On the Consistency of Naive Set Theories over Substructural and Fuzzy Logics

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Abstract

The purpose of this paper is to invite structural proof theorists to a challenging problem in substructural and fuzzy logics: the consistency of Cantor-Łukasiewicz naive set theory. To this end, we consider two logics: **FLew** (Full Lambek calculus with exchange and weakening) and its extension **Ł** (Lukasiewicz logic). The former is equivalent to !-free intuitionistic linear logic with weakening, while the latter is the most prominent system of mathematical fuzzy logic. For each of them, we consider two extensions: one with a fixed point operator (which is neither least nor greatest) and the other with a naive set theory with unrestricted comprehension, so that we end up with four systems: $\mathbf{FLew}_{\mathsf{fp}}$, $\mathbf{FLew}_{\mathsf{set}}$, $\mathbf{Ł}_{\mathsf{fp}}$ and $\mathbf{Ł}_{\mathsf{set}}$. The first two admit an easy proof of consistency by cut elimination, while the third admits a proof by the Brouwer fixed point theorem. The last system $\mathbf{Ł}_{\mathsf{set}}$ (known as Cantor-Łukasiewicz set theory) is our main target. Although there are some partial results, the consistency of the full system is still open. In this paper, we consider a restricted fragment of $\mathbf{Ł}_{\mathsf{set}}$ and prove its consistency.

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1 Introduction

This paper concerns with a technically challenging problem in nonclassical proof theory: the consistency of *Cantor-Łukasiewicz set theory*. Let us begin with some background.

Consider a formal system of naive set theory in which unrestricted comprehension is available:

$$A(t) \leftrightarrow t \in \{x \mid A(x)\},$$

where $A(x)$ is an arbitrary formula in the system. As is quite well-known, this axiom allows us to derive *self-contradiction* $R \leftrightarrow \neg R$ by setting $r := \{x \mid x \notin x\}$ and $R := r \in r$. It then implies all sentences by way of logical reasoning. That is, the system is inconsistent. Faced with this and other antinomies around the beginning of the 20th century, people looked for a firmer foundation of mathematics. Two well-known approaches are the axiomatic set theory (Zelmelo) and the type theory (Russell). However, there is yet another, lesser-known, eccentric approach, that is to weaken the base logic so that self-contradiction does not imply inconsistency. It might sound ridiculous to the reader, but notice that self-contradiction $R \leftrightarrow \neg R$ is sort of a (nonmonotone) fixed point. So, the spirit of this approach is to save fixed points at the sacrifice of logical strength. We believe that it is worth considering for a while.

A pioneer in this direction is Church [3], who proposed a type-free foundation of mathematics consisting of 37 axioms that allows unrestricted comprehension but restricts the law of excluded middle, whereas the law of double negation elimination is maintained. Unfortunately, his system turned out to be inconsistent immediately [4]. Since then, he struggled hard to obtain a consistent logical system with unrestricted comprehension [5].

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As is well known, not just classical logic **Cl**, but also intuitionistic logic **Int** is inconsistent with $R \leftrightarrow \neg R$. In fact, the sequent calculus for **Int** allows us to derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{R \Rightarrow R}}{\neg R, R \Rightarrow}}{R, R \Rightarrow}}{R \Rightarrow}\,(\mathsf{c}) \qquad \dfrac{\dfrac{\dfrac{\vdots\,(*)}{R \Rightarrow}}{\Rightarrow \neg R}}{\Rightarrow R}}{\Rightarrow}\,(\mathsf{cut}) \qquad\qquad \dfrac{\Gamma, \Gamma, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\,(\mathsf{c})$$

where subderivation (*) is identical with the subderivation on the left branch.

Looking at this, one immediately suspects that the main criminal for inconsistency is the contraction rule (c). Hence an easy way out is to remove (c) from the inference rules of **Int**. Such a logic is nowadays known as *full Lambek calculus with exchange and weakening* **FLew** in substructural logics (see [12]), that is also known as !-free intuitionistic linear logic with weakening. Naive set theories over **FLew** (or similar logics) have been investigated by several researchers, not for an alternative foundation of mathematics, but, supposedly, for technical, computational, and philosophical interests [14, 19, 36, 13, 28, 1, 29, 33, 21].

A related project has been undertaken on the basis of (infinite-valued) *Łukasiewicz logic* **Ł**, the most well-known system among mathematical fuzzy logics (see [15, 8, 9]). Since **Ł** is an axiomatic extension of **FLew** that does not admit contraction, the two lines of research are essentially the same. The naive set theory over **Ł** is known as *Cantor-Łukasiewicz set theory*, and it is this theory that we would like to call attention to. The reason is simply that its consistency is still an open problem in spite of partial achievements [30, 2, 11] and a proof theoretic attempt [35]. The former impose some restrictions on comprehension, while the latter turned out to be erroneous [34, 17].

In this paper, we begin with **FLew**, an intuitionistic logic without contraction. Rather than directly introducing a naive set theory over **FLew**, we first introduce a propositional extension $\mathbf{FLew}_{\mathsf{fp}}$ with a fixed point operator. The reason is that it is easier to deal with than the set theoretic extension, and this way we can discuss the effect of self-contradiction independently of its set theoretic implementation. There is no surprise that $\mathbf{FLew}_{\mathsf{fp}}$ admits cut elimination, because it is structurally the same as untyped affine lambda calculus with additives. So, it is consistent (Section 2).

The next step is to extend the base logic to **Ł**. So we study Łkasiewicz logic with a fixed point operator, denoted by $\mathbf{Ł}_{\mathsf{fp}}$, which admits *arbitrary* fixed points including nonmonotone and nested ones, in contrast with prior fixed point extensions of **Ł** [25, 31, 10]. $\mathbf{Ł}_{\mathsf{fp}}$ turns out to be consistent, but it is hard to give a syntactic proof due to lack of a "good" proof system. Instead, we give a semantic proof based on the *Brouwer fixed point theorem* (BFT). The proof demonstrates a prototypical use of BFT, which we wish to apply to naive set theories. Interestingly, we also have a partial converse: the consistency of $\mathbf{Ł}_{\mathsf{fp}}$ implies a special case of BFT. This motivates us to study $\mathbf{Ł}_{\mathsf{fp}}$ independently of BFT but based on proof theory (Section 3).

Having studied fixed points enough, we at last introduce naive set theories over **FLew** and **Ł**. We first study the naive set theory $\mathbf{FLew}_{\mathsf{set}}$ over **FLew**. As it is obviously consistent by cut elimination, our focus is rather on its expressive power. Although it is extremely poor as a set theory, it does enjoy the recursion theorem, that helps us develop some elementary arithmetic within.

We finally address the consistency of $\mathbf{Ł}_{\mathsf{set}}$ (Cantor-Łukasiewicz set theory). Our approach is to use BFT again, but there are certain obstacles to adapt the previous argument to $\mathbf{Ł}_{\mathsf{set}}$. We therefore analyze the reason why it does not work, and propose a suitable restriction on logical inference: when applying the $\forall$ left rule

$$\frac{A(t), \Gamma \Rightarrow \Pi}{\forall x.A(x), \Gamma \Rightarrow \Pi}$$

t must be either a variable or a closed term. With this restriction, the resulting system $\mathbf{Ł}^{-}_{\mathsf{set}}$ is amenable to the BFT technique, so that we are able to prove its consistency. Although the restriction is rather strong, $\mathbf{Ł}^{-}_{\mathsf{set}}$ still enjoys the recursion theorem so that one can encode and prove all true Σ_1 sentences of arithmetic within. Our slogan here is, unlike prior work [30, 2, 11], to restrict *logic* rather than *comprehension* (see Remark 32 for a detailed comparison), inheriting the original spirit of Church [3] (Section 4).

2 Substructural logics over FLew

The starting point of our quest is **FLew**, a prototypical system of logic without contraction [27]. See [12] for a comprehensive account on substructural logics in general. After setting up the syntax (Subsection 2.1) and the algebraic semantics (Subsection 2.2), we discuss how to extend **FLew** with fixed points. By lack of contraction, the resulting system $\mathbf{FLew}_{\mathsf{fp}}$ admits cut elimination quite obviously (Subsection 2.3). We are thus led to extend $\mathbf{FLew}_{\mathsf{fp}}$ with additional rules and discuss their consistency (Subsection 2.4).

2.1 Syntax of FLew

We begin with the syntax of *Full Lambek calculus with exchange and weakening* **FLew**, also known as *!-free intuitionistic linear logic with weakening*. We could alternatively begin with related systems such as !-free classical linear logic, but we prefer **FLew** since it serves as a *prebasis* of mathematical fuzzy logics, and the presence of weakening helps us hide some tedious details.

Let Var be a fixed countable set of *propositional variables*. The set Fm of *formulas* is defined by the following grammar:

$$A, B \ ::= \ p \mid 1 \mid 0 \mid A \wedge B \mid A \vee B \mid A \cdot B \mid A \to B \qquad (p \in \mathsf{Var}).$$

Here we adopt the standard notation in substructural logics. In linear logic, the connectives $1, 0, \wedge, \vee, \cdot, \to$ are respectively written as $\mathbf{1}, \bot, \&, \oplus, \otimes, \multimap$. We write $\neg A$ for $A \to 0$, and $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$.

A *sequent* is of the form $\Gamma \Rightarrow \Pi$, where Γ is a multiset of formulas and Π is either a singleton of a formula or an empty set. Inference rules of the sequent calculus for **FLew** are given in Figure 1.

Note that rules (1ℓ) and $(0r)$ are superfluous in the presence of (i) and (o). If we drop (i) and (o), what we obtain is **FLe**, that is nothing but !-free intuitionistic linear logic. On the other hand, if we add rule (c) to this calculus, we obtain $A \wedge B \leftrightarrow A \cdot B$, so the result is intuitionistic logic **Int**.

Given a formula $A \in \mathsf{Fm}$, we write A^{seq} for the sequent $\Rightarrow A$ with empty antecedent. Given a set $\Phi \cup \{A\}$ of formulas, we write $\Phi \vdash_{\mathbf{FLew}} A$ if A^{seq} is derivable from some sequents in $\Phi^{seq} := \{B^{seq} \mid B \in \Phi\}$ by rules in Figure 1. The binary relation $\vdash_{\mathbf{FLew}} \subseteq \wp(\mathsf{Fm}) \times \mathsf{Fm}$ should not be confused with symbol $\Rightarrow$. For instance, we have $\{p\} \vdash_{\mathbf{FLew}} p \cdot p$ since sequent $\Rightarrow p \cdot p$ is derivable from two copies of $\Rightarrow p$ by rule $(\cdot r)$, but we do not have $p \Rightarrow p \cdot p$ in the sequent calculus.

$$\frac{}{A \Rightarrow A}\ (\mathsf{init}) \qquad \frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{cut}) \qquad \frac{\Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{i}) \qquad \frac{\Gamma \Rightarrow}{\Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{o})$$

$$\frac{A_i, \Delta \Rightarrow \Pi}{A_1 \wedge A_2, \Delta \Rightarrow \Pi}\ (\wedge\ell) \qquad \frac{\Gamma \Rightarrow A_1 \quad \Gamma \Rightarrow A_2}{\Gamma \Rightarrow A_1 \cdot A_2}\ (\wedge r) \qquad \frac{\Delta \Rightarrow \Pi}{1, \Delta \Rightarrow \Pi}\ (1\ell) \qquad \frac{}{\Rightarrow 1}\ (1r)$$

$$\frac{A_1, \Delta \Rightarrow \Pi \quad A_2, \Delta \Rightarrow \Pi}{A_1 \vee A_2, \Delta \Rightarrow \Pi}\ (\vee\ell) \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (\vee r) \qquad \frac{}{0 \Rightarrow}\ (0\ell) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (0r)$$

$$\frac{A, B, \Delta \Rightarrow \Pi}{A \cdot B, \Delta \Rightarrow \Pi}\ (\cdot\,\ell) \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \cdot B}\ (\cdot\, r) \qquad \frac{\Gamma \Rightarrow A \quad B, \Delta \Rightarrow \Pi}{\Gamma, A \to B, \Delta \Rightarrow \Pi}\ (\to \ell) \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ (\to r)$$

Figure 1 Sequent calculus for **FLew**.

We are not only interested in **FLew** itself, but also in its axiomatic extensions. For that purpose, it is convenient to identify a logic with the set of provable formulas in it, following a long tradition in nonclassical logics.

▶ **Definition 1.** *A* substructural logic over **FLew** *is a set* $\mathrm{L} \subseteq \mathsf{Fm}$ *which is closed under deduction in* **FLew** *and substitution:*
- $\mathrm{L} \vdash_{\mathbf{FLew}} A \implies A \in \mathrm{L}$,
- $A \in \mathrm{L} \implies A' \in \mathrm{L}$, *where* A' *is a substitution instance of* A.

We denote by $\mathsf{Ext}(\mathbf{FLew})$ *the set of substructural logics over* **FLew**.

It is known that $(\mathsf{Ext}(\mathbf{FLew}), \subseteq)$ is a complete lattice of cardinality $2^{\aleph_0}$. The least member is **FLew**, while the greatest one is Fm (the inconsistent logic), and the second greatest one is classical logic **Cl** [12].

Given $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$, we write $\mathrm{L} \oplus A$ to denote the least substructural logic that includes $\mathrm{L} \cup \{A\}$. See Figure 2 below for some typical substructural logics.

2.2 Algebraic semantics of FLew

We next proceed to algebraic semantics: FLew algebras (also known as commutative integral bounded residuated lattices). From a categorical point of view, an FLew algebra is just a small posetal monoidal closed category with finite products and coproducts in which the monoidal unit is terminal. A plain definition follows.

▶ **Definition 2.** *An* FLew algebra *is a tuple* $\mathrm{P} = (P, \wedge, \vee, \cdot, \to, 1, 0)$ *such that*
- $(P, \wedge, \vee, 1, 0)$ *is a bounded lattice with* 1 *greatest and* 0 *least,*
- $(P, \cdot, 1)$ *is a commutative monoid, and*
- $a \cdot (-) : P \to P$ *is a left adjoint of* $a \to (-) : P \to P$ *for every* $a \in P$*:*

$$a \cdot x \leq y \iff x \leq a \to y, \tag{1}$$

where $x \leq y$ *is a shorthand for* $x \vee y = y$ *(or equivalently* $x = x \wedge y$*).*

In particular, we have $x \to y = 1$ iff $x \leq y$. Let us write $f_a(x) := a \cdot x$ and $g_a(z) := a \to z$ for conciseness. Because of adjointness, f_a distributes over $\vee$ and g_a over $\wedge$:

$$f_a(x \vee y) = f_a(x) \vee f_a(y), \qquad g_a(z \wedge w) = g_a(z) \wedge g_a(w). \tag{2}$$

We also have the unit and counit laws:

$$x \leq g_a(f_a(x)), \qquad f_a(g_a(z)) \leq z. \tag{3}$$

Conversely, (1) is a consequence of (2) and (3) together with lattice axioms. As a consequence, the class $\mathcal{FL}ew$ of FLew algebras is definable by a set of equations. By Birkhoff's HSP theorem, it forms a *variety*, that is a class of algebras closed under H (homomorphic images), S (subalgebras) and P (products). We write $\mathsf{Sub}(\mathcal{FL}ew)$ for the lattice of subvarieties of $\mathcal{FL}ew$. Its greatest member is $\mathcal{FL}ew$ itself, while the smallest one is the class of trivial (one-element) FLew algebras.

A formula $A \in \mathsf{Fm}$ can be seen as a term of FLew algebras. Hence it makes sense to consider a homomorhism $v : \mathsf{Fm} \to \mathrm{P}$ for any FLew algebra P. Such v is called a *valuation*. We say that v *satisfies* $\Phi \subseteq \mathsf{Fm}$ if $v(A) = 1$ holds for every $A \in \Phi$.

Given a class $\mathcal{K}$ of FLew algebras and a set $\Phi \cup \{A\}$ of formulas, we write $\Phi \models_{\mathcal{K}} A$ if for every $\mathrm{P} \in \mathcal{K}$ and every valuation $v : \mathsf{Fm} \to \mathrm{P}$ with $\mathrm{P} \in \mathcal{K}$, if v satisfies Φ, then it also satisfies A. When $\Phi = \emptyset$, we write $\models_{\mathcal{K}} A$. When $\mathcal{K} = \{\mathrm{P}\}$, we write $\Phi \models_{\mathrm{P}} A$.

As is expected, logic **FLew** is complete with respect to variety $\mathcal{FL}ew$, but we can say slightly more. Given a set $\Phi \subseteq \mathsf{Fm}$ and a class $\mathcal{K} \subseteq \mathcal{FL}ew$, define

$$\mathsf{Mod}(\Phi) := \{\mathrm{P} \in \mathcal{FL}ew \mid \forall A \in \Phi.\ \models_{\mathrm{P}} A\}, \qquad \mathsf{Log}(\mathcal{K}) := \{A \in \mathsf{Fm} \mid \forall \mathrm{P} \in \mathcal{K}.\ \models_{\mathrm{P}} A\}.$$

Then we have a Galois connection $\mathcal{K} \subseteq \mathsf{Mod}(\Phi) \iff \mathsf{Log}(\mathcal{K}) \supseteq \Phi$. We have:

- $\Phi = \mathsf{Log}(\mathsf{Mod}(\Phi)) \iff \Phi \in \mathsf{Ext}(\mathbf{FLew})$,
- $\mathcal{K} = \mathsf{Mod}(\mathsf{Log}(\mathcal{K})) \iff \mathcal{K} \in \mathsf{Sub}(\mathcal{FL}ew)$,
- $(\mathsf{Mod}, \mathsf{Log})$ gives a lattice isomorphism $\mathsf{Ext}(\mathbf{FLew}) \cong \mathsf{Sub}(\mathcal{FL}ew)^{op}$.

This in particular allows us to define a logic L by a class $\mathcal{K} \subseteq \mathcal{FL}ew$ of algebras: $\mathrm{L} = \mathsf{Log}(\mathcal{K})$. The equation means that L is *complete* with respect to $\mathcal{K}$: $\vdash_{\mathrm{L}} A \iff \models_{\mathcal{K}} A$ for every $A \in \mathsf{Fm}$. We also have strong soundness: for any set $\Phi \cup \{A\} \subseteq \mathsf{Fm}$,

$$\Phi \vdash_{\mathrm{L}} A \implies \Phi \models_{\mathcal{K}} A.$$

If the converse direction holds for any set (resp. any finite set) $\Phi \cup \{A\} \subseteq \mathsf{Fm}$, we say that L is *strongly complete* (resp. *finitely strongly complete*) with respect to $\mathcal{K}$. It is known that when $\mathcal{K}$ is a variety, the corresponding logic $\mathrm{L} = \mathsf{Log}(\mathcal{K})$ is strongly complete with respect to $\mathcal{K}$ (see [12]).

The following lemma will be useful when proving (in)consistency by an algebraic means.

▶ **Lemma 3.** *Suppose that* $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$ *is finitely strongly complete with respect to* $\mathcal{K} \subseteq \mathcal{FL}ew$, *and let* $\Phi \subseteq \mathsf{Fm}$. *Then* $\Phi \not\vdash_{\mathrm{L}} 0$ *if and only if for any finite subset* $\Phi_0 \subseteq \Phi$, *there is a valuation* $v : \mathsf{Fm} \to \mathrm{P}$ *with* $\mathrm{P} \in \mathcal{K}$ *satisfying* Φ_0.

Proof. Let us prove the contraposition. We have $\Phi \vdash_{\mathrm{L}} 0$ iff $\Phi_0 \vdash_{\mathrm{L}} 0$ for some finite subset $\Phi_0 \subseteq \Phi$. By finite strong completeness, $\Phi_0 \vdash_{\mathrm{L}} 0$ is equivalent to $\Phi_0 \models_{\mathcal{K}} 0$, which means that there is no valuation $v : \mathsf{Fm} \to \mathrm{P}$ with $\mathrm{P} \in \mathcal{K}$ satisfying Φ_0. ◀

2.3 Substructural logics with fixed points

In the introduction, we focused on the self-contradiction $R \leftrightarrow \neg R$ that arises in naive set theory, but it is just one of many fixed points, and it appears in other contexts too. Hence we isolate fixed points from their actual constructions, and study the effect of adding them all to a logic. Concretely, we equip each substructural logic L with a fixed point operator to obtain a system L_{fp}. Although a standard way is to extend the language Fm with an explicit fixed point operator, we rather *implement* it without extending Fm. This approach will be useful when discussing the semantics of fixed points.

Let Var° be a copy of Var. Given a set C of expressions such that $\mathsf{Var}^\circ \cap \mathsf{C} = \emptyset$, let us tentatively write $\mathsf{Fm}(\mathsf{C})$ for the set of formulas built from $\mathsf{Var}^\circ \cup \mathsf{C}$ as the set of propositional variables (C is informally regarded as a set of propositional constants). We define a set C_n of expressions for each $n < \omega$ as follows. $\mathsf{C}_0 := \emptyset$, and $\mathsf{C}_{n+1} := \{\xi p.A \mid p \in \mathsf{Var}^\circ \text{ and } A \in \mathsf{Fm}(\mathsf{C}_n)\}$. Let $\mathsf{C} := \bigcup_{n<\omega} \mathsf{C}_n$ and update the variable set by $\mathsf{Var} := \mathsf{Var}^\circ \cup \mathsf{C}$.

This way the set Fm of formulas contains $\xi p.A \in \mathsf{C}$ (a propositional variable but informally regarded as a constant) for every $p \in \mathsf{Var}^\circ$ and $A \in \mathsf{Fm}$. To express a fixed point axiom $\xi p.A \leftrightarrow A[\xi p.A/p]$, we need to substitute *into* such a variable/constant. Given $A, B \in \mathsf{Fm}$ and $p \in \mathsf{Var}^\circ$ the substitution $A[B/p]$ is defined as follows:

$$p[B/p] := B, \qquad q[B/p] := q, \qquad \ell[B/p] := \ell \ (\ell = 0, 1), \qquad (\xi q.C)[B/p] := \xi q.(C[B/p]),$$
$$(\xi p.C)[B/p] := \xi p.C, \qquad (A_1 \star A_2)[B/p] := A_1[B/p] \star A_2[B/p],$$

where $q \in \mathsf{Var}^\circ$, $q \not\equiv p$, and $\star \in \{\wedge, \vee, \cdot, \to\}$. By adopting a standard variable convention, it is ensured that substitution $A[\xi p.A/p]$ never causes a variable clash.

▶ **Definition 4.** *Let* Φ_{fp} *be the set of formulas of the form*

$$\xi p.A \leftrightarrow A[\xi p.A/p],$$

called fixed point axioms. *Given* $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$*, we define* $\mathrm{L}_{\mathsf{fp}} := \{A \in \mathsf{Fm} \mid \Phi_{\mathsf{fp}} \vdash_{\mathrm{L}} A\}$*, and write* $\vdash_{\mathrm{L}_{\mathsf{fp}}} A$ *if* $A \in \mathrm{L}_{\mathsf{fp}}$. L *is said to be* consistent with fixed points *if* $\mathrm{L}_{\mathsf{fp}} \not\vdash 0$ *(or equivalently* $\mathrm{L}_{\mathsf{fp}} \neq \mathsf{Fm}$*).*

To avoid confusion, let us stress that our fixed points are not intended to be least or greatest ones. They have nothing to do with *Knaster-Tarski's fixed points*, which are very often employed in semantics of fixed point logics. As we will see later, ours is rather related to *Russellian* and *Brouwerian fixed points*.

Let us now move on to proof theory. Note that the fixed point axioms can be equivalently expressed by a pair of inference rules:

$$\frac{\Gamma \Rightarrow A[\xi p.A/p]}{\Gamma \Rightarrow \xi p.A}\ (\mathsf{fp}r) \qquad \frac{A[\xi p.A/p], \Delta \Rightarrow \Pi}{\xi p.A, \Delta \Rightarrow \Pi}\ (\mathsf{fp}\ell)$$

Hence $\mathbf{FLew}_{\mathsf{fp}}$ can be regarded as a sequent calculus for $\mathbf{FLew}$ extended by these two rules. $(\mathsf{fp}r)$ and $(\mathsf{fp}\ell)$ are symmetric, so a *principal cut* on $\xi p.A$ can be easily reduced:

$$\frac{\dfrac{\Gamma \Rightarrow A[\xi p.A/p]}{\Gamma \Rightarrow \xi p.A}\ (\mathsf{fp}r) \qquad \dfrac{A[\xi p.A/p], \Delta \Rightarrow \Pi}{\xi p.A, \Delta \Rightarrow \Pi}\ (\mathsf{fp}\ell)}{\Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{cut}) \quad \Longrightarrow \quad \frac{\Gamma \Rightarrow A[\xi p.A/p] \quad A[\xi p.A/p], \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{cut})$$

Cuts of other types are reducible too, just as in $\mathbf{FLew}$. Notice however that the cut formula may become bigger by reduction. Hence when one proves cut elimination, one cannot argue by induction on the size of a cut formula. The only available measure is the *size* $|\pi|$ of a derivation π. Thanks to the absence of contraction, there is indeed a proof of cut elimination that only relies on $|\pi|$ (see Appendix A.1).

▶ **Theorem 5.** $\mathbf{FLew}_{\mathsf{fp}}$ *admits cut elimination, so is consistent. In particular,* $\mathbf{FLew}$ *with self-contradiction* $R \leftrightarrow \neg R$ *is consistent.*

▶ Remark 6. Our fixed point axioms yield simultaneous fixed points too by the standard Bekič construction. Given two formulas $A(p, q)$ and $B(p, q)$, define $P := \xi p.A(p, \xi q.B(p, q))$ and $Q := \xi q.B(P, q)$. $\mathbf{FLew}_{\mathsf{fp}}$ then proves $P \leftrightarrow A(P, Q)$ and $Q \leftrightarrow B(P, Q)$ simultaneously.

2.4 Extensions of $\mathbf{FLew}$ (in)consistent with fixed points

Although $\mathbf{FLew}_{\mathsf{fp}}$ is consistent, its logical strength is miserably poor. It is therefore natural to look for a stronger logic that is still consistent with fixed points.

As a warm-up, let us consider two weaker forms of contraction: *knotted rules* and *parallel contraction.*

$$\frac{\Gamma^{(n)}, \Delta \Rightarrow \Pi}{\Gamma^{(m)}, \Delta \Rightarrow \Pi}\ (\mathsf{knot}^n_m) \qquad \frac{\Gamma, \Gamma, \Delta \Rightarrow \Pi \quad \Sigma, \Sigma, \Delta \Rightarrow \Pi}{\Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (\mathsf{pc})$$

where $\Gamma^{(m)}$ stands for the m-fold concatenation $\Gamma, \dots, \Gamma$.

Notice that rule (knot^n_m) is axiomatized by formula $\mathbf{knot}^n_m := p^m \to p^n$, where p^m stands for the m-fold multiplication $p \cdots p$, and (pc) is axiomatized by $\mathbf{pc} := p \cdot q \to p^2 \vee q^2$.

Now, can you guess which of $\mathbf{FLew} \oplus \mathbf{knot}^n_m$ and $\mathbf{FLew} \oplus \mathbf{pc}$ is consistent with fixed points?

▶ **Proposition 7.** $\mathbf{FLew} \oplus \mathbf{knot}^n_m$ *is inconsistent with fixed points for every* $n > m \geq 0$.

Proof. We may assume $m > 0$, since $\mathbf{FLew} \oplus \mathbf{knot}^n_0$ is obviously inconsistent. Furthermore, we may assume $m = n-1$, since (knot^n_{n-1}) is derivable from any (knot^n_m) with $n > m$ by weakening. Now let $Q := \xi p.\neg p^{n-1}$ so that $Q \leftrightarrow \neg Q^{n-1}$ is a fixed point axiom. We have a derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{Q \Rightarrow Q \quad \cdots \quad Q \Rightarrow Q}{Q^{(n-1)} \Rightarrow Q^{n-1}}\ (\cdot\, r)^{n-1}}{\neg Q^{n-1}, Q^{(n-1)} \Rightarrow}\ (\neg \ell)}{Q^{(n)} \Rightarrow}\ (\mathsf{fp}\ell)}{Q^{(n-1)} \Rightarrow}\ (\mathsf{knot}^n_{n-1})$$

where $(\cdot\, r)^{n-1}$ stands for the iterated applications of rule $(\cdot\, r)$ ($n-1$ times), and $(\neg \ell)$ is a derived rule for the defined connective $\neg$. So we have proved $Q^{(n-1)} \Rightarrow$. By successively applying $(\cdot\, \ell)$, $(\neg r)$ and $(\mathsf{fp}r)$, we obtain $\Rightarrow Q$. By applying (cut) between $Q^{(n-1)} \Rightarrow$ and $\Rightarrow Q$ ($n-1$ times), we obtain the empty sequent $\Rightarrow$. ◀

The above theorem has an immediate consequence which applies to a wide range of logics. A logic $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$ is *finite-valued* if there is a finite algebra $\mathrm{P} \in \mathcal{FL}ew$ such that $\mathrm{L} = \mathsf{Log}(\{\mathrm{P}\})$.

▶ **Corollary 8.** *Any finite-valued logic* $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$ *admits* (knot^n_{n-1}) *for a sufficiently large* n*. Hence it is inconsistent with fixed points.*

Proof. Suppose that $\mathrm{L} = \mathsf{Log}(\{\mathrm{P}\})$ where P is an n-element FLew algebra. We claim that $a^n = a^{n-1}$ holds for every $a \in P$. Indeed, since $a \leq 1$, we have $a^n \leq a^{n-1} \leq \cdots \leq a^1 \leq a^0 = 1$. Hence $a^k = a^{k-1}$ for some $1 \leq k \leq n$, which implies $a^n = a^{n-1}$. As a consequence, P validates $p^{n-1} \leq p^n$, that is, $\models_{\mathrm{P}} \mathbf{knot}^n_{n-1}$. So $\vdash_{\mathrm{L}} \mathbf{knot}^n_{n-1}$ by completeness. Therefore, L is inconsistent with fixed points by Proposition 7. ◀

Turning to parallel contraction (pc), proving cut elimination in the presence of (pc) is not straightforward. In fact, if we naively apply a permutative reduction step

$$\cfrac{\begin{matrix}\vdots\ \pi \\ \Gamma \Rightarrow A\end{matrix} \quad \cfrac{A, A \Rightarrow \Pi \quad \Delta, \Delta \Rightarrow \Pi}{A, \Delta \Rightarrow \Pi}\ (\mathsf{pc})}{\Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{cut}) \quad \Longrightarrow \quad \cfrac{\cfrac{\begin{matrix}\vdots\ \pi \\ \Gamma \Rightarrow A\end{matrix} \quad \begin{matrix}\vdots\ \pi \\ \Gamma \Rightarrow A\end{matrix} \quad A, A \Rightarrow \Pi}{\Gamma, \Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{cut})^2 \quad \Delta, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\ (\mathsf{pc})$$

the subderivation π is duplicated. However, there is a nice cut elimination procedure if we restrict the derivations to those of the empty sequent.

pl :	$(p \to q) \vee (q \to p)$	(prelinearity)	**MTL**	:=	**FLew** $\oplus$ **pl**	(monoidal t-norm logic)
div :	$p \wedge q \to p \cdot (p \to q)$	(divisibility)	**BL**	:=	**MTL** $\oplus$ **div**	(basic fuzzy logic)
c :	$p \to p \cdot p$	(contraction)	**G**	:=	**BL** $\oplus$ **c**	(Gödel logic)
in :	$\neg\neg p \to p$	(involutivity)	**Ł**	:=	**BL** $\oplus$ **in**	(Łukasiewicz logic)
prod :	$\neg p \vee (p \to p \cdot q) \to q$	(product)	**Π**	:=	**BL** $\oplus$ **prod**	(product logic)

Figure 2 Five major systems of mathematical fuzzy logics.

▶ **Proposition 9.** **FLew** $\oplus$ **pc** *is consistent with fixed points.*

This is a consequence of Theorem 14, which will be semantically proved later. So let us just give an intuition of a syntactic proof. Suppose that there is a derivation π of $\Rightarrow$ of the form:

$$\dfrac{\begin{matrix}\vdots\;\pi_1\\ \Rightarrow A\end{matrix}\qquad \begin{matrix}\vdots\;\pi_2\\ \Rightarrow B\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\;\pi_{11}\\ A, A \Rightarrow\end{matrix}\qquad \begin{matrix}\vdots\;\pi_{22}\\ B, B \Rightarrow\end{matrix}}{A, B \Rightarrow}\,(\mathsf{pc})}{\Rightarrow}\,(\mathsf{cut})^2$$

It can be reduced in two ways thanks to the empty conclusion:

$$\dfrac{\begin{matrix}\vdots\;\pi_1\\ \Rightarrow A\end{matrix}\qquad \begin{matrix}\vdots\;\pi_1\\ \Rightarrow A\end{matrix}\qquad \begin{matrix}\vdots\;\pi_{11}\\ A, A \Rightarrow\end{matrix}}{\Rightarrow}\,(\mathsf{cut})^2 \quad \text{or} \quad \dfrac{\begin{matrix}\vdots\;\pi_2\\ \Rightarrow B\end{matrix}\qquad \begin{matrix}\vdots\;\pi_2\\ \Rightarrow B\end{matrix}\qquad \begin{matrix}\vdots\;\pi_{22}\\ B, B \Rightarrow\end{matrix}}{\Rightarrow}\,(\mathsf{cut})^2$$

Call them π_1' and π_2', respectively. If $|\pi_1| \leq |\pi_2|$, choose π_1' since $|\pi_1'| = 2|\pi_1| + |\pi_{11}| < |\pi|$. Otherwise choose π_2'. In any case, we obtain a smaller derivation. Although the general case is much more involved, the basic idea is the same: choose a derivation of smaller size in the course of cut elimination.

To sum up, **FLew** is not the only logic consistent with fixed points. It has an axiomatic extension **FLew** $\oplus$ **pc** which is still consistent. It is then natural to look for a *maximal* extension. Neither intuitionistic logic nor finite-valued logics are fine. We are thus led to look into another direction: the realm of (infinite-valued) mathematical fuzzy logics.

3 Mathematical fuzzy logics with fixed points

Mathematical fuzzy logics constitute a huge and diverse research field which has been extensively studied and still is growing. A classic monograph is [15], while more recent handbooks are [8, 9]. We also recommend [24] which provides an excellent proof theoretic account.

After a quick introduction to fuzzy logics (Subsection 3.1), we consider extensions with fixed points and prove some fundamental (in)consistency results (Subsection 3.2). In particular, we prove the consistency of $\mathbf{Ł}_{\mathsf{fp}}$ by employing the Brouwer fixed point theorem (BFT). Conversely, a special case of BFT follows from the consistency of $\mathbf{Ł}_{\mathsf{fp}}$. This provides a good motivation to give an independent, syntactic proof of consistency (Subsection 3.3).

3.1 Basics of mathematical fuzzy logics

Figure 2 displays five major systems of mathematical fuzzy logics, all of which are obtained by adding axioms to **FLew**.

Name	$x * y$	$x \to y$ (when $x > y$)	FLew algebra
Gödel norm	$\min(x, y)$	y	$[0,1]_{\mathbf{G}}$
Łukasiewicz norm	$\max(x + y - 1, 0)$	$1 - x + y$	$[0,1]_{\mathbf{Ł}}$
Product norm	xy	y/x	$[0,1]_{\mathbf{\Pi}}$

Figure 3 Fundamental t-norms. We have $x \to y = 1$ when $x \le y$.

Recall the parallel contraction rule (pc) discussed in Subsection 2.4. It is derivable in **MTL**, because we have the following derivation in **FLew**:

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,(\mathsf{init}) \quad B, B \Rightarrow \Pi}{A \to B, A, B \Rightarrow \Pi}\,(\to \ell) \qquad \dfrac{\dfrac{}{B \Rightarrow B}\,(\mathsf{init}) \quad A, A \Rightarrow \Pi}{B \to A, A, B \Rightarrow \Pi}\,(\to \ell)}{(A \to B) \vee (B \to A), A, B \Rightarrow \Pi}\,(\vee \ell)$$

Hence **MTL** is an extension of **FLew** $\oplus$ **pc**.

Each of the five logics can be semantically characterized in terms of t-norms. A *t-norm* is a binary operation $*$ on the unit interval $[0, 1]$ such that $([0, 1], *, 1)$ is a commutative monoid and $*$ is monotone in both arguments. Suppose that $*$ is further *left continuous*, that is, operation $a * (-) : [0, 1] \to [0, 1]$ preserves arbitrary joins $\bigvee$ for every $a \in [0, 1]$. It then has a right adjoint $a \to (-) : [0, 1] \to [0, 1]$ by the (degenerated) adjoint functor theorem:

$$a \to z \;:=\; \bigvee\{x \in [0, 1] \mid a * x \le z\},$$

leading to an FLew algebra $[0, 1]_* \;:=\; ([0, 1], \min, \max, *, \to, 1, 0)$.

We are mostly interested in (left) continuous t-norms. So let $\mathcal{LCTN} := \{[0, 1]_* \mid * \text{ is left continuous}\}$ and $\mathcal{CTN} := \{[0, 1]_* \mid * \text{ is continuous}\}$. Let us also consider three t-norms in Figure 3. We then have:

▶ **Theorem 10** (Completeness of major fuzzy logics)**.**
Strong completeness holds for the following pairs: $(\mathbf{MTL}, \mathcal{LCTN})$*,* $(\mathbf{G}, [0, 1]_{\mathbf{G}})$*.*
Finite strong completeness holds for the following pairs: $(\mathbf{BL}, \mathcal{CTN})$*,* $(\mathbf{Ł}, [0, 1]_{\mathbf{Ł}})$*,* $(\mathbf{\Pi}, [0, 1]_{\mathbf{\Pi}})$*.*

▶ Remark 11. **BL**, **Ł** and **Π** only satisfy finite strong completeness, because of the badly behaved axiom **div**. On the other hand, **G** can be axiomatized as **Int** $\oplus$ **pl** without using **div**, that is why it satisfies strong completeness.

Note that implication $\to$ is continuous in $[0, 1]_{\mathbf{Ł}}$, while it is discontinuous at $(0, 0)$ in $[0, 1]_{\mathbf{G}}$ and $[0, 1]_{\mathbf{\Pi}}$. In fact, we have $0 \to 0 = 1$, while $x \to 0 = 0$ for any $x > 0$. Since any continuous t-norm is an ordinal sum of Gödel, Łukasiewicz and product norms [26] (see also [15, 8, 24]), a well-known fact is obtained.

▶ **Theorem 12.** *Among the infinitely many t-norms, Łukasiewicz norm is the only one that makes all logical connectives* $\wedge, \vee, \cdot, \to$ *continuous.*

3.2 Fuzzy logics with fixed points

We are now ready to discuss consistency of fuzzy logics with fixed points. Below is a folklore (see, e.g., [16]).

▶ **Proposition 13.** *Both* **G** *and* **Π** *are inconsistent with fixed points.*

Proof. Clearly $\mathbf{G}_{\mathsf{fix}}$ is inconsistent. For $\mathbf{\Pi}$, let $R := \xi p.\neg p$ so that $R \leftrightarrow \neg R$ is a fixed point axiom. Since $[0,1]_{\mathbf{\Pi}}$ satisfies $\neg x = 0$ when $x > 0$ and $\neg x = 1$ when $x = 0$, there is no valuation $v : \mathsf{Fm} \to [0,1]_{\mathbf{\Pi}}$ such that $v(R) = v(\neg R)$. Hence by Theorem 10 and Lemma 3, we obtain $\Phi_{\mathsf{fp}} \vdash_{\mathbf{\Pi}} 0$. ◀

The next result is proved by using BFT.

▶ **Theorem 14.** **Ł** *is consistent with fixed points.*

Proof. We will make use of Lemma 3. So let $\Phi \subseteq \Phi_{\mathsf{fp}}$ be a finite subset, and let $\overline{\Phi} := \{A_1, \dots, A_n\}$ be the set of all subformulas of formulas in Φ, where variable $\xi p.A$ does not have a proper subformula by definition. $\overline{\Phi}$ is still finite.

We now define a function $f = (f_1, \dots, f_n) : [0,1]^n \to [0,1]^n$. For $\overline{x} = (x_1, \dots, x_n) \in [0,1]^n$,

$$\begin{array}{lcll} f_i(\overline{x}) & := & \ell & \text{if } A_i \equiv \ell \text{ for } \ell \in \{0,1\}, \\ & := & x_j \star x_k & \text{if } A_i \equiv A_j \star A_k \text{ for } \star \in \{\wedge, \vee, \cdot, \to\}, \\ & := & x_j & \text{if } A_i \equiv \xi p.B \text{ and } B[\xi p.B/p] \equiv A_j \text{ for some } 1 \leq j \leq n, \\ & := & 0 & \text{if } A_i \text{ is a variable but the above case does not apply.} \end{array}$$

f is continuous by Theorem 12. Hence **by BFT**, it has a fixed point $\overline{c} = (c_1, \dots, c_n) \in [0,1]^n$.

Define a (partial) valuation $v : \overline{\Phi} \to [0,1]_{\text{Ł}}$ by setting $v(A_i) := c_i$. It is homomorphic for each $\star \in \{\wedge, \vee, \cdot, \to\}$. In fact, if $A_i \equiv A_j \star A_k$,

$$v(A_i) \;=\; c_i \;=\; f_i(\overline{c}) \;=\; c_j \star c_k \;=\; v(A_j) \star v(A_k).$$

Moreover, v satisfies Φ. Suppose $A_i \equiv \xi p.B$ and $A_j \equiv B[\xi p.B/p]$. Then:

$$v(\xi p.B) \;=\; c_i \;=\; f_i(\overline{c}) \;=\; c_j \;=\; v(B[\xi p.B/p]).$$

v can be extended to a total valuation $v' : \mathsf{Fm} \to [0,1]_{\text{Ł}}$ by setting $v'(p) := 0$ for $p \notin \overline{\Phi}$ and homomorphically extending it to Fm. By Theorem 10 and Lemma 3, we conclude $\Phi_{\mathsf{fp}} \nvdash_{\text{Ł}} 0$. ◀

So all logics $\mathrm{L} \subseteq$ **Ł** are consistent with fixed points too. What then about extensions? The answer is negative. In Appendix A.2, we prove:

▶ **Theorem 15.** **Ł** *is a maximal logic in* $\mathsf{Ext}(\mathbf{FLew})$ *consistent with fixed points.*

▶ Remark 16. There are other fixed point extensions of **Ł** in the literature, but as far as we know, all of them impose certain restrictions on the application of the fixed point operator. Concretely, [31, 32, 10] does not allow nested fixed points motivated by the fact that an application of BFT may cause discontinuity, while [25] applies the operators only to monotone formulas just as in modal μ-calculus. In contrast, we do not impose any such restriction. Our proof of consistency is still correct, since we use BFT only once and for all, no matter how fixed point operators are nested in a formula.

3.3 Consistency of Ł$_{\mathsf{fp}}$ implies BFT for quasi-McNaughton functions

We have shown that BFT implies the consistency of **Ł**$_{\mathsf{fp}}$. It is also possible to give a converse direction to some extent.

▶ **Definition 17.** *A* McNaughton function $f : [0,1]^n \to [0,1]$ *is a continuous, piecewise linear function with integer coefficients. More precisely,* f *is McNaughton if it is continuous and there are linear functions* $g_i(\overline{x}) = \overline{a}_i\overline{x} + b_i$ *with* $\overline{a}_i, b_i \in \mathbb{Z}$ *(*$i = 1, \dots, k$*) such that for every* $\overline{x} \in [0,1]$*,* $f(\overline{x}) = g_i(\overline{x})$ *holds for some* $1 \leq i \leq k$*.*

A formula $A = A(p_1, \dots, p_n)$ in Fm *represents* a function $f : [0,1]^n \to [0,1]$ if for any valuation $v : \mathsf{Fm} \to [0,1]_{\text{Ł}}$ we have $f(v(p_1), \dots, v(p_n)) = v(A)$. The following result is classic.

▶ **Theorem 18** (McNaughton representation theorem [22]). *A function $f : [0,1]^n \to [0,1]$ is McNaughton if and only if it is represented by a formula in* Fm.

In addition to McNaughton functions, rational numbers are also representable in Ł_{fp}.

▶ **Lemma 19.** *For every rational number $b \in (0,1]$, there is a formula $B \in \mathsf{Fm}$ such that $v(\xi p.B) = v(B[\xi p.B/p])$ implies $v(\xi p.B) = b$ for every valuation $v : \mathsf{Fm} \to [0,1]_{\text{Ł}}$.*

Proof. Suppose that $b = \frac{m}{n}$ for natural numbers m, n with $1 \le m \le n$. Define a McNaughton function $f : [0,1] \to [0,1]$ by $f(x) := (1-n)x + m$ (for $x \ge \frac{m-1}{n-1}$) and $f(x) := 1$ (otherwise). Then there is a formula $B(p)$ representing f by Theorem 18. Let $Q := \xi p.B(p)$. If valuation v satisfies $v(Q) = v(B(Q)) = f(v(Q))$, then $v(Q) \ge \frac{m-1}{n-1}$, since otherwise $f(v(Q)) = 1$ and $v(Q) < \frac{m-1}{n-1} \le 1$. Hence $v(Q) = (1-n)v(Q) + m$, from which we obtain $v(Q) = \frac{m}{n}$. ◀

Hence one can represent slightly more than McNaughton functions in Ł_{fp}. A *quasi-McNaughton function* $f : [0,1]^n \to [0,1]$ is obtained from a McNaughton function $g : [0,1]^{n+m} \to [0,1]$ by letting $f(\overline{x}) := g(\overline{x}, \overline{b})$ for some rational numbers $\overline{b} \in [0,1]^m$.

We are now ready to derive a special case of BFT from the consistency of Ł_{fp}.

▶ **Theorem 20.** *Every tuple $f = (f_1, \dots, f_n) : [0,1]^n \to [0,1]^n$ of quasi-McNaughton functions has a fixed point.*

Proof. Suppose that $f(\overline{x}) = g(\overline{x}, \overline{b})$ holds for a tuple of McNaughton functions $g = (g_1, \dots, g_n) : [0,1]^{n+m} \to [0,1]^n$ and rational numbers $\overline{b} = (b_1, \dots, b_m) \in [0,1]^m$. By Theorem 18, there are formulas $\overline{A} = A_1, \dots, A_n$ representing $g_1, \dots, g_n$. There are also formulas $\overline{B} = B_1, \dots, B_m$ representing $\overline{b}$ in the above sense.

Let $\overline{P}$ and $\overline{Q}$ be simultaneous fixed points of $\overline{A}, \overline{B}$, that is,

$$P_i \leftrightarrow A_i(\overline{P}, \overline{Q}), \quad Q_j \leftrightarrow B_j(Q_j) \tag{4}$$

is provable from Φ_{fp} in Ł for every $1 \le i \le n$ and $1 \le j \le m$ (see Remark 6). Clearly a finite set $\Phi \subseteq \Phi_{\mathsf{fp}}$ suffices to prove them all. **By the consistency of Ł_{fp}**, $\Phi \not\vdash_{\text{Ł}} 0$. Hence there is a valuation $v : \mathsf{Fm} \to [0,1]_{\text{Ł}}$ that satisfies Φ by Lemma 3. By soundness, v satisfies the formulas in (4) as well. Therefore, $v(Q_j) = b_j$ by Lemma 19, and $v(P_i) = v(A_i(\overline{P}, \overline{Q})) = g_i(v(\overline{P}), v(\overline{Q})) = g_i(v(\overline{P}), \overline{b}) = f_i(v(\overline{P}))$ for every $1 \le i \le n$. We have thus obtained a fixed point of $f = (f_1, \dots, f_n)$. ◀

If it is the case that every continuous function $f : [0,1]^n \to [0,1]$ can be approximated by a sequence $\{f_i\}_{i<\omega}$ of quasi-McNaughton functions, then the full BFT will follow. But we are not sure about this. Even if this is true, it does not mean that the consistency of Ł_{fp} is equivalent to BFT, and hence to WKL_0 (weak König lemma), in the sense of reverse mathematics, as correctly pointed out in the appendix of [10].

Still, Theorem 20 motivates us to prove the consistency of Ł_{fp} independently of BFT, because, in the author's opinion, BFT is not as easy as the Knaster-Tarski or the Banach fixed point theorems in that it requires a clever combinatorial/topological idea such as Sperner's lemma or algebraic topology. Hence we expect that it will be an equally clever proof if one manages to find a consistency proof for Ł_{fp} in proof theory. That is an exciting challenge for proof theorists, isn't it?

▶ Remark 21. Although there are some proof systems for **Ł** with cut elimination [23, 24], none of them seem to work well with fixed points. This might be related to a fundamental limitation on the proof theory of **Ł** imposed by an algebraic fact: the variety $\mathsf{Mod}(\mathbf{Ł})$ of MV algebras is not closed under completions [20]. The latter is equivalent to saying that **Ł** does not admit conservative extension (in a strong sense) with infinitary conjunction $\bigwedge$ and disjunction $\bigvee$. That is, there cannot be a proof system for **Ł** that allows us to eliminate any intermediate use of $\bigwedge$ and $\bigvee$ in a derivation. See [6, 7], where this point is discussed in more depth.

4 Naive set theories over substructural logics

Let us now turn to naive set theories. We first introduce a naive set theory over **FLew** and discuss its expressive power. In particular, we discuss how to develop elementary arithmetic within $\mathbf{FLew}_{\mathsf{set}}$ with the aid of the cut elimination theorem and the recursion theorem (Subsection 4.1). We then proceed to our main target: Cantor-Łukasiewicz set theory $\mathbf{Ł}_{\mathsf{set}}$ (Subsection 4.2). Although its consistency is still open, we consider a restricted fragment $\mathbf{Ł}^-_{\mathsf{set}}$ and prove consistency by employing the Brouwer fixed point theorem (Subsection 4.3). A comparison with prior work is made in Remark 32.

4.1 Naive set theory $\mathbf{FLew}_{\mathsf{set}}$

We here collect some basic facts on the naive set theory over **FLew**, which are mostly known but scattered in the literature. See in particular [1, 33], which contain proofs of most theorems below.

Let $\mathsf{Var}_{\mathsf{set}} := \{x_0, x_1, x_2, \dots\}$ be a fixed countable set of *term variables*. The set $\mathsf{Tm}_{\mathsf{set}}$ of *terms* and the set $\mathsf{Fm}_{\mathsf{set}}$ of *formulas* are simultaneously defined by the following grammar:

$$\begin{array}{lcl} t, u & ::= & x \mid \{x \mid A\}, \\ A, B & ::= & t \in u \mid 1 \mid 0 \mid A \wedge B \mid A \vee B \mid A \cdot B \mid A \to B, \end{array}$$

where variable x is bound in formula $\forall x.A$ and in term $\{x \mid A\}$. Existential quantifier is defined by $\exists y.A := \forall x.\,(\forall y.\,(A \to x^\bullet)) \to x^\bullet$, where x is fresh and $x^\bullet := \bullet \in x$ for a fixed closed term $\bullet$. In the sequel, we will also use notation $A \otimes B$ instead of $A \cdot B$ for readability.

The sequent calculus for $\mathbf{FLew}_{\mathsf{set}}$ consists of the inference rules in Figure 1 together with:

$$\frac{A[t/x], \Delta \Rightarrow \Pi}{t \in \{x|A\}, \Delta \Rightarrow \Pi}\ (\in \ell) \qquad \frac{\Gamma \Rightarrow A[t/x]}{\Gamma \Rightarrow t \in \{x|A\}}\ (\in r) \qquad \frac{A[t/x], \Delta \Rightarrow \Pi}{\forall x.A, \Delta \Rightarrow \Pi}\ (\forall \ell) \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \forall x.A}\ (\forall r)$$

where rule $(\forall r)$ is subject to the eigenvariable condition.

Given a set $\Phi \cup \{A\} \subseteq \mathsf{Fm}_{\mathsf{set}}$, we write $\Phi \vdash_{\mathbf{FLew}_{\mathsf{set}}} A$ if A^{seq} is derivable from Φ^{seq} by the above inference rules. Given $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$, let $\tilde{\mathrm{L}} \subseteq \mathsf{Fm}_{\mathsf{set}}$ be the set of all substitution instances of propositional formulas in L (in the language $\mathsf{Fm}_{\mathsf{set}}$). We define $\mathrm{L}_{\mathsf{set}} := \{A \in \mathsf{Fm}_{\mathsf{set}} \mid \tilde{\mathrm{L}} \vdash_{\mathbf{FLew}_{\mathsf{set}}} A\}$, and write $\vdash_{\mathrm{L}_{\mathsf{set}}} A$ if $A \in \mathrm{L}_{\mathsf{set}}$.

As is expected, one can define the fixed point operator $\xi p.A$ in $\mathbf{FLew}_{\mathsf{set}}$. Let $A(p)$ be a formula which tentatively contains a propositional variable p. Defining $r := \{x \mid A(x \in x)\}$ and $\xi p.A(p) := r \in r$, we obtain

$$\xi p.A(p) \ \equiv\ r \in r \ \leftrightarrow\ A(r \in r) \ \equiv\ A(\xi p.A),$$

which we call a *Russellian fixed point*. Hence one can translate each formula A in $\mathbf{FLew}_{\mathsf{fp}}$ to a formula $A^{\in}$ in $\mathbf{FLew}_{\mathsf{set}}$. Let us record this fact for later reference.

▶ **Theorem 22** (Russelian fixed points). *For every* $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$, $\vdash_{\mathrm{L}_{\mathsf{fp}}} A$ *implies* $\vdash_{\mathrm{L}_{\mathsf{set}}} A^{\in}$. *As a consequence, the consistency of* $\mathrm{L}_{\mathsf{set}}$ *implies that of* L_{fp}.

Turning to proof thoery, note that rules $(\in \ell)$, $(\in r)$ of $\mathbf{FLew}_{\mathsf{set}}$ are symmetric just as rules $(\mathsf{fp}\ell)$, $(\mathsf{fp}r)$ of $\mathbf{FLew}_{\mathsf{fp}}$. Hence a principal cut on $t \in u$ can be easily reduced:

$$\dfrac{\dfrac{\Gamma \Rightarrow A[t/x]}{\Gamma \Rightarrow t \in \{x|A\}}\,(\in r) \quad \dfrac{A[t/x], \Delta \Rightarrow \Pi}{t \in \{x|A\}, \Delta \Rightarrow \Pi}\,(\in \ell)}{\Gamma, \Delta \Rightarrow \Pi}\,(\mathsf{cut}) \quad \Longrightarrow \quad \dfrac{\Gamma \Rightarrow A[t/x] \quad A[t/x], \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi}\,(\mathsf{cut})$$

▶ **Theorem 23** ([14]). $\mathbf{FLew}_{\mathsf{set}}$ *enjoys cut elimination, hence is consistent.*

Due to the absence of the right contraction rule, our system is of constructive nature.

▶ **Corollary 24.** $\mathbf{FLew}_{\mathsf{set}}$ *has the disjunction property and the existence property:*

- $\vdash_{\mathbf{FLew}_{\mathsf{set}}} A \vee B \implies \vdash_{\mathbf{FLew}_{\mathsf{set}}} A$ *or* $\vdash_{\mathbf{FLew}_{\mathsf{set}}} B$,
- $\vdash_{\mathbf{FLew}_{\mathsf{set}}} \exists x.A \implies \vdash_{\mathbf{FLew}_{\mathsf{set}}} A[t/x]$ *for some term* t.

In $\mathbf{FLew}_{\mathsf{set}}$, equality can be defined by $t = u \iff \forall x.(t \in x \to u \in x)$. This definition, known as *Leibniz equality*, works fine at least superficially, as one proves reflexivity, symmetricity, transitivity, substitution invariance $(t = u \to A(t) \to A(u))$, and contraction $(t = u \to t = u \otimes t = u)$. Hence equational reasoning can be done as in intuitionistic logic.

Furthermore, it allows us to build various basic sets:

$$\begin{array}{lcllcl} \emptyset & := & \{x \mid 0\}, & \{t, u\} & := & \{x \mid x = t \vee x = u\}, \\ \langle t, u\rangle & := & \{\{t\}, \{t, u\}\}, & t \cup u & := & \{x \mid x \in t \vee x \in u\}. \end{array}$$

It looks ok, but the cut elimination theorem reveals that Leibniz equality is too strict.

▶ **Theorem 25** ([33]). $\vdash_{\mathbf{FLew}_{\mathsf{set}}} t = u$ *if and only if* t *and* u *are syntactically identical.*

Proof. If $\vdash_{\mathbf{FLew}_{\mathsf{set}}} t = u$, one obtains a derivation of $t \in x \Rightarrow u \in x$ by inversion. The cut elimination theorem implies that it is a conclusion of (init). Hence t and u must be identical. ◀

Thus, one cannot even prove $t \cup u = u \cup t$ unless extensionality is further incorporated. However, it turns out that $\mathbf{FLew}$ is inconsistent with the extensionality axiom [14, 1]. Hence it is hopeless to regard $\mathbf{FLew}_{\mathsf{set}}$ as an alternative of a ZF-like set theory. That is so bad, but there is a good news too.

▶ **Theorem 26** (recursion theorem [13, 1]). *For every formula* $A(x, y)$, *there is a term* T *such that* $\vdash_{\mathbf{FLew}_{\mathsf{set}}} \forall x.(x \in T \leftrightarrow A(x, T))$.

Proof. Just define $Q(y) := \{w \mid \langle w, y\rangle \in y\}$, $S := \{z \mid \exists x.\exists y.\, \langle x, y\rangle = z \otimes A(x, Q(y))\}$ and $T := Q(S)$. ◀

The recursion theorem allows us to develop some little portion of arithmetic. Let N be a term such that $\mathbf{FLew}_{\mathsf{set}}$ proves

$$\forall x.(x \in \mathsf{N} \;\leftrightarrow\; x = \mathsf{0} \vee \exists y \in \mathsf{N}.\, x = \mathsf{s}(y)),$$

where $\mathsf{0} := \emptyset$, $\mathsf{s}(y) := \langle \emptyset, y\rangle$, and $\exists y \in \mathsf{N}.\, A$ is an abbreviation of $\exists y.\, y \in \mathsf{N} \otimes A$. Term N then works as the set of natural numbers. Let us call a term of the form $\mathsf{s}(\cdots \mathsf{s}(\mathsf{0}) \cdots)$ a *numeral.*

▶ **Theorem 27** ([33]). $\vdash_{\mathbf{FLew}_{\mathsf{set}}} t \in \mathsf{N}$ *if and only if* t *is a numeral.*

One can further develop arithmetic with the aid of the recursion theorem. Indeed, there are terms add and mult such that

$$\begin{aligned}\langle x, y, z\rangle \in \mathsf{add} &\leftrightarrow (y = \mathsf{0} \otimes x = z) \vee \exists y', z'.\, y = \mathsf{s}(y') \otimes \langle x, y', z'\rangle \in \mathsf{add} \otimes z = \mathsf{s}(z'),\\ \langle x, y, z\rangle \in \mathsf{mult} &\leftrightarrow (y = \mathsf{0} \otimes z = \mathsf{0}) \vee \exists y', z'.\, y = \mathsf{s}(y') \otimes \langle x, y', z'\rangle \in \mathsf{mult} \otimes \langle x, z', z\rangle \in \mathsf{add}.\end{aligned}$$

These terms represent addition and multiplication for numerals. This leads to a translation of any first order arithmetical formula A into a formula $A^\star$ in $\mathsf{Fm}_{\mathsf{set}}$. For instance, one translates $(\exists x.A)^\star := \exists x \in \mathsf{N}.A^\star$.

▶ **Theorem 28** (Σ_1 completeness)**.** *For every Σ_1 sentence A of arithmetic, A is true if and only if $\vdash_{\mathbf{FLew}_{\mathsf{set}}} A^\star$.*

As a consequence, provability in $\mathbf{FLew}_{\mathsf{set}}$ is undecidable [1].

It is routine to prove the completeness direction, while the soundness is basically due to Corollary 24, Theorem 25 and Theorem 27. Remarkably, all of them are consequences of the cut elimination theorem.

Note that term N does not admit mathematical induction, since it is not defined as a least fixed point. By lack of induction, the arithmetical strength of $\mathbf{FLew}_{\mathsf{set}}$ is severely limited.

▶ Remark 29. Let us mention that there is a computationally meaningful extension of $\mathbf{FLew}_{\mathsf{set}}$: *light affine set theory* **LAST** [13, 33]. The basic idea is to equip $\mathbf{FLew}_{\mathsf{set}}$ with two modalities ! and §, the former controlling contraction (as in linear logic) and the latter bounding !. The set theory precisely captures the polynomial time functions: a function is polynomial time computable if and only if it is provably total in **LAST**.

4.2 Cantor-Łukasiewicz set theory and its fragment

We finally address $\mathbf{Ł}_{\mathsf{set}}$, known as Cantor-Łukasiewicz set theory.[1] For the consistency of this theory, there are some partial results by Skolem [30], Chang [2], and Fenstad [11]. They introduced their own fragments of $\mathbf{Ł}_{\mathsf{set}}$ and proved consistency. We will make a comparison with our work in Remark 32.

We must also mention a paper by White [35], where he “proved” the consistency of full $\mathbf{Ł}_{\mathsf{set}}$. He introduced a rather eccentric natural deduction system for $\mathbf{Ł}_{\mathsf{set}}$, and then “proved” a normalization theorem for that. The result had been naively believed correct for a long while, until the author found a gap and circulated a note [34]. The gap is evident to the author, and seems to be more or less accepted among fuzzy logicians [17].

In the sequel, we introduce a new fragment $\mathbf{Ł}^-_{\mathsf{set}}$ and prove its consistency. To motivate the definition of $\mathbf{Ł}^-_{\mathsf{set}}$, we first remark that there are at least three approaches to tackle the consistency problem.

1. **By proof theory**. In spite of the failure of [35], it is still possible to give a syntactic proof. A main obstacle is, however, lack of a “good” proof system for **Ł** (see Remark 21).
2. **By BFT**. This is also possible, but there are again hard obstacles. Recall that the consistency of $\mathbf{Ł}_{\mathsf{fp}}$ was successfully proved mainly because there were only finitely many (sub)formulas involved. This allowed us to work on a hypercube $[0, 1]^n$ of finite dimension, to which BFT applies directly. In contrast, $\mathbf{Ł}_{\mathsf{set}}$ has quantifiers, meaning that a hypercube $[0, 1]^\omega$ of infinite dimension is needed. Even more problematic is the interpretation of

[1] It is called *weak* Cantor-Łukasiewicz set theory in [16, 37], while Cantor-Łukasiewicz set theory itself refers to a stronger system extended by arithmetical induction, which turns out to be inconsistent. We here follow the terminology in [8], which is becoming the standard.

quantifiers. A natural algebraic way of interpreting $\forall$ is by an infinite meet $\bigwedge$. However, it easily breaks continuity, a key property that makes everything work.
Although there are various generalizations of Brouwer's theorem such as Schauder's one, we have not yet encountered a suitable one that helps us solve the problem (in spite of suggestions by several colleagues).

3. **By proving inconsistency**. This is not ridiculous, as it is quite possible that $\mathbf{Ł}_{\mathsf{set}}$ is inconsistent. Hence this direction has to be pursued too. In passing, let us mention an interesting inconsistency result. If we extend $\mathbf{Ł}_{\mathsf{set}}$ with mathematical induction on N, the resulting system is inconsistent [16, 37].

What we will now take is the second approach. To apply the BFT argument, we have to find a suitable fragment of $\mathbf{Ł}_{\mathsf{set}}$, for which a hypercube of finite dimension suffices. The following fragment meets this requirement.

▶ **Definition 30.** *System $\mathbf{FLew}^{-}_{\mathsf{set}}$ is obtained from $\mathbf{FLew}_{\mathsf{set}}$ by imposing the following restriction:*

(*) when applying rule $(\forall\ell)$, t must be either a variable or a closed term.

We define $\mathbf{Ł}^{-}_{\mathsf{set}} := \{A \in \mathsf{Fm}_{\mathsf{set}} \mid \tilde{\mathbf{Ł}} \vdash_{\mathbf{FLew}^{-}_{\mathsf{set}}} A\}$.

Before getting into $\mathbf{Ł}^{-}_{\mathsf{set}}$, let us make some comments on $\mathbf{FLew}^{-}_{\mathsf{set}}$. Even with the restriction (*), $\mathbf{FLew}^{-}_{\mathsf{set}}$ still enjoys the cut elimination theorem and the recursion theorem restricted to two-variable formulas $A(x, y)$. The latter is sufficient for developing arithmetic. Hence all arithmetic results in Subsection 4.1 hold for $\mathbf{FLew}^{-}_{\mathsf{set}}$ too. In particular, $\mathbf{FLew}^{-}_{\mathsf{set}}$ is Σ_1 complete.

On the other hand, there is a serious difficulty in proving a Π_2 sentence. In fact, one cannot even prove an obvious fact $\forall x.\exists y.\, y = \mathsf{s}(x)$ in $\mathbf{FLew}^{-}_{\mathsf{set}}$, since inferring $\Rightarrow \exists y.y = \mathsf{s}(x)$ from $\Rightarrow \mathsf{s}(x) = \mathsf{s}(x)$ is prohibited by condition (*).

4.3 Consistency of $\mathbf{Ł}^{-}_{\mathsf{set}}$

Let us now prove the consistency of $\mathbf{Ł}^{-}_{\mathsf{set}}$. We begin with some preparations. Suppose that a finite set $\Phi \subseteq \mathsf{Fm}_{\mathsf{set}} \cup \mathsf{Tm}_{\mathsf{set}}$ of formulas and terms is given. We consider its closure $\overline{\Phi}$ under subformulas and subterms. Explicitly, $\overline{\Phi}$ is the least set such that (i) $\Phi \subseteq \overline{\Phi}$, (ii) $A \star B \in \overline{\Phi}$ implies $A, B \in \overline{\Phi}$ for $\star \in \{\wedge, \vee, \cdot, \to\}$, (iii) $\forall x.A \in \overline{\Phi}$ implies $A \in \overline{\Phi}$, (iv) $(t \in u) \in \overline{\Phi}$ implies $t, u \in \overline{\Phi}$, and (v) $\{x \mid A\} \in \overline{\Phi}$ implies $A \in \overline{\Phi}$.

Let Φ_{ctm} be the set of closed terms in $\overline{\Phi}$, and let

$$\Phi_{\mathsf{cfm}} := \{A^{\sigma} \in \mathsf{Fm}_{\mathsf{set}} \mid A \in \overline{\Phi},\ \sigma : \mathsf{Var}_{\mathsf{set}} \to \Phi_{\mathsf{ctm}}\},$$

where A^{σ} is a closed formula obtained by applying substitution σ to A. As is easily observed, $\forall x.B \in \Phi_{\mathsf{cfm}}$ implies $B[t/x] \in \Phi_{\mathsf{cfm}}$ for every $t \in \Phi_{\mathsf{ctm}}$. Since $\overline{\Phi}$ is a finite set, so are Φ_{ctm} and Φ_{cfm}.

Suppose that $\Phi_{\mathsf{cfm}} = \{A_1, \dots, A_n\}$. We define a function $f = (f_1, \dots, f_n) : [0,1]^n \to [0,1]^n$ as in the proof of Theorem 14. Concretely, given $\overline{x} = (x_1, \dots, x_n) \in [0,1]^n$, let

$$\begin{array}{lcll}
f_i(\overline{x}) & := & \ell & \text{if } A_i \equiv \ell \quad (\ell = 0, 1),\\
 & := & x_j \star x_k & \text{if } A_i \equiv A_j \star A_k \quad (\star \in \{\wedge, \vee, \cdot, \to\}),\\
 & := & \bigwedge_{t \in \Phi_{\mathsf{ctm}}} x_{j_t} & \text{if } A_i \equiv \forall x.B \text{ and } B[t/x] \equiv A_{j_t} \text{ for each } t \in \Phi_{\mathsf{ctm}}\ (1 \leq j_t \leq n),\\
 & := & x_j & \text{if } A_i \equiv t \in \{x \mid B\} \text{ and } B[t/x] \equiv A_j \text{ for some } 1 \leq j \leq n,\\
 & := & 0 & \text{if } A_i \equiv t \in u \text{ but the previous case does not apply.}
\end{array}$$

Then f is continuous by Theorem 12 and finiteness of Φ_{ctm}. **By BFT**, there is $\overline{c} = (c_1, \ldots, c_n) \in [0,1]^n$ such that $f(\overline{c}) = \overline{c}$. This gives us a function $v : \Phi_{\mathsf{cfm}} \to [0,1]_{\text{Ł}}$ defined by $v(A_i) := c_i$.

v is homomorphic with respect to propositional connectives $1, 0, \wedge, \vee, \cdot, \to$ as in the proof of Theorem 14. Moreover, if $A_i \equiv t \in \{x \mid B\}$ and $A_j \equiv B[t/x]$, we have

$$v(t \in \{x \mid B\}) \;=\; c_i \;=\; f_i(\overline{c}) \;=\; c_j \;=\; v(B[t/x]), \tag{5}$$

so v validates rules $(\in \ell)$ and $(\in r)$ applied to formulas in Φ_{cfm}. Finally if $A_i \equiv \forall x.B$ and $A_{j_t} \equiv B[t/x]$ for each $t \in \Phi_{\mathsf{ctm}}$, we have

$$v(\forall x.B) \;=\; c_i \;=\; f_i(\overline{c}) \;=\; \bigwedge_{t \in \Phi_{\mathsf{ctm}}} c_{j_t} \;=\; \bigwedge_{t \in \Phi_{\mathsf{ctm}}} v(B[t/x]). \tag{6}$$

We are now ready to give a consistency proof.

▶ **Theorem 31.** $\text{Ł}^{-}_{\mathsf{set}}$ *is consistent.*

Proof. Suppose that $\text{Ł}^{-}_{\mathsf{set}}$ is inconsistent. This means that there are a finite set of formulas $\Psi \subseteq \tilde{\text{Ł}}$ and a derivation π of $\Rightarrow 0$ from Ψ^{seq}. Let Φ be the finite set of all formulas occurring in π, and define Φ_{ctm}, Φ_{cfm} and valuation $v : \Phi_{\mathsf{cfm}} \to [0,1]_{\text{Ł}}$ as above.

We prove by induction on the derivation the following:

- if sequent $\Gamma \Rightarrow \Pi$ occurs in π and $\sigma : \mathsf{Var}_{\mathsf{set}} \to \Phi_{\mathsf{ctm}}$, then $v(\Gamma^\sigma) \leq v(\Pi^\sigma)$.

Here $v(\Gamma) := v(A_1) \cdots v(A_m)$ if $\Gamma \equiv A_1, \ldots, A_m$, and $v(\Gamma) := 1$ if Γ is empty. Likewise, $v(\Pi) := v(A)$ if $\Pi \equiv \{A\}$, and $v(\Pi) := 0$ otherwise.

- If the derivation just consists of $\Rightarrow B$ with $B \in \Psi$, then we have $1 \leq v(B^\sigma)$ as B^σ is a substitution instance of a propositional formula in **Ł**.
- If the derivation ends with a rule for a propositional connective or $(\in \ell)$ or $(\in r)$, the claim easily follows from the induction hypothesis and (5).
- Suppose that the derivation ends with rule $(\forall r)$ inferring $\Gamma \Rightarrow \forall x.B$ from $\Gamma \Rightarrow B$. Given $\sigma : \mathsf{Var}_{\mathsf{set}} \to \Phi_{\mathsf{ctm}}$, let σ' be a substitution that agrees with σ on $\mathsf{Var}_{\mathsf{set}} \backslash \{x\}$ but leaves x unchanged. By the induction hypothesis and the eigenvariable condition, we have $v(\Gamma^\sigma) = v(\Gamma^{\sigma'}) \leq v(B^{\sigma'}[t/x])$ for every $t \in \Phi_{\mathsf{ctm}}$. Hence by (6),

  $$v(\Gamma^\sigma) \leq \bigwedge_{t \in \Phi_{\mathsf{ctm}}} v(B^{\sigma'}[t/x]) = v(\forall x.B^{\sigma'}) = v((\forall x.B)^\sigma).$$

- Now suppose that the derivation ends with rule $(\forall \ell)$ inferring $\forall x.B, \Gamma \Rightarrow \Pi$ from $B[t/x], \Gamma \Rightarrow \Pi$. By the induction hypothesis we have $v(B[t/x]^\sigma) \cdot v(\Gamma^\sigma) \leq v(\Pi^\sigma)$. If t is closed, then $t^\sigma = t \in \Phi_{\mathsf{ctm}}$. If t is a variable y, then $t^\sigma = \sigma(y) \in \Phi_{\mathsf{ctm}}$. In any case, we have $t^\sigma \in \Phi_{\mathsf{ctm}}$. So

  $$v((\forall x.B)^\sigma) = v(\forall x.B^{\sigma'}) = \bigwedge_{u \in \Phi_{\mathsf{ctm}}} v(B^{\sigma'}[u/x]) \leq v(B^{\sigma'}[t^\sigma/x]) = v(B[t/x]^\sigma),$$

  where σ' is as above. Hence we conclude $v((\forall x.B)^\sigma) \cdot v(\Gamma^\sigma) \leq v(\Pi^\sigma)$.

This is a contradiction because derivation π ends with sequent $\Rightarrow 0$. ◀

▶ Remark 32 (Comparison with prior work). Unlike [16] and this work, set theories in [30, 2, 11] are based on the first order Łukasiewicz logic with binary predicates $\in$ and $=$, but without abstraction term $\{x \mid A\}$. They consider comprehension axioms of the form:

$$\mathbf{comp}(A) \;:=\; \forall \overline{x}.\, \exists y.\, \forall z.\, (z \in y \leftrightarrow A(\overline{x}, y, z)),$$

where all free variables of A are indicated. Note that y may occur in A, so it subsumes the recursion theorem. They then prove consistency under the following restrictions on A.

- Skolem [30]: A is quantifier-free and y does not occur in A.
- Chang(a) [2]: A is parameter-free, that is, the list $\overline{x}$ is empty.
- Chang(b) [2]: A does not contain an atomic formula $u \in v$ with $u \not\equiv v$ and u a bound variable.
- Fenstad [11]: A does not contain an atomic formula $u \in z$

Our $\mathbf{Ł}^-_{\mathsf{set}}$ proves $\mathbf{comp}(A)$ precisely when A satisfies Chang(a). However, this does not mean that our result is subsumed by [2], because we can freely use $t \in \{x \mid A\} \leftrightarrow A[t/x]$ without any restriction on A (althoguh use of rule $(\forall\ell)$ is restricted). In particular, Russellian fixed points (Theorem 22) are available in $\mathbf{Ł}^-_{\mathsf{set}}$ in contrast to Chang(a).

It is hard to tell which system is most natural from an objective perspective, but in the author's opinion, Skolem is too weak, while Chang(b) and Fenstad are somewhat artificial. Chang(a) is a natural fragment, but it does not admit Russellian fixed points.

5 Conclusion

We have proved the consistency of four systems: $\mathbf{FLew}_{\mathsf{fp}}$, $\mathbf{FLew}_{\mathsf{set}}$, $\mathbf{Ł}_{\mathsf{fp}}$ and $\mathbf{Ł}^-_{\mathsf{set}}$. The first two are proved by cut elimination, while the other two are by the Brouwer fixed point theorem. We have motivated a syntactic proof of the consistency of $\mathbf{Ł}_{\mathsf{fp}}$ by pointing out that it reversely implies a special case of BFT. We have also established that $\mathbf{Ł}$ is a maximal logic in $\mathsf{Ext}(\mathbf{FLew})$ which is consistent with fixed points. This, of course, does not exclude the existence of other maximals. This could be an interesting direction to pursue.

It might be conceptually interesting to point out that $\mathbf{Ł}^-_{\mathsf{set}}$ admits encoding of $\mathbf{Ł}_{\mathsf{fp}}$ by means of *Russellian fixed points*, while the consistency of $\mathbf{Ł}^-_{\mathsf{set}}$ relies on *Brouwerian fixed points*. Altogether we could say that Russellian fixed points are justified by Brouwerian ones.

When defining $\mathbf{Ł}^-_{\mathsf{set}}$, our approach is to restrict *logic*, in contrast to prior work [30, 2, 11] which restrict *comprehension*. We believe that our approach is closer to the original type-free approach of Church [3].

Let us conclude by restating the main open problem: *is Cantor-Łukasiewicz set theory* $\mathbf{Ł}_{\mathsf{set}}$ *consistent?*

References

1 Andrea Cantini. The undecidability of Grišin's set theory. *Studia Logica*, 74(3):345–368, 2003. doi:10.1023/A:1025159016268.

2 C. C. Chang. The axiom of comprehension in infinite valued logic. *Math. Scand.*, 13:9–30, 1963. doi:10.7146/math.scand.a-10685.

3 Alonzo Church. A set of postulates for the foundation of logic. *Ann. of Math. (2)*, 33(2):346–366, 1932. doi:10.2307/1968337.

4 Alonzo Church. A set of postulates for the foundation of logic. *Ann. of Math. (2)*, 34(4):839–864, 1933. doi:10.2307/1968702.

5 Alonzo Church. A proof of freedom from contradiction. In *Proceedings of the National Academy of Sciences*, volume 21 (5), pages 275–281, 1935. doi:10.1073/pnas.21.5.275.

6 Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. Algebraic proof theory for substructural logics: cut-elimination and completions. *Ann. Pure Appl. Logic*, 163(3):266–290, 2012. doi:10.1016/j.apal.2011.09.003.

7 Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui. Algebraic proof theory: hypersequents and hypercompletions. *Ann. Pure Appl. Logic*, 168(3):693–737, 2017. doi:10.1016/j.apal.2016.10.012.

8 Petr Cintula, Petr Hájek, and Carles Noguera, editors. *Handbook of mathematical fuzzy logic. Volume 1*, volume 37 of *Studies in Logic (London)*. College Publications, London, 2011. Mathematical Logic and Foundations.

9 Petr Cintula, Petr Hájek, and Carles Noguera, editors. *Handbook of mathematical fuzzy logic. Volume 2*, volume 38 of *Studies in Logic (London)*. College Publications, London, 2011. Mathematical Logic and Foundations.

10 Abhishek De. A Note on Fixed Points in Łukasiewicz Logic. *Log. Univers.*, 19(4):603–618, 2025. doi:10.1007/s11787-025-00395-1.

11 Jens Erik Fenstad. On the consistency of the axiom of comprehension in the Łukasiewicz infinite valued logic. *Math. Scand.*, 14:65–74, 1964. doi:10.7146/math.scand.a-10706.

12 Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono. *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2007.

13 Jean-Yves Girard. Light linear logic. *Inform. and Comput.*, 143(2):175–204, 1998. doi:10.1006/inco.1998.2700.

14 V. N. Grišin. Predicate and set-theoretic calculi based on logic without the contraction rules. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(1):47–68, 239, 1981.

15 Petr Hájek. *Metamathematics of fuzzy logic*, volume 4 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 1998. doi:10.1007/978-94-011-5300-3.

16 Petr Hájek. On arithmetic in the Cantor-Łukasiewicz fuzzy set theory. *Arch. Math. Logic*, 44(6):763–782, 2005. doi:10.1007/s00153-005-0284-0.

17 Petr Hájek. Some remarks on Cantor-Łukasiewicz fuzzy set theory. *Log. J. IGPL*, 21(2):183–186, 2013. doi:10.1093/jigpal/jzs014.

18 Yuichi Komori. Super-Łukasiewicz propositional logics. *Nagoya Math. J.*, 84:119–133, 1981.

19 Yuichi Komori. Illative combinatory logic based on bck-logic. *Mathematica Japonica*, 34(4), 1989.

20 Tomasz Kowalski and Tadeusz Litak. Completions of GBL-algebras: negative results. *Algebra Universalis*, 58(4):373–384, 2008. doi:10.1007/s00012-008-2056-2.

21 Richard McKinley. Soft linear set theory. *J. Log. Algebr. Program.*, 76(2):226–245, 2008. doi:10.1016/j.jlap.2008.02.010.

22 Robert McNaughton. A theorem about infinite-valued sentential logic. *J. Symbolic Logic*, 16:1–13, 1951. doi:10.2307/2268660.

23 George Metcalfe, Nicola Olivetti, and Dov Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Trans. Comput. Log.*, 6(3):578–613, 2005. doi:10.1145/1071596.1071600.

24 George Metcalfe, Nicola Olivetti, and Dov Gabbay. *Proof theory for fuzzy logics*, volume 36 of *Applied Logic Series*. Springer, New York, 2009.

25 Matteo Mio and Alex Simpson. łukasiewicz μ-calculus. *Fund. Inform.*, 150(3-4):317–346, 2017. doi:10.3233/FI-2017-1472.

26 Paul S. Mostert and Allen L. Shields. On the structure of semigroups on a compact manifold with boundary. *Ann. of Math. (2)*, 65:117–143, 1957. doi:10.2307/1969668.

27 Hiroakira Ono and Yuichi Komori. Logics without the contraction rule. *J. Symbolic Logic*, 50(1):169–201, 1985. doi:10.2307/2273798.

28 Uwe Petersen. Logic without contraction as based on inclusion and unrestricted abstraction. *Studia Logica*, 64(3):365–403, 2000. doi:10.1023/A:1005293713265.

29 Masaru Shirahata. Linear set theory with strict comprehension. In *Proceedings of the Sixth Asian Logic Conference (Beijing, 1996)*, pages 223–245. World Sci. Publ., River Edge, NJ, 1998.

30 Thoralf Skolem. Bemerkungen zum Komprehensionsaxiom. *Z. Math. Logik Grundlagen Math.*, 3:1–17, 1957.

31 Luca Spada. ŁΠ logic with fixed points. *Arch. Math. Logic*, 47(7-8):741–763, 2008. doi:10.1007/s00153-008-0105-3.

32 Luca Spada. μmV-algebras: an approach to fixed points in Łukasiewicz logic. *Fuzzy Sets and Systems*, 159(10):1260–1267, 2008. doi:10.1016/j.fss.2007.12.010.
33 Kazushige Terui. Light affine set theory: a naive set theory of polynomial time. *Studia Logica*, 77(1):9–40, 2004. doi:10.1023/B:STUD.0000034183.33333.6f.
34 Kazushige Terui. A flaw in R.B. White's article "the consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz". Unpublished note, 2014. URL: https://www.kurims.kyoto-u.ac.jp/~terui/whitenew.pdf.
35 Richard B. White. The consistency of the axiom of comprehension in the infinite-valued predicate logic of Łukasiewicz. *J. Philos. Logic*, 8(4):509–534, 1979. doi:10.1007/bf00258447.
36 Richard B. White. A demonstrably consistent type-free extension of the logic BCK. *Math. Japon.*, 32(1):149–169, 1987.
37 Shunsuke Yatabe. Comprehension contradicts to the induction within Łukasiewicz predicate logic. *Arch. Math. Logic*, 48(3-4):265–268, 2009. doi:10.1007/s00153-009-0127-5.

A Omitted proofs

A.1 Proof of Theorem 5

It is intuitively clear that $\mathbf{FLew}_{\mathsf{fp}}$ admits cut elimination. There is, however, a subtlety in permutative reduction steps for additive conjunction and disjunction, a common problem frequently encountered in linear logic. This can be resolved by carefully defining the size of a derivation.

We write $\pi = (r)[\pi_1, \dots, \pi_n]$ if derivation π ends with an application of inference rule (r) to subderivations $\pi_1, \dots, \pi_n$. Suppose that $\pi = (r)[\pi_1, \dots, \pi_n]$ $(0 \leq n \leq 2)$. The *size* $|\pi|$ of π is defined by induction.

$$\begin{array}{rcll} |\pi| & = & 1 & \text{if } n = 0, \\ & = & |\pi_1| + 1 & \text{if } n = 1, \\ & = & |\pi_1| + |\pi_2| & \text{if } (r) \text{ is } (\mathsf{cut}), \\ & = & |\pi_1| + |\pi_2| + 1 & \text{if } (r) \text{ is } (\cdot\, r) \text{ or } (\to \ell), \\ & = & \max(|\pi_1|, |\pi_2|) + 1 & \text{if } (r) \text{ is } (\wedge r) \text{ or } (\vee \ell). \end{array}$$

This notion of size works perfectly. Consider a derivation of the form $\pi = (\mathsf{cut})[\pi_1, \pi_2]$ with π_1 and π_2 cut-free. One can then prove by induction on $|\pi|$ that π reduces to a cut-free derivation π' such that $|\pi'| \leq |\pi|$. We here consider two cases.

If $\pi_1 = (\to r)[\pi_{11}]$, $\pi_2 = (\to \ell)[\pi_{21}, \pi_{22}]$ and the cut is principal, then π looks like:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_{11}\\ A, \Gamma \Rightarrow B\end{array}}{\Gamma \Rightarrow A \to B}\,(\to r) \qquad \dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ \Delta_1 \Rightarrow A\end{array} \quad \begin{array}{c}\vdots\ \pi_{22}\\ B, \Delta_2 \Rightarrow \Pi\end{array}}{A \to B, \Delta_1, \Delta_2 \Rightarrow \Pi}\,(\to \ell)}{\Gamma, \Delta_1, \Delta_2 \Rightarrow \Pi}\,(\mathsf{cut})$$

We have $|\pi| = |\pi_{11}| + |\pi_{21}| + |\pi_{22}| + 2$. We first consider $(\mathsf{cut})[\pi_{11}, \pi_{21}]$, which by the induction hypothesis reduces to a cut-free derivation π_1' of size $\leq |\pi_{11}| + |\pi_{21}|$. We then consider $(\mathsf{cut})[\pi_1', \pi_{22}]$, which reduces to a cut-free derivation π' such that $|\pi'| \leq |\pi_1'| + |\pi_{22}| \leq |\pi_{11}| + |\pi_{21}| + |\pi_{22}| < |\pi|$ as required.

If $\pi_2 = (\wedge r)[\pi_{21}, \pi_{22}]$ and the cut is not principal, then π looks like:

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \Rightarrow A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\ \pi_{21}\\ A, \Delta \Rightarrow B_1\end{array} \quad \begin{array}{c}\vdots\ \pi_{22}\\ A, \Delta \Rightarrow B_2\end{array}}{A, \Delta \Rightarrow B_1 \wedge B_2}\,(\wedge r)}{\Gamma, \Delta \Rightarrow B_1 \wedge B_2}\,(\mathsf{cut})$$

We have $|\pi| = |\pi_1| + \max(|\pi_{21}|, |\pi_{22}|) + 1$. We first consider derivation $(\mathsf{cut})[\pi_1, \pi_{2i}]$ for each $i = 1, 2$, which reduces to a cut-free derivation π'_i of size $\leq |\pi_1| + |\pi_{2i}|$. Finally we obtain $\pi' = (\wedge r)[\pi'_1, \pi'_2]$. Its size is $|\pi'| = \max(|\pi'_1|, |\pi'_2|) + 1 \leq |\pi_1| + \max(|\pi_{21}|, |\pi_{22}|) + 1 = |\pi|$ as required. The other cases are similar or easier.

A.2 Proof of Theorem 15

We begin with an algebraic characterization of proper extensions of $\mathbf{Ł}$.

Let $n > 0$. We define $Ł_{n+1}$ to be the subalgebra of $[0,1]_{\mathbf{Ł}}$ on the underlying set $\{\frac{0}{n}, \frac{1}{n}, \dots, \frac{n}{n}\}$. The logic $\mathbf{Ł}_{n+1} := \mathsf{Log}(Ł_{n+1})$, known as $(n+1)$*-valued Łukasiewicz logic*, is a proper extension of $\mathbf{Ł}$ for every $n > 0$, but there are other extensions too. To capture them all, we introduce an algebra K_{n+1} for every $n > 0$.

We denote by $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{\leq 0}$ the integers, the nonnegative integers, and the nonpositive integers, respectively. Define $P_n := \{n\} \times \mathbb{Z}_{\geq 0}$, $P_i := \{i\} \times \mathbb{Z}$ for $n > i > 0$, $P_0 := \{0\} \times \mathbb{Z}_{\leq 0}$, and $P := P_n \cup P_{n-1} \cup \dots \cup P_1 \cup P_0$.

P is totally ordered as follows: $(i, j) < (k, l)$ if $i > k$, or $i = k$ and $j < l$. Hence the least element is $\bot := (n, 0) \in P_n$, while the greatest one is $\top := (0, 0) \in P_0$.

Multiplication and negation are defined as follows:

$$\begin{array}{lcll} (i,j)\cdot(k,l) & := & (i+j, k+l) & \text{if } i+j<n, \text{ or } i+j=n \text{ and } k+l\geq 0, \\ & := & (n,0) & \text{otherwise,} \\ \neg(i,j) & := & (n-i, -j). & \end{array}$$

Finally by defining $a \to b := \neg(a \cdot \neg b)$, we obtain an FLew algebra $\mathrm{K}_{n+1} := (P, \min, \max, \cdot, \to, \top, \bot) \in \mathsf{Mod}(\mathbf{Ł})$, known as a *Komori algebra.*

$Ł_{n+1}$ and K_{n+1} together give a complete characterization of proper extensions of $\mathbf{Ł}$ (see, e.g., Theorem 7.1.23 of [9]).

▶ **Theorem 33** ([18]). *Let* $\mathrm{L} \in \mathsf{Ext}(\mathbf{FLew})$ *be a consistent proper extension of* $\mathbf{Ł}$. *Then there are finite sets of indices* $I, J \subseteq \mathbb{Z}_{\geq 1}$ *with* $I \cup J \neq \emptyset$ *such that* $\mathrm{L} = \mathsf{Log}(\{Ł_{n+1}\}_{n\in I} \cup \{\mathrm{K}_{n+1}\}_{n\in J})$.

It is easy to deal with $Ł_{n+1}$, hence let us focus on K_{n+1} with $n > 0$. Suppose that $m \geq n + 1$. Then for every $a \in P$ we have:

- $a^m \in P_0$ iff $a \in P_0$.
- $a \notin P_0$ implies $a^m = \bot$.

▶ **Lemma 34.** $(a \leftrightarrow \neg a^m) \notin P_0$ *for every* $a \in P$.

Proof. Suppose that $(a \leftrightarrow \neg a^m) \in P_0$. Since $(a \leftrightarrow \neg a^m) = (a \to \neg a^m) \wedge (\neg a^m \to a) = \neg a^{m+1} \wedge \neg(\neg a^m \cdot \neg a)$, we have $\neg a^{m+1} \in P_0$ and $\neg(\neg a^m \cdot \neg a) \in P_0$. From the former, we have $a^{m+1} \notin P_0$, so $a \notin P_0$. Hence $a^m = \bot$, which implies $\neg a^m = \top$. Therefore $a = \neg\neg a = \neg(\neg a^m \cdot \neg a) \in P_0$. That is a contradiction. ◀

As a consequence, $(a \leftrightarrow \neg a^m)^m = \bot$, hence $\neg(a \leftrightarrow \neg a^m)^m = \top$ holds for every $a \in P$.

▶ **Theorem 35.** *There is no proper extension of* $\mathbf{Ł}$ *which is consistent with fixed points.*

Proof. By Theorem 33, any proper extension L of $\mathbf{Ł}$ is characterized as $\mathrm{L} = \mathsf{Log}(\{Ł_{n+1}\}_{n\in I} \cup \{\mathrm{K}_{n+1}\}_{n\in J})$ for some finite sets I, J of indices. Let $m := \max(I \cup J) + 1$ and let $A_m(R)$ be formula $\neg(R \leftrightarrow \neg R^m)^m \in \mathsf{Fm}$, where R is an arbitrary formula in Fm. We then have $\models_{\mathrm{K}_{n+1}} A_m(R)$ for every $n \in J$ by the above consideration. It is easy to show that $\models_{Ł_{n+1}} A_m(R)$ for every $n \in I$. Hence by completeness, we obtain $\vdash_{\mathrm{L}} A_m(R)$.

On the other hand, by letting $R := \xi p.\neg p^m$, we obtain a fixed point axiom $R \leftrightarrow \neg R^m$. Therefore $\Phi_{\mathsf{fp}} \vdash_{\mathrm{L}} 0$, that is, L is inconsistent with fixed points. ◀