Relational Dualities and Bisimulation

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Abstract

The Kripke semantics of various logics arises via categorical dualities between a category of relational frames and their maps, and a category of algebras and logical homomorphisms. When the relational frames are considered as computational systems (e.g. the states of a machine), the corresponding algebra is one of logical predicates on these systems (e.g. predicates on these states, i.e. program logics). Our aim is to extend this phenomenon to relations, putting well-behaved relations between systems (e.g. bisimulations) in correspondence with relations between predicates. This is achieved by constructing particular relational extensions of Tarski duality (for infinitary classical propositional logic) and Thomason duality (for infinitary classical modal logic). We sketch how these dualities give rise to a proof system that relates formulae between different systems.

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1 Introduction

It is a well-known secret that relational semantics (e.g. Kripke semantics) arises from *dualities*, i.e. categorical equivalences of the form $F : \mathcal{C}^{\mathsf{op}} \simeq \mathcal{D}$. The idea is that $\mathcal{C}$ is a "spatial" category whose objects are *frames*, and $\mathcal{D}$ is an "algebraic" category whose objects are *algebras of predicates*, which allow reasoning about the elements of the frames.

For example, truth table semantics for infinitary classical propositional logic arises from the *Tarski duality* $\mathbf{Set}^{\mathsf{op}} \simeq \mathbf{CABA}$ between the category of sets and the category of *complete atomic Boolean algebras* (CABAs) [22, §1].[1] The Kripke semantics of infinitary classical modal logic [23, 24] arises from the *Thomason duality* $\mathbf{Frm}^{\mathsf{op}}_{\mathrm{open}} \simeq \mathbf{CABAO}$ between the category of Kripke frames and the category of CABAs with operators (i.e. modalities) [42] [22, §2.4]. The Kripke semantics of infinitary intuitionistic logic [25] arises from the duality $\mathbf{Pos}^{\mathsf{op}} \simeq \mathbf{PrAlgLatt}$ between posets and prime algebraic lattices [34, 21]. Famous dualities of this form include Stone duality (for Boolean algebras) [19], Jónsson-Tarski duality [20, 12, 37] (for Boolean algebras with operators), and Esakia duality [9] (for Heyting algebras). See the chapter by Kishida [22] for further references.

[1] The semantics of *finitary* classical propositional logic arises from the duality $\mathbf{Set}^{\mathsf{op}}_{\mathrm{fin}} \simeq \mathbf{Bool}_{\mathrm{fin}}$ between finite sets and finite Boolean algebras. In fact, we can arrive at the Tarski duality $\mathbf{Set}^{\mathsf{op}} \simeq \mathbf{CABA}$ by taking the Pro-completion of this duality [27].

The overwhelming majority of work in this area has not paid much attention to the logical rôle of *morphisms* of these categories.[2] In many cases the "frame" morphisms $f : W \to W'$ of $\mathcal{C}$ can be thought of as "truth-preserving": they map worlds to worlds in a way that preserves and reflects truth, i.e. $w \vDash \varphi$ iff $f(w) \vDash \varphi$. The corresponding morphism $f^* : F(W') \to F(W)$ of $\mathcal{D}$ then preserves the logical structure, e.g. $f^*(\varphi \wedge \psi) = f^*(\varphi) \wedge f^*(\psi)$. Thus f^* can be considered as mapping predicates over W' to predicates over W, in a manner that preserves logical connectives.

Obtaining this property in the case of intuitionistic and modal logics requires strong properties on frame morphisms. For example, in the Kripke semantics of modal logic the induced map $f^* : \mathcal{P}(W') \to \mathcal{P}(W)$ preserves the modality $\Box$ iff $f : W \to W'$ is an *open map*, i.e. a transition-preserving function whose graph is a *(functional) bisimulation* [22, §2.4].[3]

However, functionality is incidental to the idea of bisimulation. It is natural to wonder what happens if we relax the morphisms of $\mathcal{C}$, requiring them to merely be *relations* instead of functions. Answering this question would allow us to have categories $\mathcal{C}$ consisting of frames and well-behaved relations, e.g. bisimulations. A duality of this form would then put bisimulations in correspondence with a different kind of morphism between algebras, enabling reasoning "across" a bisimulation.

Such results, which may be called *relational dualities*, can already be found in the literature. However, most of them put relations between frames in correspondence with *hemimorphisms*, i.e. morphisms that preserve most – but not all! – logical connectives. For example, a relation $R \subseteq W \times W'$ uniquely corresponds to a function $W \to \mathcal{P}(W')$, which uniquely corresponds to a function $R^* : \mathcal{P}(W') \to \mathcal{P}(W)$ that preserves all joins (but not necessarily all meets) [20] [22, §2.3].

This is unsatisfactory from a computational perspective. Traditional dualities allow us to compute the "action" of a morphism of frames $f : W \to W'$ on the *syntax* of a formula. Given a formula φ (over W') we can define a formula $f^*(\varphi)$ (over W) by recursion. For example, in Tarski duality this would be computed by defining $f^*(\varphi \wedge \psi) \stackrel{\text{def}}{=} f^*(\varphi) \wedge f^*(\psi)$ and so on. Thus, f^* can be pushed through all connectives until it reaches a propositional variable p, where it returns a predicate $f^{-1}(p)$ which is true at w exactly when p is true at $f(w)$. Hence, a function $f : W \to W'$ allows us to map predicates on W' to predicates on W in a logic-preserving manner. As hemimorphisms do *not* preserve all logical connectives, it is impossible to adapt this definition to a relation $R \subseteq W \times W'$.

To tackle this difficulty this paper develops a different type of duality, viz. one that puts relations $R \subseteq W \times W'$ between frames in correspondence with relations $\mathcal{F}(R) \subseteq F(W) \times F(W')$ between predicates. This allows us to view $\mathcal{F}(R)$ as a *relational judgment* between formulae pertaining to different systems, with R playing the rôle of a "background theory" which specifies how propositions over the two systems are related. The ultimate goal is to develop a formal system for synthetically reasoning about the *relation between formulae over different frames*. It is not difficult to imagine that this might have applications in program logics.

If the relation $R \subseteq W \times W'$ is also a *bisimulation* between frames, it can be thought of as a notion of equivalence between the states of two systems, with $w \mathrel{R} w'$ meaning that w and w' have the same behaviours. Viewed as a judgment, the induced relation $\mathcal{F}(R) \subseteq F(W) \times F(W')$ then satisfies "inference rules," which we wish to identify.

[2] One exception is the work of Abramsky [1] on domain theory in logical form.
[3] Open maps are also called *p-morphisms* [5, §2.3] or *bounded morphisms* [4, §2.1].

As we are interested in bisimulations, we take a hint from the relevant literature on coalgebra [41, 30, 18] and base all the results in this paper on the *lower lifting*, which maps a relation $R \subseteq W \times W'$ to the *lower relation* $\mathfrak{L}(R) \subseteq \mathcal{P}(W) \times \mathcal{P}(W')$, defined by

$$S \,\mathfrak{L}(R)\, T \stackrel{\text{def}}{\equiv} \forall s \in S.\ \exists t \in T.\ s\, R\, t.$$

Famously, this construction is *not* a functor $\mathbf{Rel} \longrightarrow \mathbf{Rel}$, as it does not preserve identities. However, we can define an appropriate category $\mathbf{CABARel}$ with CABAs as objects, so that it becomes a functor $\mathfrak{L} : \mathbf{Rel} \longrightarrow \mathbf{CABARel}$. Surprisingly, this functor is an equivalence. Moreover, $\mathfrak{L}$ "extends" Tarski duality, in the sense that the diagram

$$\begin{tikzcd}
\mathbf{Set}^{\mathsf{op}} \arrow[r, hook] \arrow[d, "\simeq"'] & \mathbf{Rel}^{\mathsf{op}} \arrow[d, "\simeq"', "\mathfrak{L}\circ(-)^{\dagger}"] \\
\mathbf{CABA} \arrow[r, hook] & \mathbf{CABARel}
\end{tikzcd}$$

commutes, where $(-)^{\dagger} : \mathbf{Rel}^{\mathsf{op}} \cong \mathbf{Rel}$ is the obvious formal duality that reverses a relation. This follows a known pattern wherein relational dualities are not really "dualities" per se, but become so only after composition with a formal duality [22, §2.3] [26, Remark 4.1]. In this light the $(-)^{\mathsf{op}}$ is merely an artifact of functionality.

After recalling some definitions to fix notation (Section 2), we briefly recap the elements of the Tarski duality $\mathbf{Set}^{\mathsf{op}} \simeq \mathbf{CABA}$ (Section 3). Then, we show that the lower lifting given above can be used to extend it to a *relational Tarski duality* (Section 4). We achieve this by precisely characterising the relations between CABAs that are obtained as lower liftings, and hence establish an equivalence. Furthermore, we show that this is an extension of the usual Tarski duality, and sketch the formal system that this duality induces.

Following this, we show how the same lifting gives rise to a relational version of the Thomason duality $\mathbf{Frm}^{\mathsf{op}}_{\mathsf{open}} \simeq \mathbf{CABAO}$ (Section 5). This is somewhat more challenging: while the Tarski duality acts on arbitrary functions, the Thomason duality associates open maps (i.e. functional bisimulations) with modal complete Boolean homomorphisms. We show that this has an analogue on the relational level, putting (bi)simulations in unique correspondence with what we call *(bi)simulatory relations* between CABAs. We show that this *relational Thomason duality* extends the duality originally given by Thomason, and again sketch the formal system that this duality induces. We close with a discussion of related work (Section 6).

2 Preliminaries

A *relation* $R : A \twoheadrightarrow B$ from a set A to a set B is a subset $R \subseteq A \times B$. We write $a\,R\,b$ to mean $(a, b) \in R$. If $R : A \twoheadrightarrow B$ and $S : B \twoheadrightarrow C$ their composition $R\,;S : A \twoheadrightarrow C$ is defined in the usual manner, i.e. $a\,(R\,;S)\,c$ just if there exists some $b \in B$ with $a\,R\,b\,S\,c$. Note that we write the composition of relations using diagrammatic order. Given a relation $R : A \twoheadrightarrow B$ we define its *opposite* $R^{\dagger} : B \twoheadrightarrow A$ by $b\,R^{\dagger}\,a$ iff $a\,R\,b$. Sets and relations form a category $\mathbf{Rel}$ under composition, with the identity $\mathsf{Id}_A : A \twoheadrightarrow A$ given by $a_1\,\mathsf{Id}_A\,a_2$ iff $a_1 = a_2$. Moreover, the opposite construction extends to a functor $(-)^{\dagger} : \mathbf{Rel}^{\mathsf{op}} \to \mathbf{Rel}$ which constitutes a formal duality $\mathbf{Rel}^{\mathsf{op}} \cong \mathbf{Rel}$ given by taking the opposite. This shows that $\mathbf{Rel}$ is a *dagger category*, i.e. equipped with an involutive contravariant functor [14, §2.3.1].

A partial order $(P, \sqsubseteq_P)$ is a set P equipped with a relation $\sqsubseteq_P : P \twoheadrightarrow P$ which is reflexive, transitive, and anti-symmetric. We will commonly refer to partial orders by their carriers $(P,\ Q,\ \dots)$ and often elide the subscript of $\sqsubseteq$. A function $f : P \to Q$ is *monotonic* just if $x \sqsubseteq_P y$ implies $f(x) \sqsubseteq_Q f(y)$. A *lattice* is a partial order that has all finite joins and meets.

We will use $\mathcal{L}, \mathcal{L}', \ldots$ to denote lattices. A *complete lattice* $\mathcal{L}$ is a partial order that has all joins, which also implies that it has all meets. A pair of monotonic functions $f : P \to Q$ and $g : Q \to P$ is an *adjunction* (which in this setting is also called a *Galois connection*) just if $f(x) \sqsubseteq_Q y$ iff $x \sqsubseteq_P g(y)$. This is often denoted by $f \dashv g$. It is a consequence that f preserves any joins that exist in P and g preserves any meets that exist in Q. The *adjoint functor theorem* [7, §7.34] [19, §I.4.2] says that if $f : P \to Q$ is monotonic, P is a complete lattice, and f preserves all joins, then f has a right adjoint $g : Q \to P$. Dually, if $g : Q \to P$ is monotonic, Q is a complete lattice, and g preserves all meets, then g has a left adjoint $f : P \to Q$. We refer the reader to the book by Davey and Priestley [7] for background on orders, and to the books by Mac Lane, Awodey, and Riehl [31, 2, 36] for category theory.

3 A Primer on Tarski Duality

We recap the Tarski duality $\mathbf{Set}^{\mathsf{op}} \simeq \mathbf{CABA}$, and show how it induces an infinitary classical propositional logic. Recall that a *Boolean algebra* is a distributive lattice $\mathcal{B}$ in which every element $x \in \mathcal{B}$ has a *complement* $\neg x \in \mathcal{B}$ for which $x \vee \neg x = \top$ and $x \wedge \neg x = \bot$ [19].

For any set X, its powerset $\mathcal{P}(X)$ is a Boolean algebra [7, §5.2]. In fact, it is a *complete* Boolean algebra, i.e. a complete lattice with joins and meets of $(S_i)_{i \in I}$ given by $\bigcup_{i \in I} S_i$ and $\bigcap_{i \in I} S_i$ respectively. Moreover, $\mathcal{P}(X)$ is *atomic*, in the following sense.

▶ **Definition 1** (Atom). *Let $(P, \sqsubseteq)$ be a partial order with a bottom element $\bot$. An element $a \in P$ is an* atom *of P if $a \neq \bot$ and $x \sqsubseteq a$ implies that either $x = \bot$ or $x = a$. We write $\mathsf{At}(P)$ for the set of atoms of P.*

▶ **Definition 2** (Atomic complete lattice). *A complete lattice $\mathcal{L}$ is* atomic *just if every element is equal to the join of atoms below it, i.e. for every $x \in \mathcal{L}$ we have $x = \bigsqcup \{a \in \mathsf{At}(\mathcal{L}) \mid a \sqsubseteq x\}$.*

A *complete atomic Boolean algebra (CABA)* is then a complete Boolean algebra $\mathcal{B}$ which is moreover atomic. For any set X, the powerset $\mathcal{P}(X)$ is a CABA: its atoms are exactly the singleton sets $\{x\}$ for $x \in X$: every $S \subseteq X$ can be reconstructed as $S = \bigcup_{x \in S}\{x\}$.

A *morphism of CABAs* $f^* : \mathcal{B} \to \mathcal{B}'$ is a monotonic function that preserves all meets and joins. This definition has some significant implications. First, as $\mathcal{B}$ and $\mathcal{B}'$ are Boolean algebras, every such $f^* : \mathcal{B} \to \mathcal{B}'$ is a *complete Boolean homomorphism*, i.e. preserves *all* Boolean connectives, including negation. Second, by the adjoint functor theorem the map $f^* : \mathcal{B} \to \mathcal{B}'$ has both a left and a right adjoint:

$$\begin{tikzcd}
\mathcal{B}' \arrow[r, bend left, "f_!"] \arrow[r, bend right, "f_*"'] & \mathcal{B} \arrow[l, "f^*"' description] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"]
\end{tikzcd}$$

CABAs and their morphisms form a category $\mathbf{CABA}$.

Given any function $f : X \to Y$ we can define a complete Boolean homomorphism $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$ by $f^*(B) = \{x \in X \mid f(x) \in B\}$ [29, 33, 22]. By the adjoint functor theorem, f^* has both a left and a right adjoint, respectively given by

$$\begin{aligned}
f_!(A) &= \{y \in Y \mid \exists x \in X.\ f(x) = y \wedge x \in A\} &&= \{f(x) \mid x \in A\} \\
f_*(A) &= \{y \in Y \mid \forall x \in X.\ f(x) = y \Rightarrow x \in A\} &&= \{y \in Y \mid f^*(\{y\}) \subseteq A\}\,.
\end{aligned}$$

Thus, we obtain a functor $\mathcal{P} : \mathbf{Set}^{\mathsf{op}} \longrightarrow \mathbf{CABA}$.

In order to invert this functor we use some properties of its adjoints. The following lemma is standard: it follows from the facts that atoms and primes coincide in CABAs, and that the left adjoint of a complete lattice homomorphism preserves primes [10, Lemma 1.23 and Exercise 1.3.10.e].

▶ **Lemma 3.** *Let $f^* : \mathcal{B} \to \mathcal{B}'$ be a morphism of CABAs. Then its left adjoint $f_! : \mathcal{B}' \to \mathcal{B}$ maps atoms to atoms.*

We can thus define a functor $\mathsf{At} : \mathbf{CABA} \to \mathbf{Set}^{\mathsf{op}}$, which maps a CABA $\mathcal{B}$ to its set of atoms $\mathsf{At}(\mathcal{B})$, and use Lemma 3 to map a morphism $f^* : \mathcal{B} \to \mathcal{B}'$ to the restriction $f_!|_{\mathsf{At}(\mathcal{B}')} : \mathsf{At}(\mathcal{B}') \to \mathsf{At}(\mathcal{B})$ of its left adjoint to atoms.

This is a pseudo-inverse to $\mathcal{P}$, and we obtain a categorical equivalence.

▶ **Theorem 4** (Tarski duality)**.** $\mathbf{Set}^{\mathsf{op}} \simeq \mathbf{CABA}$.

The gist is that every CABA $\mathcal{B}$ is isomorphic to the powerset $\mathcal{P}(\mathsf{At}(\mathcal{B}))$ of its atoms: its elements are uniquely determined by the atoms below them.

This duality induces (infinitary) classical propositional logic. Let W be a set of *worlds*. This set corresponds to the powerset $\mathcal{P}(W)$, which can be seen as a set of *predicates*. A world $w \in W$ satisfies the predicate $\varphi \in \mathcal{P}(W)$ just if $w \in \varphi$. As $\mathcal{P}(W)$ is a Boolean algebra, predicates are closed under all Boolean operations.[4] Individual worlds $w \in W$ correspond to singleton predicates $\{w\} \in \mathcal{P}(W)$ which uniquely characterise them. If we interpret W as the *set of states* of a computer, logics of this ilk have been in continuous employment since the pioneering work of Dijkstra [8].

4 A Relational Tarski Duality

If instead of a function $f : W \to W'$ we were to have an arbitrary *relation* $R : W \looparrowright W'$, what would the corresponding morphism between $\mathcal{P}(W)$ and $\mathcal{P}(W')$ be?

It is possible to obtain such a duality by enlarging $\mathbf{CABA}$ to $\mathbf{CABA}_{\bigvee}$, whose morphisms only preserve joins, and then show that $\mathbf{Rel} \simeq \mathbf{CABA}_{\bigvee}$ [20, 39] [22, §2.3]. Indeed, if $h : \mathcal{P}(X) \to \mathcal{P}(Y)$ preserves all joins, then – by writing any set as a union of singletons – we see that h is completely determined by a function $X \to \mathcal{P}(Y)$, i.e. a relation $X \looparrowright Y$. Hofmann and Nora [16, §4.5] show that this duality is a special case of a general one. However, these more general morphisms do *not* preserve the logical structure of predicates. For example it may be that $h(\varphi \wedge \psi) \neq h(\varphi) \wedge h(\psi)$. We have thereby lost the benefit of being able to recursively compute h on the syntax of formulae.

Instead, we would like a duality whose logical side consists of predicates and *relations*. To construct one we consider a simple way of *lifting* relations to powersets [28].

▶ **Definition 5** (Lower Relation)**.** *Given a relation $R : X \looparrowright Y$, the associated* lower relation *$\mathfrak{L}(R) : \mathcal{P}(X) \looparrowright \mathcal{P}(Y)$ is defined by*

$$S\,\mathfrak{L}(R)\,T \overset{def}{\equiv} \forall s \in S.\ \exists t \in T.\ s\,R\,t.$$

This lifting has a long history. It is one half of the *Egli-Milner lifting*, which first appeared in the powerdomain literature [35, 13]. It then resurfaced in the coalgebra literature [41], where post-fixed-points amount to *simulations* between transition systems [18, §3]. Intuitively, $S\,\mathfrak{L}(R)\,T$ means every state of S can be "R-simulated" by a state of T.

[4] The fact that $\mathcal{P}(W)$ is a *complete* Boolean algebra means that predicates are also closed under *infinite* conjunction $\bigwedge_{i \in I} \varphi_i$ and disjunction $\bigvee_{i \in I} \varphi_i$.

$\mathfrak{L}$ maps a relation between sets to a relation between CABAs. However, it is *not* a functor $\mathbf{Rel} \longrightarrow \mathbf{Rel}$, as it maps the identity relation $\mathsf{id} : X \nrightarrow X$ to the containment relation $\subseteq : \mathcal{P}(X) \nrightarrow \mathcal{P}(X)$. We will define a category $\mathbf{CABARel}$ so that $\mathfrak{L}$ becomes a functor $\mathbf{Rel} \longrightarrow \mathbf{CABARel}$. The relations between CABAs will be characterised as follows.

▶ **Definition 6** (Directionally atomic relation). *Let P, Q be partial orders, $\mathcal{L}$ be a complete lattice, and $\mathcal{B}, \mathcal{B}'$ be CABAs.*

1. *A relation $R : P \nrightarrow Q$ is a* bimodule *just when $p' \sqsubseteq p \mathrel{R} q \sqsubseteq q'$ implies $p' \mathrel{R} q'$.*
2. *A relation $R : \mathcal{L} \nrightarrow Q$ is* left-disjunctive *just if $a_i \mathrel{R} b$ for all $i \in I$ implies $\left(\bigsqcup_{i\in I} a_i\right) \mathrel{R} b$.*
3. *A relation $R : \mathcal{B} \nrightarrow \mathcal{B}'$ is* atomic-founded *when for any atom $a \in \mathcal{B}$, if $a \mathrel{R} b$ then there exists an atom $b' \sqsubseteq b$ with $a \mathrel{R} b'$.*
4. *A relation $R : \mathcal{B} \nrightarrow \mathcal{B}'$ is* directionally atomic *if it is left-disjunctive, atomic-founded, and a bimodule.*

It is simple to show that directionally atomic relations compose, and that $\sqsubseteq$ is the identity.

▶ **Proposition 7.** *If $R : \mathcal{B} \nrightarrow \mathcal{B}'$ and $S : \mathcal{B}' \nrightarrow \mathcal{B}''$ are directionally atomic, then so is their composition $R \,;\, S : \mathcal{B} \nrightarrow \mathcal{B}''$.*

▶ **Proposition 8.** *$\sqsubseteq : \mathcal{B} \nrightarrow \mathcal{B}$ is directionally atomic, and is the identity for composition of directionally atomic relations.*

We can therefore define a category $\mathbf{CABARel}$ with CABAs as objects, and directionally atomic relations as morphisms. We can then construct a functor:

▶ **Lemma 9.** *$\mathfrak{L}(R) : \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$ is directionally atomic for any relation $R : X \nrightarrow Y$.*

▶ **Lemma 10.** *$\mathfrak{L}$ is a functor $\mathbf{Rel} \longrightarrow \mathbf{CABARel}$.*

Proof. By Lemma 9 every $\mathfrak{L}(R)$ is a morphism in $\mathbf{CABARel}$.

First, we show that $\mathfrak{L}$ preserves the identity. If $\mathsf{id} : X \nrightarrow X$ then $\mathfrak{L}(\mathsf{id})$ is just the relation $\subseteq$, which is the identity $\mathcal{P}(X) \nrightarrow \mathcal{P}(X)$ in $\mathbf{CABARel}$.

Second, we show that $\mathfrak{L}$ preserves composition. Let $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$ be relations. Suppose $A \mathrel{\mathfrak{L}(R\,;\,S)} C$. For any $a \in A$ define

$$B_a \stackrel{\text{def}}{=} \{b \in Y \mid a \mathrel{R} b \wedge \exists c \in C.\ b \mathrel{S} c\}, \qquad B \stackrel{\text{def}}{=} \bigcup_{a\in A} B_a.$$

Then $A \mathrel{\mathfrak{L}(R)} B$ and $B \mathrel{\mathfrak{L}(S)} C$, so $A \mathrel{\mathfrak{L}(R)\,;\,\mathfrak{L}(S)} C$.

For the converse, suppose instead that $A \mathrel{\mathfrak{L}(R)\,;\,\mathfrak{L}(S)} C$, so there exists some B with $A \mathrel{\mathfrak{L}(R)} B \mathrel{\mathfrak{L}(S)} C$. Then for every $a \in A$ there must exist $b \in B$ and $c \in C$ with $a \mathrel{R} b \mathrel{S} c$. Hence $A \mathrel{\mathfrak{L}(R\,;\,S)} C$. ◀

▶ **Lemma 11.** *$\mathfrak{L} : \mathbf{Rel} \longrightarrow \mathbf{CABARel}$ is faithful.*

Proof. Suppose $R, S : X \nrightarrow Y$ are two relations and that $\mathfrak{L}(R) = \mathfrak{L}(S)$. Then $x \mathrel{R} y$ iff $\{x\} \mathrel{\mathfrak{L}(R)} \{y\}$ iff $\{x\} \mathrel{\mathfrak{L}(S)} \{y\}$ iff $x \mathrel{S} y$. ◀

▶ **Lemma 12.** *$\mathfrak{L} : \mathbf{Rel} \longrightarrow \mathbf{CABARel}$ is full. In other words, any directionally atomic $R : \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$ is $\mathfrak{L}(S)$ for some relation $S : X \nrightarrow Y$.*

Proof. We will prove that $R = \mathfrak{L}(S_R)$ where $S_R : X \nrightarrow Y$ is given by $x \mathrel{S_R} y \stackrel{\text{def}}{\equiv} \{x\} \mathrel{R} \{y\}$.

Suppose that $A \mathrel{R} B$. Then for any $a \in A$ we must have that $\{a\} \mathrel{R} B$, as R is a bimodule. But then we must also have that there exists some $b \in B$ such that $\{a\} \mathrel{R} \{b\}$, because R is atomic-founded, i.e. $a \mathrel{S_R} b$. Therefore $A \mathrel{\mathfrak{L}(S_R)} B$.

Suppose now that $A\,\mathfrak{L}(S_R)\,B$, which is to say that for all $a \in A$ there exists a $b \in B$ such that $a\,S_R\,b$, which is to say $\{a\}\,R\,\{b\}$. Because R is a bimodule this means that for all $a \in A$, $\{a\}\,R\,B$. But $A = \bigcup_{a\in A}\{a\}$, so $A\,R\,B$ as R is left-disjunctive. ◀

One can identify a pseudo-inverse for $\mathfrak{L}$, viz. the functor $\mathsf{At} : \mathbf{CABARel} \longrightarrow \mathbf{Rel}$ which restricts every relation $R : \mathcal{L} \to\!\!\!\!\!\to \mathcal{L}'$ to atoms:

$$\mathsf{At}(\mathcal{L}) \overset{\text{def}}{=} \{x \in \mathcal{L} \mid x \text{ is an atom of } \mathcal{L}\}\,, \qquad\qquad \mathsf{At}(R) \overset{\text{def}}{=} R|_{\mathsf{At}(\mathcal{L})\times\mathsf{At}(\mathcal{L}')}.$$

This is a functor: when a and c are atoms, any witness for $a\,(R\,;S)\,c$ can be replaced by an atomic witness using atomic-foundedness of R and bimodularity of S.

▶ **Lemma 13.** $\mathfrak{L} : \mathbf{Rel} \longrightarrow \mathbf{CABARel}$ *is essentially surjective.*

Proof. Given a CABA $\mathcal{L}$, define relations $R_{\mathcal{L}} : \mathcal{L} \to\!\!\!\!\!\to \mathcal{P}(\mathsf{At}(\mathcal{L}))$ and $R_{\mathcal{L}}^{-1} : \mathcal{P}(\mathsf{At}(\mathcal{L})) \to\!\!\!\!\!\to \mathcal{L}$ by $x\,R_{\mathcal{L}}\,X$ just if $x \sqsubseteq \bigsqcup X$, and $X\,R_{\mathcal{L}}^{-1}\,x$ just if $\bigsqcup X \sqsubseteq x$. These relations are bimodules and left-disjunctive by monotonicity of joins. Atomic-foundedness of $R_{\mathcal{L}}$ uses the fact that atoms in a CABA are completely join-prime: if $a \sqsubseteq \bigsqcup X$, then $a \sqsubseteq a'$ for some $a' \in X$, hence $a = a' \in X$, so $\{a\} \sqsubseteq X$ witnesses the property. Atomic-foundedness of $R_{\mathcal{L}}^{-1}$ follows by choosing the same atom.

Their composites are the order identities. On $\mathcal{L}$, this says $x \sqsubseteq y$ iff $x \sqsubseteq \bigsqcup X \sqsubseteq y$ for some set X of atoms; the backward implication is immediate, and the forward one follows by taking the atoms below y. On $\mathcal{P}(\mathsf{At}(\mathcal{L}))$, it says $X \subseteq Y$ iff $\bigsqcup X \sqsubseteq \bigsqcup Y$, again by complete join-primality of atoms. ◀

▶ **Theorem 14.** $\mathbf{Rel} \simeq \mathbf{CABARel}$.

Of course, Theorem 14 is not explicitly a duality; to make it one we have to compose it with the formal duality $(-)^\dagger : \mathbf{Rel}^{\mathsf{op}} \simeq \mathbf{Rel}$ in the style of Kishida [22, §2.3], obtaining

▶ **Theorem 15** (Relational Tarski duality)**.** $\mathbf{Rel}^{\mathsf{op}} \simeq \mathbf{CABARel}$.

Intuitively, the fact that $\mathbf{Rel}$ is a dagger category means that its morphisms have *no inherent direction*. Directionality appears when we restrict it to functions, making this equivalence a "proper" duality [22, §2.3] [26, Remark 4.1]. Indeed, it is possible to show that this duality is an extension of the usual Tarski duality, in the sense that there is a commutative diagram

$$\begin{tikzcd}
\mathbf{Set}^{\mathsf{op}} \arrow[r, hook, "\mathrm{Gr}^{\mathsf{op}}"] \arrow[d, "\mathcal{P}"', "\simeq"] & \mathbf{Rel}^{\mathsf{op}} \arrow[d, "\simeq"', "\mathfrak{L}\circ(-)^\dagger"] \\
\mathbf{CABA} \arrow[r, hook, "j"'] & \mathbf{CABARel}
\end{tikzcd} \tag{1}$$

where Gr and j are faithful and injective-on-objects.

We define j by taking each CABA $\mathcal{B}$ to itself, and each complete Boolean homomorphism $f^* : \mathcal{B} \to \mathcal{B}'$ to $j(f^*) : \mathcal{B} \to\!\!\!\!\!\to \mathcal{B}'$ given by $b\,j(f^*)\,b'$ just if $b \sqsubseteq f_!(b')$, where $f_! \dashv f^*$.

▶ **Lemma 16.** $j : \mathbf{CABA} \longrightarrow \mathbf{CABARel}$ *is faithful and injective-on-objects.*

Proof. To begin we show that $j(f^*)$ is a morphism of $\mathbf{CABARel}$. First, $j(f^*)$ is left-disjunctive: suppose that for each $i \in I$ we have that $x_i\,j(f^*)\,y$, i.e. $x_i \sqsubseteq f_!(y)$. Then $\bigsqcup_{i\in I} x_i \sqsubseteq f_!(y)$, and hence $\bigsqcup_{i\in I} x_i j(f^*) y$. Second, $j(f^*)$ is atomic-founded. Suppose $x j(f^*) y$, i.e. $x \sqsubseteq f_!(y)$, with x an atom. Write $y = \bigsqcup_{i\in I} a_i$ for a_i atoms. Then $f_!(y) = \bigsqcup_{i\in I} f_!(a_i)$, because $f_!$ is a left adjoint and preserves joins. As x is an atom, $x \sqsubseteq f_!(a_i)$ for a particular $i \in I$, so $x\,j(f^*)\,a_i$ for some $a_i \sqsubseteq y$. Third, $j(f^*)$ is a bimodule, by monotonicity of $f_!$.

j preserves identities: we have $x j(\mathsf{id}) y$ iff $x \sqsubseteq \mathsf{id}_!(y) = y$, viz. the identity in **CABARel**. j preserves composition: let $f^* : X \to Y$ and $g^* : Y \to Z$ be complete Boolean homomorphisms. Then $x\, j(g^* \circ f^*)\, z$ iff $x \sqsubseteq f_!(g_!(z))$ iff $x\, j(f^*)\, g_!(z)$. But by reflexivity $g_!(z) \sqsubseteq g_!(z)$, i.e. $g_!(z) j(g^*) z$, so $x(j(f^*)\, ;\, j(g^*))z$. For the converse, suppose $x j(f^*) y j(g^*) z$, i.e. $x \sqsubseteq f_!(y)$ and $y \sqsubseteq g_!(z)$. By the monotonicity of $f_!$ we have $x \sqsubseteq f_!(y) \sqsubseteq f_!(g_!(z))$, therefore $x\, j(g^* \circ f^*)\, z$.

Finally, j is faithful because $j(f^*)$ determines $f_!$ by a Yoneda-type argument, and hence determines f^* too. It is injective-on-objects because it is the identity on objects. ◀

We define Gr by taking each set X to itself, and each function $f : X \to Y$ to its *graph* $\mathrm{Gr}\,(f) : X \nrightarrow Y$, viz. $x\, \mathrm{Gr}\,(f)\, y \stackrel{\text{def}}{\equiv} (f(x) = y)$.

▶ **Lemma 17.** *$Gr : \mathbf{Set} \longrightarrow \mathbf{Rel}$ is faithful and injective-on-objects.*

Proof. Gr clearly preserves identity and composition. It is faithful because functions are entirely determined by their graph. ◀

Finally, (1) commutes: for $f : X \to Y$, both routes send f, seen as a morphism $Y \to X$ in $\mathbf{Set}^{\mathsf{op}}$, to the relation $A \subseteq f_!(B)$ between $A \in \mathcal{P}(Y)$ and $B \in \mathcal{P}(X)$.

Sketch of a Formal System

Having developed this duality we can now see how it can be used for relating formulae. Suppose we have a relation $R : X \nrightarrow Y$. Under the relational Tarski duality this induces a relation $\mathfrak{L}(R) : \mathcal{P}(X) \nrightarrow \mathcal{P}(Y)$ between predicates. We denote this relation by the judgment

$$\varphi \overset{R}{\searrow} \psi$$

where $\varphi \in \mathcal{P}(X)$ is a predicate over X and $\psi \in \mathcal{P}(Y)$ over Y. Intuitively, this judgment says that *every state of X that satisfies φ is R-related to some state of Y that satisfies ψ.* The fact $\mathfrak{L}(R)$ is a bimodule means that the rule

$$\frac{\varphi' \vdash \varphi \qquad \varphi \overset{R}{\searrow} \psi \qquad \psi \vdash \psi'}{\varphi' \overset{R}{\searrow} \psi'}$$

is sound. This is reminiscent of the *consequence rule* of Hoare logic [15].

The fact $\mathfrak{L}(R)$ is left-disjunctive means that the rule

$$\frac{\varphi_1 \overset{R}{\searrow} \psi \qquad \varphi_2 \overset{R}{\searrow} \psi}{(\varphi_1 \vee \varphi_2) \overset{R}{\searrow} \psi}$$

is sound. This allows us to reason by cases on the left. The final characteristic property of $\mathfrak{L}(R)$, namely its atomic-foundedness, would require introducing judgments that capture atomicity of predicates, in the style of Abramsky [1].

Finally, the fact $\mathfrak{L}$ is a functor can be formally expressed by the rules

$$\frac{\varphi \overset{R}{\searrow} \psi \qquad \psi \overset{S}{\searrow} \chi}{\varphi \overset{R;S}{\searrow} \chi} \qquad\qquad \frac{\varphi \vdash \psi}{\varphi \overset{\mathsf{id}}{\searrow} \psi} \qquad\qquad \frac{\varphi \overset{\mathsf{id}}{\searrow} \psi}{\varphi \vdash \psi}$$

5 A Modal Relational Duality

In this section we will extend the relational Tarski duality to (infinitary) classical modal logic. The basis for this is the *Thomason duality* [42] [22, §2.4], which uniquely associates each Kripke frame with a CABA equipped with a modal operator (CABAO).

A *Kripke frame* (X, R) (also called a *transition system*) is a set X equipped with a relation $R : X \twoheadrightarrow X$. The intuition is that X is a set of *worlds*, and the relation R encodes *transitions* from one world to another. We will sometimes write $x_1 \to_R x_2$ instead of $x_1 \mathrel{R} x_2$. A morphism of Kripke frames $f : (X, R) \to (Y, S)$ is a function $f : X \to Y$ which preserves transitions, meaning $x_1 \to_R x_2$ implies $f(x_1) \to_S f(x_2)$. Kripke frames and their morphisms form a category **Frm**.

The relation R can also be seen as a map $\lambda R : X \to \mathcal{P}(X)$ by currying. Then there is a unique join-preserving function $\blacklozenge_R : \mathcal{P}(X) \to \mathcal{P}(X)$ that makes the following diagram commute, where $\mathbf{y} : X \to \mathcal{P}(X)$ maps every element $x \in X$ to the singleton set $\{x\}$.[5]

$$\begin{tikzcd} X \arrow[r, "\mathbf{y}"] \arrow[dr, "\lambda R"'] & \mathcal{P}(X) \arrow[d, dashed, bend right, "\blacklozenge_R"'] \arrow[d, phantom, "\dashv"] \\ & \mathcal{P}(X) \arrow[u, dotted, bend right, "\Box_R"'] \end{tikzcd}$$

As $\blacklozenge_R$ preserves joins and $\mathcal{P}(X)$ is a complete lattice, the adjoint functor theorem implies that it has a right adjoint $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$, which preserves all meets.

These maps are explicitly given by

$$\blacklozenge_R(A) = \{w \in X \mid \exists v \in A.\ v \mathrel{R} w\}, \quad \Box_R(A) = \{w \in X \mid \forall v \in X.\ w \mathrel{R} v \implies v \in A\}.$$

Having one of these three pieces of data (the relation R; a join-preserving map $\blacklozenge_R$; or a meet-preserving map $\Box_R$) uniquely determines the other two [21].

It is possible to define morphisms of frames purely in terms of operators:

▶ **Lemma 18.** *A function $f : X \to Y$ is a morphism of frames $f : (X, R) \to (Y, S)$ if and only if $f^* \circ \Box_S \subseteq \Box_R \circ f^*$.*

▶ **Notation.** *In a CABAO $\mathcal{B}$, we will usually write the operator(s) as $\blacklozenge_{\mathcal{B}} \dashv \Box_{\mathcal{B}}$, unless we explicitly want to draw attention to the relation R (as above), or it is clear from context which CABA and transition relation are relevant.*

Let $\mathbf{CABAO}^-$ be the category whose objects $(\mathcal{B}, \Box_{\mathcal{B}})$ are CABAs $\mathcal{B}$ equipped with an *operator* $\Box_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ that preserves all meets,[6] and morphisms $f^* : (\mathcal{B}, \Box_{\mathcal{B}}) \to (\mathcal{B}', \Box_{\mathcal{B}'})$ are complete Boolean homomorphisms $f^* : \mathcal{B} \to \mathcal{B}'$ that satisfy $f^* \circ \Box_{\mathcal{B}} \sqsubseteq \Box_{\mathcal{B}'} \circ f^*$ (with the pointwise order). This yields a modal duality

▶ **Theorem 19** (Weak Thomason duality)**.** $\mathbf{Frm}^{\mathsf{op}} \simeq \mathbf{CABAO}^-$.

The maps of CABAOs preserve the Boolean structure, but they only preserve $\Box$ weakly. To strengthen this we need the notion of an *open map*.

▶ **Definition 20.** *A morphism of frames $f : (X, R) \to (Y, S)$ is* open *iff $f(x) \to_S y'$ implies that there exists an $x' \in X$ with $x \to_R x'$ and $f(x') = y'$.*

[5] This is also known as the *left Kan extension* of λR along the Yoneda embedding.
[6] Or, equivalently, a join-preserving operator $\blacklozenge_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$.

We let $\mathbf{Frm}_{\text{open}}$ be the wide subcategory of $\mathbf{Frm}$ whose morphisms are open. It is easy to show that such an f is open iff $f^* \circ \Box = \Box \circ f^*$. Then, letting $\mathbf{CABAO}$ be the wide subcategory of $\mathbf{CABAO}^-$ whose morphisms f^* preserve $\Box$ (i.e. $f^* \circ \Box = \Box \circ f^*$) we obtain a refinement of Theorem 19:

▶ **Theorem 21** (Thomason duality). $\mathbf{Frm}_{\text{open}}^{\mathsf{op}} \simeq \mathbf{CABAO}$.

To obtain a relational version of this duality we must relax the functionality of morphisms of Kripke frames. To achieve that, notice that the notion of an open map is precisely a *functional bisimulation* [38, §3.2]. Hence, we replace open maps with general *(bi)simulations*.

▶ **Definition 22** (Simulations, Cosimulations, and Bisimulations). *Let (X, R) and (Y, S) be Kripke frames, and let $Q : X \to\!\!\!\!\!\!+\ Y$ be a relation.*

1. *Q is a* simulation *whenever $x\,Q\,y$ and $x \to_R x'$ imply that we have a $y' \in Y$ with $y \to_S y'$ and $x'\,Q\,y'$.*
2. *Q is a* cosimulation *whenever $Q^\dagger$ is a simulation.*
3. *Q is a* bisimulation *whenever it is both a simulation and a cosimulation.*

We illustrate the definitions of simulation and cosimulation pictorially:

$$\begin{tikzcd} x' \arrow[r, dashed, "Q"] & \exists y' \\ x \arrow[u, "R"] \arrow[r, "Q"] & y \arrow[u, dotted, "S"'] \end{tikzcd} \qquad \begin{tikzcd} \exists x' \arrow[r, dashed, "Q"] & y' \\ x \arrow[u, dotted, "R"] \arrow[r, "Q"] & y \arrow[u, "S"'] \end{tikzcd}$$

These two conditions are often called the *forth* and the *back* conditions respectively.

It is easy to see that an open map is exactly a function whose graph is a bisimulation. Indeed, the fact it preserves transitions means it is a simulation (i.e. it satisfies the "forth" condition). The fact it is open means it is a cosimulation (i.e. it satisfies the "back" condition).

(Co)simulations are closed under relational composition, and the identity relation is a bisimulation. We thus obtain two categories $\mathbf{FrmSim}$ and $\mathbf{FrmBisim}$ with Kripke frames as objects, and simulations and bisimulations as morphisms respectively. Note that the opposite of a simulation is a cosimulation, and vice versa. Therefore, there is a formal duality $(-)^\dagger : \mathbf{FrmBisim}^{\mathsf{op}} \simeq \mathbf{FrmBisim}$, meaning $\mathbf{FrmBisim}$ is also a dagger category.

It is possible to characterise simulations purely in terms of the lower relation.

▶ **Lemma 23.** *For a relation $Q : (X, R) \to\!\!\!\!\!\!+\ (Y, S)$ between Kripke frames, the following are equivalent:*

(i) *$Q : (X, R) \to\!\!\!\!\!\!+\ (Y, S)$ is a simulation.*
(ii) *$A\,\mathfrak{L}(Q)\,B$ implies $\blacklozenge A\,\mathfrak{L}(Q)\,\blacklozenge B$ for any $A \subseteq X$ and $B \subseteq Y$.*
(iii) *$A\,\mathfrak{L}(Q)\,\Box B$ implies $\blacklozenge A\,\mathfrak{L}(Q)\,B$ for any $A \subseteq X$ and $B \subseteq Y$.*

Proof. To show (i) implies (ii), suppose that $Q : (X, R) \to\!\!\!\!\!\!+\ (Y, S)$ is a simulation and that $A\,\mathfrak{L}(Q)\,B$ for some $A \subseteq X$ and $B \subseteq Y$. If $x \in \blacklozenge A$, then there exists some $a \in A$ with $a \to_R x$. We know that there exists some $b \in B$ for which $a\,Q\,b$, so by simulation there must exist some $y \in Y$ such that $b \to_S y$, so $y \in \blacklozenge B$, and $x\,Q\,y$. So $\blacklozenge A\,\mathfrak{L}(Q)\,\blacklozenge B$.

To show (ii) implies (iii), suppose the premise of (ii) and that $A\,\mathfrak{L}(Q)\,\Box B$ for some $A \subseteq X$ and $B \subseteq Y$. Then $\blacklozenge A\,\mathfrak{L}(Q)\,\blacklozenge\Box B$ by (ii). But $\blacklozenge \dashv \Box$, so $\blacklozenge\Box B \subseteq B$, and hence $\blacklozenge A\,\mathfrak{L}(Q)\,B$ as $\mathfrak{L}(Q)$ is a bimodule.

Finally, to show (iii) implies (i), suppose the premise of (iii) and that $x\,Q\,y$ for some $x \in X$ and $y \in Y$, equally $\{x\}\,\mathfrak{L}(Q)\,\{y\}$. Note that $\{y\} \subseteq \Box\blacklozenge\,\{y\}$ and $\mathfrak{L}(Q)$ is a bimodule, therefore $\{x\}\,\mathfrak{L}(Q)\,\Box\blacklozenge\,\{y\}$, then by (iii) $\blacklozenge\,\{x\}\,\mathfrak{L}(Q)\,\blacklozenge\,\{y\}$. So for any $x' \in X$ for which $x \to_R x'$, there exists some $y' \in Y$ for which $y \to_R y'$ and $x'\,Q\,y'$, i.e. Q is a simulation. ◀

▶ **Definition 24.** *A* simulatory relation $Q : (\mathcal{B}, \Box_{\mathcal{B}}) \to\!\!\!\!\!\!+ (\mathcal{B}', \Box_{\mathcal{B}'})$ *between CABAOs is a directionally atomic* $Q : \mathcal{B} \to\!\!\!\!\!\!+ \mathcal{B}'$ *for which the condition of Lemma 23 holds, i.e.*

$$A\, Q\, \Box_{\mathcal{B}'} B \implies \blacklozenge_{\mathcal{B}} A\, Q\, B.$$

▶ **Proposition 25.** *If* $R : (\mathcal{B}, \Box_{\mathcal{B}}) \to\!\!\!\!\!\!+ (\mathcal{B}', \Box_{\mathcal{B}'})$ *and* $S : (\mathcal{B}', \Box_{\mathcal{B}'}) \to\!\!\!\!\!\!+ (\mathcal{B}'', \Box_{\mathcal{B}''})$ *are simulatory relations, then their relational composition is a simulatory relation* $R \,;\, S : (\mathcal{B}, \Box_{\mathcal{B}}) \to\!\!\!\!\!\!+ (\mathcal{B}'', \Box_{\mathcal{B}''})$.

Define **CABAOSim** to have CABAs with operators $(\mathcal{B}, \Box_{\mathcal{B}})$ as objects and simulatory relations $Q : (\mathcal{B}, \Box_{\mathcal{B}}) \to\!\!\!\!\!\!+ (\mathcal{B}', \Box_{\mathcal{B}'})$ as morphisms. We will often write these morphisms simply as $Q : \mathcal{B} \to\!\!\!\!\!\!+ \mathcal{B}'$ and omit subscripts. Using the preceding proposition, this is a category, with $\sqsubseteq$ as an identity morphism.

▶ **Lemma 26.** $\mathfrak{L}$ *is a functor* **FrmSim** $\longrightarrow$ **CABAOSim**.

▶ **Lemma 27.** $\mathfrak{L}$: **FrmSim** $\longrightarrow$ **CABAOSim** *is faithful.*

▶ **Lemma 28.** $\mathfrak{L}$: **FrmSim** $\longrightarrow$ **CABAOSim** *is full. In other words, every simulatory relation* $Q : (\mathcal{P}(X), \Box_R) \to\!\!\!\!\!\!+ (\mathcal{P}(Y), \Box_S)$ *is* $\mathfrak{L}(T)$ *for some simulation* $T : (X, R) \to\!\!\!\!\!\!+ (Y, S)$.

We can define a functor At : **CABAOSim** $\longrightarrow$ **FrmSim** by restricting a simulatory relation to atoms, as before. Using Lemma 23 it is easy to check that $\mathsf{At}(R)$ is a simulation.

▶ **Lemma 29.** $\mathfrak{L}$: **FrmSim** $\longrightarrow$ **CABAOSim** *is essentially surjective.*

Proof. Following the proof of Lemma 13, given a CABAO $\mathcal{B}$, define the relations $R_{\mathcal{B}} : \mathcal{B} \to\!\!\!\!\!\!+ \mathcal{P}(\mathsf{At}(\mathcal{B}))$ and $R_{\mathcal{B}}^{-1} : \mathcal{P}(\mathsf{At}(\mathcal{B})) \to\!\!\!\!\!\!+ \mathcal{B}$ by $x\, R_{\mathcal{B}}\, X$ iff $x \sqsubseteq \bigsqcup X$ and $X\, R_{\mathcal{B}}^{-1}\, x$ iff $\bigsqcup X \sqsubseteq x$ respectively. As in the proof of Lemma 13 these are both directionally atomic and evidently inverses to each other. Thus, it suffices to show that they are simulatory.

Suppose that $x \sqsubseteq \bigsqcup \Box X$, then

$$x \sqsubseteq \bigsqcup \Box X \sqsubseteq \Box \blacklozenge \bigsqcup \Box X = \Box \bigsqcup \blacklozenge \Box X \sqsubseteq \Box \bigsqcup X$$

by properties of the adjunction $\blacklozenge \dashv \Box$. Hence $\blacklozenge x \sqsubseteq \bigsqcup X$, and $R_{\mathcal{B}}$ is simulatory.

Similarly, if $\bigsqcup X \sqsubseteq \Box x$, then $\blacklozenge \bigsqcup X \sqsubseteq x$ and therefore $\bigsqcup \blacklozenge X \sqsubseteq x$, so $R_{\mathcal{B}}^{-1}$ is simulatory. ◀

In summary, we obtain an equivalence

▶ **Theorem 30.** **FrmSim** $\simeq$ **CABAOSim**.

As in our previous relational duality, the directionality disappears in the relational case. We could turn this result into a duality by composing it with a formal duality between **FrmSim** and the category of *co*simulations. However, this is somewhat awkward.

Our final objective is to restrict this equivalence by analogy to the restriction of $\mathbf{Frm}^{\mathsf{op}} \simeq \mathbf{CABAO}^{-}$ to $\mathbf{Frm}^{\mathsf{op}}_{\text{open}} \simeq \mathbf{CABAO}$, where the maps in $\mathbf{Frm}^{\mathsf{op}}_{\text{open}}$ are open, and correspondingly the maps in **CABAO** preserve all logical connectives. In our relational duality the morphisms are no longer functional, but merely simulations. Thus, openness will turn them into *bisimulations*.

Unfortunately, expressing bisimulations using our current vocabulary is not immediately possible. For that we will need the following "dual" relational lifting.

▶ **Definition 31** (Upper Relation). *Given a relation $R : X \rightarrow\!\!\!\!\rightarrow Y$, the associated upper relation $\mathfrak{U}(R) : \mathcal{P}(X) \rightarrow\!\!\!\!\rightarrow \mathcal{P}(Y)$ is defined by*

$$S\, \mathfrak{U}(R)\, T \overset{def}{\equiv} \forall t \in T.\ \exists s \in S.\ s\, R\, t.$$

It is easy to see that $\mathfrak{U}(R) = \mathfrak{L}(R^\dagger)^\dagger$ for any $R : X \rightarrow\!\!\!\!\rightarrow Y$. This makes the theory dual, in the sense that $\mathfrak{U}(R)$ is *opdirectionally atomic* (i.e. right-disjunctive, atomic op-founded, and an opbimodule). To avoid this proliferation of concepts we will simply work with the opposite relation $\mathfrak{L}(R^\dagger)$. We can construct this relation directly on relations between CABAs.

▶ **Definition 32.** *For a relation $R : \mathcal{B} \rightarrow\!\!\!\!\rightarrow \mathcal{B}'$ between CABAs we define its* variant relation *$R^\bullet : \mathcal{B}' \rightarrow\!\!\!\!\rightarrow \mathcal{B}$ to be*

$$b'\, R^\bullet\, b \overset{def}{\equiv} \forall atom\ a' \sqsubseteq b'.\ \exists atom\ a \sqsubseteq b.\ a\, R\, a'.$$

This acts in the expected way between lifted relations.

▶ **Lemma 33.** $\mathfrak{L}(R)^\bullet = \mathfrak{L}(R^\dagger)$ *for any $R : X \rightarrow\!\!\!\!\rightarrow Y$, and hence $\mathfrak{U}(R) = \mathfrak{L}(R)^{\bullet\dagger}$.*

Proof. Let $R : X \rightarrow\!\!\!\!\rightarrow Y$ be a relation between sets. Then for any $T \subseteq Y$ and $S \subseteq X$ we have $T\, \mathfrak{L}(R)^\bullet\, S$ iff $\forall t \in T.\ \exists s \in S.\ s\, R\, t$. Note that this is exactly the same as $T\, \mathfrak{L}(R^\dagger)\, S$. ◀

As every directionally atomic relation is in the image of $\mathfrak{L}$, Lemma 33 implies that the variant of a directionally atomic relation is directionally atomic. Hence, the variant construction forms a functor $(-)^\bullet : \mathbf{CABARel}^{\mathsf{op}} \longrightarrow \mathbf{CABARel}$ which sends every CABA to itself, and every directionally atomic relation to its variant. Moreover,

▶ **Theorem 34.** $(-)^\bullet : \mathbf{CABARel}^{\mathsf{op}} \longrightarrow \mathbf{CABARel}$ *is an equivalence.*

Thus, we have a self-duality on **CABARel**, which makes the following claims straightforward.

As with simulations and Lemma 23, it is possible to characterise cosimulations purely in terms of the variant relation.

▶ **Lemma 35.** *For a relation $Q : (X, R) \rightarrow\!\!\!\!\rightarrow (Y, S)$ between Kripke frames, the following are equivalent:*
(i) *$Q : (X, R) \rightarrow\!\!\!\!\rightarrow (Y, S)$ is a cosimulation.*
(ii) *$B\, \mathfrak{L}(Q)^\bullet\, A$ implies $\blacklozenge B\, \mathfrak{L}(Q)^\bullet \blacklozenge A$ for any $B \subseteq Y$ and $A \subseteq X$.*
(iii) *$B\, \mathfrak{L}(Q)^\bullet\, \Box A$ implies $\blacklozenge B\, \mathfrak{L}(Q)^\bullet\, A$ for any $B \subseteq Y$ and $A \subseteq X$.*

Proof. Q is a cosimulation iff $Q^\dagger$ is a simulation, so just by Lemmas 23 and 33. ◀

▶ **Definition 36.** *A* cosimulatory relation *$Q : (\mathcal{B}, \Box_\mathcal{B}) \rightarrow\!\!\!\!\rightarrow (\mathcal{B}', \Box_{\mathcal{B}'})$ between CABAOs is a directionally atomic relation $Q : \mathcal{B} \rightarrow\!\!\!\!\rightarrow \mathcal{B}'$ for which the condition of Lemma 35 holds, i.e.*

$$B\, Q^\bullet\, \Box_\mathcal{B} A \implies \blacklozenge_{\mathcal{B}'} B\, Q^\bullet\, A.$$

The following proposition is analogous to Proposition 25.

▶ **Proposition 37.** *If $R : (\mathcal{B}, \Box_\mathcal{B}) \rightarrow\!\!\!\!\rightarrow (\mathcal{B}', \Box_{\mathcal{B}'})$ and $S : (\mathcal{B}', \Box_{\mathcal{B}'}) \rightarrow\!\!\!\!\rightarrow (\mathcal{B}'', \Box_{\mathcal{B}''})$ are cosimulatory relations, then their composition is a cosimulatory relation $R\,;S : (\mathcal{B}, \Box_\mathcal{B}) \rightarrow\!\!\!\!\rightarrow (\mathcal{B}'', \Box_{\mathcal{B}''})$.*

Call a relation *bisimulatory* if it is both simulatory and cosimulatory. Let **CABAOBisim** be the category whose objects are CABAOs and whose morphisms are bisimulatory relations. As before, $\sqsubseteq$ is the identity.

The next four lemmas are the bisimulation analogues of Lemmas 26–29.

▶ **Lemma 38.** *$\mathfrak{L}$ is a functor* $\mathbf{FrmBisim} \longrightarrow \mathbf{CABAOBisim}$.

▶ **Lemma 39.** $\mathfrak{L} : \mathbf{FrmBisim} \longrightarrow \mathbf{CABAOBisim}$ *is faithful.*

▶ **Lemma 40.** $\mathfrak{L} : \mathbf{FrmBisim} \longrightarrow \mathbf{CABAOBisim}$ *is full. In other words, every bisimulatory relation $Q : (\mathcal{P}(X), \Box_R) \rightarrow\!\!\!\!\!\rightarrow (\mathcal{P}(Y), \Box_S)$ is $\mathfrak{L}(T)$ for some bisimulation $T : (X, R) \rightarrow\!\!\!\!\!\rightarrow (Y, S)$.*

▶ **Lemma 41.** $\mathfrak{L} : \mathbf{FrmBisim} \longrightarrow \mathbf{CABAOBisim}$ *is essentially surjective.*

Proof. We augment the proof of Lemma 29 by showing that both $R_{\mathcal{B}} : \mathcal{B} \rightarrow\!\!\!\!\!\rightarrow \mathcal{P}(\mathsf{At}(\mathcal{B}))$ and $R_{\mathcal{B}}^{-1} : \mathcal{P}(\mathsf{At}(\mathcal{B})) \rightarrow\!\!\!\!\!\rightarrow \mathcal{B}$, given by $x \mathrel{R_{\mathcal{B}}} X$ iff $x \sqsubseteq \bigsqcup X$ and $X \mathrel{R_{\mathcal{B}}^{-1}} x$ iff $\bigsqcup X \sqsubseteq x$ respectively, are also cosimulatory.

First, $X \mathrel{R_{\mathcal{B}}^{\bullet}} x$ iff for every atom $\{c\} \subseteq X$ there exists an atom $a \sqsubseteq x$ with $a \mathrel{R_{\mathcal{B}}} \{c\}$. By definition this means $a \sqsubseteq c$, and since a and c are atoms, $a = c$. Thus every element of X (all are atoms) is below x, or equivalently $\bigsqcup X \sqsubseteq x$, i.e. iff $X \mathrel{R_{\mathcal{B}}^{-1}} x$. Hence $R_{\mathcal{B}}^{\bullet} = R_{\mathcal{B}}^{-1}$.

Similarly, $x \mathrel{R_{\mathcal{B}}^{-1\bullet}} X$ iff $x \mathrel{R_{\mathcal{B}}} X$. Hence $R_{\mathcal{B}}^{-1\bullet} = R_{\mathcal{B}}$.

A relation is cosimulatory precisely when its variant is simulatory, and $R_{\mathcal{B}}$ and $R_{\mathcal{B}}^{-1}$ are both simulatory by Lemma 29. ◀

We thus obtain the following equivalence.

▶ **Theorem 42.** $\mathbf{FrmBisim} \simeq \mathbf{CABAOBisim}$.

As before, this is not explicitly a duality, but we can obtain one from the formal duality $(-)^{\dagger} : \mathbf{FrmBisim}^{\mathsf{op}} \simeq \mathbf{FrmBisim}$.

▶ **Theorem 43** (Relational Thomason duality)**.** $\mathbf{FrmBisim}^{\mathsf{op}} \simeq \mathbf{CABAOBisim}$.

We can show that Theorem 43 is an extension of the Thomason duality; as with (1), there is a commutative diagram of functors

$$\begin{tikzcd}
\mathbf{Frm}^{\mathsf{op}}_{\mathrm{open}} \arrow[r, hook, "\mathrm{Gr}^{\mathsf{op}}"] \arrow[d, "\mathcal{P}"', "\simeq"] & \mathbf{FrmBisim}^{\mathsf{op}} \arrow[d, "\simeq"', "\mathfrak{L}\circ(-)^{\dagger}"] \\
\mathbf{CABAO} \arrow[r, hook, "j"'] & \mathbf{CABAOBisim}
\end{tikzcd} \tag{2}$$

where Gr and j are faithful and injective-on-objects.

We define j as before, by taking each CABAO $(\mathcal{B}, \Box_{\mathcal{B}})$ to itself, and each complete Boolean homomorphism $f^* : \mathcal{B} \to \mathcal{B}'$ to $j(f^*) : \mathcal{B} \rightarrow\!\!\!\!\!\rightarrow \mathcal{B}'$ given by $b \mathrel{j(f^*)} b'$ iff $b \sqsubseteq f_!(b')$.

▶ **Lemma 44.** $j : \mathbf{CABAO} \longrightarrow \mathbf{CABAOBisim}$ *is faithful and injective-on-objects.*

Proof. By extension of Lemma 16, we know that $j(f^*)$ is a directionally atomic relation, and that j preserves identities and composition, and that it is faithful. It remains only to show that $j(f^*)$ is bisimulatory.

First, to show that $j(f^*)$ is simulatory, suppose that $x \mathrel{j(f^*)} \Box y$, that is, $x \sqsubseteq f_!(\Box y)$. Then

$$x \sqsubseteq f_!(\Box y) \sqsubseteq \Box \blacklozenge f_!(\Box y) \sqsubseteq \Box f_!(\blacklozenge \Box y) \sqsubseteq \Box f_!(y)$$

by the properties of the adjunction $\blacklozenge \dashv \Box$. Hence $\blacklozenge x \sqsubseteq f_!(y)$, so $j(f^*)$ is simulatory.

Second, to show that $j(f^*)$ is cosimulatory, notice first that $b \mathrel{j(f^*)^{\bullet}} a$ iff, for any atom $y \sqsubseteq b$ there exists some atom $x \sqsubseteq a$ for which $x \sqsubseteq f_!(y)$, which is the same as $x = f_!(y)$ by the properties of atoms and recalling that $f_!$ preserves atoms. Therefore, $b \mathrel{j(f^*)^{\bullet}} a$ iff $b \sqsubseteq f^*(a)$. Hence, if $b \sqsubseteq f^*(\Box a) = \Box f^*(a)$, then $\blacklozenge b \sqsubseteq f^*(a)$ by adjunction, and $j(f^*)$ is cosimulatory. The functor is injective-on-objects because it is the identity on objects. ◀

As usual, Gr takes each frame (X, R) to itself and each open map $f : (X, R) \to (Y, S)$ to the graph $\mathrm{Gr}\,(f) : X \nrightarrow Y$ defined by $x\,\mathrm{Gr}\,(f)\,y$ iff $f(x) = y$.

▶ **Lemma 45.** *The functor* $Gr : \mathbf{Frm}_{\mathrm{open}} \longrightarrow \mathbf{FrmBisim}$ *is faithful and injective-on-objects.*

Proof. Gr clearly takes open maps to bisimulations, preserves identity and composition. It is faithful because any function is determined entirely by its graph, and injective-on-objects because it is the identity on objects. ◀

Finally, a direct calculation shows that (2) commutes.

Sketch of a Formal System

Recall the formal system of the judgment $\overset{Q}{\searrow}$ for a relation $Q : X \nrightarrow Y$ that we sketched in Section 4. Now assume that (X, R) and (Y, S) are Kripke frames. We can extend this system to a modal logic where the logics over X and Y have corresponding modalities $\blacklozenge$ and $\Box$. If Q is a simulation then by Lemma 23 we can extend the system with the rules

$$\frac{\varphi \overset{Q}{\searrow} \psi}{\blacklozenge\varphi \overset{Q}{\searrow} \blacklozenge\psi} \qquad\qquad \frac{\varphi \overset{Q}{\searrow} \Box\psi}{\blacklozenge\varphi \overset{Q}{\searrow} \psi}$$

If Q is a cosimulation, then by Lemmas 33 and 35 we obtain the rules

$$\frac{\varphi \overset{Q^\dagger}{\searrow} \psi}{\blacklozenge\varphi \overset{Q^\dagger}{\searrow} \blacklozenge\psi} \qquad\qquad \frac{\varphi \overset{Q^\dagger}{\searrow} \Box\psi}{\blacklozenge\varphi \overset{Q^\dagger}{\searrow} \psi}$$

which could also be obtained just by noting $Q^\dagger$ is a simulation.

▶ **Example 46** (A Worked Example: Bisimulation Between Buffer Systems)**.** We will now show how to use these rules to reason over a particular bisimulation. The two frames both model a *buffer* holding a single natural number. However, the two buffers have *internal branching*: the first has two different “ways” of holding a number (the left and right slots), while the second one has three (slots A, B, and C). This might happen if the buffer is provided by a distributed cluster of machines whose internal structure is not observable.

Define $(X, \to_X)$ by letting $X = \{\mathsf{empty}_X\} \cup \{\mathsf{L}(n), \mathsf{R}(n) \mid n \in \mathbb{N}\}$ with

$$\mathsf{empty}_X \to_X \mathsf{L}(n) \qquad \mathsf{empty}_X \to_X \mathsf{R}(n) \qquad \mathsf{L}(n) \to_X \mathsf{empty}_X \qquad \mathsf{R}(n) \to_X \mathsf{empty}_X$$

for all $n \in \mathbb{N}$. Similarly, define $(Y, \to_Y)$ by $Y = \{\mathsf{empty}_Y\} \cup \{\mathsf{A}(n), \mathsf{B}(n), \mathsf{C}(n) \mid n \in \mathbb{N}\}$ with

$$\mathsf{empty}_Y \to_Y \mathsf{A}(n) \qquad \mathsf{empty}_Y \to_Y \mathsf{B}(n) \qquad \mathsf{empty}_Y \to_Y \mathsf{C}(n)$$

$$\mathsf{A}(n) \to_Y \mathsf{empty}_Y \qquad \mathsf{B}(n) \to_Y \mathsf{empty}_Y \qquad \mathsf{C}(n) \to_Y \mathsf{empty}_Y$$

for all $n \in \mathbb{N}$. There is a bisimulation $Q : X \nrightarrow Y$ between these frames given by

$$Q = \{(\mathsf{empty}_X, \mathsf{empty}_Y)\} \cup \{(w, v) \mid n \in \mathbb{N},\ w \in F_X(n),\ v \in F_Y(n)\}$$

where $F_X(n) = \{\mathsf{L}(n), \mathsf{R}(n)\}$ and $F_Y(n) = \{\mathsf{A}(n), \mathsf{B}(n), \mathsf{C}(n)\}$. In other words, full states holding the same value are related, and so are empty states. This is evidently a non-functional bisimulation.

We define the following predicates, parametrised by $n \in \mathbb{N}$. On X let $\mathit{empty}_X = \{\mathsf{empty}_X\}$, $L_n = \{\mathsf{L}(n)\}$, and $R_n = \{\mathsf{R}(n)\}$. On Y let $\mathit{empty}_Y = \{\mathsf{empty}_Y\}$.

We will use the following facts as the "background theory":

$$\begin{array}{lll} \blacklozenge\, \mathit{empty}_X = \bigvee_n F_X(n) & \qquad \Box\, \mathit{empty}_X = \bigvee_n F_X(n) & \qquad L_n \overset{Q}{\searrow} F_Y(n) \\ \blacklozenge\, \mathit{empty}_Y = \bigvee_n F_Y(n) & \qquad \Box\, \mathit{empty}_Y = \bigvee_n F_Y(n) & \qquad R_n \overset{Q}{\searrow} F_Y(n) \end{array}$$

The first two columns arise by direct observation. The last column holds because every left/right state is simulated by a full state holding the same value.

We derive the fact that if the previous state in the first buffer was a left or right one holding n, then the current state is simulated by an empty state in the second buffer:

$$(\blacklozenge\, L_n \vee \blacklozenge\, R_n) \overset{Q}{\searrow} \mathit{empty}_Y$$

The derivation proceeds in two symmetric branches, combined by disjunction:

$$\dfrac{\dfrac{\dfrac{L_n \overset{Q}{\searrow} F_Y(n) \qquad F_Y(n) \vdash \Box\, \mathit{empty}_Y}{L_n \overset{Q}{\searrow} \Box\, \mathit{empty}_Y}}{\blacklozenge L_n \overset{Q}{\searrow} \mathit{empty}_Y} \qquad \dfrac{\dfrac{R_n \overset{Q}{\searrow} F_Y(n) \qquad F_Y(n) \vdash \Box\, \mathit{empty}_Y}{R_n \overset{Q}{\searrow} \Box\, \mathit{empty}_Y}}{\blacklozenge R_n \overset{Q}{\searrow} \mathit{empty}_Y}}{(\blacklozenge L_n \vee \blacklozenge R_n) \overset{Q}{\searrow} \mathit{empty}_Y}$$

6 Related Work

A few relational dualities have been previously described in the literature. Most are of the form $\mathcal{C}^{\mathsf{op}} \simeq \mathcal{D}$ where $\mathcal{C}$ is a category whose morphisms are relations, whereas $\mathcal{D}$ is a category of algebras with some form of *hemimorphism*, i.e. a morphism preserving most – but not all! – of the logical structure. The earliest duality of this form is $\mathbf{Rel} \simeq \mathbf{CABA}_{\bigvee}$. This duality has been rediscovered multiple times, but is likely due to Jónsson [20]. Kishida [22, §2.3] argues that this extends to a "2-duality," as both of these are (strict) 2-categories. Halmos [11] extended it to the continuous case, i.e. a duality between Stone spaces with continuous relations on the one hand, and Boolean algebras with hemimorphisms on the other. Cignoli et al. [6] do something similar for Priestley spaces and continuous monotone relations. Hofmann and Nora [16] have proposed a general framework for such dualities, obtaining the relational side as the Kleisli category of a suitable monad. In later work they extended such dualities to metric structures and quantale-enriched categories [17].

Kurz, Moshier, and Jung [26] present dualities that are much closer to the flavour we employ in this paper. They achieve this by working in an order-enriched setting, where relations can be presented as both spans and cospans in a 2-categorical manner. They use this to extend (well-behaved) dualities to dualities between categories of relations, and even adjunctions of categories to adjunctions between framed bicategories [40]. For example, if their recipe is applied to the category **Pos** of posets and monotone functions it lifts a bimodule $R : X \twoheadrightarrow Y$ to the relation $2^R : [X, 2] \twoheadrightarrow [Y, 2]$ between upper sets that is defined by letting $A\, 2^R\, B$ just if $x \in A$ and $x\, R\, y$ implies $y \in B$. The type of lifting they obtain is very different: as pointed out by one of our reviewers, $A\, 2^R\, B$ just if $R[A] \subseteq B$, whereas $A\, \mathfrak{L}(R)\, B$ just if $A \subseteq R^{-1}[B]$.

Birkmann, Urbat, and Milius [3] present extensions of categorical dualities through monoidal adjunctions and apply them to algebraic language theories. As a corollary they obtain some relational dualities, e.g. between well-behaved relations of profinite ordered monoids and natural morphisms of residuation algebras.

Malacaria [32] presents the Thomason duality in a new light, and shows how it can be used to give bisimulation an algebraic meaning. In particular, the main theorem shows that two Kripke frames are bisimilar just if their associated (dual) algebras have an isomorphic subalgebra.

References

1 Samson Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51(1):1–77, 1991. doi:10.1016/0168-0072(91)90065-T.

2 Steve Awodey. *Category Theory*. Oxford Logic Guides. Oxford University Press, 2010.

3 Fabian Birkmann, Henning Urbat, and Stefan Milius. Monoidal extended stone duality. In Naoki Kobayashi and James Worrell, editors, *Foundations of Software Science and Computation Structures*, volume 14574 of *Lecture Notes in Computer Science*, pages 144–165. Springer Nature Switzerland, 2024. doi:10.1007/978-3-031-57228-9_8.

4 Patrick Blackburn, Maarten de Rijke, and Yde Venema. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001. doi:10.1017/CBO9781107050884.

5 Alexander Chagrov and Michael Zakharyaschev. *Modal Logic*. Number 35 in Oxford Logic Guides. Oxford University Press, 1996. doi:10.1093/oso/9780198537793.001.0001.

6 R. Cignoli, S. Lafalce, and A. Petrovich. Remarks on priestley duality for distributive lattices. *Order*, 8(3):299–315, 1991. doi:10.1007/BF00383451.

7 B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2nd edition, 2002. doi:10.1017/CBO9780511809088.

8 Edsger W. Dijkstra. *A Discipline of Programming*. Prentice-Hall, 1976.

9 Leo Esakia. *Heyting Algebras: Duality Theory*, volume 50 of *Trends in Logic*. Springer International Publishing, 2019. doi:10.1007/978-3-030-12096-2.

10 Mai Gehrke and Sam van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*. Number 61 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2024. arXiv:2203.03286.

11 Paul R. Halmos. Algebraic logic, I. Monadic boolean algebras. *Compositio Mathematica*, 12:217–249, 1954-1956. Publisher: Kraus Reprint. URL: https://www.numdam.org/item/CM_1954-1956__12__217_0/.

12 G. Hansoul. A duality for boolean algebras with operators. *Algebra Universalis*, 17(1):34–49, 1983. doi:10.1007/BF01194512.

13 M. C. B. Hennessy and G. D. Plotkin. Full abstraction for a simple parallel programming language. In Jiří Bečvář, editor, *Mathematical Foundations of Computer Science 1979*, volume 74 of *Lecture Notes in Computer Science*, pages 108–120. Springer Berlin Heidelberg, 1979. doi:10.1007/3-540-09526-8_8.

14 Chris Heunen and Jamie Vicary. *Categories for Quantum Theory: An Introduction*. Oxford University Press, 2019. doi:10.1093/oso/9780198739623.001.0001.

15 C. A. R. Hoare. An axiomatic basis for computer programming. *Communications of the ACM*, 12(10):576–580, 1969. doi:10.1145/363235.363259.

16 Dirk Hofmann and Pedro Nora. Dualities for modal algebras from the point of view of triples. *Algebra universalis*, 73(3):297–320, 2015. doi:10.1007/s00012-015-0324-5.

17 Dirk Hofmann and Pedro Nora. Duality theory for enriched priestley spaces. *Journal of Pure and Applied Algebra*, 227(3):107231, 2023. doi:10.1016/j.jpaa.2022.107231.

18 Bart Jacobs. *Introduction to Coalgebra: Towards Mathematics of States and Observation*. Cambridge University Press, 2016. doi:10.1017/CBO9781316823187.

19 Peter T. Johnstone. *Stone Spaces*. Number 3 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1982.

20 Bjarni Jonsson and Alfred Tarski. Boolean algebras with operators. part i. *American Journal of Mathematics*, 73(4):891, 1951. doi:10.2307/2372123.

21 G. A. Kavvos. Two-Dimensional Kripke Semantics I: Presheaves. In Jakob Rehof, editor, *9th International Conference on Formal Structures for Computation and Deduction (FSCD 2024)*, volume 299 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 14:1–14:23, Dagstuhl, Germany, 2024. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. `doi:10.4230/LIPIcs.FSCD.2024.14`.

22 Kohei Kishida. Categories and modalities. In Elaine Landry, editor, *Categories for the Working Philosopher*. Oxford University Press, 2018. `doi:10.1093/oso/9780198748991.003.0009`.

23 Saul Kripke. Semantical Considerations on Modal Logic. *Acta Philosophica Fennica*, 16:83–94, 1963.

24 Saul A. Kripke. Semantical Analysis of Modal Logic I. Normal Modal Propositional Calculi. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9(5-6):67–96, 1963. `doi:10.1002/malq.19630090502`.

25 Saul A. Kripke. Semantical Analysis of Intuitionistic Logic I. In J. N. Crossley and M. A. E. Dummett, editors, *Formal Systems and Recursive Functions*, volume 40 of *Studies in Logic and the Foundations of Mathematics*, pages 92–130. Elsevier, 1965. `doi:10.1016/S0049-237X(08)71685-9`.

26 Alexander Kurz, Andrew Moshier, and Achim Jung. Stone Duality for Relations. In Alessandra Palmigiano and Mehrnoosh Sadrzadeh, editors, *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, pages 159–215. Springer International Publishing, Cham, 2023. `doi:10.1007/978-3-031-24117-8_5`.

27 Alexander Kurz and Jiri Rosicky. Strongly complete logics for coalgebras. *Logical Methods in Computer Science*, Volume 8, Issue 3:1231, 2012. `doi:10.2168/LMCS-8(3:14)2012`.

28 Alexander Kurz and Jiří Velebil. Relation lifting, a survey. *Journal of Logical and Algebraic Methods in Programming*, 85(4):475–499, 2016. `doi:10.1016/j.jlamp.2015.08.002`.

29 F. William Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In *Proceedings of the AMS Symposium on Pure Mathematics*, volume 17, pages 1–14, 1970. URL: `https://ncatlab.org/nlab/files/LawvereComprehension.pdf`.

30 Paul Blain Levy. Similarity quotients as final coalgebras. In Martin Hofmann, editor, *Foundations of Software Science and Computational Structures*, volume 6604 of *Lecture Notes in Computer Science*, pages 27–41. Springer Berlin Heidelberg, 2011. `doi:10.1007/978-3-642-19805-2_3`.

31 Saunders Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer New York, 2 edition, 1978. `doi:10.1007/978-1-4757-4721-8`.

32 Pasquale Malacaria. Studying equivalences of transition systems with algebraic tools. *Theoretical Computer Science*, 139(1-2):187–205, 1995. `doi:10.1016/0304-3975(94)00047-M`.

33 Paul-André Melliès and Noam Zeilberger. A bifibrational reconstruction of lawvere's presheaf hyperdoctrine. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 555–564. Association for Computing Machinery, 2016. `doi:10.1145/2933575.2934525`.

34 Mogens Nielsen, Gordon Plotkin, and Glynn Winskel. Petri nets, event structures and domains, Part I. *Theoretical Computer Science*, 13(1):85–108, 1981. `doi:10.1016/0304-3975(81)90112-2`.

35 Gordon D. Plotkin. A powerdomain construction. *SIAM Journal on Computing*, 5(3):452–487, 1976. `doi:10.1137/0205035`.

36 Emily Riehl. *Category Theory in Context*. Dover Publications, 2016. URL: `http://www.math.jhu.edu/~eriehl/context.pdf`.

37 Giovanni Sambin and Virginia Vaccaro. Topology and duality in modal logic. *Annals of Pure and Applied Logic*, 37(3):249–296, 1988. `doi:10.1016/0168-0072(88)90021-8`.

38 Davide Sangiorgi. On the origins of bisimulation and coinduction. *ACM Transactions on Programming Languages and Systems*, 31(4):1–41, 2009. `doi:10.1145/1516507.1516510`.

39 Dana S. Scott. Relating Theories of the Lambda Calculus. In Jonathan P. Seldin and J. Roger Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*. Academic Press, London, 1980.

40 Michael A. Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738, 2008.

41 Albert Marchienus Thijs. *Simulation and Fixpoint Semantics*. PhD thesis, University of Groningen, 1996. URL: `https://hdl.handle.net/11370/13d08025-29ff-4193-a7f2-ea5bcd20f15d`.

42 S. K. Thomason. Categories of frames for modal logic. *The Journal of Symbolic Logic*, 40(3):439–442, 1975. `doi:10.2307/2272167`.