Proof Identity and Categorical Models of BV

Matteo Acclavio
FORM, University of Southern Denmark, Odense, Denmark
University of Sussex, UK

Lutz Straßburger
Inria, LIX, Ecole Polytechnique, Institut Polytechnique Paris, France

Vladimir Zamdzhiev
Université Paris-Saclay, CNRS, ENS Paris-Saclay, Inria, Laboratoire Méthodes Formelles, 91190, Gif-sur-Yvette, France

Abstract

BV-categories are a recent development that aims to give categorical semantics to proofs in the logic BV. However, due to the absence of a coherence theorem on one side and a well-defined notion of proof identity for BV on the other side, the precise relation between BV-categories and the logic BV is still not clear. To improve on this situation, we define in this paper a notion of proof identity for BV, based on the notion of atomic flows, which can be seen as a special form of string diagrams. Based on this notion of proof identity, we then strengthen the existing notion of BV-category and prove that it is sound with respect to the logic.

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1 Introduction

The logic BV [30, 26] is an extension of multiplicative linear logic (MLL) with a self-dual non-commutative connective $\triangleleft$, called *seq*. It came to life because of an attempt to give a deductive proof system to *pomset logic* [50, 52, 53], which is defined in terms of proof nets and shares with BV the same language for formulas. Only recently, it has been shown that the two logics are different [47, 46]. Nonetheless, the logic BV has successfully been used to model sequentiality in the context of concurrency [13, 34, 35] and causality in the context of quantum computing [9, 10]. On the one hand, the syntax and proof theory of BV is well-developed [26, 59, 30], and it has a wide range of applications [3, 13, 35], but on the other hand, the semantics and possible models of BV are not so well investigated. This is currently changing through the increasing interest from the quantum computing community [56, 55, 57, 43] and the development of the notion of BV-category [9, 32]. Some non-trivial examples of BV-categories are the *higher-order causal theories* of [56] and the *finite-dimensional operator spaces* of [43].

A natural question to ask is whether the aforementioned BV-categories are *sound* models of BV. This question is less silly than it seems, as the logic BV and the notion of BV-category have been investigated by essentially disjoint communities. The precise relation between BV-categories and the logic BV is not known and neither is soundness.

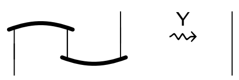
In this paper we want to improve on this situation and bring the two communities together by proving a soundness result: whenever two proofs are equal in BV, then they have the same interpretation in the categorical model. But for that, we need to answer another question: When are two BV-proofs equal? One of the most widely accepted notions of proof identity is via proof normalization, i.e., cut elimination. However, BV does not have a sequent calculus proof system [59, 46], but is presented in a deep inference setting. This means that cut elimination for BV is not proved by the standard methods, but via a *splitting lemma* [26, 5], which is hard to mimic in a categorical setting. Furthermore, even in the cut-free case, it is not clear when two BV-proofs should be the same, as deep inference allows for more rule permutations than a sequent system.

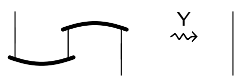
For this reason, we introduce in this paper a notion of *atomic flows* [27, 29] for BV, that we use to identify proofs, i.e., two BV proofs are the same if and only if they are mapped to the same atomic flow. In that respect, atomic flows share properties of *proof nets* [23, 24] and *string diagrams* [40, 6, 11, 12].

The next step, after having established a notion of proof identity for BV, is to find the right notion for a categorical model of BV. However, it is not clear if the existing proposals for BV-categories [9, 32] are invariant with respect to cut elimination and they also do not seem to match well with the notion of proof identity that we introduce here. In order to provide a categorical semantics for BV that is sound in the sense discussed above, we propose a new notion of BV-category that we call *strong* BV*-category.* Indeed, a strong BV-category is also a BV-category in the sense of [9, 32] (Theorem 18), but the converse implications are unlikely to hold, in general. The key idea behind the definition of a strong BV-category (Definition 17) is the following: our atomic flows are string diagrams that are known to be the internal language of *strict compact closed categories* [38, 54], and since the notion of proof identity is based on atomic flows, we may simply lift the coherence properties of strict compact closed categories (which are already known [38]) to the relevant diagrams of the BV-category in which we interpret our proofs by assuming the existence of a suitable functor.

In summary, our paper makes the following contributions:

1. We introduce atomic flows for BV and use them to define proof equivalence. We show that the *splitting lemma* [26, 5], which is at the core of the cut elimination procedure, preserves the atomic flows, and therefore cut elimination in BV corresponds to *yanking* in atomic flows:

   and
   

   (1)

   Therefore, if we impose (1) as an equality on atomic flows, then atomic flows are preserved under cut elimination. Note that this is not trivial: whereas in linear logic yanking corresponds to cut elimination [39, 36], this is not the case in classical logic [29].
2. We introduce a notion of *strong* BV*-category* which has a suitable functor into a strict compact closed category as part of the definition. This gives us a sufficient amount of coherence properties which we use to prove our main semantic result: if two BV proofs are equivalent, then they have the same interpretation in the model. We also show that the strictified versions of the aforementioned examples of higher-order causal theories [56] and finite-dimensional operator spaces [43] form strong BV-categories in our sense.

$$\equiv \frac{A}{B}\ \text{if } A \equiv B \qquad \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr a^{\perp}} \qquad \mathsf{q}{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)}$$

$$\mathsf{ai}{\uparrow}\,\frac{a \otimes a^{\perp}}{\mathbb{I}} \qquad \mathsf{q}{\uparrow}\,\frac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)}$$

equivalence ***switch*** ***atomic interaction*** ***sequentiality***

Figure 1 The inference rules of the proof systems $\mathsf{BV} = \{\equiv, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}\}$ and $\mathsf{SBV} = \mathsf{BV} \cup \{\mathsf{ai}{\uparrow}, \mathsf{q}{\uparrow}\}$.

Outline of the paper. In Section 2, we recall the logic BV, then, in Section 3, we introduce atomic flows, and in Section 4 we prove that they are preserved under cut elimination in BV. Then, in Section 5 we introduce our *strong* BV*-categories* and prove their soundness for BV in Section 6. Finally, in Section 7 we discuss some concrete examples of strong BV-categories.

2 Preliminaries on BV

The set $\mathcal{F} = \{A, B, C, \ldots\}$ of formulas is generated by a countable set of ***atoms*** $\mathcal{A}$ and a ***unit*** $\mathbb{I}$ using the following grammar:

$$A, B \coloneqq \mathbb{I} \mid a \mid a^{\perp} \mid (A \parr B) \mid (A \lhd B) \mid (A \otimes B) \qquad \text{with } a \in \mathcal{A} \tag{2}$$

using the binary connectives $\parr$ (***par***), $\lhd$ (***seq***), and $\otimes$ (***tensor***). The ***negation*** $(\cdot)^{\perp}$ can be extended to all formulas via the following *De Morgan laws*:

$$\mathbb{I}^{\perp} = \mathbb{I}\,,\ (a^{\perp})^{\perp} = a\,,\ (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}\,,\ (A \lhd B)^{\perp} = A^{\perp} \lhd B^{\perp}\,,\ (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp} \tag{3}$$

It follows that $(A^{\perp})^{\perp} = A$ for all A. A ***context*** is a formula with a unique occurrence of the ***hole*** $[\cdot]$ occurring at the place of an atom:

$$C[\cdot] \ \coloneqq\ [\cdot] \mid B \parr C[\cdot] \mid C[\cdot] \parr B \mid B \lhd C[\cdot] \mid C[\cdot] \lhd B \mid B \otimes C[\cdot] \mid C[\cdot] \otimes B$$

Then $C[A]$ stands for the formula obtained from the context $C[\cdot]$ by replacing the hole $[\cdot]$ by the formula A. We also employ a ***formula equivalence*** $\equiv$ to be the smallest congruence relation generated by:

$$\begin{array}{ccc} (A \otimes B) \otimes C \equiv A \otimes (B \otimes C) & A \otimes B \equiv B \otimes A & A \otimes \mathbb{I} \equiv A \\ (A \parr B) \parr C \equiv A \parr (B \parr C) & A \parr B \equiv B \parr A & A \parr \mathbb{I} \equiv A \\ (A \lhd B) \lhd C \equiv A \lhd (B \lhd C) & & A \lhd \mathbb{I} \equiv A \equiv \mathbb{I} \lhd A \end{array} \tag{4}$$

An ***inference rule*** is a scheme $\mathsf{r}\,\frac{A}{B}$ where r is its name, A its ***premise*** and B its ***conclusion***. A ***proof system*** is a set of inference rules, and the rules we use in this paper are shown in Figure 1. For the inference rule $\equiv$, we have the side condition that $A \equiv B$ via one of the equations in (4) above. We write it with a dotted line to ease readability of longer derivations, and we also often condense several such steps into a single one. We call BV the proof system $\{\equiv, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}\}$ and the proof system $\mathsf{SBV} = \mathsf{BV} \cup \{\mathsf{ai}{\uparrow}, \mathsf{q}{\uparrow}\}$.

To build *derivations* we use the syntax of *open deduction* [28], which allows to use the same primitives that are used to construct formulas also for constructing derivations.

▶ **Definition 1.** *We define the set $\mathcal{D} = \{\Phi, \Psi, \ldots\}$ of **derivations** together with two functions* $\mathsf{pr}, \mathsf{cn}\colon \mathcal{D} \to \mathcal{F}$ *(called **premise** and **conclusion**, respectively) inductively as follows:*

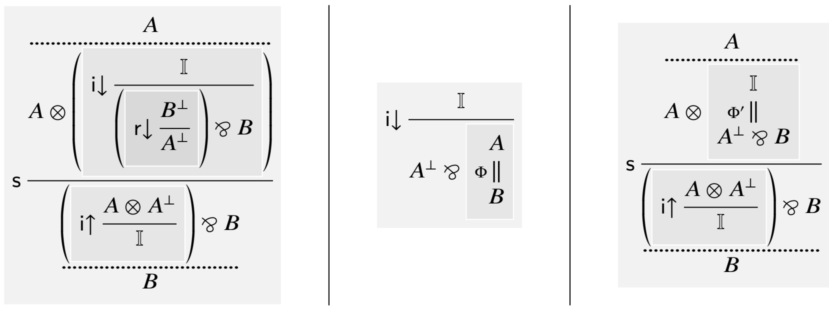


Figure 2 Derivations to prove Proposition 3.3 and the two directions of Proposition 4.2.

- *For every formula A, we have that A is a derivation and* $\mathsf{pr}(A) = \mathsf{cn}(A) = A$.
- *If* Φ *and* Ψ *are derivations and* $\odot \in \{\parr, \lhd, \otimes\}$, *then* $\Phi \odot \Psi$ *is a derivation and* $\mathsf{pr}(\Phi \odot \Psi) = \mathsf{pr}(\Phi) \odot \mathsf{pr}(\Psi)$ *and* $\mathsf{cn}(\Phi \odot \Psi) = \mathsf{cn}(\Phi) \odot \mathsf{cn}(\Psi)$.
- *If* Φ *and* Ψ *are derivations and* $\mathsf{r}\,\dfrac{A}{B}$ *is an instance of an inference rule with* $\mathsf{cn}(\Phi) = A$ *and* $\mathsf{pr}(\Psi) = B$ *then* $\Theta = \mathsf{r}\,\dfrac{\Phi}{\Psi}$ *is a derivation and* $\mathsf{pr}(\Theta) = \mathsf{pr}(\Phi)$ *and* $\mathsf{cn}(\Theta) = \mathsf{cn}(\Psi)$.

If X is a proof system, we often write $\begin{array}{c} A \\ \Phi \Vert \mathsf{X} \\ B \end{array}$ to denote a derivation Φ with premise $\mathsf{pr}(\Phi) = A$ and conclusion $\mathsf{cn}(\Phi) = B$ where all inference rules used in Φ are in the proof system X. We may denote by $\begin{array}{c} \Phi \,⫴\, \mathsf{X} \\ B \end{array}$ a derivation in X with premise $\mathbb{I}$, and by $\mathsf{r}\,\begin{array}{c} A \\ \hline\hline B \end{array}$ a derivation using only instances of the rule r. We also write $A \vdash_{\mathsf{X}} B$ (resp. $\vdash_{\mathsf{X}} B$) if there is a derivation in the system X with conclusion B and premise A (resp. premise $\mathbb{I}$).

Now consider the following two rules, called ***axiom*** and ***cut***.

$$\mathsf{i}{\downarrow}\,\frac{\mathbb{I}}{A^\perp \parr A} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\,\frac{A \otimes A^\perp}{\mathbb{I}} \tag{5}$$

They are the generalized form of the two inference rules $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$ respectively. The cut elimination in BV is the following statement, whose proof will be discussed in Section 4.

▶ **Theorem 2.** *Let A be a formula. If* $\vdash_{\mathsf{BV}\cup\{\mathsf{i}\uparrow\}} A$ *then* $\vdash_{\mathsf{BV}} A$.

We conclude this section by recalling some basic and easy to prove properties of BV.

▶ **Proposition 3.** *The following hold:*
1. *The rule* $\mathsf{i}{\downarrow}$ *is derivable in* $\{\equiv, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}\} = \mathsf{BV}$.
2. *The rule* $\mathsf{i}{\uparrow}$ *is derivable in* $\{\equiv, \mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\uparrow}\} = \mathsf{SBV} \setminus \{\mathsf{ai}{\downarrow}, \mathsf{q}{\downarrow}\}$.
3. *Every rule* $\mathsf{r}{\uparrow}$ *is derivable in* $\{\equiv, \mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \mathsf{r}{\downarrow}\}$.

Proof. The first two statements are proved by induction on A, and the third statement by the derivation on the left of Figure 2. ◀

▶ **Proposition 4.** *Let A and B be formulas. Then,*
1. $\vdash_{\mathsf{SBV}} A$ *iff* $\vdash_{\mathsf{BV}\cup\{\mathsf{i}\uparrow\}} A$, *and*
2. $A \vdash_{\mathsf{SBV}} B$ *iff* $\vdash_{\mathsf{SBV}} A^\perp \parr B$.

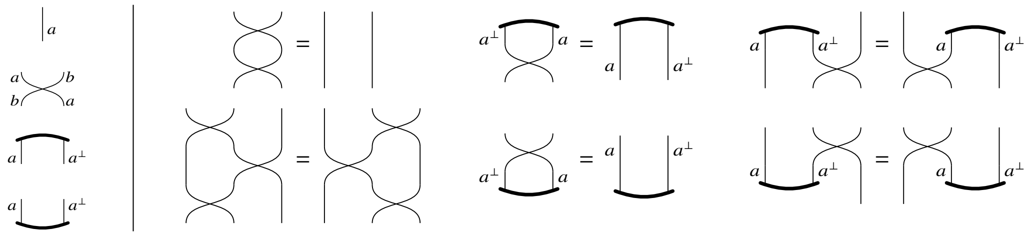


Figure 3 Generators (left) and equivalences (right) for atomic flows, with a and b literals.

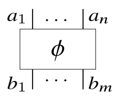
Proof. The first statement follows immediately from Proposition 3. To prove left-to-right (resp. right-to-left) implication in the second statement, we assume the existence of a derivation Φ with premise A and conclusion B (resp. a derivation Φ' with premise $\mathbb{I}$ and conclusion $A^\perp \parr B$) in SBV, and we construct the derivations in the center (resp. right) of Figure 2. ◀

3 Atomic Flows

Atomic flows were originally introduced in [27, 29] as a graphical formalism to study normalization properties of proofs for classical logic. Because of their "graphical nature", atomic flows can be seen as relatives of Girard's *proof nets* [23, 17] and Buss' *logical flow graphs* [49, 14]. Atomic flows have never been considered for linear logic because of the existence of proof nets. However, for BV, we do not have proof nets,[1] and therefore, atomic flows for BV can also be seen as a first step towards the development of proof nets for BV.

The basic idea of atomic flows is to track atom occurrences in a derivation, abstracting away from the logical structure of the formulas (i.e., the connectives and their arrangement). In that respect, they can also be seen as a special case of *string diagrams* [6, 11, 12], or simply *two-dimensional diagrams* [40].

We write $\mathcal{A}^\perp = \{a^\perp, b^\perp, \ldots\}$ for the set of negated atoms and we let $Q = \{p, q, s, t, \ldots\}$ be the set of finite strings over the set $\mathcal{A} \cup \mathcal{A}^\perp$ of ***literals***. We write ε for the ***empty string*** and $p \boxtimes q$ for ***string concatenation***.

▶ **Definition 5.** *An **atomic flow** $\phi\colon p \to q$ is a two-dimensional diagram written as*  *where $p = \mathsf{in}_\phi = a_1 \boxtimes \cdots \boxtimes a_n$ is its **input** and $q = \mathsf{out}_\phi = b_1 \boxtimes \cdots \boxtimes b_m$ its **output**, with $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathcal{A} \cup \mathcal{A}^\perp$.*

The set of atomic flows is generated from the basic flows shown on the left of Figure 3 (the identity flow, the simple crossing, and the cap and the cups making interact dual literals) via following two operations:

- ***vertical composition:** if $\phi\colon p \to q$ and $\psi\colon q \to s$ are atomic flows, then so is $\theta = \phi \mathbin{;} \psi\colon p \to s$, depicted on the left of Equation (6) below, and*

[1] The proof nets of *pomset logic* [51, 52] do not have a suitable correctness criterion for BV [47, 46].

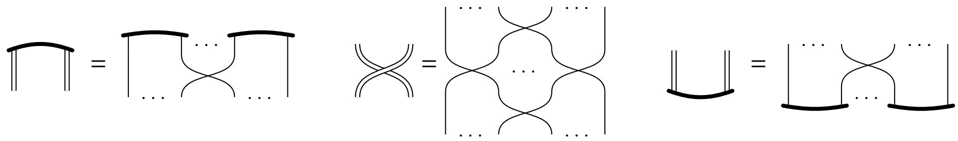
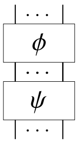
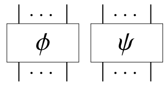


Figure 4 Abbreviations for atomic flows.

- ***horizontal composition:*** *if $\phi\colon p \to q$ and $\psi\colon s \to t$ are atomic flows, then so is $\theta = \phi \boxtimes \psi\colon p \boxtimes s \to q \boxtimes t$, depicted on the right of Equation (6) below,*

$\phi \mathbin{;} \psi$ =  *if* $\mathsf{out}_\phi = \mathsf{in}_\psi$ *and* $\phi \boxtimes \psi$ =  (6)

*quotiented by the equivalence relation generated by $(\phi \mathbin{;} \phi') \boxtimes (\psi \mathbin{;} \psi') = (\phi \boxtimes \psi) \mathbin{;} (\phi' \boxtimes \psi')$ and the equations shown on the right of Figure 3. We denote the **empty atomic flow** by $\varnothing\colon \varepsilon \to \varepsilon$.*

We are now going to show how SBV-derivations are translated into atomic flows. For this we associate to each formula A it string $\|A\|$ of atom occurrences, which is formally defined as follows:

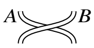
$$\|\mathbb{I}\| = \varepsilon \qquad \|a\| = a \qquad \|a^\perp\| = a^\perp \qquad \|A \mathbin{⅋} B\| = \|A \mathbin{\triangleleft} B\| = \|A \otimes B\| = \|A\| \boxtimes \|B\| \tag{7}$$

For each formula A, we may denote the ***identity flow*** $\mathsf{id}_A\colon \|A\| \to \|A\|$ as $|\cdots|$ or as $\|_A$. Moreover, we may also use the abbreviations in Figure 4 when writing atomic flows.

We associate to each inference rule in SBV an atomic flow as shown below:

$\mathsf{AF}(\mathsf{ai}{\downarrow})$ =  $\mathsf{AF}(\mathsf{s}) = \|_A \; \|_B \; \|_C$

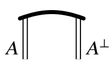
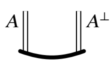
$\mathsf{AF}(\mathsf{ai}{\uparrow})$ =  $\mathsf{AF}(\mathsf{q}{\downarrow}) = \mathsf{AF}(\mathsf{q}{\uparrow})$ =  (8)

where $\mathsf{AF}(\equiv)$ is $\|_A \; \|_B \; \|_C$, or  , or $\|_A$, for the first, second, and last columns in (4), respectively.

We associate to each derivation Φ in SBV its atomic flow $\mathsf{AF}(\Phi)$ inductively as follows:

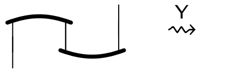
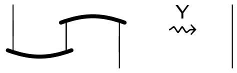
- For a formula A, we have $\mathsf{AF}(A) = \mathsf{id}_A$, in particular, $\mathsf{AF}(\mathbb{I}) = \varnothing$;
- for $\odot \in \{⅋, \triangleleft, \otimes\}$, we define $\mathsf{AF}(\Phi \odot \Psi) = \mathsf{AF}(\Phi) \boxtimes \mathsf{AF}(\Psi)$.
- For $\mathsf{r} \in \mathsf{SBV}$, and $\Theta = \mathsf{r}\,\dfrac{\Phi}{\Psi}$ then $\mathsf{AF}(\Theta) = \mathsf{AF}(\Phi) \mathbin{;} \mathsf{AF}(\mathsf{r}) \mathbin{;} \mathsf{AF}(\Psi)$, where $\mathsf{AF}(\mathsf{r})$ is defined as above.

▶ **Proposition 6.** *The derivations constructed in the proof of Proposition 3.1 and Proposition 3.2 have the following atomic flows:*

$\mathsf{AF}\left(\mathsf{i}{\downarrow}\,\dfrac{\mathbb{I}}{A \mathbin{⅋} A^\perp}\right)$ =  $\mathsf{AF}\left(\mathsf{i}{\uparrow}\,\dfrac{A \otimes A^\perp}{\mathbb{I}}\right)$ =  (9)

Proof. By the same induction as in the proof of Proposition 3. ◀

▶ **Definition 7.** *We call **yanking** the rewrite relation $\overset{\mathsf{Y}}{\rightsquigarrow}$ on atomic flows that is generated by the following two rewrite rules:*

 *and*  (10)

▶ **Proposition 8.** *Yanking is terminating and confluent.*

Proof. Termination is immediate because the atomic flow gets smaller at each step, and confluence follows because the only critical pairs are given by the two ways of reducing  to  and the two ways of reducing  to  . ◀

This means that each atomic flow ϕ has a unique normal form under $\overset{\mathsf{Y}}{\rightsquigarrow}$, and we denote this normal form by $\mathsf{Y}(\phi)$ and call it the ***yanking of ϕ***.

4 Cut Elimination is Yanking. Also in BV

Cut elimination in a deep inference system usually means that the *up-fragment* (i.e., the rules with the ↑ in the name) can be eliminated. Propositions 3 and 4 above say that BV also follows this pattern. Cut elimination in BV is proved via a *splitting lemma* that allows to decompose a derivation of the premise of an up-rule into a derivation of the conclusion. This *splitting* (proved for BV in [26]; see also [31, 35] for alternative presentations) is a global property of a derivation and provides, *a priori*, no information about the atomic flows. But in order to establish the relation between cut elimination and yanking in BV, which is a local rewrite rule, we need to strengthen the splitting lemma to also talk about the atomic flows.

▶ **Lemma 9** (Splitting, improved formulation)**.** *Let A, B, and K be formulas, and let a be an atom.*

1. *If there is a derivation* $\begin{array}{c}\Phi \,\|\, \mathsf{BV} \\ (A \otimes B) \parr K\end{array}$, *then there are formulas K_A and K_B and derivations*

$$\begin{array}{c}K_A \parr K_B \\ \Phi_K \,\|\, \mathsf{BV} \\ K\end{array} \quad \textit{and} \quad \begin{array}{c}\Phi_A \,\|\, \mathsf{BV} \\ A \parr K_A\end{array} \quad \textit{and} \quad \begin{array}{c}\Phi_B \,\|\, \mathsf{BV} \\ B \parr K_B\end{array}, \ \textit{s.t.}\ \mathsf{AF}\left(\mathsf{s}\,\dfrac{\left(\begin{array}{c}\Phi_A \,\| \\ A \parr K_A\end{array}\right) \otimes \left(\begin{array}{c}\Phi_B \,\| \\ B \parr K_B\end{array}\right)}{(A \otimes B) \parr \left(\begin{array}{c}K_A \parr K_B \\ \Phi_K \,\| \\ K\end{array}\right)}\right) = \mathsf{AF}(\Phi)$$

2. *If there is a derivation* $\begin{array}{c}\Phi \,\|\, \mathsf{BV} \\ (A \triangleleft B) \parr K\end{array}$, *then there are formulas K_A and K_B and derivations*

$$\begin{array}{c}K_A \triangleleft K_B \\ \Phi_K \,\|\, \mathsf{BV} \\ K\end{array} \quad \textit{and} \quad \begin{array}{c}\Phi_A \,\|\, \mathsf{BV} \\ A \parr K_A\end{array} \quad \textit{and} \quad \begin{array}{c}\Phi_B \,\|\, \mathsf{BV} \\ B \parr K_B\end{array}, \ \textit{s.t.}\ \mathsf{AF}\left(\mathsf{q}{\downarrow}\,\dfrac{\left(\begin{array}{c}\Phi_A \,\| \\ A \parr K_A\end{array}\right) \triangleleft \left(\begin{array}{c}\Phi_B \,\| \\ B \parr K_B\end{array}\right)}{(A \triangleleft B) \parr \left(\begin{array}{c}K_A \triangleleft K_B \\ \Phi_K \,\| \\ K\end{array}\right)}\right) = \mathsf{AF}(\Phi)$$

3. *If there is a derivation* $\begin{array}{c}\Phi \,\|\, \mathsf{BV} \\ a \parr K\end{array}$, *then there is a formula K and a derivation*

$$\begin{array}{c}a^{\perp} \\ \Phi_a \,\|\, \mathsf{BV} \\ K\end{array}, \ \textit{s.t.}\ \mathsf{AF}\left(\mathsf{ai}{\downarrow}\,\dfrac{\mathbb{I}}{a \parr \left(\begin{array}{c}a^{\perp} \\ \Phi_a \,\| \\ K\end{array}\right)}\right) = \mathsf{AF}(\Phi)$$

Proof. The proof is the same as for the standard splitting lemma for BV [26, 58], which proceeds by induction on the size of Φ and a case analysis on the bottommost rule instance in Φ. The only difference here is the additional observation that each transformation preserves the atomic flow. See the extended version [4] for more details. ◀

In the next step, the context $[\cdot] \parr K$ of the splitting lemma is generalized to an arbitrary context $C[\cdot]$.

▶ **Lemma 10** (Context Reduction, improved formulation)**.** *If there is a derivation* $\begin{array}{c}\Phi \,\|\, \mathsf{BV}\\ C[A]\end{array}$, *then there is a formula K and derivations*

$$\begin{array}{c}\Phi_A \,\|\, \mathsf{BV}\\ A \parr K\end{array} \quad \textit{and} \quad \begin{array}{c}X \parr K\\ \Phi_{C[X]} \,\|\, \mathsf{BV}\\ C[X]\end{array} \ \textit{for every formula } X\textit{, such that } \mathsf{AF}\left(\begin{array}{c}\Phi_A \,\|\, \mathsf{BV}\\ A \parr K\\ \Phi_{C[A]} \,\|\, \mathsf{BV}\\ C[A]\end{array}\right) = \mathsf{AF}\,(\Phi)$$

Proof. As in the previous lemma, the proof is a straightforward adaptation of the standard context reduction lemma for BV [26, 58], which proceeds by induction on the structure of $C[\cdot]$, by repeatedly applying splitting, until the base case $[\cdot] \parr K$ is reached. It follows immediately that each transformation preserves the atomic flow. ◀

These two lemmas are enough to eliminate the rules $\mathsf{q}\uparrow$ and $\mathsf{ai}\uparrow$ from a derivation. Again, this has already been shown in [26, 58], and we only need to observe what happens to the atomic flows under this transformation.

▶ **Lemma 11.** *If there is a derivation* $\Phi = \begin{array}{c}\Phi' \,\|\, \mathsf{BV}\\ C\left[\mathsf{q}\uparrow \dfrac{(A \triangleleft B) \otimes (C \triangleleft D)}{(A \otimes C) \triangleleft (B \otimes D)}\right]\end{array}$ *then there is a derivation* $\begin{array}{c}\Phi^* \,\|\, \mathsf{BV}\\ C[(A \otimes C) \triangleleft (B \otimes D)]\end{array}$ *such that* $\mathsf{AF}\left(\Phi^*\right) = \mathsf{AF}\,(\Phi)$.

Proof. By the previous two lemmas, we have derivations

$$\begin{array}{c}((A \otimes C) \triangleleft (B \otimes D)) \parr (K_A \triangleleft K_B) \parr (K_C \triangleleft K_D)\\ \Phi_{C[(A\otimes C)\triangleleft(B\otimes D)]} \,\|\, \mathsf{BV}\\ C[(A \otimes C) \triangleleft (B \otimes D)]\end{array}, \quad \begin{array}{c}\Phi_A \,\|\, \mathsf{BV}\\ A \parr K_A\end{array}, \quad \begin{array}{c}\Phi_B \,\|\, \mathsf{BV}\\ B \parr K_B\end{array}, \quad \begin{array}{c}\Phi_C \,\|\, \mathsf{BV}\\ C \parr K_C\end{array}, \quad \begin{array}{c}\Phi_D \,\|\, \mathsf{BV}\\ D \parr K_D\end{array}$$

Which can be put together to get a derivation Φ^*. The fact that $\mathsf{AF}\left(\Phi^*\right) = \mathsf{AF}\,(\Phi)$ follows from the fact that atomic flows are equivalent modulo the identities in Figure 3. See Appendix A for more details. ◀

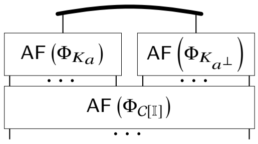
▶ **Lemma 12.** *If there is a derivation* $\Phi = \begin{array}{c}\Phi' \,\|\, \mathsf{BV}\\ C\left[\mathsf{ai}\uparrow \dfrac{a \otimes a^\perp}{\mathbb{I}}\right]\end{array}$ *then there is a derivation* $\begin{array}{c}\Phi^* \,\|\, \mathsf{BV}\\ C[\mathbb{I}]\end{array}$ *such that* $\mathsf{AF}\,(\Phi) \overset{\mathsf{Y}}{\rightsquigarrow} \mathsf{AF}\left(\Phi^*\right)$.

Proof. By context reduction and splitting, we get two formulas K_a and $K_{a^\perp}$ and derivations

$$\begin{array}{c}(a \otimes a^\perp) \parr K_a \parr K_{a^\perp}\\ \Phi_{C[a\otimes a^\perp]} \,\|\, \mathsf{BV}\\ C[a \otimes a^\perp]\end{array} \quad \text{and} \quad \begin{array}{c}\mathbb{I} \parr K_a \parr K_{a^\perp}\\ \Phi_{C[\mathbb{I}]} \,\|\, \mathsf{BV}\\ C[\mathbb{I}]\end{array} \quad \text{and} \quad \begin{array}{c}a^\perp\\ \Phi_a \,\|\, \mathsf{BV}\\ K_a\end{array} \quad \text{and} \quad \begin{array}{c}a\\ \Phi_{a^\perp} \,\|\, \mathsf{BV}\\ K_{a^\perp}\end{array}$$

We can then build the derivations Φ' and Φ^* shown below.

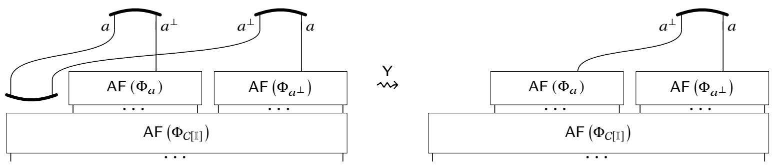
$$\Phi' = \mathsf{s},\equiv\,\dfrac{\left(\mathsf{ai}{\downarrow}\,\dfrac{\mathbb{I}}{a \parr \Phi_a}\right) \otimes \left(\mathsf{ai}{\downarrow}\,\dfrac{\mathbb{I}}{a^\perp \parr \Phi_{a^\perp}}\right)}{\dfrac{(a \otimes a^\perp) \parr K_a \parr K_{a^\perp}}{C\left[\mathsf{ai}{\uparrow}\,\dfrac{a \otimes a^\perp}{\mathbb{I}}\right]}\,\Phi_{C[a\otimes a^\perp]}\,\Vert} \qquad \Phi^* = \mathsf{ai}{\downarrow}\,\dfrac{\mathbb{I}}{\equiv\,\dfrac{\left(\dfrac{a^\perp}{K_a}\,\Phi_a \Vert \mathsf{BV}\right) \parr \left(\dfrac{a}{K_{a^\perp}}\,\Phi_{a^\perp} \Vert \mathsf{BV}\right)}{\dfrac{\mathbb{I} \parr K_a \parr K_{a^\perp}}{C[\mathbb{I}]}\,\Phi'_{C[\mathbb{I}]}\,\Vert}}$$

We have $\mathsf{AF}(\Phi) = \mathsf{AF}(\Phi')$ and $\mathsf{AF}(\Phi^*) =$ 

by Lemmas 9 and 10, and

$$\mathsf{AF}\left(\dfrac{(a \otimes a^\perp) \parr K_a \parr K_{a^\perp}}{C\left[\mathsf{ai}{\uparrow}\,\dfrac{a \otimes a^\perp}{\mathbb{I}}\right]}\,\Phi_{C[a\otimes a^\perp]}\,\Vert\right) = \mathsf{AF}\left(\dfrac{\left(\mathsf{ai}{\uparrow}\,\dfrac{a \otimes a^\perp}{\mathbb{I}}\right) \parr K_a \parr K_{a^\perp}}{C[\mathbb{I}]}\,\Phi_{C[\mathbb{I}]}\,\Vert\right)$$

by rewriting via the equalities in Figure 3. Therefore $\mathsf{AF}(\Phi)$ is as shown on the left below. This rewrites with a single $\overset{\mathsf{Y}}{\rightsquigarrow}$ step, into the atomic flow on the right below, which is equal to $\mathsf{AF}(\Phi^*)$.

$\mathsf{AF}(\Phi) =$ 

$= \mathsf{AF}(\Phi^*)$ ◀

Now Theorem 2 follows immediately from the previous two lemmas and Proposition 4. However, with atomic flows, we have a stronger result.

▶ **Notation 13.** Let Φ be an SBV derivation with premise $\mathbb{I}$. We write $\Phi \overset{\mathsf{CE}}{\rightsquigarrow} \Phi'$ if Φ' is a BV derivation that is obtained from Φ by eliminating the $\mathsf{q}{\uparrow}$- and $\mathsf{ai}{\uparrow}$-rule instances according to the proofs of Lemmas 11 and 12.

We can now state and prove our main result.

▶ **Theorem 14.** *If* $\Phi \overset{\mathsf{CE}}{\rightsquigarrow} \Phi'$ *then* $\mathsf{AF}(\Phi') = \mathsf{Y}(\mathsf{AF}(\Phi))$.

Proof. By Lemmas 11 and 12, we have $\mathsf{AF}(\Phi) \overset{\mathsf{Y}}{\rightsquigarrow} \mathsf{AF}(\Phi')$. Since Φ' has no $\mathsf{ai}{\uparrow}$-instance (and no $\mathsf{i}{\uparrow}$-instance), we have that $\mathsf{AF}(\Phi')$ has no yanking redex. Hence $\mathsf{AF}(\Phi') = \mathsf{Y}(\mathsf{AF}(\Phi))$. ◀

▶ **Corollary 15.** *Let* Φ *be a derivation of* A *in* $\mathsf{BV} \cup \{\mathsf{i}{\uparrow}\}$. *Then, there is a derivation* Φ' *of* A *in* BV *such that* $\mathsf{AF}(\Phi') = \mathsf{Y}(\mathsf{AF}(\Phi))$.

Proof. By Proposition 3, we can construct a derivation Φ' by replacing each $\mathsf{i}\!\uparrow$-instance in Φ by a derivation in $\{\equiv, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\uparrow\}$. By Proposition 6 the atomic flow of Φ' is the same as the one of Φ. Finally, by Theorem 14, we can eliminate the up-rules from Φ' obtaining a derivation Φ'' in BV such that $\mathsf{AF}(\Phi'') = \mathsf{Y}(\mathsf{AF}(\Phi')) = \mathsf{Y}(\mathsf{AF}(\Phi))$. ◀

We can now formally define proof identity in BV.

▶ **Definition 16.** *Let Φ and Ψ be two derivations of A in* SBV*. We say that Φ and Ψ are* ***equivalent*** *(denoted $\Phi \sim \Psi$) if there are derivations Φ' and Ψ' such that $\Phi \overset{\mathsf{CE}}{\rightsquigarrow} \Phi'$ and $\Psi \overset{\mathsf{CE}}{\rightsquigarrow} \Psi'$ and $\mathsf{AF}(\Phi') = \mathsf{AF}(\Psi')$.*

5 What is a BV-Category?

It is commonly expected from a categorical model that it identifies proofs that are considered equal by the logic. A typical example are $*$-autonomous categories that identify proofs of multiplicative linear logic (MLL) up to rule permutations in the sequent calculus [8, 41, 36].

Recall that if $\langle \mathbf{C}, \otimes, \mathbf{1}, \cdot^{\perp} \rangle$ is a $*$-autonomous category, then we can define another symmetric monoidal structure $A \parr B \triangleq (A^{\perp} \otimes B^{\perp})^{\perp}$ with unit $\perp \triangleq \mathbf{1}^{\perp}$ by duality. As BV is a conservative extension of MLL (with mix and mix_0)[2], it immediately follows that any categorical model of BV should be a $*$-autonomous category that is also isomix (this allows us to identify the two units $\mathbf{1}$ and $\perp$). Following the notation of the previous sections, we denote this unit by $\mathbb{I}$. Furthermore, since BV has an additional non-commutative binary connective $\triangleleft$ (with unit $\mathbb{I}$), we also use an additional monoidal structure $\langle \triangleleft, \mathbb{I} \rangle$.

The first proposal for what a BV-category should be was made in [9] followed by another one in [32]. We invite the reader to consult Appendix B for the definition of a BV-*category with negation* in the sense of [9]. The definition of a BV-category with negation in [9] is equivalent to the definition of a BV-category in [32][3] and henceforth we often refer to both of these (equivalent) concepts as BV-*category* for simplicity. The presentation of BV-categories in [9] is based on linearly distributive categories whereas the presentation in [32] is based on $*$-autonomous categories.

Unfortunately, soundness has not been demonstrated for the two aforementioned proposals for BV-categories. Furthermore, we conjecture that the notion of proof identity, based on atomic flows, that we consider in this paper does not match well with these proposals. Note that atomic flows in the logic BV correspond to the string diagrams of a strict compact closed category (see previous section). The BV-categories of [9, 32] are not built around compact closed categories and we conjecture that, in general, arbitrary BV-categories in the sense of [9, 32] cannot be embedded in a compact closed category in the way we need (Definition 17).

Because of this, we believe that our notion of proof equivalence does not match well with the BV-categories of [9, 32]. This is our justification for introducing a new notion of BV-category (Definition 17) for which we prove soundness (Theorem 23) and which behaves well with respect to our notion of proof equivalence based on atomic flows (Corollary 24). We give our proposal the name *strong* BV-*category*, because every strong BV-category is also a BV-category in the sense of [9, 32] (Theorem 18), but we conjecture that the converse implication does not hold.

Before we formulate our main definition, recall that every compact closed category $(\mathbf{D}, \boxtimes, J, \cdot^{\perp})$ is $*$-autonomous with $A \parr B \cong A \boxtimes B$. In this paper we work with *strict* compact closed categories. By this we mean that: (1) the symmetric monoidal structure is strict so

[2] mix is $\perp \multimap \mathbf{1}$ and mix_0 is $\mathbf{1} \multimap \perp$. See [22, 1].
[3] James Hefford, personal communication.

that $A \boxtimes (B \boxtimes C) = (A \boxtimes B) \boxtimes C$ and $J \boxtimes A = A = A \boxtimes J$; (2) the canonical De Morgan isomorphism is the identity so that $(A \boxtimes B)^\perp = A^\perp \boxtimes B^\perp$; (3) the double dual isomorphism is the identity so that $A = A^{\perp\perp}$. We also recall that a strict $*$-autonomous functor $U \colon \mathbf{C} \to \mathbf{D}$ is a functor that preserves all of the $*$-autonomous structure up to equality, i.e., $U(A \otimes B) = UA \otimes UB$, $U(A^\perp) = (UA)^\perp$, $U(f \otimes g) = Uf \otimes Ug$, etc.

▶ **Definition 17.** *A **strong** BV**-category** is a tuple $\langle \mathbf{C}, \otimes, \triangleleft, \mathbb{I}, \cdot^\perp, U \rangle$ such that:*

1. *$\langle \mathbf{C}, \otimes, \mathbb{I}, \cdot^\perp \rangle$ is a $*$-autonomous category together with an isomix isomorphism $\mathbb{I}^\perp \cong \mathbb{I}$;*
2. *$\langle \mathbf{C}, \triangleleft, \mathbb{I} \rangle$ is a monoidal category;*
3. *there is a natural isomorphism $\kappa_{AB} \colon A^\perp \triangleleft B^\perp \cong (A \triangleleft B)^\perp$;*
4. *there is a natural transformation $q_{ABCD} \colon (A \parr B) \triangleleft (C \parr D) \to (A \triangleleft C) \parr (B \triangleleft D)$;*
5. *$U \colon \mathbf{C} \hookrightarrow \mathbf{D}$ is a faithful strict $*$-autonomous functor into a strict compact closed category $(\mathbf{D}, \boxtimes, \cdot^\perp, J)$ such that:*
   - *U is strict monoidal with respect to $\triangleleft$;*
   - *$U(\kappa_{AB}) = \mathrm{id}_X$, where $X = (UA)^\perp \boxtimes (UB)^\perp = (UA \boxtimes UB)^\perp$ in $\mathbf{D}$;*
   - *the following diagram commutes for any choice of objects A, B, C, D:*

$$\begin{tikzcd} U((A \parr B) \triangleleft (C \parr D)) \arrow[r, equal] \arrow[d, "Uq"'] & UA \boxtimes UB \boxtimes UC \boxtimes UD \arrow[d, "\mathrm{id} \boxtimes \sigma \boxtimes \mathrm{id}"] \\ U((A \triangleleft C) \parr (B \triangleleft D)) \arrow[r, equal] & UA \boxtimes UC \boxtimes UB \boxtimes UD \end{tikzcd} \tag{11}$$

where σ stands for the monoidal symmetry.

By using the canonical De Morgan isomorphisms in the $*$-autonomous category $\mathbf{C}$, the double isomorphism, and the self-duality of $\triangleleft$ via the κ natural isomorphism, one may now define a natural transformation

$$q'_{ABCD} \colon (A \triangleleft B) \otimes (C \triangleleft D) \xrightarrow{\cong \circ q^\perp \circ \cong} (A \otimes C) \triangleleft (B \otimes D)$$

which is therefore principally determined by q via duality. The importance of condition 5 from the above definition is that it may be understood as specifying a coherence property. Note that coherence for compact closed categories is already known, so the faithfulness of U and the fact that U preserves all the relevant structure provide us with a simple criterion to check if a diagram in a strong BV-category commutes. Less formally, condition 5 allows us to inherit many of the coherence properties of the compact closed category $\mathbf{D}$ (which are already known) into $\mathbf{C}$.

▶ **Theorem 18.** *Let $\langle \mathbf{C}, \otimes, \triangleleft, \mathbb{I}, \cdot^\perp, U \rangle$ be a strong* BV*-category. Then $\langle \mathbf{C}, \otimes, \triangleleft, \mathbb{I}, \cdot^\perp, U \rangle$ is also a* BV*-category (with negation) in the sense of [9] and [32].*

Proof (Sketch). The proof follows easily from the fact that U is faithful and strictly preserves all the relevant data which allows us to reduce the commutativity of all the relevant diagrams in $\mathbf{C}$ to their commutativity in $\mathbf{D}$ under U. The latter is very easy to verify using the string diagram language (or coherence theorem) for strict compact closed categories.

To illustrate this, let us show why diagram (20) from Appendix B commutes in $\mathbf{C}$. Since U is faithful and strictly preserves the two monoidal structures $\otimes$ and $\triangleleft$, it follows that (20), i.e.,

$$\begin{tikzcd}
((A \lhd B) \otimes (C \lhd D)) \otimes (E \lhd F) \arrow[r, "\alpha"] \arrow[d, "q' \otimes \mathrm{id}"'] & (A \lhd B) \otimes ((C \lhd D) \otimes (E \lhd F)) \arrow[d, "\mathrm{id} \otimes q'"] \\
((A \otimes C) \lhd (B \otimes D)) \otimes (E \lhd F) \arrow[d, "q'"'] & (A \lhd B) \otimes ((C \otimes E) \lhd (D \otimes F)) \arrow[d, "q'"] \\
((A \otimes C) \otimes E) \lhd ((B \otimes D) \otimes F) \arrow[r, "\alpha \lhd \alpha"'] & (A \otimes (C \otimes E)) \lhd (B \otimes (D \otimes F))
\end{tikzcd}$$

commutes in $\mathbf{C}$, iff the diagram

$$\begin{tikzcd}
UA \boxtimes UB \boxtimes UC \boxtimes UD \boxtimes UE \boxtimes UF \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id} \boxtimes \sigma \boxtimes \mathrm{id} \boxtimes \mathrm{id}"'] & UA \boxtimes UB \boxtimes UC \boxtimes UD \boxtimes UE \boxtimes UF \arrow[d, "\mathrm{id} \boxtimes \mathrm{id} \boxtimes \sigma \boxtimes \mathrm{id}"] \\
UA \boxtimes UC \boxtimes UB \boxtimes UD \boxtimes UE \boxtimes UF \arrow[d, "\mathrm{id} \boxtimes \sigma \boxtimes \mathrm{id}"'] & UA \boxtimes UB \boxtimes UC \boxtimes UE \boxtimes UD \boxtimes UF \arrow[d, "\mathrm{id} \boxtimes \sigma \boxtimes \mathrm{id}"] \\
UA \boxtimes UC \boxtimes UE \boxtimes UB \boxtimes UD \boxtimes UF \arrow[r, "\mathrm{id}"'] & UA \boxtimes UC \boxtimes UE \boxtimes UB \boxtimes UD \boxtimes UF
\end{tikzcd}$$

commutes in $\mathbf{D}$ (which we recall is *strict* compact closed) and this follows immediately from the coherence theorem of symmetric monoidal categories.

The remaining diagrams can be checked in a similar way. ◀

Note that the definition of strong BV-category requires checking considerably fewer coherence diagrams compared to that of BV-category (see Appendix B). This is largely thanks to condition 5, which is indeed a strong assumption.

6 Denotational Semantics

We now describe the denotational semantics of BV in an arbitrary, but fixed, strong BV-category with $U \colon \mathbf{C} \hookrightarrow \mathbf{D}$ the forgetful functor. Because of Definition 17.1, we may choose $\mathbb{I}$ to be the monoidal unit for the $⅋$ monoidal structure as well and then we can define $(\cdot)^\perp = (\cdot) \multimap \mathbb{I}$. We do so in the sequel for simplicity. A BV-formula A is interpreted as an object $[\![A]\!] \in \mathrm{Ob}(\mathbf{C})$ as follows:

$$[\![\mathbb{I}]\!] = \mathbb{I} \qquad [\![a]\!] = c_a \qquad [\![a^\perp]\!] = [\![a]\!]^\perp \qquad [\![A \odot B]\!] = [\![A]\!] \odot [\![B]\!] \tag{12}$$

where c_a is some chosen object of $\mathbf{C}$, and $\odot \in \{⅋, \lhd, \otimes\}$. A BV-formula A also admits an interpretation $(\!|A|\!) \in \mathrm{Ob}(\mathbf{D})$ which can be defined by interpreting all tensors as $\boxtimes$:

$$(\!|\mathbb{I}|\!) = J \qquad (\!|a|\!) = Uc_a \qquad (\!|a^\perp|\!) = (\!|a|\!)^\perp \qquad (\!|A \odot B|\!) = (\!|A|\!) \boxtimes (\!|B|\!) \tag{13}$$

where again $\odot \in \{⅋, \lhd, \otimes\}$. It now immediately follows that $U[\![A]\!] = (\!|A|\!)$, because the functor U strictly preserves all this data. The interpretation of formula equivalence $A \equiv B$ is given by a (symmetric) monoidal natural isomorphism $[\![A \equiv B]\!] \colon [\![A]\!] \cong [\![B]\!]$ in $\mathbf{C}$ defined in the obvious way. Likewise, formula equivalence can be interpreted in $\mathbf{D}$ as a symmetric monoidal natural isomorphism $(\!|A \equiv B|\!) \colon (\!|A|\!) \cong (\!|B|\!)$ in $\mathbf{D}$. Note that in $\mathbf{D}$, all the isomorphisms are identities, except for the two symmetries of $⅋$ and $\otimes$.

▶ **Definition 19.** *Let* $\Phi = \begin{array}{c} A \\ \Phi \| \mathsf{SBV} \\ B \end{array}$ *be a derivation in* SBV *with premise* A *and conclusion* B*. The interpretation of* Φ *in* $\mathbf{C}$ *is the morphism* $\llbracket \Phi \rrbracket \colon \llbracket A \rrbracket \to \llbracket B \rrbracket$ *defined as follows:*

$$\llbracket \Phi \odot \Psi \rrbracket := \llbracket \Phi \rrbracket \odot \llbracket \Psi \rrbracket \qquad \left\llbracket \mathsf{r}\,\frac{\Phi}{\Psi} \right\rrbracket := \llbracket \Psi \rrbracket \circ \llbracket \mathsf{r} \rrbracket \circ \llbracket \Phi \rrbracket \qquad \left\llbracket \equiv \begin{array}{c} A \\ \cdots\cdots \\ B \end{array} \right\rrbracket := \llbracket A \rrbracket \xrightarrow{\llbracket A \equiv B \rrbracket} \llbracket B \rrbracket$$

where $\llbracket \mathsf{r} \rrbracket$ *is defined as follows*

$$\begin{aligned}
&\left\llbracket \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr a^\perp} \right\rrbracket := \mathbb{I} \xrightarrow{\Lambda(\mathrm{ev}\circ\sigma\circ l)} \left(\llbracket a \rrbracket^\perp \otimes \llbracket a \rrbracket^{\perp\perp}\right)^\perp \qquad \left(\textit{where } \left(\llbracket a \rrbracket^\perp \otimes \llbracket a \rrbracket^{\perp\perp}\right)^\perp = \llbracket a \parr a^\perp \rrbracket \textit{ by definition}\right) \\
&\left\llbracket \mathsf{ai}{\uparrow}\,\frac{a \otimes a^\perp}{\mathbb{I}} \right\rrbracket := \left(\llbracket a \rrbracket \otimes \llbracket a \rrbracket^\perp\right) \xrightarrow{\sigma} \left(\llbracket a \rrbracket^\perp \otimes \llbracket a \rrbracket\right) \xrightarrow{\mathrm{ev}} \mathbb{I} \\
&\left\llbracket \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \right\rrbracket := \left(\llbracket A \rrbracket \otimes (\llbracket B \rrbracket \parr \llbracket C \rrbracket)\right) \xrightarrow{\Lambda(\mathrm{ev}\circ(\mathrm{id}\otimes\mathrm{ev})\circ\cong)} \left((\llbracket A \rrbracket \otimes \llbracket B \rrbracket)^\perp \multimap \llbracket C \rrbracket\right) \xrightarrow{\cong} \left((\llbracket A \rrbracket \otimes \llbracket B \rrbracket) \parr \llbracket C \rrbracket\right) \qquad (14) \\
&\left\llbracket \mathsf{q}{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \right\rrbracket := \left((\llbracket A \rrbracket \parr \llbracket B \rrbracket) \lhd (\llbracket C \rrbracket \parr \llbracket D \rrbracket)\right) \xrightarrow{q} \left((\llbracket A \rrbracket \lhd \llbracket C \rrbracket) \parr (\llbracket B \rrbracket \lhd \llbracket D \rrbracket)\right) \\
&\left\llbracket \mathsf{q}{\uparrow}\,\frac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)} \right\rrbracket := \left((\llbracket A \rrbracket \lhd \llbracket B \rrbracket) \otimes (\llbracket C \rrbracket \lhd \llbracket D \rrbracket)\right) \xrightarrow{q'} \left((\llbracket A \rrbracket \otimes \llbracket C \rrbracket) \lhd (\llbracket B \rrbracket \otimes \llbracket D \rrbracket)\right)
\end{aligned}$$

where $\odot \in \{\parr, \lhd, \otimes\}$*;* $\Lambda_{ABC} \colon \mathbf{C}(A \otimes B, C) \cong \mathbf{C}(A, B \multimap C)$ *is the usual natural bijection;* $\mathrm{ev}_{AB} \colon (A \multimap B) \otimes A \to B$ *is the evaluation morphism in* $\mathbf{C}$*;* $l_A \colon \mathbb{I} \otimes A \cong A$ *is the left unitor; the left unnamed isomorphism in the* s *rule has type* $(\llbracket B \rrbracket^\perp \multimap \llbracket C \rrbracket) \otimes (\llbracket A \rrbracket \multimap \llbracket B \rrbracket^\perp) \otimes \llbracket A \rrbracket \cong \llbracket A \rrbracket \otimes (\llbracket B \rrbracket \parr \llbracket C \rrbracket) \otimes (\llbracket A \rrbracket \otimes \llbracket B \rrbracket)^\perp$ *and it is constructed using the obvious isomorphisms from the* $*$*-autonomous structure.*

The interpretation of Φ *in* $\mathbf{D}$ *is the morphism* $\llparenthesis \Phi \rrparenthesis \colon \llparenthesis A \rrparenthesis \to \llparenthesis B \rrparenthesis$ *defined as follows:*

$$\llparenthesis \Phi \odot \Psi \rrparenthesis := \llparenthesis \Phi \rrparenthesis \boxtimes \llparenthesis \Psi \rrparenthesis \qquad \left\llparenthesis \mathsf{r}\,\frac{\Phi}{\Psi} \right\rrparenthesis := \llparenthesis \Psi \rrparenthesis \circ \llparenthesis r \rrparenthesis \circ \llparenthesis \Phi \rrparenthesis \qquad \left\llparenthesis \equiv \begin{array}{c} A \\ \cdots\cdots \\ B \end{array} \right\rrparenthesis := \llparenthesis A \rrparenthesis \xrightarrow{\llparenthesis A \equiv B \rrparenthesis} \llparenthesis B \rrparenthesis$$

$$\begin{aligned}
&\left\llparenthesis \mathsf{ai}{\downarrow}\,\frac{\mathbb{I}}{a \parr a^\perp} \right\rrparenthesis := J \xrightarrow{\eta_{\llparenthesis a \rrparenthesis^\perp}} \llparenthesis a \rrparenthesis \boxtimes \llparenthesis a \rrparenthesis^\perp \qquad \left\llparenthesis \mathsf{ai}{\uparrow}\,\frac{a \otimes a^\perp}{\mathbb{I}} \right\rrparenthesis := \llparenthesis a \rrparenthesis \boxtimes \llparenthesis a \rrparenthesis^\perp \xrightarrow{\epsilon_{\llparenthesis a \rrparenthesis}} J \\
&\left\llparenthesis \mathsf{s}\,\frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \right\rrparenthesis := \llparenthesis A \rrparenthesis \boxtimes \llparenthesis B \rrparenthesis \boxtimes \llparenthesis C \rrparenthesis \xrightarrow{\mathrm{id}} \llparenthesis A \rrparenthesis \boxtimes \llparenthesis B \rrparenthesis \boxtimes \llparenthesis C \rrparenthesis \qquad (15) \\
&\left\llparenthesis \mathsf{q}{\downarrow}\,\frac{(A \parr B) \lhd (C \parr D)}{(A \lhd C) \parr (B \lhd D)} \right\rrparenthesis := \llparenthesis A \rrparenthesis \boxtimes \llparenthesis B \rrparenthesis \boxtimes \llparenthesis C \rrparenthesis \boxtimes \llparenthesis D \rrparenthesis \xrightarrow{\mathrm{id}\boxtimes\sigma\boxtimes\mathrm{id}} \llparenthesis A \rrparenthesis \boxtimes \llparenthesis C \rrparenthesis \boxtimes \llparenthesis B \rrparenthesis \boxtimes \llparenthesis D \rrparenthesis \\
&\left\llparenthesis \mathsf{q}{\uparrow}\,\frac{(A \lhd B) \otimes (C \lhd D)}{(A \otimes C) \lhd (B \otimes D)} \right\rrparenthesis := \llparenthesis A \rrparenthesis \boxtimes \llparenthesis B \rrparenthesis \boxtimes \llparenthesis C \rrparenthesis \boxtimes \llparenthesis D \rrparenthesis \xrightarrow{\mathrm{id}\boxtimes\sigma\boxtimes\mathrm{id}} \llparenthesis A \rrparenthesis \boxtimes \llparenthesis C \rrparenthesis \boxtimes \llparenthesis B \rrparenthesis \boxtimes \llparenthesis D \rrparenthesis
\end{aligned}$$

where $\odot \in \{\parr, \lhd, \otimes\}$*, and where* $\eta_A \colon J \to A^\perp \boxtimes A$ *and* $\epsilon_A \colon A \boxtimes A^\perp \to J$ *are the unit and counit of the compact closed structure.*

▶ Remark 20. The definition of $\llparenthesis \Phi \rrparenthesis$ can be seen as directly interpreting the atomic flow of Φ as a string diagram representing the morphism in a strict compact closed category $\mathbf{D}$, where represents the morphism η and represents the morphism ϵ, which are usually drawn as cups and caps, respectively, in string diagrams (see, e.g., [12]).

The relationship between the two semantic interpretations is given by the following lemma.

▶ **Lemma 21.** *For any* SBV*-derivation* $\Phi = \begin{array}{c} A \\ \Phi \, \| \, \mathsf{SBV} \\ B \end{array}$*, the following diagram commutes in* $\mathbf{D}$.

$$\begin{tikzcd} U[\![A]\!] \arrow[r, equal] \arrow[d, "U[\![\Phi]\!]"'] & (\!|A|\!) \arrow[d, "(\!|\Phi|\!)"] \\ U[\![B]\!] \arrow[r, equal] & (\!|B|\!) \end{tikzcd} \tag{16}$$

Proof. This follows by induction on Φ. The two most interesting cases are $\mathsf{q}\!\downarrow$ and $\mathsf{q}\!\uparrow$. The case for $\mathsf{q}\!\downarrow$ follows immediately from the definition of a strong BV-category. The case for $\mathsf{q}\!\uparrow$ follows easily, because $U[\![\mathsf{q}\!\uparrow]\!] = U(\cong) \circ U([\![q]\!]^{\perp}) \circ U(\cong) = \mathrm{id} \circ (\mathrm{id} \boxtimes \sigma^{\perp} \boxtimes \mathrm{id}) \circ \mathrm{id} = \mathrm{id} \boxtimes \sigma \boxtimes \mathrm{id} = (\!|\mathsf{q}\!\uparrow|\!)$ where we used the fact that U strictly preserves all the relevant structure and maps the two unnamed isomorphisms to identities in $\mathbf{D}$. The remaining cases are straightforward and follow easily from standard results about $*$-autonomous and compact closed categories. More details can be found in the extended version [4]. ◀

▶ **Proposition 22.** *Let* Φ *and* Φ' *be two* SBV*-derivations such that* $\mathsf{AF}(\Phi') = \mathsf{Y}(\mathsf{AF}(\Phi))$. *Then* $(\!|\Phi|\!) = (\!|\Phi'|\!)$.

Proof. This follows easily from Remark 20, because the semantic interpretation in the strict compact closed category $\mathbf{D}$ is sound (i.e., invariant) with respect to yanking. ◀

▶ **Theorem 23** (Soundness). *Let* Φ *and* Φ' *be derivations with* $\Phi \overset{\mathsf{CE}}{\rightsquigarrow} \Phi'$. *Then* $[\![\Phi]\!] = [\![\Phi']\!]$.

Proof. Since U is a faithful functor, it suffices to prove that $U[\![\Phi]\!] = U[\![\Phi']\!]$. By Lemma 21, it suffices to prove that $(\!|\Phi|\!) = (\!|\Phi'|\!)$. Using Theorem 14, we know that $\mathsf{AF}(\Phi') = \mathsf{Y}(\mathsf{AF}(\Phi))$ and Proposition 22 concludes the proof. ◀

▶ **Corollary 24.** *If* $\Phi \sim \Psi$*, then* $[\![\Phi]\!] = [\![\Psi]\!]$*, i.e., the semantic interpretation is invariant with respect to proof equivalence.*

Proof. By the previous theorem, it suffices to prove this for normalized proofs. Since U is faithful, it suffices to prove $U[\![\Phi]\!] = (\!|\Phi|\!) = (\!|\Psi|\!) = U[\![\Psi]\!]$. This now follows trivially from Proposition 22. ◀

7 Concrete Models of BV

In this section we describe several categories that have the structure of a strong BV-category, thus providing concrete models of the logic BV. All the concrete models that we consider are based on existing categories that have already been studied in prior work. Because of the strictness requirement of Definition 17, a common pattern in three of the models that we consider is that we take the *skeleton* subcategory of a category that has already been considered in the literature. This allows us to easily satisfy the strictness requirement. However, we conjecture that the strictness requirement is not necessary – see Section 8 for more discussion related to this.

The most obvious, but rather degenerate, class of models is given by strict compact closed categories (Subsection 7.1) where one can interpret all three multiplicative connectives of BV using the same monoidal tensor. The most mathematically interesting and natural example is based on finite-dimensional operator spaces (Subsection 7.2), with strong links to quantum theory, where mathematicians have identified three different operator space tensor products that allow us to interpret BV. Our two remaining examples are based on gluing and orthogonality [37]: the Caus[-] construction on a category of completely positive maps (Subsection 7.3) and finite-dimensional probabilistic coherence spaces (Subsection 7.4). The former has strong links to quantum theory whereas the latter is relevant to classical probabilistic computation.

7.1 Strict Compact Closed Categories

Every strict compact closed category can be seen as a strong BV-category by taking $\mathbf{C} = \mathbf{D}$ and U to be the identity functor.

A relevant example is the *skeleton subcategory of finite-dimensional vector spaces* $\mathbf{FdVect_{sk}}$ whose objects are vector spaces of the form $\mathbb{C}^n$ and morphisms $\mathbf{FdVect_{sk}}(\mathbb{C}^n, \mathbb{C}^m)$ are $m \times n$ complex matrices, where the multiplication of matrices serves as composition of morphisms, the tensor product $\boxtimes$ is the Kronecker product of matrices (with unit $\mathbb{C}$), and where duality is defined as transposition of matrices, denoted $\cdot^{\mathrm{t}}$. Note that $\mathbf{FdVect_{sk}} \simeq \mathbf{FdVect}$, because every $m \times n$ matrix determines a linear map $\mathbb{C}^n \to \mathbb{C}^m$ (with respect to the standard basis), so we may view the morphisms as linear maps as well.

▶ **Theorem 25.** *The tuple* $\langle \mathbf{FdVect_{sk}}, \boxtimes, \boxtimes, \mathbb{C}, \cdot^{\mathrm{t}}, \mathsf{Id} \rangle$ *is a strong* BV*-category.*

Proof. It is well-known that $\mathbf{FdVect_{sk}}$ is *strict* compact closed. See for example [33]. ◀

7.2 Finite-dimensional Operator Spaces

A *finite-dimensional operator space* [19] is a pair $(X, \mathcal{O})$ consisting of a finite-dimensional vector space X and an *operator space structure* $\mathcal{O} = \left\{ \|\cdot\|_{\mathbb{M}_n(X)} \mid n \in \mathbb{N} \right\}$ given by a sequence of norms $\|\cdot\|_{\mathbb{M}_n(X)} : \mathbb{M}_n(X) \to \mathbb{R}_+$ on the vector space of $n \times n$ matrices with entries in X that satisfy specific axioms (omitted here). If $(X, \mathcal{O})$ and $(Y, \mathcal{O}')$ are two operator spaces, we say that a linear map $f \colon X \to Y$ is a *complete contraction* [19] if $\|f(A)\|_{\mathbb{M}_n(Y)} \leq \|A\|_{\mathbb{M}_n(X)}$ for every $n \in \mathbb{N}$ and $A \in \mathbb{M}_n(X)$, where by $f(A)$ we denote the matrix obtained by applying f to each entry of A.

The category $\mathbf{FdOS}$ of finite-dimensional operator spaces and (linear) complete contractions was recently studied in [43], where it was shown that it is a BV-category. Duals (in the categorical/logical sense) coincide with the operator space duals [19], written $\cdot^*$ in $\mathbf{FdOS}$. The *projective tensor* $\widehat{\otimes}$ [19, §7.1] serves as the multiplicative conjunction $\otimes$, the *injective tensor* $\check{\otimes}$ [7, (1.5.1)] can be identified with the multiplicative disjunction $⅋$, and the *Haagerup tensor* $\overset{\mathrm{h}}{\otimes}$ [19, §9], [7, pp. 30–34], [48, §5] as the seq $\triangleleft$ of BV. By [18, Theorem 6.1] (see also [43]), we can define a complete contraction

$$q_{ABCD} \colon (A \check{\otimes} B) \overset{\mathrm{h}}{\otimes} (C \check{\otimes} D) \to (A \overset{\mathrm{h}}{\otimes} C) \check{\otimes} (B \overset{\mathrm{h}}{\otimes} D)$$

which gives us the q natural transformation. It is well-known that the Haagerup tensor is not symmetric and it is self-dual in $\mathbf{FdOS}$ [19, §9], thus giving us $\kappa_{AB} \colon A^* \overset{\mathrm{h}}{\otimes} B^* \cong (A \overset{\mathrm{h}}{\otimes} B)^*$.

The ***skeleton subcategory*** $\mathbf{FdOS_{sk}}$ is the subcategory of $\mathbf{FdOS}$ whose objects are operator spaces of the form $(\mathbb{C}^n, \mathcal{O})$ for some operator space structure $\mathcal{O}$ on $\mathbb{C}^n$ and whose morphisms are the complex matrices that represent linear complete contractions between such spaces. All of the aforementioned constructions can be easily adapted to $\mathbf{FdOS_{sk}}$ and the obvious forgetful functor $U \colon \mathbf{FdOS_{sk}} \to \mathbf{FdVect_{sk}}$ strictly preserves all this data.

▶ **Theorem 26.** *The tuple* $\langle \mathbf{FdOS_{sk}}, \widehat{\otimes}, \overset{\mathrm{h}}{\otimes}, \mathbb{C}, \cdot^*, U \rangle$ *is a strong* BV*-category.*

Proof. Straightforward verification by using (the adapted) results from [43]. ◂

The theory of operator spaces has strong links to quantum theory. Von Neumann algebras (e.g., $B(H)$ – the algebra of bounded operators on a Hilbert space H) are operator spaces and so are their preduals (e.g., $T(H)$ – the trace class operators on a Hilbert space H). The former can be used to formulate quantum computation in the Heisenberg picture of quantum theory, whereas the latter can be used to formulate quantum computation in the Schrödinger picture. The authors in [43] use the category **FdOS** to construct a model of MALL (Multiplicative Additive Linear Logic) in which the polarised linear logic duality coincides with the Heisenberg-Schrödinger duality of quantum theory. The Haagerup tensor, which is used to interpret the seq connective $\triangleleft$ of BV, plays an essential role in this development. The work in [43] builds on prior work [45] on (infinite-dimensional) operator spaces, linear logic and the Heisenberg-Schrödinger duality where the authors also consider the Haagerup tensor and discuss its relation to the logic BV and some of its other properties relevant to quantum computation.

7.3 Completely Positive Maps and the Caus[-] Construction

In [56], the authors describe a categorical construction, called $\mathbf{Caus}[-]$, that produces a new $*$-autonomous category $\mathbf{Caus}[\mathbf{C}]$ with some additional structure (in particular an extra tensor $<$) from a given compact closed category $\mathbf{C}$ satisfying specific conditions. This allows them to construct a BV-category where the tensor $<$ represents the connective $\triangleleft$ of BV. There is an evident forgetful functor $U\colon \mathbf{Caus}[\mathbf{C}] \to \mathbf{C}$ which strictly preserves all of the relevant structure. The main concrete example is the category $\mathbf{Caus}[\mathbf{CP}^*[\mathbf{FHilb}]]$, where $\mathbf{FHilb}$ is the category of finite-dimensional Hilbert spaces with linear maps as morphisms and $\mathbf{CP}^*$ is another categorical construction which gives us a category of completely-positive maps in this case.

If we consider the skeletal subcategory $\mathbf{FHilb_{sk}}$ whose objects are the Hilbert spaces $\mathbb{C}^n$ and the morphisms are the complex matrices, then we can construct a strong BV-category.

▶ **Theorem 27.** *The tuple* $\langle \mathbf{Caus}[\mathbf{CP}^*[\mathbf{FHilb_{sk}}], \otimes, <, \mathbb{I}, \cdot^*, U\rangle$ *is a strong* BV*-category.*

Proof. Straightforward verification using the results from [56] and [33] (see the extended version [4] for more details). ◂

This example also has strong links to quantum theory. The category $\mathbf{CP}^*[\mathbf{FHilb}]$ is equivalent to the category of finite-dimensional von Neumann algebras (equivalently finite-dimensional C*-algebras) and completely positive maps between them. The category $\mathbf{Caus}[\mathbf{CP}^*[\mathbf{FHilb}]]$ then gives a model of higher-order quantum computation (for finite-dimensional systems).

7.4 Probabilistic Coherence Spaces

Our next example is based on a category of finite-dimensional probabilistic coherence spaces [25]. The category is described in [9] and it is $*$-autonomous. The authors in [9] also define a third tensor product that we use. We adapt the results from [9] by considering the skeletal versions of these constructions. Let $\mathbf{P_{sk}}$ be the category whose objects are pairs $A = (\mathbb{R}_+^n, P_A)$, where $n \in \mathbb{N}$ and $P_A \subseteq \mathbb{R}_+^n$, where $\mathbb{R}_+$ is the set of non-negative reals, such that:

- $P_A = P_A^{\perp\perp}$, where $P_A^{\perp} = \{w \in \mathbb{R}_+^n \mid \langle w, v\rangle := \sum_{i=1}^n w_i v_i \leq 1 \text{ for all } v \in P_A\}$;
- and certain other conditions hold, but we omit them here for brevity.

The homsets $\mathbf{P_{sk}}\big((\mathbb{R}^n_+, P_A), (\mathbb{R}^m_+, P_B)\big)$ are given by the $m \times n$ matrices with coefficients in $\mathbb{R}_+$, such that for all $v \in P_A$ we have that $Mv \in P_B$, where Mv stands for the multiplication of the matrix M with the vector v. We write $\otimes$ for the tensor product in $\mathbf{P_{sk}}$ which can be adapted in an obvious way from the usual definition in [16, 9, 25]. We write $\oslash$ for the (again obvious) adapation of the tensor defined in [9] to $\mathbf{P_{sk}}$. We note that the action of $\otimes$ and $\oslash$ on morphisms coincides with the Kronecker product of matrices. Duals are given by $(\mathbb{R}^n_+, P_A)^\perp = (\mathbb{R}^n_+, P_A^\perp)$ and their action on morphisms is given by transposition of matrices. Let $\mathbf{Mat}_{\mathbb{R}_+}$ be the category whose objects are given by $\mathbb{R}^n_+$, for $n \in \mathbb{N}$, and whose homsets $\mathbf{Mat}_{\mathbb{R}_+}(\mathbb{R}^n_+, \mathbb{R}^m_+)$ are given by the $m \times n$ matrices with coefficients in $\mathbb{R}_+$. This category is strict compact closed with all the constructions fully analogous to $\mathbf{FdVect_{sk}}$. We write $U \colon \mathbf{P_{sk}} \hookrightarrow \mathbf{Mat}_{\mathbb{R}^+}$ for the obvious forgetful functor.

▶ **Theorem 28.** *The tuple $\langle \mathbf{P_{sk}}, \otimes, \oslash, \mathbb{R}^+, \cdot^\perp, U\rangle$ is a strong* BV*-category.*

Proof. Straightforward verification using the (adapted) results from [9]. ◀

Probabilistic coherence spaces have been studied in the context of (classical) probabilistic computation [21, 20].

8 Conclusion and Future Work

In this paper we proposed a notion of *proof identity* for BV, and for this we introduced *atomic flows* for BV. We have shown that the global operation of cut elimination in BV via *splitting* coincides on atomic flows with the local operation of *yanking*. This allowed us to provide a refined notion of *strong* BV*-category* that is sound for the logic BV, and we provided some concrete non-trivial examples of such categories.

This work brings together two communities: those who study the proof theory of BV and those who study its categorical semantics. This opens up new directions of research on which we elaborate below.

Relation to Pomset Logic. An independent justification for our choice of proof identity for BV is that it coincides with the one induced by the proof nets of pomset logic. It has been shown in [47, 46], that every BV-proof can be translated into a correct pomset logic proof net. In the cut-free case, this is just the formula in the conclusion together with the axiom links. But the atomic flow of a cut-free BV-derivation with premise $\mathbb{I}$ is also just the axiom links on the conclusion. Therefore, two BV proofs have the same atomic flow if and only if they have the same pomset logic proof net. This raises some questions for future research: What is the precise relation between cut elimination in pomset logic proof nets and yanking in atomic flows? And can we have a correctness criterion for BV atomic flows? As shown in [47, 46], the correctness criterion for pomset logic is not suitable for BV.

Strong BV-categories vs BV-categories. We proved that every strong BV-category is a BV-category in the sense of [9, 32]. We conjecture that the converse is not true, i.e. there exists a BV-category which cannot be embedded into a compact closed one in the way that we require. The embedding of a strong BV-category into a strict compact closed category ensures that we get all the coherence properties for the proof of soundness (Theorem 23) and invariance with respect to proof equivalence (Corollary 24), for which we also use results based on atomic flows. Even though the original definition of a BV-category gives us many coherence diagrams (see Appendix B), we do not know if it gives us *all* the ones we need, because the splitting lemma (used for cut elimination / proof normalisation) is difficult to work with categorically, and we doubt that BV-categories give us enough coherence diagrams for atomic flows in relation to Corollary 24.

We would also like to formulate a more general notion of proof equivalence compared to the one that we used here (which is based on atomic flows) and then check if the original definition of BV-category gives us enough coherence diagrams for this notion of proof equivalence. If not, then it would be important to identify the right set of coherence laws for BV-categories and modify the definition of a BV-category accordingly. Another open problem related to this is whether one can formulate a coherence theorem for BV-categories (in a similar way to Mac Lane's coherence theorem for monoidal categories [42]). The massive complexity discrepancy between pomset logic and BV suggests that this is a non-trivial question.

Yanking without Cut Elimination. Our main result (Theorem 14) only speaks about derivations with premise $\mathbb{I}$. However, we conjecture that this can be generalized as follows:

▶ **Conjecture 29.** *Given a derivation* $\begin{array}{c} A \\ \Phi \,\Vert\, \mathsf{SBV} \\ B \end{array}$ *, then there is a derivation* $\begin{array}{c} A \\ \Phi' \,\Vert\, \mathsf{SBV} \\ B \end{array}$ *, such that* $\mathsf{AF}\left(\Phi'\right) = \mathsf{Y}\left(\mathsf{AF}\left(\Phi\right)\right)$.

This would then also strengthen our second result Theorem 23. We also conjecture that the methods that we need to develop to prove Theorem 14 would also help to make progress towards a coherence theorem for BV-categories as discussed above.

Strictness. Definition 17 has a strictness condition on the compact closed category $\mathbf{D}$ and on the forgetful functor $U \colon \mathbf{C} \to \mathbf{D}$. One reason for this is that in atomic flows the associativity and unitality of the tensor is "on the nose" and it is not immediately clear how to relax the strictness condition. It would certainly require checking many coherence conditions and possibly modifying the proofs in Section 4, but we conjecture that Theorem 23 can also be extended to the non-strict case. In particular, we conjecture that the non-strict versions of the concrete categorical models in Section 7 are sound. Another reason for the strictness assumption is that it is currently unclear (to us) what is a good notion of a (non-strict) $*$-autonomous functor (compare [15] and [32]). We conjecture that the strictness assumption on the functor U can be removed, but this requires performing a more careful analysis of all of the coherence properties that are required to hold in relation to it.

Models of Intuitionistic BV. Another direction for future work is to investigate suitable categorical models for Intuitionistic BV [3, 2]. It would be interesting to see if natural models can be found by considering categories of (possibly infinite-dimensional) operator spaces [45, 44] where the seq connective $\lhd$ of BV is interpreted by variants of the Haagerup tensor (see [18]).

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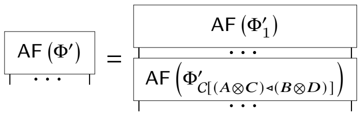
A Omitted Proof for Lemma 11

▶ **Lemma 11.** *If there is a derivation* $\Phi = C\left[\begin{array}{c}\Phi' \,\Vert\, \mathsf{BV} \\ \mathsf{q}\!\uparrow \dfrac{(A \vartriangleleft B) \otimes (C \vartriangleleft D)}{(A \otimes C) \vartriangleleft (B \otimes D)}\end{array}\right]$ *then there is a derivation* $\begin{array}{c}\Phi^* \,\Vert\, \mathsf{BV} \\ C[(A \otimes C) \vartriangleleft (B \otimes D)]\end{array}$ *such that* $\mathsf{AF}(\Phi^*) = \mathsf{AF}(\Phi)$.

Proof. We start by applying context reduction to Φ', obtaining a formula K and derivations

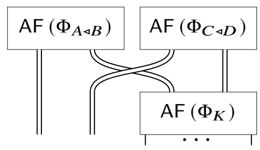
$$\begin{array}{c}\Phi'_1 \,\Vert\, \mathsf{BV} \\ ((A \otimes C) \vartriangleleft (B \otimes D)) \parr K\end{array} \quad \text{and} \quad \begin{array}{c} X \parr K \\ \Phi'_{C[X]} \,\Vert\, \mathsf{BV} \\ C[X]\end{array} \quad \text{for every formula } X \tag{17}$$

such that 

We then apply the splitting lemma to Φ'_1, obtaining formulas K_l and K_r and derivations

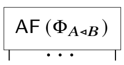
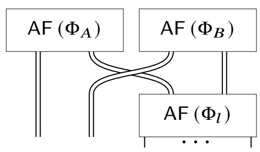
$$\begin{array}{c} K_l \parr K_r \\ \Phi_K \,\Vert\, \mathsf{BV} \\ K\end{array} \quad \text{and} \quad \begin{array}{c}\Phi_{A \vartriangleleft B} \,\Vert\, \mathsf{BV} \\ (A \vartriangleleft B) \parr K_l\end{array} \quad \text{and} \quad \begin{array}{c}\Phi_{C \vartriangleleft D} \,\Vert\, \mathsf{BV} \\ (C \vartriangleleft D) \parr K_r\end{array} \quad \text{such that}$$

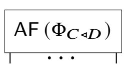
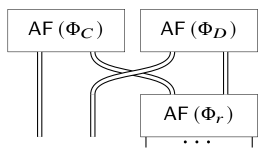
$$\mathsf{AF}\left(\Phi'_1\right) = \mathsf{AF}\left(\mathsf{s}\,\dfrac{\Phi_{A \vartriangleleft B} \otimes \Phi_{C \vartriangleleft D}}{((A \vartriangleleft B) \otimes (C \vartriangleleft D)) \parr \Phi_K}\right) = \tag{18}$$



Then, we can apply splitting on both $\Phi_{A \vartriangleleft B}$ and $\Phi_{C \vartriangleleft D}$, obtaining formulas K_A, K_B, K_C, and K_D and derivations

$$\begin{array}{c} K_A \vartriangleleft K_B \\ \Phi_l \,\Vert\, \mathsf{BV} \\ K_l\end{array}, \quad \begin{array}{c} K_C \vartriangleleft K_D \\ \Phi_r \,\Vert\, \mathsf{BV} \\ K_r\end{array}, \quad \begin{array}{c}\Phi_A \,\Vert\, \mathsf{BV} \\ A \parr K_A\end{array}, \quad \begin{array}{c}\Phi_B \,\Vert\, \mathsf{BV} \\ B \parr K_B\end{array}, \quad \begin{array}{c}\Phi_C \,\Vert\, \mathsf{BV} \\ C \parr K_C\end{array} \quad \text{and} \quad \begin{array}{c}\Phi_D \,\Vert\, \mathsf{BV} \\ D \parr K_D\end{array} \quad \text{such that}$$

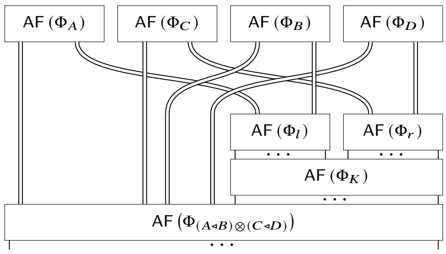
 $= \mathsf{AF}\left(\Phi_{A \vartriangleleft B}\right) = \mathsf{AF}\left(\mathsf{q}\!\downarrow \dfrac{\Phi_A \vartriangleleft \Phi_B}{(A \vartriangleleft B) \parr \Phi_l}\right) =$  (19)

 $= \mathsf{AF}\left(\Phi_{C \vartriangleleft D}\right) = \mathsf{AF}\left(\mathsf{q}\!\downarrow \dfrac{\Phi_C \vartriangleleft \Phi_D}{(C \vartriangleleft D) \parr \Phi_r}\right) =$ 

We can now build the derivation Φ^* as shown below, whose atomic flow below is such that

AF (Φ^*) = AF (Φ) by the equations on atomic flows in Equations (18) and (19) and Figure 3.

$$
\mathsf{q}{\downarrow}\,\dfrac{\left(\mathsf{s}\,\dfrac{\boxed{\Phi_A} \otimes \boxed{\Phi_C}}{(A \otimes C) \parr K_A \parr K_C}\right) \triangleleft \left(\mathsf{s}\,\dfrac{\boxed{\Phi_B} \otimes \boxed{\Phi_D}}{(B \otimes D) \parr K_B \parr K_D}\right)}{((A \otimes C) \triangleleft (B \otimes D)) \parr \left(\dfrac{\boxed{\Phi_l} \parr \boxed{\Phi_r}}{\boxed{\Phi_K}}\right)}
\quad \Phi'_{C[(A\otimes C)\triangleleft(B\otimes D)]} \, \| \quad C[(A \otimes C) \triangleleft (B \otimes D)]
$$



◀

B Definition of BV-category

In this appendix we give the definition of a BV-category with negation based on the approach taken in [9] with some small adaptations related to presentation. The presentation and source code for most of the diagrams is taken from the appendix in [43] (arxiv version) with some adaptations on our end.

▶ **Definition 30** (Normal Duoidal Structure). *Let* $\mathbf{C}$ *be a category with a monoidal structure* $(\mathbf{C}, I, \otimes)$ *and another monoidal structure* $(\mathbf{C}, J, \triangleleft)$*. We say that* $(J, \triangleleft)$ *is* normal duoidal *to* $(I, \otimes)$ *if there exists a natural transformation:*

$$w : (A \triangleleft B) \otimes (C \triangleleft D) \longrightarrow (A \otimes C) \triangleleft (B \otimes D)$$

called weak interchange, *together with morphisms* $i : I \cong J$, $\delta_I : I \to I \triangleleft I$, *and* $\mu_J : J \otimes J \to J$ *which satisfy the following coherence properties:*

the weak interchange respects associativity in the sense that the following diagrams

$$
\begin{tikzcd}
((A \triangleleft B) \otimes (C \triangleleft D)) \otimes (E \triangleleft F) \arrow[r, "\alpha"] \arrow[d, "w\otimes\mathrm{id}"'] & (A \triangleleft B) \otimes ((C \triangleleft D) \otimes (E \triangleleft F)) \arrow[d, "\mathrm{id}\otimes w"] \\
((A \otimes C) \triangleleft (B \otimes D)) \otimes (E \triangleleft F) \arrow[d, "w"'] & (A \triangleleft B) \otimes ((C \otimes E) \triangleleft (D \otimes F)) \arrow[d, "w"] \\
((A \otimes C) \otimes E) \triangleleft ((B \otimes D) \otimes F) \arrow[r, "\alpha\triangleleft\alpha"'] & (A \otimes (C \otimes E)) \triangleleft (B \otimes (D \otimes F))
\end{tikzcd}
\tag{20}
$$

$$
\begin{tikzcd}
((A \triangleleft C) \triangleleft E) \otimes ((B \triangleleft D) \triangleleft F) \arrow[r, "\alpha\otimes\alpha"] \arrow[d, "w"'] & (A \triangleleft (C \triangleleft E)) \otimes (B \triangleleft (D \triangleleft F)) \arrow[d, "w"] \\
((A \triangleleft C) \otimes (B \triangleleft D)) \triangleleft (E \otimes F) \arrow[d, "w\triangleleft\mathrm{id}"'] & (A \otimes B) \triangleleft ((C \triangleleft E) \otimes (D \triangleleft F)) \arrow[d, "\mathrm{id}\triangleleft w"] \\
((A \otimes B) \triangleleft (C \otimes D)) \triangleleft (E \otimes F) \arrow[r, "\alpha"'] & (A \otimes B) \triangleleft ((C \otimes D) \triangleleft (E \otimes F))
\end{tikzcd}
\tag{21}
$$

commute; the weak interchange respects unitality in the sense that the following diagrams

$$
\begin{tikzcd}
I \otimes (A \triangleleft B) \arrow[r, "\delta_I\otimes id"] & (I \triangleleft I) \otimes (A \triangleleft B) \arrow[d, "w"] \\
A \triangleleft B \arrow[u, "\lambda_{A\triangleleft B}"] \arrow[r, "\lambda_A\triangleleft\lambda_B"'] & (I \otimes A) \triangleleft (I \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \triangleleft B) \otimes I \arrow[r, "id\otimes\delta_I"] & (A \triangleleft B) \otimes (I \triangleleft I) \arrow[d, "w"] \\
A \triangleleft B \arrow[u, "\rho_{A\triangleleft B}"] \arrow[r, "\rho_A\triangleleft\rho_B"'] & (A \otimes I) \triangleleft (B \otimes I)
\end{tikzcd}
\tag{22}
$$

$$\begin{tikzcd} J \lhd (A \otimes B) & (J \otimes J) \lhd (A \otimes B) \arrow[l, "\mu_J \lhd id"'] \\ A \otimes B \arrow[u, "\lambda_{A\otimes B}"] \arrow[r, "\lambda_A \otimes \lambda_B"'] & (J \lhd A) \otimes (J \lhd B) \arrow[u, "w"'] \end{tikzcd} \qquad \begin{tikzcd} (A \otimes B) \lhd J & (A \otimes B) \lhd (J \otimes J) \arrow[l, "id \lhd \mu_J"'] \\ A \otimes B \arrow[u, "\rho_{A\otimes B}"] \arrow[r, "\rho_A \otimes \rho_B"'] & (A \lhd J) \otimes (B \lhd J) \arrow[u, "w"'] \end{tikzcd} \tag{23}$$

commute; we furthermore require that the isomorphism i is an isomix map, i.e. the diagram

$$\begin{tikzcd} J \otimes J \arrow[r, "\mathrm{id}\otimes i^{-1}"] \arrow[d, "i^{-1}\otimes \mathrm{id}"] & J \otimes I \arrow[d, "\cong"] \\ I \otimes J \arrow[r, "\cong"] & J \end{tikzcd}$$

commutes; we also require that (I, i, δ_I) is a comonoid object with respect to the $(C, J, \lhd)$ monoidal structure and we require that (J, i^{-1}, μ_J) is a monoid object with respect to the $(C, I, \otimes)$ monoidal structure.

► **Definition 31.** *A* BV-category with negation *is a $*$-autonomous category $(\mathbf{C}, I, \otimes, \multimap)$ with an additional monoidal structure $(\mathbf{C}, J, \lhd)$, such that*

- $(\mathbf{C}, J, \lhd)$ *is* normal duoidal *to* $(\mathbf{C}, I, \otimes)$ *and we write*

  $$w\colon (A \lhd B) \otimes (C \lhd D) \to (A \otimes C) \lhd (B \otimes D)$$

  for the weak interchange natural transformation; we also require that w commutes with the symmetry of $\otimes$ in the sense that the following diagram commutes

  $$\begin{tikzcd} (A \lhd B) \otimes (C \lhd D) \arrow[r, "\cong"] \arrow[d, "w"'] & (C \lhd D) \otimes (A \lhd B) \arrow[d, "w"] \\ (A \otimes C) \lhd (B \otimes D) \arrow[r, "\cong"'] & (C \otimes A) \lhd (D \otimes B) \end{tikzcd} \tag{24}$$

- $(\mathbf{C}, \bot, \parr)$ *is* normal duoidal *to* $(\mathbf{C}, J, \lhd)$ *and we write*

  $$w'\colon (A \parr B) \lhd (C \parr D) \to (A \lhd C) \parr (B \lhd D)$$

  for the weak interchange natural transformation; we also require that w' commutes with the symmetry of $\parr$ in the sense that the following diagram commutes

  $$\begin{tikzcd} (A \parr B) \lhd (C \parr D) \arrow[r, "\cong"] \arrow[d, "w'"'] & (B \parr A) \lhd (D \parr C) \arrow[d, "w'"] \\ (A \lhd C) \parr (B \lhd D) \arrow[r, "\cong"'] & (B \lhd D) \parr (A \lhd C) \end{tikzcd} \tag{25}$$

- *We also require that the following diagram*

  $$\begin{tikzcd} (A \lhd B) \otimes ((C \parr E) \lhd (D \parr F)) \arrow[r, "w"] \arrow[d, "\mathrm{id}\otimes w'"'] & (A \otimes (C \parr E)) \lhd (B \otimes (D \parr F)) \arrow[d, "a \lhd a"] \\ (A \lhd B) \otimes ((C \lhd D) \parr (E \lhd F)) \arrow[d, "a"'] & ((A \otimes C) \parr E) \lhd ((B \otimes D) \parr F) \arrow[d, "w'"] \\ ((A \lhd B) \otimes (C \lhd D)) \parr (E \lhd F) \arrow[r, "w \parr \mathrm{id}"'] & ((A \otimes C) \lhd (B \otimes D)) \parr (E \lhd F) \end{tikzcd} \tag{26}$$

  and its symmetric dual commute. Here we write $a_{ABC}\colon A \otimes (B \parr C) \to (A \otimes B) \parr C$ for the canonical morphism that may be defined via the $*$-autonomous structure.