing reductions), rather than the more commonly used *Karp reductions* (i.e., polynomial-time many-one reductions). Recall that a many-one reduction is a reduction that maps each instance of a problem to exactly one instance of another problem whereas a Turing reduction is allowed to use the oracle of the target problem multiple times on different instances. Thus, saying that a problem is NP-complete under Cook reductions is a weaker statement than saying that it is NP-complete under Karp reductions, but it still implies that there is no polynomial-time algorithm for the problem unless P = NP.

3 Main Result

Having formally defined our various problems, we now present in this section the main result of this paper, which classifies the complexity of the S-Hamiltonian cycle problem, up to some remaining open cases. Recall that sets S are considered to be finite, unless stated otherwise – we revisit this choice in Section 7. Results on the other variants (S-Hamiltonian path, with or without specified endpoints), and on the generalization of our problems when S can be infinite, will also be presented in Section 7.

▶ **Theorem 3.1.** *For every finite and non-empty set $S \subseteq \mathbb{N}^+$ the S-Hamiltonian cycle-problem is either NP-complete under Cook reductions, in P, trivial (i.e., true on all graphs), or open, as depicted in Figure 1. Further, when the problem is in P or trivial, we can compute a witnessing S-Hamiltonian cycle in linear time in the input graph.*

We explain in the remaining of this section the roadmap to prove Theorem 3.1. We start by recalling the relevant known results from the literature.

▶ **Proposition 3.2** ([19])**.** *The {1}-Hamiltonian cycle problem is NP-complete.*

▶ **Proposition 3.3** ([25])**.** *The {1,2}-Hamiltonian cycle problem is NP-complete.*

The following will be particularly useful to us:

▶ **Proposition 3.4** ([18, 24])**.** *The {1,2,3}-Hamiltonian cycle problem is trivial. More precisely, for every graph G and $u, v \in V(G)$ with $u \neq v$, there exists a {1,2,3}-Hamiltonian path from u to v in G.*

Now, we present the new results that complete the classification of Theorem 3.1.

▶ **Theorem 3.5.** *The {2}-Hamiltonian cycle problem is NP-complete.*

▶ **Theorem 3.6.** *The {2,4}-Hamiltonian cycle problem is NP-complete under Cook reductions.*

▶ **Theorem 3.7.** *Every connected graph admits a {1,2,4}-Hamiltonian cycle. Furthermore, such a cycle can be found in linear time.*

▶ **Theorem 3.8.** *The graphs that have a {2,4,6}-Hamiltonian cycle are exactly the non-bipartite graphs, which implies that the {2,4,6}-Hamiltonian cycle problem can be solved in linear time. Furthermore, on graphs having a {2,4,6}-Hamiltonian cycle, we can construct one in linear time.*

From these results, we can decide the complexity of any set S thanks to the decision process illustrated by the decision tree in Figure 1, where we assume without loss of generality that S is closed under subtraction of 2. In this decision tree, we denote by P that the problem corresponding to the sets S considered is polynomial, by *NP-c* that it is NP-complete (possibly under Cook reductions), by *Trivial* that every graph admits such a cycle, and by *open* that we do not know yet. The new results are in the bold nodes. The detailed proof of the correctness of that decision process, which is also a proof of Theorem 3.1, is in of [2, Appendix A]. That said, the only non-obvious reasoning in this proof is the following. First, for sets containing 6, we know by Theorem 3.8 that the lengths 2, 4, and 6 are enough to find a S-Hamiltonian cycle in any non-bipartite graph. On the other hand, having more even lengths will never allow to get a S-Hamiltonian cycle in a bipartite graph for obvious reasons, thus sets S with only even numbers that are supersets of {2,4,6} can be grouped together. Second, all sets S that are supersets of {1,2,3} can be grouped together because we know by Proposition 3.4 that the lengths $1, 2$ and 3 are enough to find an S-Hamiltonian cycle in any graph. Third, all sets S that are supersets of {1,2,4} can be grouped together for the same reason thanks to Theorem 3.7.

The next three sections are dedicated to proving the four main theorems of this section.

4 NP-Complete Cases

In this section, we start the proof of Theorem 3.1 by giving the NP-hardness results for the cases $S = \{2\}$ (in Section 4.1) and $S = \{2, 4\}$ (in Section 4.2).

4.1 The Case $S = \{2\}$

To show that the {2}-Hamiltonian cycle problem is NP-hard, we proceed in two steps. We first reduce the {1}-Hamiltonian path problem with specified endpoints which is known to be NP-complete [19], to the {2}-Hamiltonian path problem with specified endpoints. Then, we reduce the latter problem to the {2}-Hamiltonian cycle problem. Let us show the first step:

▶ **Theorem 4.1.** *The {2}-Hamiltonian path problem with specified endpoints is NP-complete.*

Proof sketch. Let G be a graph and $\alpha \neq \beta$ be the specified endpoints in $V(G)$ for the Hamiltonian path problem. Construct a new graph H by first taking the incidence graph of G (which features an *edge-vertex* for each edge of G) and then connecting β to a dangling triangle. Pick an arbitrary edge-vertex β' adjacent to α in H. We can then conclude by establishing the following: G has a $\{1\}$-Hamiltonian path connecting α and β if and only if H has a $\{2\}$-Hamiltonian path connecting α and β'.

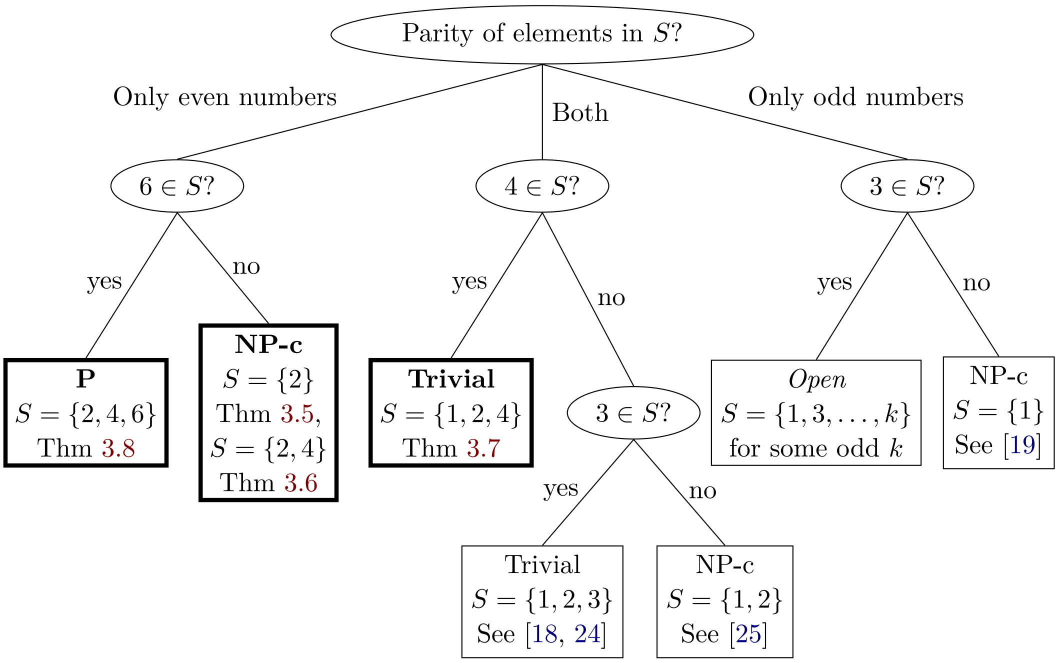

Figure 1 Decision tree for the complexity of the S-Hamiltonian cycle problem for S a finite and non-empty subset of $\mathbb{N}^+$.

In particular, for the forward direction, to obtain a $\{2\}$-Hamiltonian path in H we first follow the original $\{1\}$-Hamiltonian path of G, which is now a simple $\{2\}$-path from α to β in H, then goes through the triangle to reach an edge-vertex. From there, we use a strengthening of [4, Theorem 6.5.4] that states that the line graph of a Hamiltonian graph is Hamiltonian, which allows us to visit all edge-vertices. ◀

We now show the hardness of the $\{2\}$-Hamiltonian cycle problem by reducing from the $\{2\}$-Hamiltonian path with specified endpoints:

▶ **Theorem 3.5.** *The $\{2\}$-Hamiltonian cycle problem is NP-complete.*

Proof sketch. Let G be an input graph and $\alpha, \beta \in V(G)$ be two different vertices. The reduction is to attach the gadget D from Figure 2 to a copy of G by adding the edges $\{\alpha, d_0\}$ and $\{\beta, d_1\}$. We then prove that G has a $\{2\}$-Hamiltonian path from α to β if and only if the new graph H has a $\{2\}$-Hamiltonian cycle. This is done by observing that the vertices d_3, d_5, d_7 of the gadget D have very constrained neighborhoods in any $\{2\}$-Hamiltonian cycle of H, which forces a specific ordering of the vertices of D in such a cycle. ◀

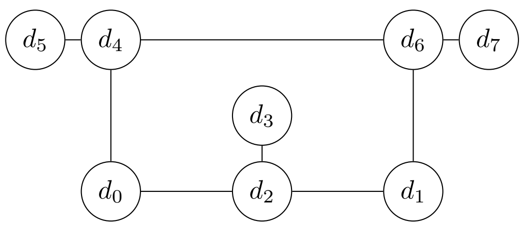

Figure 2 Gadget D for the proof of Theorem 3.5, found with the help of a computer program.

4.2 The Case $S = \{2, 4\}$

We now turn to the case of the {2,4}-Hamiltonian cycle problem. We show NP-hardness via a Cook reduction from the {1,2}-Hamiltonian cycle problem; formally:

▶ **Theorem 3.6.** *The {2,4}-Hamiltonian cycle problem is NP-complete under Cook reductions.*

Our hardness proof uses Cook reductions for their ability to reduce an instance to multiple instances with multiple oracle calls. While we suspect that {2,4}-Hamiltonian cycle should also be NP-hard under the more standard notion of Karp reductions, we have not been able to show this.

Theorem 3.6 will directly follow from the following result.

▶ **Proposition 4.2.** *Given an input graph G and two vertices $x \neq y$ at distance at most 2 in G, we can construct in polynomial time a graph $H_{x,y}$ such that:*

- *if G has a {1,2}-Hamiltonian cycle where x and y are consecutive, then $H_{x,y}$ has a {2,4}-Hamiltonian cycle,*
- *if $H_{x,y}$ has a {2,4}-Hamiltonian cycle, then G has a {1,2}-Hamiltonian cycle.*

Note that, when $H_{x,y}$ has a {2,4}-Hamiltonian cycle, then the {1,2}-Hamiltonian cycle in G is not necessarily one where x and y are consecutive. Nevertheless, the result of Proposition 4.2 suffices to design a Cook reduction that shows the NP-hardness of the {2,4}-Hamiltonian cycle problem:

Proof of Theorem 3.6. We do a Cook reduction from the {1,2}-Hamiltonian cycle problem, which as we already mentioned in Proposition 3.2 is NP-hard. Given an input G to the {1,2}-Hamiltonian cycle problem, assuming without loss of generality that it has at least two vertices, we consider every pair x, y of vertices at distance at most 2 in G and construct the corresponding graph $H_{x,y}$: this amounts to polynomially many graphs which are all built in polynomial time. Now, if G has a {1,2}-Hamiltonian cycle C then for any choice of contiguous vertices x, y of C, we know that $H_{x,y}$ has a {2,4}-Hamiltonian cycle. Conversely, if some $H_{x,y}$ has a {2,4}-Hamiltonian cycle then G has a {1,2}-Hamiltonian cycle. This implies that G has a {1,2}-Hamiltonian cycle if and only if some $H_{x,y}$ has a {2,4}-Hamiltonian cycle, so we can indeed solve {1,2}-Hamiltonian cycle in polynomial time with access to an oracle for {2,4}-Hamiltonian cycle. This establishes the Cook reduction and concludes the proof. ◀

Thus, all that is left to show Theorem 3.6 is to prove Proposition 4.2. To do so, we will need the following intermediate lemma which is a stronger version of [21, Theorem 2]. This lemma originally proves that the square of a line graph is always Hamiltonian, i.e., that the line graph of any (connected) graph always has a {1,2}-Hamiltonian cycle, and we need a slightly stronger version that allows us to choose two vertices of the line graph to be consecutive in the {1,2}-Hamiltonian cycle.

▶ **Lemma 4.3** (Strengthening of Theorem 2 of [21]). *Let G be a graph with at least 3 edges. Let $L(G)$ be the line graph of G. Let α and β be two vertices of $L(G)$ connected by a walk of length 2 in $L(G)$. Then there exists a {1,2}-Hamiltonian cycle in $L(G)$ where α and β are consecutive.*

With this lemma in hand, we are now ready to prove Proposition 4.2.

Proof sketch of Proposition 4.2. Given an input graph G on which we want to determine the existence of a {1,2}-Hamiltonian cycle and two of its vertices x and y, we build $H_{x,y}$ by taking the incidence graph of G (formed of node-vertices and edge-vertices) and attaching

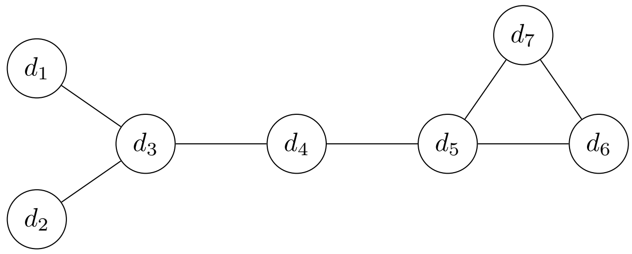

Figure 3 Gadget D for the proof of Proposition 4.2.

the gadget from Figure 3 with edges from d_1 and d_2 respectively to the vertices x' and z' corresponding to x and to a vertex z which is a neighbor of x towards y (specifically, we take $z = y$ if x and y are at distance 1, and we take z to be a common neighbor of x and y if they are at distance 2).

For the forward direction, we show that a {1,2}-Hamiltonian cycle C of G where x and y are adjacent can be lifted to a {2,4}-Hamiltonian cycle C' of $H_{x,y}$. At a high level, C shows us how to visit the node-vertices of $H_{x,y}$, and we use D to toggle to the edge-vertices and back. Further, we use Lemma 4.3 to build the portion of the cycle that visits the edge-vertices of $H_{x,y}$ while starting and ending at appropriate vertices.

For the backward direction, given a {2,4}-Hamiltonian cycle C' of $H_{x,y}$, we use the fact that $H_{x,y}$ is almost bipartite except for the triangle $\{d_5, d_6, d_7\}$ in D. This allows us to show that C' can only switch one time from node-vertices to edge-vertices and one time back, so that C' must contain a contiguous {2,4}-Hamiltonian path N that visits all the node-vertices, which is preceded and succeeded by visits to D. Further considerations on the structure of $H_{x,y}$ ensure that N must in fact be a {1,2}-Hamiltonian cycle of G, i.e., its starting and ending vertices correspond to vertices at a distance of at most 2 in G. ◀

5 Trivial Case: $S = \{1, 2, 4\}$

In this section we show that every graph admits a {1,2,4}-Hamiltonian cycle which we can construct in linear time.

▶ **Theorem 3.7.** *Every connected graph admits a {1,2,4}-Hamiltonian cycle. Furthermore, such a cycle can be found in linear time.*

Up to taking a spanning tree, this is equivalent to showing that every tree admits a {1,2,4}-Hamiltonian cycle, which we do in the following lemma (with an additional property that is needed for its own proof).

▶ **Lemma 5.1.** *Let T be a tree rooted at r such that $|V(T)| \geq 2$. Then there exists a {1,2,4}-Hamiltonian cycle $(r, v_1, \ldots, v_k)$ of T such that v_1 is a child of r. Furthermore, such a cycle can be computed in linear time.*

Proof sketch. The idea of the proof is to do an induction on the number of vertices of T. We distinguish the children of r in T between those children that are leaves and those children that are roots of non-trivial subtrees. If there are leaf children, then we can simply visit them in order with a 1-hop from r and 2-hops between each of them, easily ensuring the requirement that the second vertex of the cycle is a child of r. For each non-leaf child v_i of r, we use the induction hypothesis to get a cycle C_i which visits the subtree of T rooted at v_i while starting at v_i and visiting a child of v_i immediately afterwards. Up to conjugating these cycles and reversing every other cycle, we can stitch these cycles together to visit the subtrees rooted at non-leaf children of r: we start at the first such vertex v_1 (reached via a 1-hop from r, or a 2-hop from the last leaf child of r), follow C_1 (after conjugation and reversal), and finish on a child w_1 of v_1, then we do 4-hop to a child w_2 of v_2 and do C_2 in reverse order to finish on v_2. At the end of this process, we finish either on the last non-leaf child of r or a child thereof, and we can close the cycle and go back to r with a 1-hop or a 2-hop.

This inductive proof can be transformed into a linear-time algorithm that constructs the desired {1,2,4}-Hamiltonian cycle of T. ◀

6 Polynomial Case: $S = \{2, 4, 6\}$

In this section, we show that the {2,4,6}-Hamiltonian cycle problem can be solved in linear time because the graphs with a {2,4,6}-Hamiltonian cycle are precisely the non-bipartite graphs:

▶ **Theorem 3.8.** *The graphs that have a {2,4,6}-Hamiltonian cycle are exactly the non-bipartite graphs, which implies that the {2,4,6}-Hamiltonian cycle problem can be solved in linear time. Furthermore, on graphs having a {2,4,6}-Hamiltonian cycle, we can construct one in linear time.*

To prove this theorem, we will first show that every non-bipartite graph admits a {2,4,6}-Hamiltonian cycle.

▶ **Lemma 6.1.** *Given a non-bipartite graph G, we can construct a {2,4,6}-Hamiltonian cycle of G in linear time.*

Proof sketch. Let G be a non-bipartite graph. Then G must contain a simple cycle of odd length. Let $C = (v_0, v_1, \dots, v_{2k})$ be a shortest odd cycle in G. We construct a spanning forest rooted at each v_i by performing a simultaneous breadth-first search starting from all v_i. This way, we obtain a collection of trees $T_0, T_1, \dots, T_{2k}$, each rooted at their respective v_i, and covering all vertices of G. We can use Proposition 3.4 about the triviality of the {1,2,3}-Hamiltonian path problem to create, in each T_i, a {1,2,3}-path of the tree obtained from taking only vertices at odd depth (resp., at even depth) in T_i, which translates to a {2,4,6}-path of those vertices in T_i. We traverse the cycle C twice. During the first traversal, we visit the vertices v_i with even index i, and during the second traversal, we visit the vertices v_i with odd index i. When visiting a vertex v_i, we also enumerate all vertices of T_i at even depth, and all vertices of T_{i+1} at odd depth. (The indices are taken modulo $2k + 1$.) This yields a {2,4,6}-Hamiltonian cycle of G. ◀

We can now prove Theorem 3.8, which concludes the proof of Theorem 3.1:

Proof of Theorem 3.8. Given a graph G, we first determine in linear time whether G is bipartite. If G is bipartite, then every walk of length in $S = \{2, 4, 6\}$ will keep us in the same part of the bipartition, so we cannot hope to ever visit the other side. Otherwise, Lemma 6.1 gives us a {2,4,6}-Hamiltonian cycle in linear time. ◀

7 Variants

In this section we consider variants of the S-Hamiltonian cycle problem studied in Theorem 3.1. We start with the case of infinite sets S, for which we can easily show that the S-Hamiltonian cycle problem is either trivial or solvable in linear time. We then move from cycles to paths, and discuss the complexity of the S-Hamiltonian path problem and of the S-Hamiltonian path problem with specified endpoints: we talk in more detail about the case $S = \{1, 2, 4\}$ for paths with specified endpoints, which requires some new reasoning. Last, we discuss the question of whether we can make S-Hamiltonian cycle tractable restricting the class of input graphs.

7.1 Infinite Sets

In this section, we discuss the complexity of the S-Hamiltonian cycle problem for infinite sets S. The crucial observation is that most infinite sets S closed under subtraction of 2 are trivial, because they contain either $\{1, 2, 3\}$ or $\{1, 2, 4\}$. (We give the formal details about this classification in [2, Appendix E.1].) The two non-trivial sets are the set of all even integers, and the set of all odd integers. The case of the set of all even integers follows easily from Theorem 3.8. As for the set of all odd integers, we give its characterization below:

▶ **Theorem 7.1.** *Let S denote the set of all odd integers, and let G be a graph. Then G admits an S-Hamiltonian cycle if and only one of the two following cases holds: G is non-bipartite, or G is bipartite with parts of equal cardinality.*

Proof sketch. Let G be a graph. If G is non-bipartite, then we can connect any two vertices by a walk of odd length. Indeed, we can go through an odd cycle as many times as needed to adjust the parity of the length of that walk. Hence, any ordering of the vertices of G is an S-Hamiltonian cycle.

If G is bipartite, the only odd-length walks are the ones connecting vertices from one part to the other. Hence, in order to have an S-Hamiltonian cycle, the ordering of the vertices must alternate between the two parts, which implies that the two parts must have equal size. Conversely, if the two parts have equal size, then any ordering that alternates between the two parts is an S-Hamiltonian cycle because any pair of vertices from different parts are connected by a walk of odd length. ◀

7.2 Hamiltonian Path Variants

In this section, we discuss the complexity of the two variants of the S-Hamiltonian cycle problem introduced in Section 2: the S-Hamiltonian path problem, and the S-Hamiltonian path problem with specified endpoints. The complexity of these problems on the various possible finite sets S does not always follow from our results on the S-Hamiltonian cycle problem, so we explain the situation in this subsection and in the next subsection.

First, for any set S, it is clearly always possible to reduce from the S-Hamiltonian path problem or from the S-Hamiltonian cycle problem to the S-Hamiltonian path problem with specified endpoints, under Cook reductions. The former is achieved by calling the oracle for S-Hamiltonian path with specified endpoints on all pairs of vertices, and the latter is achieved by calling that oracle on all pairs of vertices connected by a walk having a length in S. However, some of the proofs presented in this paper can be modified to show a better complexity for the S-Hamiltonian path problems.

As most of these problems behave the same way as their S-Hamiltonian cycle counterpart, we will highlight some cases here: we present the detailed explanations of every set for both variants (and their proofs when applicable) in [2, Appendix E.2]. The only result that has been proved to be different is for the {1,2,4}-Hamiltonian path with specified endpoints. Whereas {1,2,4}-Hamiltonian cycle was trivial (Theorem 3.7), for {1,2,4}-Hamiltonian path with specified endpoints the problem can be solved in linear time by excluding a small explicit family of negative instances. We present this result in the next section.

The only other sets S where the complexities of the path and cycle problems are not known to coincide is $S = \{2,4\}$ and $S = \{2,4,6\}$, where we leave some problems open: see [2, Appendix E.2] for more details.

7.3 {1,2,4}-Hamiltonian Path Problem with Specified Endpoints

In this section, we prove that the {1,2,4}-Hamiltonian path problem with specified endpoints can be solved in linear time. For those proofs, we will need to introduce the notions of *star* and *bistar*. A *star centered at a vertex x* is a graph with at least two vertices such that every vertex different from x is adjacent only to x. A *bistar centered at vertices a and b* is a graph such that a and b are adjacent and every vertex different from a and b is adjacent to only either a or b (but not both), and both a and b have degree at least 2. These two definitions are illustrated in Figure 4. We now introduce a lemma that will help us prove the desired theorem.

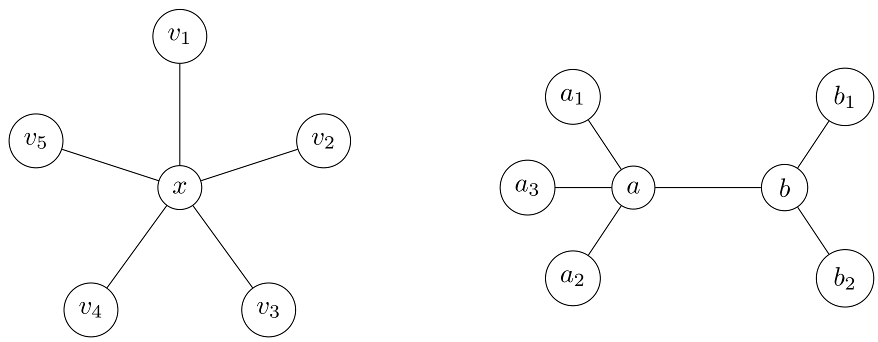

Figure 4 A star centered at x (left) and a bistar centered at a and b (right).

▶ **Lemma 7.2.** *Let T be a tree with $|V(T)| > 2$ and $a, b \in V(T)$ two vertices of T such that a and b are not adjacent. Then there exists a {1,2,4}-Hamiltonian path between a and b.*

Proof sketch. The proof is done by induction on the number of vertices of T. The idea is to divide the tree into multiple vertex disjoint subtrees depending on the shape of T, and to use the induction hypothesis as well as Lemma 5.1 to construct simple $\{1,2,4\}$-paths of these subtrees, which can then be concatenated to form the desired {1,2,4}-Hamiltonian path between a and b. ◀

We can now show our characterization for the {1,2,4}-Hamiltonian path problem with specified endpoints:

▶ **Theorem 7.3.** *Let G be a graph and a and b be two different vertices of G. Then G admits a {1,2,4}-Hamiltonian path from a to b if and only if G is not a bistar centered at a and b.*

Proof sketch. Let G be a graph and a and b be two different vertices of G. If they are non-adjacent, we conclude by Lemma 7.2. If a and b are adjacent and G is not a bistar, then up to exchanging a and b we find a neighbor v_1 of a which is not b and which has itself a neighbor v_2 which is also not b. We take a spanning tree of G including the edges $\{a, v_1\}$ and $\{v_1, v_2\}$, and we use Lemma 7.2 and finally obtain a {1,2,4}-Hamiltonian path of G.

Now, if G is a bistar, a path from a can start from neighbors of a (resp. neighbors of b) but will not be able to reach neighbors of b (resp. a) without taking b (which must be at the end). Thus a bistar centered at a and b cannot contain a {1,2,4}-Hamiltonian path from a to b, which concludes the proof. ◀

7.4 Restricted Graph Classes

In this section, we last turn to one last question: can the S-Hamiltonian cycle problem be made tractable if we assume that the input graphs are in restricted graph classes? This is motivated by the fact that the Hamiltonian cycle problem, while NP-hard in general, is known to be tractable on many restricted classes of graphs, in particular bounded-treewidth and bounded-cliquewidth graphs [5, 12]. Hence, one can ask whether the same is true of the S-Hamiltonian cycle problem for sets S other than $S = \{1\}$.

There are related results in this direction for the case $S = \{1, 2\}$ and when the input graph is a tree. Namely, it is known that a tree T has a {1,2}-Hamiltonian cycle if and only if it is a so-called caterpillar graph [14]. Further, the trees having a {1,2}-Hamiltonian path have been characterized in [22] and can be recognized in linear time.

We now claim that, for arbitrary finite sets S, tractability extends beyond trees to bounded-treewidth and even bounded-cliquewidth graphs. Namely:

▶ **Theorem 7.4.** *For any fixed finite set $S \subseteq \mathbb{N}^+$ and bound $k \in \mathbb{N}$, the S-Hamiltonian cycle problem on input graphs of cliquewidth at most k is in polynomial time.*

Proof sketch. The proof uses the fact that bounded cliquewidth is preserved by MSO interpretations [7, 6]. So, from the input graph G, we construct the graph G_S that has the same vertices as G and connects two vertices if and only if they are connected by a walk of a length in S: this is definable by an MSO interpretation, so G_S also has bounded cliquewidth. It is now easy to see that G_S admits a Hamiltonian cycle if and only if G admits an S-Hamiltonian cycle. From there, thanks to the result from [5], there exists an XP algorithm to solve the Hamiltonian cycle problem on graphs with bounded cliquewidth, which allows us to conclude. ◀

However, bounded cliquewidth is not the end of the story for the Hamiltonian cycle problem, as it is also known to be tractable on some classes of unbounded cliquewidth, e.g., proper interval graphs [17, Theorem 5]. Hence, a natural question for future work would be to explore whether the S-Hamiltonian cycle problem is also tractable on other classes of unbounded cliquewidth.

8 Conclusion and Future Work

We have studied the S-Hamiltonian cycle problem and we determined its complexity for most finite sets S. We classified the complexity of every result into a decision tree that allows to restrict our attention to fewer cases. Among those cases, some were known results, and others have been proven in this paper, namely the NP-completeness when $S = \{2\}$ and $S = \{2, 4\}$, the triviality when $S = \{1, 2, 4\}$, and the polynomial-time solvability when $S = \{2, 4, 6\}$. Furthermore, we also addressed the case of infinite sets S, which are either trivial or polynomial-time solvable; and we also studied the variants of paths, with or without specified endpoints, for which we obtained similar complexity classifications for most cases.

We now mention potential future work. The most immediate open question is to classify the complexity of the S-Hamiltonian cycle problem for $S = \{1, 3\}$ (and of other finite sets of odd integers): we suspect that this problem is NP-hard, but we have not been able to show it. However, it is important to notice that in case of bipartite graphs (and for the same reasons as in Theorem 7.1 on the infinite set of all odd numbers), a necessary condition to admit an S-Hamiltonian cycle for a finite set S of odd integers is to have both parts of the same size. That is nonetheless not a sufficient condition, as there may be an "imbalance" of vertices of a certain part of the bipartition in some local part of the graph, that hinders the alternation of parts in our cycle. Yet, we do not know how to conclude from here, and the complexity of S-Hamiltonian cycle remains open, even when the input graph is restricted to be bipartite, or to be non-bipartite. Concerning the path variants of our problems, obviously the same as above holds true for S a finite set of only odd numbers, but there are also the cases of {2,4}-Hamiltonian path and {2,4,6}-Hamiltonian path with specified endpoints that we have left open.

Another direction would be to study the alternate definition that does not allow back-and-forth travels in intermediate walks of S-paths. This definition is harder to grasp, and there are a lot more sets to consider because we can no longer assume without loss of generality that S is closed by subtraction of 2. One could also consider the more stringent definition that requires the back-and-forth travels to form simple paths, or those where the lengths in S are required to correspond to distances between the two vertices (i.e., prohibiting the existence of shorter paths).

Last, another natural question would be to look into directed graphs. However, it seems that for any given finite set S, there exist arbitrarily large graphs that do not contain an S-Hamiltonian cycle. This suggests that the general complexity answers for directed graphs might be totally different from those on undirected graphs.

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