New and Formalized Proofs for Right-Forward Closures and Core Matrix Interpretations

René Thiemann
University of Innsbruck, Austria

Dieter Hofbauer
ASW Saarland, Germany

Ulysse Le Huitouze
University of Rennes, France

Johannes Waldmann
HTWK Leipzig, Germany

Abstract

We provide new proofs of two important theorems for proving termination of term rewrite systems (TRSs), including a full formalization in Isabelle/HOL.

We first consider Dershowitz' theorem that termination starting from arbitrary terms is equivalent to termination starting from terms in the right-forward closures of right-hand sides, provided that the TRS is right-linear or orthogonal. Our new proof deviates from the original one in that no reorderings of steps in infinite derivations are required, making it more precise in its argumentation. It also subsumes a later result that one can weaken orthogonality to locally confluent overlay TRSs.

The second theorem is about matrix interpretations. These were introduced by Hofbauer and Waldmann for proving termination of string rewrite systems (SRSs), internally using the concept of a core. Subsequently, Endrullis, Waldmann and Zantema developed matrix interpretations for TRSs without using the idea of a core. Whereas matrix interpretations for TRSs have already been formalized several times, so far this was not the case for core SRS matrix interpretations. We not only provide such a formalization, but also extend core SRS matrix interpretations to TRSs. These new core matrix interpretations for TRSs generalize previous approaches.

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Supplementary Material *InteractiveResource (Website with Example Certificates and Formalization)*: `https://isafor-ceta.uibk.ac.at/experiments/rfc_core`

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1 Introduction

Consider the task of automated termination analysis for rewrite systems. Here, every year automated tools participate in the Termination Competition [18] in order to prove or disprove termination of as many rewrite systems as possible.

Since there might be bugs in the tools (or flaws in statements of published termination techniques), since 2007 there has also been the possibility to check the generated termination proofs on their correctness. In this process, *certifiers* are using formally verified algorithms to validate the output of the termination tools, where the formal verification might be conducted in proof assistants such as Isabelle/HOL [31], Lean [7], or Rocq [4].

Table 1 Successful termination proofs for TRSs and SRSs in the Termination Competition 2025.

Category	Tool	# YES	certified	in %
TRS	AProVE	1030	946	92 %
SRS	matchbox	1425	1223	86 %
SRS	MnM	1370	0	0 %
SRS	AProVE	1020	827	81 %

For this certification process to work, several prerequisites have to be met:

- Every termination technique that is mentioned in the generated proof must be supported by the certifier.
- Support by the certifier requires a formal proof of the termination technique, e.g., that a certain term ordering is strongly normalizing.
- Support by the certifier further requires some method to validate that some termination technique has been applied correctly in the concrete generated termination proof.

The coverage of certified termination proving vs. the uncertified case is clearly visible in the Termination Competition 2025, cf. Table 1. Whereas for the strongest tool for TRSs – AProVE [17] – more than 90 % of the termination proofs can be certified, this number is reduced by more than 10 % if one switches to SRSs. And also for the two strongest tools for termination analysis of SRSs – matchbox [34] and MnM [21] – we do not see a coverage that is as good as for TRSs.

When looking into those termination proofs of AProVE, matchbox and MnM that are not certified, we see two re-appearing reasons, i.e., termination techniques that are not supported by the certifier CeTA [32], but are utilized in the generated proofs.

- There are termination proofs of AProVE and matchbox that combine the match-bounds technique with Dershowitz' criterion on right-forward closures (RFC) [13, 14, 25]. The latter criterion states that under certain conditions (which are satisfied for every SRS), for proving global termination it suffices to consider all terms in right-forward closures of right-hand sides as starting terms [8].
- MnM uses a variant of matrix interpretations for SRSs (core matrix interpretations) [22] that is not covered by the version of matrix interpretations for TRSs [11] which is supported by CeTA. Since a large part of all MnM proofs build upon these matrix interpretations, there is currently no support for CeTA certificates in MnM at all.

In order to improve the situation, in this paper we make the following contributions.

- Section 3: We provide a new proof of Dershowitz' RFC-criterion, also including later generalizations of Dershowitz and Hoot, and fully formalize it in Isabelle/HOL.
- Section 4.1: We formalize the core matrix interpretations for SRSs and thereby correct a mistake in the original proof.
- Section 4.3: We generalize the concept of core matrix interpretations for SRSs to a core matrix interpretation for TRSs in such a way that it subsumes both core matrix interpretations for SRSs and matrix interpretations for TRSs.

We integrate verified algorithms for checking core matrix interpretations for TRSs into CeTA. The website `https://isafor-ceta.uibk.ac.at/experiments/rfc_core` provides access to our formalization.

2 Preliminaries

We assume familiarity with term rewriting [3] and recall important notions and notations.

We consider first order terms $s, t, \ell, r, \ldots \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ that consist of variables $x, y, z, \ldots \in \mathcal{V}$ and function applications $f(t_1, \ldots, t_n)$ for n-ary symbols $f \in \mathcal{F}$. Substitutions $\sigma, \delta, \mu, \ldots$ are mappings from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$, and we write $t\sigma$ to substitute each variable x in a term t by $\sigma(x)$. A substitution σ is a variable renaming if σ is an injective mapping from $\mathcal{V}$ to $\mathcal{V}$. Given a substitution σ and a set of variables X we write $\sigma \upharpoonright X$ for the substitution that restricts σ to X, i.e., $(\sigma \upharpoonright X)(x) = \sigma(x)$ if $x \in X$, and $(\sigma \upharpoonright X)(x) = x$ if $x \notin X$. We denote the set of positions of t by $\mathit{Pos}(t)$, i.e., $\mathit{Pos}(x) = \{\varepsilon\}$ and $\mathit{Pos}(f(t_1, \ldots, t_n)) = \{\varepsilon\} \cup \{ip \mid 1 \le i \le n, p \in \mathit{Pos}(t_i)\}$. The subterm $t|_p$ of t at position p is defined as $t|_\varepsilon = t$ and $f(t_1, \ldots, t_n)|_{ip} = t_i|_p$. The term $t[s]_p$ is obtained from t by replacing $t|_p$ by s at position p. The set $\mathit{FPos}(t)$ is the set of function positions, i.e., $\mathit{FPos}(t) = \{p \in \mathit{Pos}(t) \mid t|_p \notin \mathcal{V}\}$. The strict subterm relation is denoted by $\rhd$, i.e., $s \rhd t$ if $s \neq t$ and $s|_p = t$ for some $p \in \mathit{Pos}(s)$. The set of variables of a term t is denoted by $\mathit{Vars}(t)$, and t is linear if no variable occurs more than once in t. We write $\mathit{mgu}(s, t) = \mu$ if μ is a most general unifier of s and t, i.e., $s\mu = t\mu$ and whenever $s\sigma = t\sigma$ then $\sigma = \mu\rho$ for some ρ.

A relation $\to \subseteq T \times T$ on a set T is strongly normalizing w.r.t. a set of starting objects $T_0 \subseteq T$, written $\mathit{SN}(\to, T_0)$, if there is no infinite sequence $t_0 \to t_1 \to t_2 \to \ldots$ with $t_0 \in T_0$. If there is no restriction on the set of starting objects we just write $\mathit{SN}(\to)$. An object t is in normal form w.r.t. relation $\to$ if there is no s such that $t \to s$.

A TRS $\mathcal{R}$ is a set of rules $\ell \to r$ such that $\ell \notin \mathcal{V}$ and $\mathit{Vars}(\ell) \supseteq \mathit{Vars}(r)$. The rewrite relation of $\mathcal{R}$ is defined as $s \to_{\mathcal{R},p} t$ if $p \in \mathit{FPos}(t)$, $\ell \to r \in \mathcal{R}$, $s|_p = \ell\mu$, and $t = s[r\mu]_p$ for some p, ℓ, r, μ. There is an equivalent characterization of the rewrite relation via contexts, where $s \to_{\mathcal{R}} t$ if $s = C[\ell\sigma]$, $t = C[r\sigma]$ for some context C, substitution σ and rule $\ell \to r \in \mathcal{R}$. Here, a context C is a term which contains exactly one special variable $\Box$, and $C[u]$ substitutes $\Box$ by term u. Narrowing of $\mathcal{R}$ is defined as $s \rightsquigarrow_{\mathcal{R},\mu \upharpoonright \mathit{Vars}(s),p} t$ if $p \in \mathit{FPos}(t)$, $\ell \to r \in \mathcal{R}$, $\mathit{mgu}(s|_p, \ell) = \mu$, and $t = s[r]_p\mu$ for some p, ℓ, r, μ, where – in contrast to the rewrite relation – unification is used instead of matching. For narrowing it is always assumed that the variables of rule $\ell \to r$ are renamed apart so that they are disjoint to $\mathit{Vars}(s)$. We often omit the subscripts in the rewrite and narrowing relation.

The set of right-hand sides of a TRS $\mathcal{R}$ is denoted by $\mathit{rhs}(\mathcal{R})$, and $\mathcal{R}$ is right-linear if all terms in $\mathit{rhs}(\mathcal{R})$ are linear. Left-linearity is defined analogously. The set of (position annotated) critical pairs is $\mathit{CP}(\mathcal{R})$, and it is defined as follows. Whenever there are rules $\ell_1 \to r_1, \ell_2 \to r_2 \in \mathcal{R}$, and $p \in \mathit{FPos}(\ell_1)$, and $p \neq \varepsilon$ or $\ell_1 \to r_1$ differs from $\ell_2 \to r_2$, and $\ell_2' \to r_2'$ is a renamed variant of $\ell_2 \to r_2$ such that $\mathit{Vars}(\ell_1)$ and $\mathit{Vars}(\ell_2')$ are disjoint, and $\mathit{mgu}(\ell_1|_p, \ell_2') = \mu$, then $(r_1\mu, (\ell_1\mu)[r_2'\mu]_p, p)$ is a critical pair. A TRS is *orthogonal* if it is left-linear and there are no critical pairs. In an *overlay* TRS all critical pairs (s, t, p) satisfy $p = \varepsilon$. A TRS $\mathcal{R}$ is *locally confluent* if all its critical pairs $(s, t, p) \in \mathit{CP}(\mathcal{R})$ are joinable, i.e., $s \to_{\mathcal{R}}^* \circ {}_{\mathcal{R}}^*\!\!\leftarrow t$. Note that, by definition, each orthogonal system is a fortiori a locally confluent overlay system.

A string rewriting system (abbreviated SRS) is a binary relation in $\Sigma^* \times \Sigma^*$ given an alphabet Σ. Rewriting using a rule $\ell \to r$ in an SRS $\mathcal{R}$ is denoted by $\to_{\mathcal{R}}$, too, and consists in rewriting $u\ell v$ to urv for any $u, v \in \Sigma^*$.

Termination of a TRS $\mathcal{R}$ is defined as $\mathit{SN}(\to_{\mathcal{R}})$, and termination of $\mathcal{R}$ relative to a TRS $\mathcal{S}$ is $\mathit{SN}(\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}} \circ \to_{\mathcal{S}}^*)$, abbreviated by $\mathit{SN}(\mathcal{R}/\mathcal{S})$, analogously for SRSs.

3 Right-Forward Closures

The first result that we are going to formally verify is a result that is attributed to Dershowitz, including a later extension by Dershowitz and Hoot. Given a TRS $\mathcal{R}$, let us denote the right-forward closure of right-hand sides of $\mathcal{R}$ by $RFC(\mathcal{R})$ [15, 16]. The result states that under certain conditions on the TRS, termination of $\mathcal{R}$ is equivalent to termination when just considering starting terms $RFC(\mathcal{R})$.

▶ **Definition 1** ($RFC(\mathcal{R})$). *The right-forward closure $RFC(\mathcal{R})$ is the least set that contains $rhs(\mathcal{R})$ and is closed under narrowing:*

$$RFC(\mathcal{R}) = \{t \mid \exists s \in rhs(R).\ s \rightsquigarrow_{\mathcal{R}}^{*} t\} = \rightsquigarrow_{\mathcal{R}}^{*}(rhs(\mathcal{R}))$$

▶ **Theorem 2** ([8]). *Let $\mathcal{R}$ be a right-linear or orthogonal TRS. Then*

$$SN(\rightarrow_{\mathcal{R}}) \quad \text{if and only if} \quad SN(\rightarrow_{\mathcal{R}}, RFC(\mathcal{R})).$$

▶ **Theorem 3** ([10]). *Let $\mathcal{R}$ be a right-linear or locally confluent overlay TRS. Then*

$$SN(\rightarrow_{\mathcal{R}}) \quad \text{if and only if} \quad SN(\rightarrow_{\mathcal{R}}, RFC(\mathcal{R})).$$

These RFC-theorems (Theorems 2 and its stronger version Theorem 3) have applications in termination proving, e.g., in combination with the match-bounds technique [13, 14, 25]. Here, the change from all starting terms to just $RFC(\mathcal{R})$ can become crucial for a successful termination proof, as it can be seen in Table 1 for the tools AProVE and matchbox: the certified proofs so far can only include the match-bounds technique without the restriction of the starting terms to $RFC(\mathcal{R})$, but there is no such restriction for non-certifiable proofs, and many such proofs are visible in the difference between each of the two versions of these tools.

▶ **Example 4.** For $\mathcal{R} = \{\mathsf{a}(\mathsf{b}(x)) \rightarrow \mathsf{b}(\mathsf{b}(\mathsf{a}(x)))\}$, we have $RFC(\mathcal{R}) = \{\mathsf{b}^{2k}(\mathsf{a}(x)) \mid k > 0\}$, and $SN(\rightarrow_{\mathcal{R}}, RFC(\mathcal{R}))$ holds true since each $t \in RFC(\mathcal{R})$ is in $\mathcal{R}$-normal form. By Theorem 2 we conclude $SN(\rightarrow_{\mathcal{R}})$. Since $\mathcal{R}$ has derivations of exponential length, starting at $\mathsf{a}^k(\mathsf{b}(x))$, this example shows that the restriction of the set of starting terms to $RFC(\mathcal{R})$ can reduce the lengths of derivations and, consequently, can increase the power of termination methods. For instance, the match-bounds technique can only be applied if the maximal derivation length is bounded linearly in the size of the starting terms.

▶ **Example 5.** A subset of Toyama's TRS [33] is easily proved to be terminating by Theorems 2 or 3. For the orthogonal TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \rightarrow \mathsf{f}(x, x, x), \mathsf{g}(x, y) \rightarrow x\}$ we obtain $rhs(\mathcal{R}) = \{\mathsf{f}(x, x, x), x\}$. Since both terms in $rhs(\mathcal{R})$ are normal forms w.r.t. narrowing, in particular $RFC(\mathcal{R}) = rhs(\mathcal{R}) = \{\mathsf{f}(x, x, x), x\}$. Since both terms in $RFC(\mathcal{R})$ are normal forms of $\mathcal{R}$, by Theorem 2 (or by the stronger Theorem 3) we conclude termination of $\mathcal{R}$.

▶ **Example 6.** The full version $\{\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \rightarrow \mathsf{f}(x, x, x), \mathsf{g}(x, y) \rightarrow x, \mathsf{g}(x, y) \rightarrow y\}$ of Toyama's TRS [33] is non-terminating. This TRS shows that one cannot just drop the pre-conditions on the TRS in Theorems 2 and 3. The reason is that – as in Example 5 – $RFC(\mathcal{R}) = rhs(\mathcal{R})$ and all these terms are normal forms and thus, terminating.

There is a challenge in verifying the proof of the RFC-theorems in a formal setting. One of the problems is that the definition of $RFC(\mathcal{R})$ – based on narrowing – is not used by Dershowitz in his proof [8], but instead a different notion of an infinite chain is utilized: an infinite chain is an infinite derivation where active positions are marked and it has to satisfy certain conditions. Moreover, the original proof performs some reordering process of steps within a derivation, where it is not obvious to us how to elegantly describe this process in the formal setting of a proof assistant.

In their later work, Dershowitz and Hoot provide an alternative definition of forward closures [10, Definition 6] that is based on narrowing, and there is a strong correspondence to $\mathit{RFC}(\mathcal{R})$: a forward closure is a full sequence of narrowing steps with all intermediate terms, whereas in $\mathit{RFC}(\mathcal{R})$ we only store the final terms of such sequences. Unfortunately, the connection between infinite chains and *infinite* forward closures as they occur in [10, Theorem 6] remains at an intuitive level in this paper.

The definition of *infinite* forward closures is made formal by Geupel. He also provides a detailed proof of Theorem 2 for non-overlapping TRSs [16, Theorem 2], but he does not consider right-linear TRSs.

A detailed proof of Theorem 2 for the case of right-linear TRSs is partly provided in two other works: Zantema considers the restriction to SRSs [36, Section 6] and Geser et al. provide a proof for linear TRSs, i.e., TRSs that are both left- and right-linear [15, Section 8].

We are not aware of a proof of Theorem 3 in the literature that covers the theorem in its full generality and at the same provides sufficiently many details, so that is can be mechanized in a proof assistant.

In the remainder of this section, we present a new proof of Theorem 3 based on Definition 1. Our proof does not require any reordering of steps as in the original proof, and it has been formally verified in Isabelle/HOL, cf. the website with the formalization, where definitions and theorems in the paper are linked to their formal counterparts.

Before we start with the main proof of Theorem 3, let us first state some simple connections between rewriting and narrowing.

▶ **Lemma 7.**
- *If $s \to_p t$ then $s \rightsquigarrow_{\mu,p} t\mu$ for some variable renaming μ.*[1]
- *If $s \rightsquigarrow_{\mu,p} t$ then $s\mu \to_p t$.*
- *If $s \rightsquigarrow_{\mu,p} t$ for some variable renaming μ then $s \to_p t\mu^{-1}$.*

The first property of Lemma 7 shows that narrowing can simulate rewriting; the second property is for the reverse direction, provided that one instantiates the starting term s by the substitution μ from the narrowing step; and finally, if narrowing just uses variable renamings μ, then it is basically rewriting (modulo variable renamings).

Note that $\mathit{SN}(\to_{\mathcal{R}})$ is equivalent to $\mathit{SN}(\to_{\mathcal{R}}, \bigcup_\sigma \sigma(\mathit{rhs}(\mathcal{R})))$ with σ ranging over arbitrary substitutions [9, Lemma 1], i.e., for termination analysis it suffices to consider all instances $r\sigma$ of right-hand sides r of $\mathcal{R}$: this can be seen by using a minimal non-terminating term argument as it is used for dependency pairs [2].

The idea of the RFC termination argument is to consider an arbitrary non-terminating derivation $r\sigma = t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} \dots$ for some right-hand side r, and convert it into a non-terminating derivation with a starting term of $\mathit{RFC}(\mathcal{R})$. Here, the aim is to prove that narrowing on r can recover σ, i.e., narrowing will piecewise reconstruct σ to simulate the non-terminating sequence, and at some point all narrowing steps will become rewrite steps as in the third property of Lemma 7. To this end, we need the following result which permits us to connect a rewrite step with source $s\sigma$ to a narrowing step with source s, provided that the position of the rewrite step is within s: one can decompose σ into $\mu\sigma'$ and the rewrite step is already possible with $s\mu$.

[1] We assume that some fixed renaming algorithm and unification algorithm is used in the definition of the narrowing relation, without any restriction on these algorithms. As a consequence, it is not guaranteed that $s \to t$ implies $s \rightsquigarrow t$, because narrowing might always perform a complete renaming of the variables.

$$\begin{tikzcd}
t_0 \arrow[r] \arrow[d, equal] & t_1 \arrow[r] \arrow[d, equal] & t_2 \arrow[r] \arrow[d, equal] & \cdots \\
s_0\sigma_0 \arrow[d, equal] & s_1\sigma_1 \arrow[d, equal] & s_2\sigma_2 \arrow[d, equal] & \cdots \\
s_0\mu_1\sigma_1 \arrow[ur] & s_1\mu_2\sigma_2 \arrow[ur] & s_2\mu_3\sigma_3 \arrow[ur] &
\end{tikzcd}$$

$$s_0 \overset{\mu_1}{\rightsquigarrow} s_1 \overset{\mu_2}{\rightsquigarrow} s_2 \overset{\mu_3}{\rightsquigarrow} \cdots$$

Figure 1 Infinite Application of One-Step Simulation.

▶ **Lemma 8** (One-step simulation). *If $s\sigma \to_p t$ and $p \in FPos(s)$ then there are s' and substitutions μ and σ' such that*

$$s \rightsquigarrow_{\mu,p} s' \text{ and } \sigma = \mu\sigma' \text{ and } t = s'\sigma'$$

or illustrated a bit differently where now the rewrite step is underlined:

$$s\sigma = \underline{s\mu}\sigma' \to \underline{s'}\sigma' = t \quad \text{and} \quad s \rightsquigarrow_\mu s'$$

Now assume there is some non-terminating derivation

$$s_0\sigma_0 = t_0 \to_\mathcal{R} t_1 \to_\mathcal{R} t_2 \to_\mathcal{R} \dots \tag{1}$$

The aim is to find some invariant on s and σ such that it is satisfied for s_0 and σ_0. Moreover, the invariant must be preserved by Lemma 8, i.e., the terms s' and substitution σ' of this lemma should also satisfy it. The reason is that with such an invariant an infinite narrowing sequence can be constructed as in Figure 1. Moreover, one can prove that eventually all narrowing steps must be rewrite steps. Thus, $\neg SN(\to_\mathcal{R}, \rightsquigarrow^*_\mathcal{R}(\{s_0\}))$ can be derived, and hence $SN(\to_\mathcal{R}, \rightsquigarrow^*_\mathcal{R}(\{s_0\}))$ is a sufficient criterion for proving $SN(\to_\mathcal{R}, \{s_0\sigma_0\})$.

▶ **Lemma 9.** *For an infinite derivation (1) let I be some invariant such that:*

- $I(s_0, \sigma_0)$,
- *if $I(s, \sigma)$ and $s\sigma \to_p t$ then $p \in FPos(s)$, and*
- *if $I(s, \sigma)$ and $s \rightsquigarrow_\mu s'$ and $\sigma = \mu\sigma'$ then $I(s', \sigma')$.*

Then $\neg SN(\to_\mathcal{R}, \rightsquigarrow^*_\mathcal{R}(\{s_0\}))$.

Proof. The proof works as follows. First, by using the invariant one can indeed construct the infinite narrowing sequence as indicated in Figure 1. Afterwards, it only needs to be shown that eventually all μ_i are variable renamings, so that from a certain point onwards, all narrowing steps in Figure 1 can be turned into rewrite steps by Lemma 7.

To this end, observe that

$$t_0 = s_0\sigma_0 = s_0\mu_1\sigma_1 = s_0\mu_1\mu_2\sigma_2 = s_0\mu_1\mu_2\mu_3\sigma_3 = \dots.$$

Thus, t_0 is an instance of $s_0\mu_1\mu_2 \dots \mu_n$ for all n. Therefore, eventually all μ_i are variable substitutions, i.e., $\mu_i : \mathcal{V} \to \mathcal{V}$ for almost every i. For these variable substitutions we infer

$$\mathit{Vars}(s_{i+1}\mu_{i+1}) = \mu_{i+1}(\mathit{Vars}(s_{i+1})) \subseteq \mu_{i+1}(\mathit{Vars}(s_i\mu_i))$$

because $s_i\mu_i \to s_{i+1}$ implies $\mathit{Vars}(s_{i+1}) \subseteq \mathit{Vars}(s_i\mu_i)$, and hence

$$|\mathit{Vars}(s_{i+1}\mu_{i+1})| \leq |\mu_{i+1}(\mathit{Vars}(s_i\mu_i))| \leq |\mathit{Vars}(s_i\mu_i)|$$

shows that $|\mathit{Vars}(s_i\mu_i)|$ weakly decreases with increasing i. Thus, at some point $|\mathit{Vars}(s_i\mu_i)|$ becomes constant, and hence for all future i, μ_{i+1} is a variable renaming. ◀

We now only need to find suitable invariants I to obtain Theorem 3.

Proof of Theorem 3: locally confluent overlay TRSs. Proving this second part of Theorem 3 is quite simple with the help of Lemma 9: we define I such that $I(s, \sigma)$ is satisfied if σ is a normal form substitution, i.e., $\sigma(x)$ is in normal form w.r.t. $\to_{\mathcal{R}}$ for all x. Hence, by Lemma 9 we conclude that $SN(\to_{\mathcal{R}}, \rightsquigarrow^*_{\mathcal{R}}(\{s\}))$ implies $SN(\to_{\mathcal{R}}, \{s\sigma\})$ for all normal form substitutions σ, and replacing $\{s\}$ by $rhs(\mathcal{R})$ we arrive at:

$$SN(\to_{\mathcal{R}}, RFC(\mathcal{R})) \quad \text{implies} \quad SN(\to_{\mathcal{R}}, \sigma(rhs(\mathcal{R}))) \quad \text{for all normal form substitutions } \sigma.$$

The switch from $SN(\to_{\mathcal{R}}, \sigma(rhs(\mathcal{R})))$ for all normal form substitutions σ to full $SN(\to_{\mathcal{R}})$ is now performed using a result of Gramlich [19]: for locally confluent overlay TRSs, innermost termination and termination coincide, and thus the restriction to normal form substitutions can be assumed without loss of generality. ◀

Proof of Theorem 3: right-linear TRSs. Here we define I differently: $I(s, \sigma)$ is satisfied if s is linear and σ is a strongly normalizing substitution, i.e., $SN(\to_{\mathcal{R}}, \{\sigma(x) \mid x \in \mathcal{V}\})$.

Given some infinite derivation (1), we cannot immediately apply Lemma 9, but first have to do some preprocessing as follows.

For an arbitrary rewrite step $s\sigma \to_p t$, it is either the case that $p \in FPos(s)$ (and we use Lemma 8), or the rewrite step is completely inside σ, e.g., below variable x. If s is linear, then in the latter case the step $s\sigma \to_p t$ can be simulated by not performing any narrowing step but instead staying equal, i.e., $s = s$ and $\sigma \to^+ \sigma'$ where σ' is defined as the substitution that is obtained by rewriting $x\sigma$ to the corresponding subterm of t. In this way one obtains $s\sigma \to t = s\sigma'$. So in both cases ($p \in FPos(s)$ is satisfied or not), one can find a term s' and substitution σ' such that $t = s'\sigma'$ and $I(s', \sigma')$ is satisfied. Moreover, $s \rightsquigarrow^* s'$. Hence, we can again obtain an infinite simulation of the rewrite sequence by narrowing steps as indicated in Figure 1, except that now some of the narrowing steps need to be replaced by equalities.

Note that we further obtain $\sigma_i(Vars(s_i))\ (\to_{\mathcal{R}} \cup \rhd)^*_{mul}\ \sigma_{i+1}(Vars(s_{i+1}))$ where here $Vars(s_i)$ is interpreted as the multiset of variables of a term, and $(\to_{\mathcal{R}} \cup \rhd)_{mul}$ is the multiset extension of relation $\to_{\mathcal{R}} \cup \rhd$. Since I enforces that the substitutions are strongly normalizing, there must be some point k such that a strict decrease is not possible anymore, i.e., for all $i \geq k$, the relation $\sigma_i(Vars(s_i))\ (\to_{\mathcal{R}} \cup \rhd)^+_{mul}\ \sigma_{i+1}(Vars(s_{i+1}))$ is not satisfied. Since such a strict decrease is always obtained if the rewrite step is completely inside the substitution, we know that for each $i \geq k$, indeed the position of the rewrite step must be an element of $FPos(s_i)$. Hence, we can apply Lemma 9 on the infinite derivation $t_k = s_k\sigma_k \to_{\mathcal{R}} s_{k+1}\sigma_{k+1} \to_{\mathcal{R}} \dots$ and obtain $\neg SN(\to_{\mathcal{R}}, \rightsquigarrow^*_{\mathcal{R}}(\{s_k\}))$. In combination with $s_0 \rightsquigarrow^* s_k$ this nearly completes the proof as in the case for locally confluent overlay TRSs. There are two additional steps that still need to proven.

First, we argue that invariant I is initially satisfied. To this end, we again refer to the minimal non-terminating term argument: if $\mathcal{R}$ is not terminating, then there must be a minimal non-terminating term u w.r.t. the subterm relation. Any infinite derivation must make a root step at some point, so $u \to^*_{\mathcal{R}} \ell\sigma \to_{\mathcal{R}} r\sigma$ for some rule $\ell \to r \in \mathcal{R}$, and $r\sigma$ is non-terminating. By minimality of u it is easy to see that $I(r, \sigma)$ is satisfied, and we choose $s_0 = r$ and $\sigma_0 = \sigma$ as starting point of derivation (1).

The final issue is the preservation of the invariant, i.e., in particular we must show for every narrowing step $s_i \rightsquigarrow_{\mathcal{R}} s_{i+1}$ that linearity of s_i is carried over to linearity of s_{i+1}. This fact is stated in the upcoming Lemma 11. ◀

Before we show that narrowing preserves linearity, we first need the result that unification preserves linearity.

▶ **Lemma 10** (Unification Involving Linear Terms). *Let $Vars(s) \cap Vars(t) = Vars(u) \cap Vars(t) = \emptyset$, let t and u be linear. If s and t are unifiable with $mgu(s,t) = \mu$ then $u\mu$ is linear.*

Lemma 10 allows non-linearity of the left term s of the unification problem (s, t), whereas the right term t must be linear. This lemma is proven by generalizing (s, t) to multisets of such unification problems, and then following the computation of a swap-free unification algorithm. In such an algorithm there is no transition of the form that replaces (s, t) by (t, s), with the overhead that there must be two rules for treating substitutions, one for pairs (x, t) and one for (t, x). Being swap-free is important to maintain the invariant regarding linearity of the right-hand sides of a unification problem.

▶ **Lemma 11** (Linearity Preservation of Narrowing). *If $v \rightsquigarrow_{\mathcal{R},p} w$ and $\mathcal{R}$ is right-linear and v is linear, then also w is linear.*

Proof. We mainly need to choose suitable parameters in Lemma 10 in order to prove that w is linear: we instantiate s by ℓ, t by $v|_p$, u by $v[r]_p$, and σ by μ. Here $\ell \to r$ and μ are the applied rule and the most general unifier from the narrowing step, respectively. ◀

4 Core Matrix Interpretations

Matrix interpretations (MI) are widely used for proving termination of TRSs and SRSs. They come in several variants.

- Hofbauer and Waldmann first introduced MI for proving termination of SRSs [22]. These are based on the concept of a *core*, and we abbreviate them by core MI for SRSs in this paper. They are presented as strongly monotonic (SM) reduction orders, i.e., to be used for direct termination proofs without dependency pairs.
- Endrullis, Waldmann and Zantema developed the first version of MI for proving termination of TRSs [11], here abbreviated as EWZ MI. They are available both in a strongly monotonic and weakly monotonic (WM) setting, so that they can be additionally be combined with dependency pairs. It is known that the SM variant of EWZ MI applied on SRSs is subsumed by core MI for SRSs [11, Theorem 6].
- Whereas the previous two variants of MI are using non-negative matrices over the ring of integers, there also have been extensions to use MI over rational or real numbers [12, 1, 28, 29] in combination with δ-orderings [20, 26, 27, 30]. We denote these by EWZ MI + δ, and they fully subsume EWZ MI.
- An extension of EWZ MI + δ has been presented by Courtieu, Gbedo, and Pons [6], abbreviated here as CGP MI + δ.
- Further classes of MI use exotic semirings [23, 24] or restricted domains [35]. These are not considered in this paper.

Known relationships of the different versions of MI are given in the left side of Figure 2.

Note that prior to this work, CeTA could certify CGP MI + δ in their full generality, but it did not have support for core MI for SRSs. To this end, we initially aimed at supporting the latter by a corresponding formalization of theoretic results and verified algorithms. Here, we recall some definitions and proofs about this kind of MI for SRSs in Section 4.1. The Isabelle formalization closely follows the existing proofs and is more or less routine. Remarkably, we spotted and fixed a flawed proof. Afterwards, we started to generalize core MI for SRSs in directions that exceed the original work [22]. In Section 4.2 we combine δ-orderings with core MI, and in Section 4.3 we do a *proper* generalization of core MI to the TRS setting. Our final variant, core MI + δ for TRSs, subsumes both core MI for SRSs and EWZ MI + δ for TRSs, cf. the right side of Figure 2.

$$
\left.\begin{array}{c}
\text{core MI for SRSs (SM)} \\
\rotatebox{90}{$\subseteq$} \\
\text{EWZ MI for SRSs (SM)} \\
\rotatebox{-90}{$\subseteq$} \\
\text{EWZ MI for TRSs (SM and WM)} \\
\rotatebox{-90}{$\subseteq$} \\
\text{EWZ MI} + \delta \text{ for TRSs (SM and WM)}
\end{array}\right\} \subseteq \begin{array}{c}
\text{our new form of MI} \\
\\
\\
\text{core MI} + \delta \text{ for TRSs (SM and WM)} \\
\\
\\
\\
\end{array}
$$

$$
\begin{array}{c}
\rotatebox{-90}{$\subseteq$} \\
\text{CGP MI} + \delta \text{ for TRSs (SM and WM)}
\end{array}
$$

Figure 2 Overview of variants of matrix interpretations with their relationship.

▶ **Example 12.** Consider the following SRS `SRS_Standard/Waldmann_19/random-246` from the Termination Problems Database (see `https://termination-portal.org/wiki/TPDB`):

$$\mathsf{baaa} \to \mathsf{bbaa} \qquad \mathsf{abbb} \to \mathsf{bbba} \qquad \mathsf{abbb} \to \mathsf{bbaa}$$

During termCOMP 2024, the tool MnM generated a termination proof by a core MI. Here, and also in later examples we highlight those numbers by underlining whose positiveness is crucial for the termination proof.

$$\alpha(\mathsf{a}) = \underbrace{\begin{pmatrix} \underline{1} & 1 & 1 & 0 & 0 \\ 0 & \underline{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}}_{B_1} \qquad \alpha(\mathsf{b}) = \underbrace{\begin{pmatrix} \underline{1} & 0 & 1 & 1 & 1 \\ 0 & \underline{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{B_2}$$

This proof could not be certified by CeTA in year 2024, since it is a core MI for SRSs, cf. Example 15, but not a CGP MI + δ, cf. Example 21. After manual translation to the input format of the new version of CeTA– that now supports core MI + δ for SRSs and TRSs – this proof is certified.

Programmatic output of core MI certificates for SRSs is realized in a development version of matchbox, and is ongoing work in MnM.

For the rest of this paper, only square matrices of dimension n (often left implicit) are considered, i.e., matrices are elements of $\mathcal{C}^{n\times n}$ for some carrier $\mathcal{C}$. We stick to standard rings for matrices, i.e., the carrier $\mathcal{C}$ is either $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$, and addition and multiplication all have their standard meaning. The zero matrix is denoted by $\mathbf{0}$, the identity matrix by $\mathbf{1}$. As usual, $A_{i,j}$ refers to the component of matrix A at the i-th row and j-th column.

4.1 Core Matrix Interpretations for SRSs

Matrix interpretations are an instance of monotone algebras, which require the definition of orders on the carrier. To this end, Hofbauer and Waldmann define the non-strict order $\geq$ on matrices by a component-wise weak decrease in all matrix entries, and the strict order $>$ additionally requires a strict decrease for at least one matrix entry [22].

In this subsection, we only consider integer matrices, i.e., $\mathcal{C} = \mathbb{Z}$. Let $N = \{A \mid A \geq \mathbf{0}\}$ be the set of non-negative matrices, and $P = \{A \mid A > \mathbf{0}\}$ be the set of non-negative and non-null matrices. For sets of matrices T_1 and T_2, let $T_1T_2 = \{A_1A_2 \mid A_1 \in T_1, A_2 \in T_2\}$, and for a matrix A and a set of matrices T, let $AT = \{AB \mid B \in T\}$ and $TA = \{BA \mid B \in T\}$. As usual, $T^0 = \{\mathbf{1}\}$, $T^{k+1} = T^kT$, and $T^* = \bigcup_k T^k$.

The core MI technique in an SRS setting, as it is described in [22], ensures via constraints that every rewrite step of the SRS results in a strict decrease of matrices, and thus termination can be concluded.

To be more precise, an MI for an SRS is a mapping $\alpha : \Sigma \to \mathcal{D}$ for some domain $\mathcal{D} \subseteq \mathcal{C}^{n\times n}$, where here a domain is a set of matrices that contains $\mathbf{1}$ and is closed under multiplication. Interpretation α is extended to a function on strings $\alpha : \Sigma^* \to \mathcal{D}$ by $\alpha(\epsilon) = \mathbf{1}$ and $\alpha(as) = \alpha(a)\alpha(s)$ for letter $a \in \Sigma$ and string s. Given a SRS $\mathcal{R}$ and some MI α, if $\alpha(s) \in N$ holds for any string s, and if $s \to_{\mathcal{R}} t$ implies $\alpha(s) > \alpha(t)$ for all $s, t \in \Sigma^*$, then $\mathcal{R}$ is terminating. If in addition $\alpha(s) \geq \alpha(t)$ whenever $s \to_{\mathcal{S}} t$ for some TRS $\mathcal{S}$, then $\mathcal{R}$ terminates relative to $\mathcal{S}$.

The next step of this generic setting is to choose domains $\mathcal{D}$ and provide criteria to ensure a strict or weak decrease. For instance, a strict decrease expands to the formula $\forall s \to_{\mathcal{R}} t.\ \alpha(s) > \alpha(t)$, a condition which is not obviously decidable due to the quantification over all possible rewrite steps. Thus, the core matrix interpretation approach aims to reformulate this condition in such a way that it becomes decidable, or can be approximated by a decidable condition. Given the definition of α, a strict decrease for $\to_{\mathcal{R}}$ and a weak decrease of $\to_{\mathcal{S}}$ is equivalently expressed as Property 1.

$$\left.\begin{array}{l} \forall u, v \in \Sigma^*, \ell \to r \in \mathcal{R}.\ \alpha(u)(\alpha(\ell) - \alpha(r))\alpha(v) > \mathbf{0} \\ \forall u, v \in \Sigma^*, \ell \to r \in \mathcal{S}.\ \alpha(u)(\alpha(\ell) - \alpha(r))\alpha(v) \geq \mathbf{0} \end{array}\right\} \textbf{Property 1}$$

Now, assume $\alpha(\ell) - \alpha(r) \in T$ for all $\ell \to r \in \mathcal{R}$, and $\mathcal{D}^*T\mathcal{D}^* \subseteq P$ for some domain $\mathcal{D}$ and set T, then the first line of Property 1 can be deduced. Furthermore, if $\alpha(\ell) - \alpha(r) \in U$ for all $\ell \to r \in \mathcal{S}$, and $\mathcal{D}^*U\mathcal{D}^* \subseteq N$ for some set U, then the second line can be deduced. The latter property is usually given for free if we choose $\mathcal{D} \subseteq N$ and $U \subseteq N$, since N is closed under multiplication.

$$\left.\begin{array}{r} \mathit{range}(\alpha) \subseteq \mathcal{D};\ \mathcal{D} \subseteq N;\ \forall \ell \to r \in \mathcal{R}.\ \alpha(l) - \alpha(r) \in T; \quad \mathcal{D}^*T\mathcal{D}^* \subseteq P \\ \forall \ell \to r \in \mathcal{S}.\ \alpha(l) - \alpha(r) \in N \qquad\qquad\qquad\quad \end{array}\right\} \textbf{Property 2}$$

In the original paper, the first line of Property 2 is presented using the definition of "core of a set of matrices", namely

$$\mathit{core}(\mathcal{D}) = \{A \in N \mid \mathcal{D}^*A\mathcal{D}^* \subseteq P\}.$$

Then, if for some MI α over domain $\mathcal{D}$ the condition $\forall \ell \to r \in \mathcal{R}.\ \alpha(\ell) - \alpha(r) \in \mathit{core}(\mathcal{D})$ is satisfied, $\mathcal{R}$ is terminating. If additionally $\mathcal{D} \subseteq N$ and $\forall \ell \to r \in \mathcal{S}.\ \alpha(\ell) - \alpha(r) \in N$ is satisfied, $\mathcal{R}$ is terminating *relative* to $\mathcal{S}$.

We now introduce concrete sets of matrices, presented in the original paper, which exhibit the desired properties stated above. Here, I is a non-empty subset of matrix indices. Note that checking whether a matrix is in such a set is decidable.

$$E_I = \{A \in N \mid \forall i \in I.\ A_{i,i} > 0\} \qquad M_I = \{A \in N \mid \forall i \in I.\ \exists j \in I.\ A_{i,j} > 0\}$$
$$P_I = \{A \in N \mid \exists i, j \in I.\ A_{i,j} > 0\}$$

▶ **Lemma 13** ([22, Lemma 4]). $P_I = \mathit{core}(E_I), \quad M_I = \mathit{core}(M_I)$

▶ **Corollary 14** (Core MI for SRSs). *Let $\mathcal{R}$ and $\mathcal{S}$ be SRSs over signature Σ and let $\alpha : \Sigma \to \mathcal{D}$ be a matrix interpretation over domain $\mathcal{D} \subseteq \mathbb{Z}^{n\times n}$ and $\emptyset \subset I \subseteq \{1,\dots,n\}$. Let $\mathcal{S}$ satisfy $\alpha(\ell) - \alpha(r) \in N$ for all $\ell \to r \in \mathcal{S}$. Then relative termination $SN(\mathcal{R}/\mathcal{S})$ is ensured if one of the following conditions is satisfied.*

- *$\mathcal{D} = E_I$ and $\alpha(\ell) - \alpha(r) \in P_I$ for all $\ell \to r \in \mathcal{R}$, or*
- *$\mathcal{D} = M_I$ and $\alpha(\ell) - \alpha(r) \in M_I$ for all $\ell \to r \in \mathcal{R}$.*

▶ **Example 15** (Continuing Example 12). The given interpretation is a core MI for domain $E_{\{1,2\}}$. One can check $\alpha(\ell) - \alpha(r) \in P_{\{1,2\}}$ for all three rewrite rules of the SRS. For instance,

$$\alpha(\mathsf{baaa}) - \alpha(\mathsf{bbaa}) = \begin{pmatrix} 1 & 14 & 1 & 1 & 1 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 6 & 1 & 1 & 1 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & \underline{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{B_3}$$

Note that only the inclusions from left to right in Lemma 13 (e.g., $P_I \subseteq core(E_I)$) are required to get Corollary 14. Nevertheless, the original proofs of the inclusions from right to left were found to be flawed during our Isabelle formalization. We present an instance contradicting the original proof for the E_I-case before providing a reconstructed proof. The case of M_I has a similar flaw, and is repaired in the same way.

1. Original proof of $core(E_I) \subseteq P_I$: For showing the inclusion, assume the existence of a matrix A with $AE_I \subseteq P$ and $A \notin P_I$, so $A_{i,j} = 0$ for $i, j \in I$. Define $B \in E_I$ by $B_{i,j} = 1$ for $i = j \in I$ and $B_{i,j} = 0$, otherwise. Then $AB = \mathbf{0} \notin P$, a contradiction.

   Instance: Consider the set of indices $I = \{1\}$ and the matrix $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then, $A \notin P_I$ holds and finally, observe that $AB = A \in P$.
2. Reconstructed proof: The only change needed is that, instead of proving $AB = \mathbf{0}$ which is not true, we prove that $BAB = \mathbf{0}$ which will result in the desired contradiction.

4.2 Generalizing the Ordered Ring

We now consider MI with other ordered rings than the integers. In particular one may choose a carrier $\mathcal{C} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, where for termination analysis the strict order is no longer the standard order $>$ on numbers, but instead one considers δ-orders [20, 26, 29]. Here, some parameter $\delta > 0$ is fixed, and $\succ_\delta$ is defined as $x \succ_\delta y$ if $x - y \geq \delta$. By choosing $\delta = 1$ and $\mathcal{C} = \mathbb{Z}$, we obtain the original setting of [22], but we may also choose a different value of δ and a different carrier.

The definitions of E_I, P_I and M_I are adjusted as follows to the new setting with δ-orders, where we also add the new set $L_{I,\delta}$: for the combination $\mathcal{C} = \mathbb{Z}$ with $\delta = 1$ we have $M_I = L_{I,\delta}$, so in the original setting no differentiation between these sets is required, but in the general case we gain from a distinction between M_I and $L_{I,\delta}$.

$$\begin{aligned} E_I &= \{A \in N \mid \forall i \in I.\ A_{i,i} \geq 1\} & M_I &= \{A \in N \mid \forall i \in I.\ \exists j \in I.\ A_{i,j} \geq 1\} \\ P_{I,\delta} &= \{A \in N \mid \exists i, j \in I.\ A_{i,j} \succ_\delta 0\} & L_{I,\delta} &= \{A \in N \mid \forall i \in I.\ \exists j \in I.\ A_{i,j} \succ_\delta 0\} \end{aligned}$$

Core MI for SRSs generalize as follows: we define P_δ as the set of non-negative matrices A for which at least one entry $A_{i,j}$ satisfies $A_{i,j} \succ_\delta 0$; the definition of core is generalized to

$$core_\delta(\mathcal{D}) = \{A \in N \mid \mathcal{D}^* A \mathcal{D}^* \subseteq P_\delta\}.$$

We generalize Lemma 13 for E_I and get a sufficient approximation for the core of M_I.

▶ **Lemma 16.** $P_{I,\delta} = core_\delta(E_I), \quad core_{|I|\cdot\delta}(M_I) \subseteq L_{I,\delta} \subseteq core_\delta(M_I)$.

The proof of the lemma is similar to the integer setting. Here, we only mention why it was required to change the definitions of E_I and M_I. Multiplication with some number $A_{i,j}$ is monotone for δ-orders: if $x \succ_\delta y$ and $A_{i,j} \geq 1$ then also $A_{i,j}x \succ_\delta A_{i,j}y$. But only requiring $A_{i,j} \succ_\delta 0$ at this point – instead of $A_{i,j} \geq 1$ – is not sufficient anymore to guarantee $A_{i,j}x \succ_\delta A_{i,j}y$.

▶ **Corollary 17** (Core MI for SRSs with δ-orders)**.** *Let $\mathcal{R}$ and $\mathcal{S}$ be SRSs over signature Σ, let $\mathcal{C} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, and let $\alpha : \Sigma \to \mathcal{D}$ be a matrix interpretation over domain $\mathcal{D} \subseteq \mathcal{C}^{n\times n}$ and $\emptyset \subset I \subseteq \{1, \ldots, n\}$. Let $\delta \in \mathcal{C}$ satisfy $\delta > 0$. Let $\mathcal{S}$ satisfy $\alpha(\ell) - \alpha(r) \in N$ for all $\ell \to r \in \mathcal{S}$. Then $SN(\mathcal{R}/\mathcal{S})$ is ensured if*

- *$\mathcal{D} = E_I$ and $\alpha(\ell) - \alpha(r) \in P_{I,\delta}$ for all $\ell \to r \in \mathcal{R}$, or*
- *$\mathcal{D} = M_I$ and $\alpha(\ell) - \alpha(r) \in L_{I,\delta}$ for all $\ell \to r \in \mathcal{R}$.*

4.3 Core Matrix Interpretations for TRSs

The main difference between matrix interpretations for SRSs and TRSs is that in the SRS version only matrix multiplication is used, whereas for TRSs we see both addition and multiplication. Given some n-ary function symbol f, the interpretation of f is of the form

$$\alpha(f)(x_1, \ldots, x_k) = f_0 + f_1 \cdot x_1 + \ldots + f_k \cdot x_k$$

where $f_1, \ldots, f_k$ are matrices, and f_0 is a vector (as in [11, 29]) or a matrix (as in [6]). We consider the "f_0 is a matrix" setting as it subsumes the vector setting: one can always enlarge the vector to a matrix by filling the remaining columns with zero entries. Hence, in this setting a matrix interpretation is a linear polynomial interpretation where the domain is a set of matrices.

Consequently, the interpretation of a term t leads to a linear polynomial $\alpha(t)$ with matrix coefficients. Orienting a rewrite step is similar to what we see for the SRS case, except that the left-contexts u from string rewriting are replaced by term contexts C and right-contexts v are replaced by substitutions σ:

$$\forall C, \sigma, \ell \to r \in \mathcal{R}.\ \alpha(C[\ell\sigma]) - \alpha(C[r\sigma]) > \mathbf{0}$$

Since contexts C are just terms with a special variable $\square$, the hole, we can rewrite $\alpha(C[\ell\sigma]) - \alpha(C[r\sigma])$ to $\alpha(C)\{\square/\alpha(\ell\sigma) - \alpha(r\sigma)\}$, and by a substitution lemma the previous condition can be rewritten to a variant of Property 1.

$$\forall C, \vec{x}, \ell \to r \in \mathcal{R}.\ \alpha(C)\{\square/\alpha(\ell) - \alpha(r)\} > \mathbf{0}$$

where $\alpha(C)\{\square/\alpha(\ell) - \alpha(r)\}$ is a linear polynomial over variables $\vec{x}$ that range over the domain of the matrix interpretation.

We now again want to use domains $\mathcal{D} \in \{E_I, M_I\}$ and the concept of $core_\delta(\mathcal{D}) = \{A \in N \mid \mathcal{D}^* A \mathcal{D}^* \subseteq P_\delta\}$. In order to get rid of the context C in comparisons, the interpretation needs to be chosen such that being in the core propagates via $\alpha(C)$ for all possible contexts C. This is done by demanding $f_1 \in \mathcal{D}, \ldots, f_k \in \mathcal{D}$ for each k-ary symbol f. Moreover, we need to ensure that the interpretation $\alpha(f)$ is a function of type $\mathcal{D}^k \to \mathcal{D}$. In order to satisfy this property, we further require the following two conditions: $f_0 \in N$ for every k-ary symbol f, and whenever f is a constant, then $f_0 \in \mathcal{D}$.

By enforcing these conditions a strict decrease of a rewrite step is now ensured by the orientation condition $\alpha(\ell) - \alpha(r) \in core_\delta(\mathcal{D})$. Note that $\alpha(\ell) - \alpha(r)$ is a linear polynomial $c_0 + c_1 x_1 + \ldots c_m x_m$ with matrix coefficients $c_0, \ldots, c_m$ over variables $x_1, \ldots, x_m$ that occur in ℓ and that range over $\mathcal{D}$. So formally, for some set of matrices M we write $\alpha(\ell) - \alpha(r) \in M$ as an abbreviation for $\forall x_1, \ldots, x_m \in \mathcal{D}.\ \alpha(\ell) - \alpha(r) \in M$, and we call $c_0, \ldots, c_m$ the coefficients of $\alpha(\ell) - \alpha(r)$. We next instantiate the abstract setting with $\mathcal{D}$ and $core_\delta(\mathcal{D})$ by E_I and $P_{I,\delta}$ (resp. M_I and $L_{I,\delta}$), i.e., the sets of matrices that are known from Section 4.2.

▶ **Theorem 18** (Core matrix interpretations for TRSs)**.** *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs over signature $\mathcal{F}$, let α be a matrix interpretation with domain $\mathcal{D} \subseteq \mathcal{C}^{n \times n}$ for some carrier $\mathcal{C} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, and let $\emptyset \subset I \subseteq \{1, \ldots, n\}$ and $\mathcal{D} = E_I$ (resp. M_I). Let every k-ary symbol f be interpreted as $\alpha(f)(x_1, \ldots, x_k) = f_0 + f_1 \cdot x_1 + \ldots + f_k \cdot x_k$. Let $\delta \in \mathcal{C}$ satisfy $\delta > 0$. Relative termination $SN(\mathcal{R}/\mathcal{S})$ is ensured if all of the following conditions are satisfied.*

1. *$f_i \in \mathcal{D}$ for all $1 \leq i \leq k$ and k-ary symbols $f \in \mathcal{F}$*
2. *$\alpha(f) : \mathcal{D}^k \to \mathcal{D}$ (this is equivalently expressed as $\alpha(f) \in \mathcal{D}$) for every $f \in \mathcal{F}$;*
3. *$\alpha(\ell) - \alpha(r) \in N$ for every rule $\ell \to r \in \mathcal{S}$;*
4. *$\alpha(\ell) - \alpha(r) \in P_{I,\delta}$ (resp. $L_{I,\delta}$) for every rule $\ell \to r \in \mathcal{R}$.*

▶ **Theorem 19** (Sufficient conditions for core matrix interpretations for TRSs)**.** *Consider the same setup as in Theorem 18 except for requiring Conditions 2 – 4.*

2. *Condition 2 in Theorem 18 is satisfied if Condition 1 is satisfied and additionally*
   a. *$f_0 \in N$ for each $f \in \mathcal{F}$, and*
   b. *$f_0 \in \mathcal{D}$ for each constant $f \in \mathcal{F}$.*
3. *Condition 3 in Theorem 18 is satisfied if each coefficient of $\alpha(\ell) - \alpha(r)$ is in N.*
4. *Condition 4 in Theorem 18 is satisfied if each coefficient of $\alpha(\ell) - \alpha(r)$ is in N and $\alpha(\ell) - \alpha(r)$ has some coefficient in $P_{I,\delta}$ (resp. $L_{I,\delta}$).*

Theorems 18 and 19 are the main result of this section and they strictly generalize the SRS setting: if every unary symbol f is interpreted by $\alpha(f)(x_1) = f_1 \cdot x_1$ then Theorems 18 and 19 are equivalent to Corollary 17, and choosing $\mathcal{C} = \mathbb{Z}$ and $\delta = 1$ we arrive at Corollary 14.

▶ Remark 20. In the context of dependency pairs one often needs pairs of orders that form a reduction pair. We briefly mention that Theorem 18 can also be adjusted to deliver reduction pairs instead of proving $SN(\mathcal{R}/\mathcal{S})$. In the adjustment one can weaken Condition 1 to the requirement $f_i \in N$.

The sufficient conditions in Theorem 19 are incomparable to the monotone matrix interpretations of [11, 6, 29]. Our version has the advantage in Condition 4: we allow a strict decrease for any coefficient of the linear polynomial $\alpha(\ell) - \alpha(r)$ whereas the strict decrease must happen in the constant part of $\alpha(\ell) - \alpha(r)$ in [11, 6, 29].

▶ **Example 21** (Continuing Example 15)**.** Consider the SRS of Example 12, now formulated as a TRS $\{\mathsf{b}(\mathsf{a}(\mathsf{a}(\mathsf{a}(x)))) \to \mathsf{b}(\mathsf{b}(\mathsf{a}(\mathsf{a}(x)))), \mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(x)))) \to \mathsf{b}(\mathsf{b}(\mathsf{b}(\mathsf{a}(x)))), \mathsf{a}(\mathsf{b}(\mathsf{b}(\mathsf{b}(x)))) \to \mathsf{b}(\mathsf{b}(\mathsf{a}(\mathsf{a}(x))))\}$. The matrix interpretation in Example 12 for SRSs is similarly converted into an interpretation for TRSs, i.e., $\alpha(\mathsf{a})(x) = B_1 \cdot x$ and $\alpha(\mathsf{b})(x) = B_2 \cdot x$.

For the first rule, $\alpha(\ell) - \alpha(r)$ is the linear polynomial $\mathbf{0} + B_3 \cdot x$. Since the constant part of this polynomial is the zero matrix, the criteria in [11, 6, 29] to ensure termination are not applicable. However, termination is proven via Theorems 18 and 19, as $B_3 \in P_{\{1,2\},1}$, and a similar decrease is obtained for the two other rules, too.

On the other hand, Condition 2b is a new requirement in Theorem 19 which is not present in [11, 6, 29]. Recall, we require that constants must be interpreted by positive matrices, since our choice of domains exclude the zero matrix. This is in contrast to the domains in [11, 6, 29] where the zero matrix (or the zero vector) is contained in the domain. Note that Condition 2b cannot be dropped from Theorem 19. Consider $\mathcal{R} = \{\mathsf{f}(x) \to \mathsf{g}(x)\}$ and $\mathcal{S} = \{\mathsf{g}(\mathsf{a}) \to \mathsf{f}(\mathsf{a})\}$. Clearly, $SN(\mathcal{R}/\mathcal{S})$ does not hold, but one can find a matrix interpretation that satisfies all required conditions except for Condition 2b in Theorem 19 (and the corresponding Condition 2 of Theorem 18), namely: $\mathcal{C} = \mathbb{Z}$, $n = 1$, $I = \{1\}$, $\mathcal{D} = E_I$, $\delta = 1$, $\alpha(\mathsf{f})(x) = 2x$, $\alpha(\mathsf{g})(x) = x$, and $\alpha(\mathsf{a}) = 0$.

We show how Theorem 18 can be implemented even better, i.e., by using more relaxed sufficient criteria to ensure Conditions 2–4. It will turn out that with these improved conditions, Theorem 18 even subsumes the monotone matrix interpretations of [11, 29].

First of all, we can improve the criterion for Condition 4 in the M_I setting:

▶ **Lemma 22.** *Condition 4 in Theorem 18 is satisfied in the M_I setting, if all coefficients of $\alpha(\ell) - \alpha(r)$ are in N and $\forall i \in I.\ \exists c \in C.\ \exists j \in I.\ c_{i,j} \succ_\delta 0$, where C is the set of coefficients of $\alpha(\ell) - \alpha(r)$.*

This new condition is more general than the one of Theorem 19, since there the sufficient criterion to ensure Condition 4 is equivalent to $\exists c \in C.\ \forall i \in I.\ \exists j \in I.\ c_{i,j} \succ_\delta 0$ where one cannot choose different coefficients c for each row i.

▶ **Example 23.** Termination of the TRS $\{\mathsf{f}(y, \mathsf{f}(\mathsf{g}(x), \mathsf{b})) \to \mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a}, \mathsf{b})), y), x))\}$ is shown via the core matrix interpretation α with $n = 2$, $I = \{1, 2\}$, $\mathcal{D} = M_I$ given by

$$\alpha(\mathsf{a}) = \begin{pmatrix} 0 & \underline{1} \\ 0 & \underline{1} \end{pmatrix}, \qquad \alpha(\mathsf{g})(x_1) = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} \underline{1} & 0 \\ \underline{1} & 0 \end{pmatrix} x_1,$$

$$\alpha(\mathsf{b}) = \begin{pmatrix} \underline{1} & 0 \\ \underline{5} & 4 \end{pmatrix}, \qquad \alpha(\mathsf{f})(x_1, x_2) = \begin{pmatrix} 0 & \underline{1} \\ \underline{1} & 1 \end{pmatrix} x_1 + \begin{pmatrix} \underline{2} & 0 \\ 0 & \underline{1} \end{pmatrix} x_2.$$

For the only rule $\ell \to r$ of the TRS we evaluate

$$\alpha(\ell) - \alpha(r) = \alpha(\mathsf{f}(y, \mathsf{f}(\mathsf{g}(x), \mathsf{b}))) - \alpha(\mathsf{g}(\mathsf{f}(\mathsf{f}(\mathsf{g}(\mathsf{f}(\mathsf{a}, \mathsf{b})), y), x)) = \begin{pmatrix} \underline{1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \underline{1} & 0 \end{pmatrix} y.$$

We apply Lemma 22 and choose, for $i = 1$, the absolute coefficient, and for $i = 2$, the coefficient of y.

For the E_I setting, we further improve the sufficient criteria for Conditions 2–4 in Theorem 18. Actually, there is an easy decision procedure to check whether some linear polynomial p with variables ranging over E_I always evaluates to some element of N (or E_I or $P_{I,\delta}$). The procedure checks on weak monotonicity and then performs a single membership test w.r.t. N (or E_I or $P_{I,\delta}$), namely, for the element that is obtained when all variables are instantiated by the unique minimal element in E_I. This minimal element is 1_I, the matrix which is always 0, except that $(1_I)_{i,i} = 1$ whenever $i \in I$.

▶ **Lemma 24.** *Let p be a linear polynomial over variables $x_1, \ldots, x_k$. Let $S \in \{N, E_I, P_{I,\delta}\}$. The following two conditions are equivalent.*

- *$\forall x_1, \ldots, x_k \in E_I.\ p(x_1, \ldots, x_k) \in S$.*
- *$p(1_I, \ldots, 1_I) \in S$ and all coefficients of variables in p are in N.*

Consequently, Conditions 2–4 of Theorem 18 can be decided in the E_I setting by applying Lemma 24 for the tests $p = \alpha(f) \in E_I$ for every $f \in \mathcal{F}$ (2), $p = \alpha(l) - \alpha(r) \in N$ for every $\ell \to r \in \mathcal{S}$ (3), and $p = \alpha(l) - \alpha(r) \in P_{I,\delta}$ for every $\ell \to r \in \mathcal{R}$ (4).

▶ **Example 25.** The E_I interpretation $\alpha(f)(x) = 2x - 1$ for $n = 1$ and $I = \{1\}$ is not accepted by the sufficient criterion for Condition 2 via Theorem 19, but it is accepted via Lemma 24, since $\alpha(f)(1) = 2 \cdot 1 - 1 = 1$ is clearly in E_I, and the only coefficient of a variable of $\alpha(f)$ is 2, an element of N.

We do not know whether one can design a similar efficient decision procedure for the conditions in the M_I setting. Here, the complication arises that there is not a unique minimal element such as 1_I in the E_I case, but there are $|I|^{|I|}$ many of those: a matrix A is minimal in M_I if for every $i \in I$ there is exactly one $j \in I$ such that $A_{i,j} = 1$, and all other entries are 0. Hence, substituting each variable in a polynomial by every minimal matrix in M_I to then check membership in S (as in the second test of Lemma 24) will result in exponential runtime.

We further show that Theorem 18 in combination with Lemma 24 fully subsumes matrix interpretations as they are defined by Endrullis et al. [11] and Neurauter et al. [29]. To this end, we use a transformation similar to the one of Contejean et al. [5, Section 3.4] that is known for polynomial interpretations over integers. In our setting, we basically switch from domain E_I to N. It is easy to see that $x \in E_I$ iff $x = y + 1_I$ for some $y \in N$. We now just substitute each variable x that ranges over E_I by a variable y that ranges over N.

▶ **Lemma 26.** *Let p be some polynomial using variables $x_1, \ldots, x_k$ that range over E_I. Let $S \in \{N, E_I, P_{I,\delta}\}$. The following two conditions are equivalent.*
- *$\forall x_1, \ldots, x_k \in E_I.\ p(x_1, \ldots, x_k) \in S$.*
- *$\forall y_1, \ldots, y_k \in N.\ p(y_1 + 1_I, \ldots, y_k + 1_I) \in S$.*

The advantage of the switch to N via Lemma 26 is that the criteria are equivalences, and the new universally quantified conditions over N can easily be decided as follows, using an even simpler variant of Lemma 24 (for N instead of E_I.) This characterization is used for $I = \{1\}$ in the termination criteria of MI of Endrullis et al. and Neurauter et al.

▶ **Lemma 27.** *Let p be a linear polynomial over variables $x_1, \ldots, x_k$. Let $S \in \{N, E_I, P_{I,\delta}\}$. The following two conditions are equivalent.*
- *$\forall x_1, \ldots, x_k \in N.\ p(x_1, \ldots, x_k) \in S$.*
- *The constant part of p is in S and all coefficients of variables in p are in N.*

▶ **Theorem 28.** *If there is some relative termination proof via a matrix interpretation following [11, 29], then there also is a relative termination proof using Theorem 18 and Lemma 24.*

Proof. Given a strictly monotone MI in the style of [11, 29] with $\alpha'(f)(y_1, \ldots, y_k) = f_0 + f_1 y_1 + \cdots + f_k y_k$ – where f_0 is interpreted as matrix – we convert it into the E_I interpretation α with $I = \{1\}$ and $\alpha(f)(x_1, \ldots, x_k) = \alpha'(f)(x_1 - 1_I, \ldots, x_k - 1_I) + 1_I$. We get the relationship $\alpha(t) = \alpha'(t) + 1_I$ for all ground terms t, and both interpretations define the same ordering, since $(\alpha(\ell) - \alpha(r))\{z_1/z_1 + 1_I, \ldots, z_m/z_m + 1_I\} = \alpha'(\ell) - \alpha'(r)$. Moreover, all conditions of Theorem 18 are satisfied for α whenever α' satisfies the criteria for a relative termination proof in [11, 29], where the criteria are converted into the conditions with the help of Lemmas 27, 26 and 24. ◀

We briefly show that Theorem 19 is not sufficient to obtain this subsumption result and therefore, Lemma 24 is crucial: given $\alpha'(f)(y) = 2y$, it is transformed into $\alpha(f)(x) = \alpha'(f)(x - 1) + 1 = 2(x - 1) + 1 = 2x - 1$, and accepting this interpretation α by Theorem 18 requires Lemma 24, cf. Example 25.

Note that the very same transformation can also be applied in the other direction, so core matrix interpretations can be turned into the ones of [11, 29] for the $E_{\{1\}}$ setting.

It remains the question of whether Theorem 18 with corresponding sufficient criteria improve the power of automated termination proving, and whether it is advantageous to have a choice of the carrier. The latter question can be answered affirmative. Neurauter and Middeldorp proved that EWZ MI with carriers $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ become strictly more powerful by choosing the larger carrier [29]. The same hierarchy carries over to Theorem 18 by using exactly the same examples and arguments as in [29].

We leave it as future work, to study the relative power of CGP MI + δ and core MI + δ, where the answer might be provided theoretically or experimentally.

5 Future Work

Although crucial results about right-forward closures and core matrix interpretations have been fully formalized, there still needs to be work done for both of these parts.

Regarding right-forward closures, it remains to verify computable approximations of $\mathit{RFC}(\mathcal{R})$. Here one might start with approximations in the SRS setting [36, Section 6], extend this to linear TRSs [15, Section 8] and even further to right-linear TRSs [25, Section 8]. In a second step, one would then have to combine these approximations with CeTA's checker for match-bounds. To eventually increase the number of certified proofs, some shared certificate format for RFC match-bounds must be developed, so that it fits the need of both CeTA and the automated termination tools such as AProVE and matchbox.

For core matrix interpretations, it would be nice to know the exact relationship to the matrix interpretations as defined by Courtieu et al., and to further know whether core matrix interpretations improve automated termination analysis for TRSs in a global sense, i.e., where arbitrary alternative termination techniques may be applied instead. At the moment, we only know that the power of CeTA is improved, as it can now certify proofs using core matrix interpretations, for both SRSs and TRSs.

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