Stabilized Profunctors and Matrix Representation

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Abstract

The (bi)category of profunctors on groupoids is a categorification of the relational model of linear logic. Its objects are not just sets but rather sets whose elements are equipped with groups encoding their symmetries, and its morphisms carry actions by these symmetries. While detailed information on such symmetries helps with, e.g., adequacy proofs of profunctorial models, it makes operations such as composition more difficult to compute. A way to ease the computation is to transform a profunctor into a matrix. Although the matrix representation is not functorial in general, it is known to behave well for certain subclasses, such as the class of profunctors definable by λ-terms. The mathematical reason behind this phenomenon, however, was not understood.

This paper shows that the key is *stability*. Stability is a classical concept in domain theory, and has been extended to profunctors in Taylor's work and further developed by Fiore et al. All λ-definable profunctors are known to be stabilized, and we show that the matrix representation behaves well for stabilized profunctors. We prove that the matrix representation defines a functor from stabilized profunctors to matrices that preserves the linear logic structures.

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1 Introduction

Symmetry is a crucial property of proofs and programs. A striking and intuitive example of this symmetry appears in the exponential modality ! of linear logic [18]. The type $!A$ intuitively grants arbitrarily many copies of A-type data. The key aspect here is that these copies of A-type data are essentially identical, meaning that a consumer of $!A$-type data can use any of the copies interchangeably without affecting the behaviour. In other words, $!A$-type data is symmetric under exchange of copies. Conversely, the provider of $!A$-type data ensures that the multiple copies of A-type data are indistinguishable to the consumer.

A model that explicitly handles such symmetries is the (bi)category of profunctors on groupoids. A profunctor $\mathcal{P} \colon \mathbb{A} \nrightarrow \mathbb{B}$ between groupoids $\mathbb{A}$ and $\mathbb{B}$ is a functor $\mathbb{A}^{\mathrm{op}} \times \mathbb{B} \longrightarrow \mathbf{Set}$. This can be compared with relations: a relation R between sets A and B is a function $A \times B \longrightarrow \{0, 1\}$. Whereas a relation only tells us whether $a \in A$ and $b \in B$ are related or not, a profunctor records how and how many ways $a \in \mathbb{A}$ and $b \in \mathbb{B}$ are associated, via the set $\mathcal{P}(a, b)$. Moreover, morphisms of $\mathbb{A}$ and $\mathbb{B}$ act on these ways of association, so that, for instance, the $!A$-action encodes what happens when copies of $!A$-type are exchanged. Such additional information carried by profunctors is often useful for theoretical analysis;

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therefore, the (bi)category of profunctors is used as a tool for analysing, e.g., intersection types [29], the Taylor expansion [34, 35] of λ-terms and programs in the sense of Ehrhard and Regnier [13], and game semantics [8–11].

Unfortunately, profunctors are inherently complex structures, involving sets equipped with group actions, and their composition involves quotients under these group actions. It is therefore natural to consider transforming a profunctor into an object on which computation becomes easier, without losing the information. Two successful approaches exist.

The first example is the transformation from Joyal's species of structures [23] into formal power series. The species of structures, a special case of profunctors [15], captures the mathematical essence of the formal power series arising in combinatorics. Formally, it is a functor $\mathbf{FinBij} \longrightarrow \mathbf{FinBij}$ on the category $\mathbf{FinBij}$ of finite sets and bijections. Given $F\colon \mathbf{FinBij} \longrightarrow \mathbf{FinBij}$, the associated *exponential power series* is

$$\widehat{F}(x) \quad := \quad \sum_{n=0}^{\infty} \frac{\#F([n])}{n!}\, x^n \tag{1}$$

where $[n] := \{0, 1, \ldots, n-1\}$ and $\#X$ is the cardinality of X. This power series captures the essential features of the original species, and it is known that many operations on species can be reduced to computations on their associated power series (see a textbook [3]).

The second example is the matrix representation of profunctorial interpretations of λ-terms. Consider a profunctor $\mathcal{P}\colon \mathbb{A} \looparrowright \mathbb{B}$ between groupoids. Assume for simplicity that $\mathbb{A}$ and $\mathbb{B}$ are skeletal (i.e., $a \cong a'$ implies $a = a'$). Then the profunctor $\mathcal{P}$ induces an $(\mathit{obj}(\mathbb{A}) \times \mathit{obj}(\mathbb{B}))$-matrix $\widehat{\mathcal{P}}$ defined by

$$\widehat{\mathcal{P}}_{a,b} \quad := \quad \frac{\#\mathcal{P}(a,b)}{\Phi^{\ominus}(a)\Psi^{\oplus}(b)},$$

where $\Phi^{\ominus}(a), \Psi^{\oplus}(b) \in (0, \infty)$ are certain reals (see Section 2.2). This is actually a generalisation of the exponential power series, and the matrix representation behaves well for certain subclasses of profunctors, e.g., those definable by λ-terms [10, 35].

However, the matrix representation does not behave well for general profunctors. In particular, the matrix representation is not functorial (see Example 8). This suggests that profunctors definable by λ-terms (as well as species of structures) possess a certain property. Intuitively, the property is the one mentioned at the beginning of this section: for profunctors definable by λ-terms, the providers and consumers of $!A$-type data respect an agreement concerning symmetry, whereas a general profunctor does not necessarily do so. Nevertheless, no mathematically rigorous analysis of this phenomenon has yet been given.

This paper shows that the key property is *stability* [4], a classical concept in domain theory. As early as 1989, Taylor [32] proposed the concept of a *stable functor*, motivated by Girard's normal functors [19] and Berry's stable domain model [4], aiming for something akin to a categorification of the stable model. Recently, the stable functor model has been extensively investigated by Fiore et al. [14, 16, 17], who developed a modern treatment based on the concept of the orthogonality construction introduced by Hyland and Schalk [21]. In these works, profunctors equipped with "sufficiently free" actions (called *stabilized profunctors*) play an important role, and the "freeness" expected of a profunctor is described by a *creed*. Although these developments were completely independent of the matrix representation, we show that stability is essential to the functoriality of the matrix representation.

The technical contributions of this paper can be summarised as follows:

- We prove that the matrix representation is functorial when it is restricted to stabilized profunctors (Theorem 19). Formally, it is a functor from (the classifying category of)

the bicategory $\mathbf{SProf}^{\star}_{\omega}$ of locally finite stabilized profunctors with chosen factors to the category $\mathbf{WRel}_{[0,\infty]}$ of weighted relations [24, 25] with coefficients $[0, \infty]$. Furthermore, the stability almost completely characterises the functoriality (Theorem 20).
- We show that the matrix representation strictly preserves the linear logic structures from the stabilized profunctor model to the weighted relational model.

A consequence is that the matrix representation is well-behaved for profunctors definable by linear logic proofs, since the (bi)category **SProf** of stabilized profunctors is a model of linear logic [17] and the forgetting functor $\mathbf{SProf} \longrightarrow \mathbf{Prof}$ strictly preserves the linear logic structures. This subsumes the well-behavedness for λ-definable profunctors [10, 35]. The class of stabilized profunctors strictly includes this class.

1.1 Related Work

The symmetry of $!A$-type data can be found in the game model by Abramsky et al. [1]. In their model, ! corresponds to a family indexed by the natural numbers, where the indices are visible to the consumer of $!A$-type data but hidden from the provider. Attempts to capture such symmetry through group actions appear, for example, in the work of Melliès [26]. He considered a set with a left action and a right action of groups representing the Proponent and Opponent symmetries to control the visibility of information for each player. While his insight is profound, a technical issue remains: the actions of the Opponent and Proponent groups are assumed to commute, but this assumption actually fails for the interpretation of $!A$. Commutative actions of groups G_L and G_R are an action of their cartesian product $G_L \times G_R$, and subsequent research [9, 10, 29, 35] suggests that the right setting is to use the *bicrossed product* or *Zappa–Szép product* instead of the cartesian product. The term "bicrossed product" is not commonly used in this context, but it is essentially equivalent to a strict factorization system. The aforementioned line of work shows that strict factorization systems arise naturally.

A closely related but distinct line of research aims to construct simple models that capture the essence of game-semantic models [1, 20]. Classical studies have used variants of relational models [2, 5, 7, 27, 37], but more recently, there have been attempts to employ (pre)sheaves and profunctors [8–11, 34–36]. This work follows this line of research and suggests that stabilized profunctors, rather than profunctors, provide a more suitable framework.

The notion of a stable functor was introduced by Taylor [32]. Inspired by Girard's normal functor [19], he aimed to develop a categorification of the stable model by Berry [4] and showed that stable functors model System F. Stable functors have recently been revisited by Fiore et al. [14, 16]. Their motivation was to provide a higher-order generalisation of the notion of polynomial functors, that is, to construct a cartesian closed category containing the category of polynomial functors. This paper provides new evidence for the relevance of stable functors and stabilized profunctors.

There is another result demonstrating the importance of stabilized profunctors in the literature. Tsukada et al. [33] prove that stability provides a correctness criterion for proof-nets. This can be regarded as a profunctorial version of Retoré [30]'s result that coherence provides a correctness criterion.

2 Background: Profunctors and Matrix Representations

This section reviews the (bi)category of profunctors and their matrix representations [10, 35]. The (bi)category of profunctors has nice structure and properties (as shown in [15]), but many operations on profunctors are somewhat involved. To ease calculations, their generating series [23] and matrix representations [10, 35] have been introduced. Although the matrix representation is functorial on λ-definable profunctors [10, 35], it is not functorial on general profunctors. Section 2.3 gives a concrete counterexample to the functoriality.

▶ **Notation 1.** *We write I for the trivial category, which has one object $\star$ and one morphism $\mathrm{id}_\star$. For groups G and H, $G \leq H$ means that G is a subgroup of H.*

We write $[0, \infty]$ for the set $\{r \in \mathbb{R} \mid r \geq 0\} \cup \{\infty\}$. The sum $\sum_{j \in J} r_j$ of a possibly infinite family $(r_j)_{j \in J}$ of elements in $[0, \infty]$ is well-defined as the least upper bound of $\{\sum_{k \in K} r_k \mid K \subseteq J, K\colon \text{finite}\}$ (where $\infty + r = \infty$).

A setoid *is a set equipped with an equivalence relation. Given a setoid $(X, \approx_X)$, we write $[x]_\approx$ for the $\approx$-equivalence class $\{\, x' \in X \mid x \approx_X x' \,\}$ of x. We will often leave the equivalence relation $\approx_x$ implicit, referring to X as a setoid and writing $[x]_{\approx_X}$ simply as $[x]$.* ◀

2.1 Profunctors on Groupoids

A *groupoid* $\mathbb{A}$ is a category in which every morphism is invertible. Then $\mathbb{A}(a, a)$ is a group for every object $a \in \mathbb{A}$. We write $\mathbb{A}, \mathbb{B}, \mathbb{C}$ for groupoids, a, b, c for their objects, and α, β, γ for their morphisms. We use the diagrammatic order for composition: $(\alpha; \beta) \in \mathbb{A}(a, c)$ when $\alpha \in \mathbb{A}(a, b)$ and $\beta \in \mathbb{A}(b, c)$. Let **Grpd** be the category of small groupoids and functors.

A groupoid is a set with symmetries. Let A be a set and $(G_a)_{a \in A}$ be a family of groups indexed by $a \in A$. These data determine a groupoid $\mathbb{A}$ as follows: $\mathit{obj}(\mathbb{A}) := A$, $\mathbb{A}(a, a) := G_a$ and $\mathbb{A}(a, a') := \emptyset$ if $a \neq a'$. Every groupoid is obtained in this way up to equivalence. Conversely, forgetting the symmetries makes a groupoid a setoid. Let $|\mathbb{A}| := (\mathit{obj}(\mathbb{A}), \cong_{\mathbb{A}})$, the set of objects in $\mathbb{A}$ equipped with the equivalence relation given by isomorphism.

For groupoids $\mathbb{A}$ and $\mathbb{B}$, a *profunctor* $\mathcal{P}\colon \mathbb{A} \nrightarrow \mathbb{B}$ is a functor $\mathcal{P}\colon \mathbb{A}^{\mathrm{op}} \times \mathbb{B} \longrightarrow \mathbf{Set}$. The definition works for general categories $\mathbb{A}$ and $\mathbb{B}$, but this paper focuses on profunctors between groupoids. For $x \in \mathcal{P}(a, b)$, $\alpha \in \mathbb{A}(a', a)$ and $\beta \in \mathbb{B}(b, b')$, we write $\alpha \cdot x \cdot \beta$ for $\mathcal{P}(\alpha, \beta)(x)$. When $\beta = \mathrm{id}$, we simply write $\alpha \cdot x \cdot \mathrm{id}$ as $\alpha \cdot x$, and similarly for $x \cdot \beta := \mathrm{id} \cdot x \cdot \beta$.

A profunctor is a relation with multiplicity and symmetry action. It tells us not only whether $a \in \mathbb{A}$ and $b \in \mathbb{B}$ are related (i.e., whether $\mathcal{P}(a, b) = \emptyset$ or not) but also in how many ways a and b are related. Furthermore, the ways $x, y \in \mathcal{P}(a, b)$ to relate a and b are not independent of each other; they can be transformed by the symmetry action $\alpha \cdot x \cdot \beta$, where $\alpha \in \mathbb{A}(a, a)$ and $\beta \in \mathbb{B}(b, b)$.

Profunctors $\mathcal{P}\colon \mathbb{A} \nrightarrow \mathbb{B}$ and $\mathcal{Q}\colon \mathbb{B} \nrightarrow \mathbb{C}$ compose analogously to relations:

$$(\mathcal{P}; \mathcal{Q})(a, c) \quad := \quad \int^{b \in \mathbb{B}} \mathcal{P}(a, b) \times \mathcal{Q}(b, c).$$

Here, $\int^{b \in \mathbb{B}}$ is an operation called *coend*, which can be viewed as a kind of existential quantification $\exists b \in \mathbb{B}$. Concretely, the composition is

$$(\mathcal{P}; \mathcal{Q})(a, c) \quad := \quad \left(\textstyle\coprod_{b \in \mathbb{B}} \mathcal{P}(a, b) \times \mathcal{Q}(b, c)\right) / {\sim},$$

where $\sim$ is the equivalence relation defined by

$$\mathsf{in}_b(x, y) \sim \mathsf{in}_{b'}(x', y') \iff (x, y) = (x' \cdot \beta, \beta^{-1} \cdot y') \text{ for some } \beta \in \mathbb{B}(b', b)$$

(where $\mathsf{in}_b\colon \mathcal{P}(a, b) \times \mathcal{P}(b, c) \longrightarrow \coprod_{b \in \mathbb{B}} \mathcal{P}(a, b) \times \mathcal{P}(b, c)$ is the injection). We shall simply write $\mathsf{in}_b(x, y)$ as (x, y). Then $(x \cdot \beta, y) \sim (x, \beta \cdot y)$ for every $x \in \mathcal{P}(a, b)$, $\beta \in \mathbb{B}(b, b')$ and $y \in \mathcal{Q}(b', c)$. The groupoid action is defined by $\alpha \cdot [(x, y)]_\sim \cdot \beta := [(\alpha \cdot x,\ y \cdot \beta)]_\sim$.

Small groupoids, profunctors, and natural transformations form a bicategory, which we write as **Prof**. Fiore et al. [14, 15] showed that **Prof** is a model of linear logic.[1] The details of this model will be described later (see Section 4).

Although **Prof** is a bicategory and it would be natural to work bicategorically, we frequently treat it as a 1-category throughout this paper for technical convenience (see Remark 2). A bicategory induces 1-categories in several ways. In this work, we shall use the *classifying category* $\mathsf{C}\mathcal{B}$ of $\mathcal{B}$ in the sense of Bénabou [6, Sec. 7]. For a bicategory $\mathcal{B}$, its classifying category has 0-cells as objects and, as morphisms, 1-cells modulo the existence of an isomorphic 2-cell.[2]

If $\mathcal{B}$ is a model of linear logic, then so is $\mathsf{C}\mathcal{B}$. As the 2-dimensional notion of a model of intuitionistic linear logic in this work, we adopt that introduced by Fiore et al. [14, Def. 4.12]. Then it can be verified directly that its classifying category naturally forms a 1-dimensional model, namely, an SMCC with a linear exponential comonad (more precisely, following the definition in [31, Def. 10] or [28, Sec. 7.4]). Moreover, the 2-dimensional models in this paper (**Prof**, **SProf**, and $\mathbf{SProf}^{\star}_{\omega}$) possess sufficient 2-dimensional biproducts and a classical structure so that their classifying categories admit 1-dimensional biproducts and $*$-autonomous structures.

▶ Remark 2. We use classifying categories in this paper because the theory of 1-categorical models of linear logic is better developed than that of bicategorical models. In particular, we work with the 2-category of 1-categorical models of linear logic [12], whereas it is not yet entirely clear what the corresponding 2-category of bicategorical models of linear logic should be. We expect that, as the study of a 2-category (or, bi-, Gray- or tri-category) of bicategorical models of linear logic progresses, the discussion in this paper will be able to be developed on that basis. ◀

▶ Remark 3. A functor $\mathsf{C}\mathbf{Prof} \longrightarrow \mathcal{C}$ to a 1-category $\mathcal{C}$ induces a pseudo-functor $\mathbf{Prof}_{2\text{-}iso} \longrightarrow \mathcal{DC}$, where $\mathsf{D}\mathcal{C}$ is a bicategory in which $\mathcal{C}(a, b)$ is regarded as the discrete category generated by the set $\mathcal{C}(a, b)$, and $\mathbf{Prof}_{2\text{-}iso}$ is the bicategory of small groupoids, profunctors and natural *isomorphisms*. ◀

2.2 Relational and Matrix Representations

Since operations such as the composition of profunctors are rather involved, it is natural to consider transforming them into computationally simpler objects. The first idea that comes to mind is a transformation into relations. We show that this approach works well.

The category **Rel** of relations has setoids as objects and relations $R \subseteq (A/{\cong_A}) \times (B/{\cong_B})$ as morphisms from A to B. A groupoid $\mathbb{A}$ induces the setoid $|\mathbb{A}|$ of its objects, and a profunctor $\mathcal{P} \colon \mathbb{A} \nrightarrow \mathbb{B}$ gives a relation $|\mathcal{P}| \subseteq |\mathbb{A}| \times |\mathbb{B}|$ defined by $([a], [b]) \in |\mathcal{P}| :\Longleftrightarrow \mathcal{P}(a, b) \neq \emptyset$. It is easy to see that this is functorial and strictly preserves the linear logic structure.

▶ Remark 4. A reader may wonder why we did not adopt the following seemingly natural definition. Let **Rel** be the category of sets and relations, and $|\mathbb{A}|'$ be the set $\{\, [a]_{\cong} \mid a \in \mathbb{A} \,\}$ of isomorphism classes of objects in $\mathbb{A}$ (for profunctors, $|\mathcal{P}|' = |\mathcal{P}|$). This is functorial, but it preserves the linear logic structure only strongly, not strictly: $|\mathbb{A} \times \mathbb{B}|' =$

[1] Precisely, they showed it for the bicategory of small *categories*, profunctors, and natural transformations, and full sub-bicategory **Prof** inherits the model structure.

[2] We have been previously criticized for a potential size issue with the construction of $\mathsf{C}\mathbf{Prof}$, as it requires a quotient of $\mathbf{Prof}(\mathbb{A}, \mathbb{B})$, which is not necessarily small. A simple way to justify the quotient here is to assume a Grothendieck universe U and to use $\mathbf{Prof}_U$ of the bicategory of U-small groupoids and $\mathbf{Set}_U$-valued profunctors.

$(obj(\mathbb{A}) \times obj(\mathbb{B}))/{\cong_{\mathbb{A}\times\mathbb{B}}}$ is canonically isomorphic to but not exactly the same as $|\mathbb{A}|' \times |\mathbb{B}|' = (obj(\mathbb{A})/{\cong_{\mathbb{A}}}) \times (obj(\mathbb{B})/{\cong_{\mathbb{B}}})$. A strong functor (in the mathematical sense) can sometimes appear to preserve the structure only "weakly," relative to one's naive expectations (see Section 4.5 for examples of strong functors). For this reason, this paper mainly works with functors that strictly preserve the structures. To this end, we need a slight modification of the category **Rel** (as well as **WRel** introduced later). ◀

Although the relational representation behaves well, it can be problematic in that it discards some useful quantitative information. For example, Tsukada et al. [35] showed that profunctors provide a foundation for the semantics of probabilistic and quantum programs, where the number $\#\mathcal{P}(a, b)$ of ways relating a and b plays an essential role.

One way to simplify computation without losing this quantitative information is to represent of Joyal's combinatorial species [23] as formal power series. Recall that a species (in the sense of Joyal [23]) is a functor $\mathbf{FinBij} \longrightarrow \mathbf{FinBij}$, where **FinBij** is the category of finite sets and bijections. The *(exponential) generating series* $\widehat{\mathcal{P}}$ of a species $\mathcal{P} \colon \mathbf{FinBij} \longrightarrow \mathbf{FinBij}$ is a formal power series defined in Equation (1), i.e.,

$$\widehat{\mathcal{P}}(x) \quad := \quad \sum_{n=0}^{\infty} \frac{\#\mathcal{P}([n])}{n!} x^n,$$

where $[n]$ represents the set $\{0, 1, \ldots, n-1\}$ and $\#X$ is the cardinality of X. The generating series $\widehat{\mathcal{P}}$ is easier to handle and contains much information about the original species $\mathcal{P}$. Many operations on species induce corresponding operations on generating series.

A species $\mathcal{P}$ induces a profunctor $\mathbf{FinBij}^{\mathrm{op}} \nrightarrow I$ (which we also write as $\mathcal{P}$) by composing the embedding $\mathbf{FinBij} \to \mathbf{Set}$, By extending the generating function from species to profunctors according to this correspondence, we can assign a matrix to each profunctor by

$$\|\mathcal{P}\|_{[a]_{\cong},[b]_{\cong}} \quad := \quad \frac{\#\mathcal{P}(a, b)}{\Phi^{\ominus}(a)\Psi^{\oplus}(b)}, \tag{2}$$

where $\Phi^{\ominus}(a)$ and $\Psi^{\oplus}(b)$ are positive reals, details of which will be discussed shortly.

▶ Remark 5. When a combinatorial species $\mathcal{P} \colon \mathbf{FinBij} \longrightarrow \mathbf{FinBij}$ is viewed as the profunctor $\mathcal{P} \colon \mathbf{FinBij}^{\mathrm{op}} \nrightarrow I$, the generating series $\widehat{\mathcal{P}}(x)$ can be written in terms of the matrix representation, as $\widehat{\mathcal{P}}(x) = \sum_{n=0}^{\infty} \|\mathcal{P}\|_{n,\star}\, x^n$, where n is the isomorphism class of sets consisting of n-elements and we choose $\Phi^{\ominus}([n]) = \#\mathbf{FinBij}([n],[n]) = n!$ and $\Psi^{\oplus}(\star) = 1$. Hence, $\|\mathcal{P}\|$ is the vector consisting of the coefficients of the generating series $\widehat{\mathcal{P}}(x)$. ◀

To properly define the matrix representation, we first introduce the target of the transformation. It is the *weighted relational model* given by Lamarche [25] and Laird et al. [24],[3] with its coefficient semiring specialized to $[0, \infty] := \{r \in \mathbb{R} \mid r \geq 0\} \cup \{\infty\}$. We write $\mathbf{WRel}_{[0,\infty]}$ for the weighted relational model with coefficient $[0, \infty]$. Its objects are setoids, and its morphisms from A to B are $((A/{\cong_A}) \times (B/{\cong_B}))$-matrices $R = (R_{[a],[b]})_{[a]\in(A/\cong_A),[b]\in(B/\cong_B)}$ with $R_{[a],[b]} \in [0, \infty]$. Given $R \in \mathbf{WRel}_{[0,\infty]}(A, B)$ and $S \in \mathbf{WRel}_{[0,\infty]}(B, C)$, their composite $(R; S) \in \mathbf{WRel}_{[0,\infty]}(A, C)$ is defined by

$$(R; S)_{[a],[c]} \quad := \sum_{[b]\in(B/\cong_B)} R_{[a],[b]} S_{[b],[c]} \quad := \quad \sup \left\{ \sum_{[b]\in J} R_{[a],[b]} S_{[b],[c]} \;\middle|\; J \subseteq_{\mathrm{fin}} (B/{\cong_B}) \right\},$$

where $J \subseteq_{\mathrm{fin}} (B/{\cong_B})$ means that J is a finite subset of $(B/{\cong_B})$ (we define $0 \cdot \infty := 0$ and $r \cdot \infty := \infty$ when $r > 0$).

[3] As mentioned above, we need a slight modification: objects are setoids instead of sets.

▶ **Definition 6** (Matrix representation). *We say a groupoid* $\mathbb{A}$ *is* locally finite *if* $\#\mathbb{A}(a,a) < \infty$ *for every* a*. A* factor *of* $\mathbb{A}$ *is an assignment* $\Phi^{\oplus} : obj(\mathbb{A}) \longrightarrow (0,\infty)$ *such that* $\Phi^{\oplus}(a) = \Phi^{\oplus}(a')$ *when* $a \cong_A a'$*. We define* $\Phi^{\ominus}(a) := \#\mathbb{A}(a,a)/\Phi^{\oplus}(a)$*, so* $\Phi^{\oplus}(a)\Phi^{\ominus}(a) = \#\mathbb{A}(a,a)$*. The* matrix representation $\|\mathcal{P}\|$ *of a profunctor* $\mathcal{P}\colon \mathbb{A} \rightarrow\!\!\!\!\!\!+\,\mathbb{B}$ *between locally finite groupoids* $\mathbb{A}$ *and* $\mathbb{B}$ *with factors is a matrix* $\|\mathcal{P}\| \in \mathbf{WRel}_{[0,\infty]}(\|\mathbb{A}\|, \|\mathbb{B}\|)$ *given by Equation* (2)*, where* $\|\mathbb{A}\|$ *is the setoid of the objects of* $\mathbb{A}$*.* ◀

Let $\mathbf{Prof}_\omega$ be the restriction of $\mathbf{Prof}$ to locally finite groupoids. We define $\mathbf{Prof}^\star_\omega$ as the (bi)category essentially the same as $\mathbf{Prof}$, except that each object is associated with a factor. Concretely, its 0-cell is $(\mathbb{A}, \Phi^{\oplus})$, a locally finite groupoid with a factor, and its 1-cell between $(\mathbb{A}, \Phi^{\oplus})$ and $(\mathbb{B}, \Psi^{\oplus})$ is a profunctor $\mathcal{P}\colon \mathbb{A} \rightarrow\!\!\!\!\!\!+\,\mathbb{B}$; note that the factor plays no role in the definitions of morphisms and composition. Then $\|-\|$ maps an object (resp. a morphism) of $\mathbf{Prof}^\star_\omega$ to an object (resp. a morphism) of $\mathbf{WRel}_{[0,\infty]}$.

▶ Remark 7. The reader may find it puzzling that, although the factors appear in the definition of the matrix representation, they do not appear in the definition of morphisms or composition in $\mathbf{Prof}^\star_\omega$. This reflects the counterintuitive fact that the factors in the matrix representation can be chosen completely arbitrarily.

In previous studies [10, 35], $\Phi^{\oplus}(a)$ was defined through a strict factorization system associated to $\mathbb{A}$. Roughly speaking, in these settings, objects are not just groupoids, but groupoids $\mathbb{A}$ equipped with a strict factorization system (L, R) (that is, L and R are wide subcategories of $\mathbb{A}$ satisfying certain conditions), and a profunctor is expected to be "compatible" with this factorization system in a suitable sense; then the matrix representation is defined by taking $\Phi^{\oplus}(a) := \#R(a,a)$.

Our observation, however, is that factors need not be tied to a particular strict factorization system. The compatibility requirement shall be described in Section 3 in terms of creed, and a factor can be chosen completely independently of creed. Functoriality of the matrix representation follows from $\Phi^{\oplus}(a)\Phi^{\ominus}(a) = \#\mathbb{A}(a,a)$ and stability, and the strict preservation of the monoidal structure requires only the condition $(\Phi \otimes \Psi)^{\oplus}(a,b) = \Phi^{\oplus}(a)\Psi^{\oplus}(b)$. See also Section 4 and Section 4.5. ◀

It might be natural to expect that $\|-\|$ is a functor. Actually, the previous work [10, 35] showed that $\|-\|$ is functorial on profunctors definable by λ-terms. However, $\|-\|$ does not always preserve the composition, as we shall see.

2.3 Non-functoriality

The matrix representation is not functorial. This fact was already mentioned in [35, Section 5], but no concrete counterexample was given there. Here we give a counterexample.

▶ **Example 8.** Assume a single-object groupoid $\mathbb{A}$ and let $\mathbb{B} = \mathbb{A}^{\mathrm{op}} \times \mathbb{A}$. We write $a \in \mathbb{A}$ and $b = (a,a) \in \mathbb{B}$ for their unique objects. Let Φ and Ψ be factors associated to I and $\mathbb{B}$. We define profunctors $\mathcal{P}\colon I \rightarrow\!\!\!\!\!\!+\,\mathbb{B}$ and $\mathcal{Q}\colon \mathbb{B} \rightarrow\!\!\!\!\!\!+\,I$ as follows:

$$\begin{aligned} &\mathcal{P}(\star, b) := \mathbb{A}(a,a) && \mathrm{id}_\star \cdot x \cdot (\alpha', \alpha) := (\alpha'; x; \alpha) \quad \text{for } x \in \mathcal{P}(\star, b) \\ &\mathcal{Q}(b, \star) := \mathbb{A}(a,a) && (\alpha', \alpha) \cdot y \cdot \mathrm{id}_\star := (\alpha; y; \alpha') \quad \text{for } y \in \mathcal{Q}(b, \star). \end{aligned}$$

$$\|\mathcal{P}\|_{[\star],[b]} = \frac{\#\mathbb{A}(a,a)}{\Phi^{\ominus}(\star)\,\Psi^{\oplus}(b)} \qquad \text{and} \qquad \|\mathcal{Q}\|_{[b],[\star]} = \frac{\#\mathbb{A}(a,a)}{\Psi^{\ominus}(b)\,\Phi^{\oplus}(\star)}$$

so their matrix composition yields

$$(\|\mathcal{P}\|; \|\mathcal{Q}\|)_{[\star],[\star]} \quad = \quad \frac{\#\mathbb{A}(a,a)}{\Phi^{\ominus}(\star)\,\Psi^{\oplus}(b)} \times \frac{\#\mathbb{A}(a,a)}{\Psi^{\ominus}(b)\,\Phi^{\oplus}(\star)} \quad = \quad 1$$

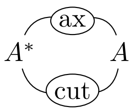
(note that $\Phi^{\ominus}(\star)\Phi^{\oplus}(\star) = \#I(\star,\star) = 1$ and $\Psi^{\ominus}(b)\Psi^{\oplus}(b) = \#\mathbb{B}(b,b) = \#\mathbb{A}^{\mathrm{op}}(a,a) \times \#\mathbb{A}(a,a)$). We compute the profunctor composition $(\mathcal{P}; \mathcal{Q})$ and its matrix representation.

The profunctor composition is given by $(\mathcal{P};\mathcal{Q})(\star,\star) \cong \big(\mathcal{P}(\star,b) \times \mathcal{Q}(b,\star)\big)/{\sim}$, where $(x,y) \sim (x',y')$ if and only if there exist $\alpha_1, \alpha_2 \in \mathbb{A}(a,a)$ such that $x' = x \cdot (\alpha_1, \alpha_2)$ and $y' = (\alpha_1^{-1}, \alpha_2^{-1}) \cdot y$. By the definitions of the actions in $\mathcal{P}$ and $\mathcal{Q}$, these conditions are equivalent to $x' = (\alpha_1; x; \alpha_2)$ and $y' = (\alpha_2; y; \alpha_1)$. By choosing $\alpha_1 = \mathrm{id}$ and $\alpha_2 = x^{-1}$, we can choose representatives from those of the form (id, y). Furthermore, $(\mathrm{id}, y) \sim (\mathrm{id}, y')$ if and only if there exists $\alpha \in \mathbb{A}(a,a)$ such that $y' = (\alpha^{-1}; y; \alpha)$, i.e., y and y' belong to the same conjugacy class. Therefore, $(\mathcal{P};\mathcal{Q})(\star,\star)$ is the set of all conjugacy classes of $\mathbb{A}(a,a)$, so

$$\|\mathcal{P};\mathcal{Q}\|_{[\star],[\star]} \quad = \quad \#\{\text{ conjugacy classes of } \mathbb{A}(a,a) \text{ }\}.$$

So $\|\mathcal{P};\mathcal{Q}\|_{[\star],[\star]} \neq (\|\mathcal{P}\|;\|\mathcal{Q}\|)_{[\star],[\star]}$ if $\mathbb{A}(a,a)$ is not abelian. ◀

Example 8 has a conceptual significance: it computes the interpretations of the trivial loop in **Prof** and $\mathbf{WRel}_{[0,\infty]}$. The trivial loop is the simplest incorrect proof-structure obtained by directly connecting an axiom and a cut:



The trivial loop can be interpreted in any compact closed category. Both $\mathbf{Prof}_\omega$ and $\mathbf{WRel}_{[0,\infty]}$ are compact closed, and $\mathcal{P}$ and $\mathcal{Q}$ (resp. $\|\mathcal{P}\|$ and $\|\mathcal{Q}\|$) are the axiom and cut nodes in **Prof** (resp. in $\mathbf{WRel}_{[0,\infty]}$). So $\mathcal{P};\mathcal{Q}$ and $\|\mathcal{P}\|;\|\mathcal{Q}\|$ are the interpretations of the trivial loop. Thus, Example 8 shows that the matrix representation of the $\mathbf{Prof}_\omega$-interpretation and the $\mathbf{WRel}_{[0,\infty]}$-interpretation of an incorrect proof structure do not necessarily coincide. By contrast, those of a correct proof-net should coincide, due to the results given in Section 4.

3 Stability Ensures Functoriality

At the end of the previous section, we saw that the matrix representation does not necessarily preserve the composition. This section discusses a sufficient condition for the preservation of composition. The key notion is the *stability* of profunctors defined in terms of *creeds* [32] and *kits* [16, 17].

To motivate the introduction of the creed/kit in the context of this paper, it is helpful to consider why the product of matrix representations and the matrix representation of the profunctor composition differ. Let $\mathcal{P} \colon \mathbb{A} \to\!\!\!\!\!\!+\; \mathbb{B}$ and $\mathcal{Q} \colon \mathbb{B} \to\!\!\!\!\!\!+\; \mathbb{C}$ be profunctors. Both the product $\|\mathcal{P}\|;\|\mathcal{Q}\|$ of matrix representations and the matrix representation $\|\mathcal{P};\mathcal{Q}\|$ of the profunctor composition $\mathcal{P};\mathcal{Q}$ involve "division": the product of matrix representations has a numerical division by the number of elements in $\mathbb{B}(b,b)$, while the composition of profunctors is defined as the quotient by the equivalence classes determined by the action of $\mathbb{B}(b,b)$. Intuitively, these two operations coincide when the equivalence classes given by the action of $\mathbb{B}(b,b)$ consist of exactly $\#\mathbb{B}(b,b)$ elements – in other words, when the $\mathbb{B}(b,b)$ action on $\mathcal{P}(a,b) \times \mathcal{Q}(b,c)$ is free. As Fiore et al. [17] have noted, creed and kit precisely capture the freeness of such actions.

3.1 Preliminaries: Creed and Kit

Creeds proposed by Taylor [32] and *kits* proposed by Fiore et al. [16, 17] are both designed to describe and control symmetries. Since these notions are essentially the same within the scope of this paper, we carry out our discussion using creeds.[4]

[4] However, our development follows Fiore et al. [16, 17].

We begin by introducing the creed that describes the symmetry of a given presheaf.

▶ **Definition 9** (Creed [32]). *Let $\mathcal{P}\colon I \to\!\!\!\mapsto \mathbb{A}$. The* creed *of $\mathcal{P}$, written $\mathfrak{C}\mathcal{P}$, is a family of subsets $(\mathfrak{C}\mathcal{P})_a \subseteq \mathbb{A}(a,a)$ parameterised by objects $a \in \mathbb{A}$ defined by*

$$(\mathfrak{C}\mathcal{P})_a \quad := \quad \{\alpha \in \mathbb{A}(a,a) \mid \exists x \in \mathcal{P}(a).\, x \cdot \alpha = x\} \quad = \quad \textstyle\bigcup_{x\in\mathcal{P}(a)} \mathsf{fix}(x),$$

where $\mathsf{fix}(x) \leq \mathbb{A}^{\mathrm{op}}(a,a) \times \mathbb{B}(b,b)$ for a profunctor $\mathcal{P}\colon \mathbb{A} \to\!\!\!\mapsto \mathbb{B}$ and $x \in \mathcal{P}(a,b)$ is the stabiliser subgroup*, i.e., $\mathsf{fix}(x) := \{\, (\alpha,\beta) \mid \alpha \in \mathbb{A}(a,a), \beta \in \mathbb{B}(b,b), \alpha \cdot x \cdot \beta = x \,\}$. The creed of $\mathcal{Q}\colon \mathbb{A} \to\!\!\!\mapsto I$ is defined similarly. A family $C = (C_a)_a$ of subsets $C_a \subseteq \mathbb{A}(a,a)$ is a* creed *if $C = \mathfrak{C}\mathcal{P}$ for some $\mathcal{P}\colon I \to\!\!\!\mapsto \mathbb{A}$.* ◀

Note that we do not define $\mathfrak{C}\mathcal{P}$ for general profunctors $\mathcal{P}\colon \mathbb{A} \to\!\!\!\mapsto \mathbb{B}$ (with $\mathbb{A} \neq I \neq \mathbb{B}$). The definitions given above differ from the original ones, but ours are equivalent to theirs as expected (Proposition 10).

▶ **Proposition 10.** *Let $\mathbb{A}$ be a groupoid. A family $C = (C_a \subseteq \mathbb{A}(a,a))_a$ is a creed if and only if (1) $\alpha \in C_a$ implies $\alpha^k \in C_a$ for every $k \in \mathbb{Z}$, and (2) $\alpha \in C_a$ and $\beta \in \mathbb{A}(a,b)$ imply $(\beta^{-1};\alpha;\beta) \in C_b$.* ◀

▶ **Definition 11.** *For a profunctor $\mathcal{P}\colon I \to\!\!\!\mapsto \mathbb{A}$ and a creed $\mathcal{A}$ on $\mathbb{A}$, we write $\mathcal{P} \models \mathcal{A}$ to mean $(\mathfrak{C}\mathcal{P})_a \subseteq \mathcal{A}_a$ for every $a \in \mathbb{A}$.*

▶ **Definition 12.** *Let C and C' be creeds on $\mathbb{A}$. They are* orthogonal*, written $C \perp C'$, if $C_a \cap C'_a \subseteq \{\mathrm{id}\}$ for every a. Given a creed C, we write $C^\perp$ for the maximum creed such that $C \perp C^\perp$. Concretely, it is given by $(C^\perp)_a = \{\alpha \in \mathbb{A}(a,a) \mid \forall k \in \mathbb{Z}.\alpha^k \notin (C_a \setminus \{\mathrm{id}\})\}$.* ◀

Once the orthogonality $\perp$ is defined, we simply follow Hyland and Schalk [22]. A creed C is *Boolean* if $C = C^{\perp\perp}$.

▶ **Definition 13.** *The category* **SProf** *of* stabilized profunctors *is defined as follows. Its objects are pairs $\mathcal{A} = (\mathbb{A}, \mathcal{A})$ of a groupoid $\mathbb{A}$ and a Boolean creed $\mathcal{A}$ on $\mathbb{A}$. Its morphisms $(\mathbb{A},\mathcal{A}) \to\!\!\!\mapsto (\mathbb{B},\mathcal{B})$ are profunctors $\mathcal{P}\colon \mathbb{A} \to\!\!\!\mapsto \mathbb{B}$ that satisfy*

- *$(\mathcal{Q};\mathcal{P}) \models \mathcal{B}$ for every $\mathcal{Q}\colon I \to\!\!\!\mapsto \mathbb{A}$ with $\mathcal{Q} \models \mathcal{A}$, and*
- *$(\mathcal{P};\mathcal{R}) \models \mathcal{A}^\perp$ for every $\mathcal{R}\colon \mathbb{B} \to\!\!\!\mapsto I$ with $\mathcal{R} \models \mathcal{B}^\perp$.*

Morphisms are composed as profunctors. We write $\mathbf{SProf}_\omega$ for the full subcategory consisting of $(\mathbb{A},\mathcal{A})$ with locally finite $\mathbb{A}$. We define $\mathbf{SProf}^\star_\omega$ to be essentially the same as $\mathbf{SProf}_\omega$, except that each object is equipped with a factor. This factor is used only for the matrix representation and plays no role in the definitions of morphisms or composition. ◀

3.2 Functoriality

This subsection proves that the matrix representation $\|{-}\|$ defines a functor from $\mathbf{SProf}^\star_\omega$ to $\mathbf{WRel}_{[0,\infty]}$, when we define $\|(\mathbb{A},\mathcal{A},\Phi)\| := \|\mathbb{A}\|$ on objects and as before on morphisms.

To simplify the discussion, we represent a profunctor as the sum of "atomic" ones as discussed in [17]. Each hom-category $\mathbf{Prof}(\mathbb{A},\mathbb{B})$ in $\mathbf{Prof}$ has sums defined as follows. For profunctors $\mathcal{P}_i\colon \mathbb{A} \to\!\!\!\mapsto \mathbb{B}$, the profunctor $\coprod_{i\in I}\mathcal{P}_i\colon \mathbb{A} \to\!\!\!\mapsto \mathbb{B}$ is defined by $(\coprod_{i\in I}\mathcal{P}_i)(a,b) := \coprod_{i\in I}(\mathcal{P}_i(a,b))$ on objects and $(\coprod_{i\in I}\mathcal{P}_i)(\alpha,\beta)(\mathsf{in}_j(x)) := \mathsf{in}_j(\mathcal{P}_j(\alpha,\beta)(x))$ on morphisms, where $\mathsf{in}_j\colon \mathcal{P}_j(a,b) \longrightarrow (\coprod_{i\in I}\mathcal{P}(a,b))$ is the injection. When $I = \{1,\dots,n\}$ is a finite set, we also write $\mathcal{P}_1 + \cdots + \mathcal{P}_n$ for $\coprod_i \mathcal{P}_i$. An "atom" with respect to $+$ is described by a group, and every profunctor is expressed as the sum of atoms, as follows.

A subgroup $G \leq \mathbb{A}(a,a)^{\mathrm{op}} \times \mathbb{B}(b,b)$, induces a profunctor $\widetilde{G} \colon \mathbb{A} \nrightarrow \mathbb{B}$ given by

$$\begin{aligned} \widetilde{G}(a',b') &:= \{\, \alpha \cdot G \cdot \beta \mid \alpha \in \mathbb{A}(a',a), \beta \in \mathbb{B}(b,b') \,\} \\ \widetilde{G}(\alpha,\beta) &:= X \mapsto (\alpha \cdot X \cdot \beta) \end{aligned}$$

where for $X \subseteq \mathbb{A}(a',a) \times \mathbb{B}(b,b')$, $\alpha \in \mathbb{A}(a'',a')$ and $\beta \in \mathbb{B}(b',b'')$, the subset $(\alpha \cdot X \cdot \beta) \subseteq \mathbb{A}(a'',a) \times \mathbb{B}(b,b'')$ is defined by $(\alpha \cdot X \cdot \beta) := \{(\alpha;\alpha_0,\ \beta_0;\beta) \mid (\alpha_0,\beta_0) \in X\}$. Note that $\widetilde{G}$ has a single orbit, i.e., for every $x \in \widetilde{G}(a,b)$ and $y \in \widetilde{G}(a',b')$, there exist $\alpha \in \mathbb{A}(a',a)$ and $\beta \in \mathbb{B}(b,b')$ such that $y = \alpha \cdot x \cdot \beta$. This profunctor cannot be decomposed by $+$ any further.

Every profunctor can be written as a sum of profunctors of the form $\widetilde{G}$. Given a profunctor $\mathcal{P} \colon \mathbb{A} \nrightarrow \mathbb{B}$, let $(x_i \in \mathcal{P}(a_i,b_i))_{i\in I}$ be a family of representatives of the equivalence classes defined by the $(\mathbb{A}^{\mathrm{op}} \times \mathbb{B})$-action. That is, for every $x \in \mathcal{P}(a,b)$, there exists a unique $i \in I$ such that $x = \alpha \cdot x_i \cdot \beta$ for some $\alpha \in \mathbb{A}(a,a_i)$ and $\beta \in \mathbb{B}(b_i,b)$. Then $\mathcal{P} \cong \coprod_{i\in I} \widetilde{\mathsf{fix}(x_i)}$.

▶ **Proposition 14.** *A profunctor $\mathcal{P} \colon \mathbb{A} \nrightarrow \mathbb{B}$ can be expressed as*

$$\mathcal{P} \;\cong\; \textstyle\coprod_{i\in I} \widetilde{G_i}, \qquad \forall i \in I.\ G_i \leq \mathbb{A}^{\mathrm{op}}(a_i,a_i) \times \mathbb{B}(b_i,b_i)$$

for some family $(G_i)_{i\in I}$. Furthermore, $(\coprod_i \widetilde{G_i});(\coprod_j \widetilde{H_j}) \cong (\coprod_{i,j} \widetilde{G_i};\widetilde{H_j})$. ◀

▶ **Proposition 15.** *For $G \leq \mathbb{A}^{\mathrm{op}}(a,a) \times \mathbb{B}(b,b)$ and $H \leq \mathbb{B}^{\mathrm{op}}(b',b') \times \mathbb{C}(c,c)$, we have $(\widetilde{G};\widetilde{H}) \neq \emptyset$ if and only if $b \cong b'$. Here $\emptyset \colon \mathbb{A} \nrightarrow \mathbb{C}$ is the profunctor such that $\emptyset(a,c) = \emptyset$.* ◀

Note that the composite $(\widetilde{G};\widetilde{H})$ for $G \leq \mathbb{A}^{\mathrm{op}}(a,a) \times \mathbb{B}(b,b)$ and $H \leq \mathbb{B}^{\mathrm{op}}(b,b) \times \mathbb{C}(c,c)$ is not necessarily of a single orbit. It can be a sum $\coprod_{i\in I} \widetilde{K_i}$ with $\#I > 1$.

We first consider the composition of single-orbit profunctors.

▶ **Lemma 16.** *Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ be locally finite groupoids equipped with factors Φ, Ψ and Θ, and $K \leq \mathbb{A}^{\mathrm{op}}(a,a) \times \mathbb{B}(b,b)$ and $H \leq \mathbb{B}^{\mathrm{op}}(b,b) \times \mathbb{C}(c,c)$. Then, for every $a' \in \mathbb{A}$ and $c' \in \mathbb{C}$,*

$$(\|\widetilde{K}\|;\|\widetilde{H}\|)_{[a'],[c']} \quad\leq\quad \|\widetilde{K};\widetilde{H}\|_{[a'],[c']}.$$

The equality holds if and only if

$$\{\beta_1;\beta;\beta_1^{-1} \mid (\mathrm{id},\beta) \in K\} \cap \{\beta_2;\beta;\beta_2^{-1} \mid (\beta,\mathrm{id}) \in H\} = \{\mathrm{id}\}$$

for every $\beta_1, \beta_2 \in \mathbb{B}(b,b)$.

Proof. Assume $a' \in \mathbb{A}$ and $c' \in \mathbb{C}$. When $\neg(a' \cong a)$, then $\widetilde{K}(a',b') = \emptyset$ for every $b' \in \mathbb{B}$. So we have $\|\widetilde{K}\|_{[a'],[b']} = 0$ for every b' and $\coprod_{b'\in\mathbb{B}} \widetilde{K}(a',b') \times \widetilde{H}(b',c') = \emptyset$. The former implies $(\|\widetilde{K}\|;\|\widetilde{H}\|)_{[a'],[c']} = 0$, and the latter implies $\|\widetilde{K};\widetilde{H}\|_{[a'],[c']} = 0$. So $\|\widetilde{K}\|;\|\widetilde{H}\|$ and $\|\widetilde{K};\widetilde{H}\|$ coincide on $([a'],[c'])$-entry when $\neg(a' \cong a)$. The case that $\neg(c' \cong c)$ is similar.

We calculate $(\|\widetilde{K}\|;\|\widetilde{H}\|)_{[a],[c]}$. Since $\widetilde{K}$ is the collection of cosets of K,

$$\#\widetilde{K}(a,b) \quad=\quad \frac{\#(\mathbb{A}^{\mathrm{op}}(a,a) \times \mathbb{B}(b,b))}{\#K}.$$

So

$$\|\widetilde{K}\|_{[a],[b]} \quad=\quad \frac{\#(\mathbb{A}^{\mathrm{op}}(a,a) \times \mathbb{B}(b,b))}{\#K\;\Phi^{\ominus}(a)\;\Psi^{\oplus}(b)} \quad=\quad \frac{\Phi^{\oplus}(a)\;\Psi^{\ominus}(b)}{\#K}.$$

The above formula, together with $\|\widetilde{K}\|_{[a],[b']} = 0$ for $\neg(b' \cong b)$, concludes

$$(\|\widetilde{K}\|;\|\widetilde{H}\|)_{[a],[c]} \quad=\quad \frac{\Phi^{\oplus}(a)\;\#\mathbb{B}(b,b)\;\Theta^{\ominus}(c)}{\#K\#H}.$$

We calculate $\|\widetilde{K};\widetilde{H}\|_{[a],[c]}$. Recall that $(\widetilde{K};\widetilde{H})(a,c) \cong (\widetilde{K}(a,b) \times \widetilde{H}(b,c))/{\sim}$, where $(x,y) \sim (x',y')$ if and only if $(x \cdot \beta, \beta^{-1} \cdot y) = (x',y')$ for some $\beta \in \mathbb{B}(b,b)$. For every $x \in \widetilde{K}(a,b)$ and $y \in \widetilde{H}(b,c)$, the equivalence class $[(x,y)]_\sim = \{(x \cdot \beta, \beta^{-1} \cdot y) \mid \beta \in \mathbb{B}(b,b)\}$ has at most $\#\mathbb{B}(b,b)$ elements. Therefore

$$\begin{aligned}\#(\widetilde{K};\widetilde{H})(a,c) \quad &\geq \quad \frac{\#\widetilde{K}(a,b)\,\#\widetilde{H}(b,c)}{\#\mathbb{B}(b,b)} \\ &= \quad \frac{\#(\mathbb{A}^{\mathrm{op}}(a,a) \times \mathbb{B}(b,b))}{\#K} \times \frac{\#(\mathbb{B}^{\mathrm{op}}(b,b) \times \mathbb{C}(c,c))}{\#H} \times \frac{1}{\#\mathbb{B}(b,b)}.\end{aligned}$$

So

$$\|\widetilde{K};\widetilde{H}\|_{[a],[c]} \quad = \quad \frac{\#(\widetilde{K};\widetilde{H})(a,c)}{\Phi^{\ominus}(a)\,\Theta^{\oplus}(c)} \quad \geq \quad \frac{\Phi^{\oplus}(a)\,\#\mathbb{B}(b,b)\,\Theta^{\ominus}(c)}{\#K\#H}.$$

Assume the additional condition in the statement. Let $x = \alpha \cdot K \cdot \beta_1$ and $y = \beta_2 \cdot H \cdot \gamma$ and suppose $(x,y) = (x \cdot \beta, \beta^{-1} \cdot y)$. Then $\alpha \cdot K \cdot \beta_1 = \alpha \cdot K \cdot \beta_1 \cdot \beta$, so $K = K \cdot \beta_1 \cdot \beta \cdot \beta_1^{-1}$. Similarly, $\beta_2 \cdot H \cdot \gamma = \beta^{-1} \cdot \beta_2 \cdot H \cdot \gamma$ implies $H = \beta_2^{-1} \cdot \beta^{-1} \cdot \beta_2 \cdot H$. This means that $(\mathrm{id}, \beta_1;\beta;\beta_1^{-1}) \in K$ and $(\beta_2^{-1};\beta^{-1};\beta_2, \mathrm{id}) \in H$. By the additional assumption, $\beta = \mathrm{id}$. Therefore, $(x,y) = (x \cdot \beta, \beta^{-1} \cdot y)$ implies $\beta = \mathrm{id}$, and thus the equivalence class $[(x,y)]_\sim$ consists of $\#\mathbb{B}(b,b)$ elements. So

$$\#(\widetilde{K};\widetilde{H})(a,c) \quad = \quad \frac{\#\widetilde{K}(a,b)\,\#\widetilde{H}(b,c)}{\#\mathbb{B}(b,b)}.$$

A straightforward calculation shows that $\|\widetilde{K};\widetilde{H}\|_{[a],[c]} = (\|\widetilde{K}\|;\|\widetilde{H}\|)_{[a],[c]}$.

Conversely, assume that $\{\beta_1;\beta;\beta_1^{-1} \mid (\mathrm{id},\beta) \in K\} \cap \{\beta_2;\beta;\beta_2^{-1} \mid (\beta,\mathrm{id}) \in H\}$ contains $\beta \neq \mathrm{id}$ for some $\beta_1, \beta_2 \in \mathbb{B}(b,b)$. Then $K = K \cdot \beta_1 \cdot \beta \cdot \beta_1^{-1}$ and hence $K \cdot \beta_1 = K \cdot \beta_1 \cdot \beta$. Similarly, since $H = \beta_2 \cdot \beta \cdot \beta_2^{-1} \cdot H$, we have $\beta^{-1} \cdot \beta_2^{-1} \cdot H = \beta_2^{-1} \cdot H$. Let $x = K \cdot \beta_1$ and $y = \beta_2^{-1} \cdot H$. Then $(x,y) = (x \cdot \beta, \beta^{-1} \cdot y)$, so the equivalence class $[(x,y)]_\sim$ consists of at most $\#\mathbb{B}(b,b)/2$ elements. Recall the inequality

$$\#(\widetilde{K};\widetilde{H})(a,c) \quad \geq \quad \frac{\#\widetilde{K}(a,b)\,\#\widetilde{H}(b,c)}{\#\mathbb{B}(b,b)}.$$

Since $\#\widetilde{K}(a,b)$ and $\#\widetilde{H}(b,c)$ are finite, the equality holds only if all the equivalence classes $[(x,y)]_\sim$ consist of $\#\mathbb{B}(b,b)$ elements. Therefore

$$\#(\widetilde{K};\widetilde{H})(a,c) \quad > \quad \frac{\#\widetilde{K}(a,b)\,\#\widetilde{H}(b,c)}{\#\mathbb{B}(b,b)}.$$

and hence $\|\widetilde{K};\widetilde{H}\|_{[a],[c]} > (\|\widetilde{K}\|;\|\widetilde{H}\|)_{[a],[c]}$. ◀

By Proposition 14, the argument for the single-orbit case extends to the general case.

▶ **Lemma 17.** *Let $\mathcal{P}\colon \mathbb{A} \nrightarrow \mathbb{B}$ and $\mathcal{Q}\colon \mathbb{B} \nrightarrow \mathbb{C}$, and assume factors for $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$. If $\mathfrak{C}(\mathcal{P}(a,-)) \perp \mathfrak{C}(\mathcal{Q}(-,c))$ for every $a \in \mathbb{A}$ and $c \in \mathbb{C}$, then $\|\mathcal{P}\|;\|\mathcal{Q}\| = \|\mathcal{P};\mathcal{Q}\|$.*

Proof. By Proposition 14, profunctors $\mathcal{P}$ and $\mathcal{Q}$ can be described as $\mathcal{P} \cong \coprod_{i \in I} \widetilde{H_i}$ and $\mathcal{Q} \cong \coprod_{j \in J} \widetilde{K_j}$ where $H_i \leq \mathbb{A}^{\mathrm{op}}(a_i,a_i) \times \mathbb{B}(b_i,b_i)$ and $K_j \leq \mathbb{B}^{\mathrm{op}}(b_j,b_j) \times \mathbb{C}(c_j,c_j)$. Note that $(\mathcal{P};\mathcal{Q}) \cong \coprod_{i,j} (\widetilde{H_i};\widetilde{K_j})$ and $\|\mathcal{P};\mathcal{Q}\| = \sum_{i,j} \|\widetilde{H_i};\widetilde{K_j}\|$. So it suffices to show that $\|\widetilde{H_i};\widetilde{K_j}\| = \|\widetilde{H_i}\|;\|\widetilde{K_j}\|$ for every i and j.

We prove that the additional condition in Lemma 16 holds for every pair (H_i, K_j) such that $b_i \cong b_j$. We can assume without loss of generality that $b_i = b_j$. Let $b = b_i = b_j$ and $\beta_1, \beta_2 \in \mathbb{B}(b,b)$. Assume that $\beta \in \mathbb{B}(b,b)$ belongs to $\{\beta_1; \beta; \beta_1^{-1} \mid (\mathrm{id}, \beta) \in H_i\} \cap \{\beta_2; \beta; \beta_2^{-1} \mid (\beta, \mathrm{id}) \in K_j\}$. Then $H_i \cdot \beta_1$ is a fixed-point of the action $(-) \cdot \beta$, so $\beta \in (\mathfrak{C}(\widetilde{H_i}(a_i, -)))_b \subseteq (\mathfrak{C}(\mathcal{P}(a_i, -)))_b$. Similarly, since $\beta_2^{-1} \cdot K_j$ is a fixed-point of the action $\beta \cdot (-)$, we have $\beta \in (\mathfrak{C}(\widetilde{K_j}(-, c_j)))_b \subseteq (\mathfrak{C}(\mathcal{Q}(-, c_j)))_b$. By the orthogonality assumption $\mathfrak{C}(\mathcal{P}(a_i, -)) \perp \mathfrak{C}(\mathcal{Q}(-, c_j))$, we have $\beta = \mathrm{id}$ as expected. ◀

It remains to show the relationship between the stability of profunctors and creeds.

▶ **Lemma 18.** *Let $\mathcal{P} \colon (\mathbb{A}, \mathcal{A}) \to\!\!\!\!\!\!\mapsto (\mathbb{B}, \mathcal{B})$ be a stabilized profunctor. Then $\mathfrak{C}\mathcal{P}(-, b) \subseteq \mathcal{A}^\perp$ and $\mathfrak{C}\mathcal{P}(a, -) \subseteq \mathcal{B}$ for every $a \in \mathbb{A}$ and $b \in \mathbb{B}$.*

Proof. Let $a \in \mathbb{A}$ and $\mathcal{Q} \colon I \to\!\!\!\!\!\!\mapsto \mathbb{A}$ be a profunctor defined by $\mathcal{Q}(\star, a') = \mathbb{A}(a, a')$. Then $(\mathfrak{C}\mathcal{Q})_{a'} = \{\mathrm{id}\}$ if $a' \cong a$ and $(\mathfrak{C}\mathcal{Q})_{a'} = \emptyset$ if $\neg(a' \cong a)$. In particular, $\mathcal{Q} \models \mathcal{A}$. By the definition of stability, $\mathcal{Q}; \mathcal{P} \models \mathcal{B}$.

Let $b \in \mathbb{B}$ and $\beta \in \mathbb{B}(b,b)$ and assume $\beta \in (\mathfrak{C}\mathcal{P}(a, -))_b$. By the definition of creed, there exists $x \in \mathcal{P}(a, b)$ such that $x \cdot \beta = x$. Then $[(\mathrm{id}, x)]_\sim \in (\mathcal{Q}; \mathcal{P})(\star, b)$ and $[(\mathrm{id}, x)]_\sim \cdot \beta = [(\mathrm{id}, x \cdot \beta)]_\sim = [(\mathrm{id}, x)]_\sim$. So $\beta \in (\mathfrak{C}(\mathcal{Q}; \mathcal{P}))_b$. Since $\mathcal{Q}; \mathcal{P} \models \mathcal{B}$, we have $\beta \in \mathcal{B}_b$. ◀

▶ **Theorem 19.** *$\|-\|$ is a functor from $\mathbf{SProf}^\star_\omega$ to $\mathbf{WRel}_{[0,\infty]}$.*

Proof. Let $\mathcal{P} \colon (\mathbb{A}, \mathcal{A}) \to\!\!\!\!\!\!\mapsto (\mathbb{B}, \mathcal{B})$ and $\mathcal{Q} \colon (\mathbb{B}, \mathcal{B}) \to\!\!\!\!\!\!\mapsto (\mathbb{C}, \mathcal{C})$ be stabilized profunctors. By Lemma 18, we have $\mathfrak{C}\mathcal{P}(a, -) \subseteq \mathcal{B}$ and $\mathfrak{C}\mathcal{Q}(-, c) \subseteq \mathcal{B}^\perp$ for every $a \in \mathbb{A}$ and $c \in \mathbb{C}$. In particular, $\mathfrak{C}\mathcal{P}(a, -) \perp \mathfrak{C}\mathcal{Q}(-, c)$. By Lemma 17, we have $\|\mathcal{P}\|; \|\mathcal{Q}\| = \|\mathcal{P}; \mathcal{Q}\|$. ◀

3.3 Completeness

We have seen that the orthogonality of the symmetries of profunctors provides a sufficient condition for the matrix representation to behave functorially. In fact, this condition is almost necessary: as long as the result of matrix multiplication is finite, the functoriality of the matrix representation entails the orthogonality of the symmetry.

▶ **Theorem 20.** *Let $\mathcal{P} \colon \mathbb{A} \to\!\!\!\!\!\!\mapsto \mathbb{B}$ and $\mathcal{Q} \colon \mathbb{B} \to\!\!\!\!\!\!\mapsto \mathbb{C}$ be profunctors, and suppose that $\|\mathcal{P}\|; \|\mathcal{Q}\| = \|\mathcal{P}; \mathcal{Q}\|$. Then, for every $a \in \mathbb{A}$ and $c \in \mathbb{C}$, we have $\mathfrak{C}(\mathcal{P}(a, -)) \perp \mathfrak{C}(\mathcal{Q}(-, c))$ provided that $\|\mathcal{P}; \mathcal{Q}\|_{[a],[c]} < \infty$.*

Proof. By Proposition 14, profunctors $\mathcal{P}$ and $\mathcal{Q}$ can be described as $\mathcal{P} \cong \coprod_{i \in I} \widetilde{H_i}$ and $\mathcal{Q} \cong \coprod_{j \in J} \widetilde{K_j}$ where $H_i \leq \mathbb{A}^{\mathrm{op}}(a_i, a_i) \times \mathbb{B}(b_i, b_i)$ and $K_j \leq \mathbb{B}^{\mathrm{op}}(b_j, b_j) \times \mathbb{C}(c_j, c_j)$. Note that $(\mathcal{P}; \mathcal{Q}) \cong \coprod_{i,j} (\widetilde{H_i}; \widetilde{K_j})$ and $\|\mathcal{P}; \mathcal{Q}\| = \sum_{i,j} \|\widetilde{H_i}; \widetilde{K_j}\|$.

Assume for contradiction that the creed orthogonality fails for some a and c. Then the orthogonality fails for $(\widetilde{H_{i_0}}, \widetilde{K_{j_0}})$ for some (i_0, j_0). By Lemma 16, $(\|\widetilde{H_{i_0}}\|; \|\widetilde{K_{j_0}}\|)_{[a],[c]} < \|\widetilde{H_{i_0}}; \widetilde{K_{j_0}}\|_{[a],[c]}$, i.e., $(\|\widetilde{H_{i_0}}\|; \|\widetilde{K_{j_0}}\|)_{[a],[c]} + \epsilon = \|\widetilde{H_{i_0}}; \widetilde{K_{j_0}}\|_{[a],[c]}$ for some $\epsilon > 0$. Since $(\|\widetilde{H_i}\|; \|\widetilde{K_j}\|)_{[a],[c]} \leq \|\widetilde{H_i}; \widetilde{K_j}\|_{[a],[c]}$ for every i and j, we have $(\sum_{i,j} \|\widetilde{H_i}\|; \|\widetilde{K_j}\|)_{[a],[c]} + \epsilon \leq (\sum_{i,j} \|\widetilde{H_i}; \widetilde{K_j}\|)_{[a],[c]}$. Since $\sum_{i,j} \|\widetilde{H_i}; \widetilde{K_j}\| < \infty$, we have $(\sum_{i,j} \|\widetilde{H_i}\|; \|\widetilde{K_j}\|)_{[a],[c]} < (\sum_{i,j} \|\widetilde{H_i}; \widetilde{K_j}\|)_{[a],[c]}$, a contradiction. ◀

4 Linear Logic Structures and their Preservation

We have seen in the previous subsection that the matrix representation $\|-\|$ is a functor from $\mathbf{SProf}^\star_\omega$ to $\mathbf{WRel}_{[0,\infty]}$. The category $\mathbf{SProf}$ is not merely a (bi)category but, in fact, a model of classical linear logic, as shown in [17]. The forgetful functor from $\mathbf{SProf}$

to **Prof** strictly preserves the model structure, and it is immediate that $\mathbf{SProf}^{\star}_{\omega}$ inherits it. This section shows that $\|-\|$ preserves the model structures (cf. Remark 2), as a *strict $*$-autonomous functor* in the sense of Cockett et al. [12]. Since we are concerned with strict preservation rather than lax or strong preservation, no extra data beyond the functor is required, and the verification is correspondingly straightforward. Accordingly, in this section, we focus mainly on describing the linear logic structures on $\mathbf{SProf}^{\star}_{\omega}$ (in particular, the choice of factor) and $\mathbf{WRel}_{[0,\infty]}$.

Finally, we discuss a functor from $\mathbf{SProf}_{\omega}$; we obtain a functor for each choice of a factor for every groupoid and, perhaps counterintuitively, all such functors preserve the linear logic structure strongly (though not strictly in general).

▶ Remark 21. Recall that $\mathbf{SProf}^{\star}_{\omega}$ is regarded in this paper as the 1-category $\mathsf{C}\mathbf{SProf}^{\star}_{\omega}$ (cf. Remark 2). So we mainly discuss a 1-functor $\mathsf{C}\mathbf{SProf}^{\star}_{\omega} \longrightarrow \mathbf{WRel}_{[0,\infty]}$, which strictly preserves the linear logic structure. Here we briefly comment on the 2-dimensional structure.

As mentioned in Remark 3, this 1-functor induces a pseudo-functor $\mathbf{Prof}^{\star}_{2\text{-}iso} \longrightarrow \mathsf{D}\mathbf{WRel}_{[0,\infty]}$. This locally discrete bicategory $\mathsf{D}\mathbf{WRel}_{[0,\infty]}$ is a bicategorical model of linear logic. We believe that this pseudofunctor $\mathbf{Prof}^{\star}_{2\text{-}iso} \longrightarrow \mathsf{D}\mathbf{WRel}_{[0,\infty]}$ would preserve the linear-logic structure in the appropriate sense, if such a notion were made mathematically precise; at present, however, there is no rigorous definition of a structure-preserving pseudo-functor between bicategorical models of linear logic. ◀

4.1 Dual

For a groupoid $(\mathbb{A}, \mathcal{A})$ with a creed, its *negation* in **SProf** is $(\mathbb{A}, \mathcal{A})^{\perp} := (\mathbb{A}^{\mathrm{op}}, \mathcal{A}^{\perp})$. This operation is strictly involutive: $(\mathbb{A}, \mathcal{A})^{\perp\perp} = (\mathbb{A}^{\perp\perp}, \mathcal{A}^{\perp\perp}) = (\mathbb{A}, \mathcal{A})$ (note that $\mathcal{A}^{\perp\perp} = \mathcal{A}$ by the definition of Boolean creed). The dual of a stabilized profunctor $\mathcal{P} \colon (\mathbb{A}, \mathcal{A}) \mathrel{\longrightarrow\!\!\!\!\!\!+} (\mathbb{B}, \mathcal{B})$ is given by $(\mathcal{P}^{\perp})(b, a) := \mathcal{P}(a, b)$.

To define the dual in $\mathbf{SProf}^{\star}_{\omega}$, it suffices to choose a factor of $\mathbb{A}^{\mathrm{op}}$. Let $(\mathbb{A}, \mathcal{A}, \Phi)^{\perp} := (\mathbb{A}^{\mathrm{op}}, \mathcal{A}^{\perp}, \Phi^{\perp})$, where $(\Phi^{\perp})^{\oplus}(a) = \Phi^{\ominus}(a)$.

The dual in $\mathbf{WRel}_{[0,\infty]}$ is defined by $X^{\perp} := X$ on objects and $(M^{\perp})_{a,b} := M_{b,a}$ on morphisms. The operation $M^{\perp}$ on matrices is usually called transposition.

The dual is strictly preserved in the sense of [12] if $\|(\mathbb{A}, \mathcal{A})\|^{\perp} = \|(\mathbb{A}, \mathcal{A})^{\perp}\|$ as objects and a certain equation holds. A direct calculation shows that this is the case.

4.2 Monoidal Structures

The monoidal structure in **SProf** is defined as follows. For objects,

$$(\mathbb{A}, \mathcal{A}) \otimes (\mathbb{B}, \mathcal{B}) := (\mathbb{A} \times \mathbb{B}, (\mathcal{A} \times \mathcal{B})^{\perp\perp})$$

where $(\mathcal{A} \times \mathcal{B})_{(a,b)} := \{ (\alpha, \beta) \mid \alpha \in \mathcal{A}_a, \beta \in \mathcal{B}_b \}$. For morphisms $\mathcal{P}_1 \colon \mathbb{A}_1 \mathrel{\longrightarrow\!\!\!\!\!\!+} \mathbb{B}_1$ and $\mathcal{P}_2 \colon \mathbb{A}_2 \mathrel{\longrightarrow\!\!\!\!\!\!+} \mathbb{B}_2$, we define $(\mathcal{P}_1 \otimes \mathcal{P}_2)((a_1, a_2), (b_1, b_2)) := \mathcal{P}_1(a_1, b_1) \times \mathcal{P}_2(a_2, b_2)$. The monoidal unit is $(I, \top)$, where I is the trivial category with the unique object $\star$ and $\top$ is the unique Boolean creed (i.e., $\top_{\star} = \{\mathrm{id}_{\star}\}$). The left unitor $l \colon (I, \top) \otimes (\mathbb{A}, \mathcal{A}) \mathrel{\longrightarrow\!\!\!\!\!\!+} (\mathbb{A}, \mathcal{A})$ is $l((\star, a), a') := \mathbb{A}(a, a')$, and the right unitor is similar. The associator $\delta \colon ((\mathbb{A}, \mathcal{A}) \otimes (\mathbb{B}, \mathcal{B})) \otimes (\mathbb{C}, \mathcal{C}) \mathrel{\longrightarrow\!\!\!\!\!\!+} (\mathbb{A}, \mathcal{A}) \otimes ((\mathbb{B}, \mathcal{B}) \otimes (\mathbb{C}, \mathcal{C}))$ is given by $\delta(((a, b), c), (a', (b', c'))) := \mathbb{A}(a, a') \times \mathbb{B}(b, b') \times \mathbb{C}(c, c')$. The symmetry $\sigma \colon (\mathbb{A}, \mathcal{A}) \otimes (\mathbb{B}, \mathcal{B}) \mathrel{\longrightarrow\!\!\!\!\!\!+} (\mathbb{B}, \mathcal{B}) \otimes (\mathbb{A}, \mathcal{A})$ is $\sigma((a, b), (a', b')) := \mathbb{A}(a, a') \times \mathbb{B}(b, b')$.

The monoidal product in $\mathbf{Prof}^{\star}_{\omega}$ is defined by $(\mathbb{A}, \mathcal{A}, \Phi) \otimes (\mathbb{B}, \mathcal{B}, \Psi) = (\mathbb{A} \times \mathbb{B}, (\mathcal{A} \times \mathcal{B})^{\perp\perp}, \Phi \otimes \Psi)$, where $(\Phi \otimes \Psi)^{\oplus}(a, b) = \Phi^{\oplus}(a)\Psi^{\oplus}(b)$.

The monoidal structure in $\mathbf{WRel}_{[0,\infty]}$ is given as follows. On objects, $A \otimes B := A \times B$ equipped with the equivalence relation given by $(a,b) \cong_{A\times B} (a',b') :\Longleftrightarrow a \cong_A a' \wedge b \cong_B b'$. On morphisms $R_1 \in \mathbf{WRel}_{[0,\infty]}(A_1, B_1)$ and $R_2 \in \mathbf{WRel}_{[0,\infty]}(A_2, B_2)$, we define $(R_1 \otimes R_2)_{([(a_1,a_2)]_{\cong},[(b_1,b_2)]_{\cong})} := (R_1)_{[a_1]_{\cong},[b_1]_{\cong}} (R_2)_{[a_2]_{\cong},[b_2]_{\cong}}$. The monoidal unit is a singleton set $I = \{\star\}$. The definitions of unitors, associator, and symmetry are similar to those for **Prof**.

It is easy to see that $\|-\|$ is strict monoidal. It suffices to show that $\|(\mathbb{A},\mathcal{A}) \otimes (\mathbb{B},\mathcal{B})\| = \|(\mathbb{A},\mathcal{A})\| \otimes \|(\mathbb{B},\mathcal{B})\|$ on objects, and that certain diagrams commute. We have slightly changed the definitions of **Rel** and **WRel** to satisfy the former; the latter is easy.

4.3 Biproduct

Both **Prof** (precisely, its classifying category **CProf**; see Remark 2) and $\mathbf{WRel}_{[0,\infty]}$ are commutative-monoid enriched. The commutative monoid structure of $\mathbf{CProf}(\mathbb{A},\mathbb{B})$ is given by the disjoint union (see Section 3.2). The sum preserves the stability, so **CSProf** is commutative-monoid enriched, as well as $\mathbf{CSProf}^{\star}_{\omega}$. The sum in $\mathbf{WRel}_{[0,\infty]}(A, B)$ is given by $(R+S)_{a,b} := R_{a,b} + S_{a,b}$. It is easy to see that $\|-\|$ preserves the sum. It also preserves the unit since $\|\emptyset\|$ is the zero matrix.

Both $\mathbf{SProf}^{\star}_{\omega}$ and $\mathbf{WRel}_{[0,\infty]}$ have biproducts [17,24]. In a commutative-monoid enriched category $\mathbf{C}$, a biproduct of $c_1, c_2 \in \mathbf{C}$ is an object $c_1 \oplus c_2$ together with morphisms $c_i \xrightarrow{\iota_i} (c_1 \oplus c_2)$ and $(c_1 \oplus c_2) \xrightarrow{\pi_i} c_i$ satisfying $(\pi_1;\iota_1) + (\pi_2;\iota_2) = \mathrm{id}$, $(\iota_i;\pi_i) = \mathrm{id}$ and $(\iota_i;\pi_j) = 0$ when $i \neq j$ (where 0 is the unit element in $\mathbf{C}(c_i, c_j)$).

The functor $\|-\|$ strongly preserves biproducts, since the biproduct is defined in terms of composition, sum, identity and zero morphisms, which are all preserved by $\|-\| : \mathbf{SProf}_{\omega} \to \mathbf{WRel}_{[0,\infty]}$. To prove strict preservation, we need to choose a biproduct, and the most naive choice achieves this. The choice of the factor is $(\Phi_1 \oplus \Phi_2)^{\oplus}(\mathsf{in}_i(a)) := \#^{\oplus}\Phi_i^{\oplus}(a)$, where $\mathsf{in}_i \colon obj(\mathbb{A}_i) \longrightarrow obj(\mathbb{A}_1 \oplus \mathbb{A}_2)$ is the injection.

4.4 Exponential

The exponential in **SProf** is given by Fiore et al. [17]. Let $!(\mathbb{A},\mathcal{A}) := (!\mathbb{A}, !\mathcal{A})$, where $!\mathbb{A}$ is the free symmetric strict monoidal category (see [15] for an exposition). Specifically, an object of $!\mathbb{A}$ is a finite sequence $\langle a_1,\dots,a_n\rangle$ of objects in $\mathbb{A}$, and a morphism between $\langle a_1,\dots,a_n\rangle$ and $\langle a'_1,\dots,a'_n\rangle$ is a pair $(\sigma,(\alpha_i)_{i=1,\dots,n})$ consisting of a permutation $\sigma \in \mathfrak{S}_n$ and a sequence of morphisms $\alpha_i \colon a_i \longrightarrow a'_{\sigma(i)}$. Here, $\mathfrak{S}_n$ is the set of all permutations of $\{1,\dots,n\}$. There is no morphism between $\langle a_1,\dots,a_n\rangle$ and $\langle a'_1,\dots,a'_m\rangle$ when $n \neq m$. The definition of the creed $!\mathcal{A}$ is somewhat complicated; the reader is referred to [17, Section 6.1] for details. For the purpose here, it suffices to know that $!\mathbb{A}$ together with an appropriate creed gives an exponential modality, and the concrete definition of the creed $!\mathcal{A}$ itself is not necessary. The stabilized profunctor $!\mathcal{P} \colon !(\mathbb{A},\mathcal{A}) \nrightarrow !(\mathbb{B},\mathcal{B})$ for $\mathcal{P} \colon (\mathbb{A},\mathcal{A}) \nrightarrow (\mathbb{B},\mathcal{B})$ is defined by:

$$(!\mathcal{P})(\langle a_1,\dots,a_n\rangle, \langle b_1,\dots,b_m\rangle) := \begin{cases} \coprod_{\sigma\in\mathfrak{S}_n} \prod_{i=1}^{n} \mathcal{P}(a_i, b_{\sigma(i)}) & \text{if } n = m \\ \emptyset & \text{if } n \neq m. \end{cases}$$

When the factor of $\mathbb{A}$ is Φ, we define $(!\Phi)^{\oplus}(\langle a_i\rangle_{i=1}^{n}) := \prod_{i=1}^{n} \Phi^{\oplus}(a_i)$. It follows that $(!\Phi)^{\ominus}(\langle a_i\rangle_{i=1}^{n}) = \#\{\sigma \mid \forall i.a_i \cong a_{\sigma(i)}\} \times \prod_{i=1}^{n} \Phi^{\ominus}(a_i)$.

The exponential in $\mathbf{WRel}_{[0,\infty]}$ is given as follows. We first give the underlying functor $! \colon \mathbf{WRel}_{[0,\infty]} \longrightarrow \mathbf{WRel}_{[0,\infty]}$. For objects, $!A := A^*$, the set of all finite sequences over A, equipped with the equivalence relation given by $\langle a_i\rangle_{i=1}^{n} \cong_{!A} \langle a'_j\rangle_{j=1}^{m} :\Longleftrightarrow \exists\sigma.\forall i.a_i \cong_A a'_{\sigma(i)}$. Note that $A^*/{\cong_{!A}}$ is essentially equivalent to the set of multisets over $A/{\cong_A}$. For morphisms,

$!R \in \mathbf{WRel}_{[0,\infty]}(!A, !B)$ for $R \in \mathbf{WRel}_{[0,\infty]}(A, B)$ is defined by

$$(!R)_{\mu,[b_1,\dots,b_m]} := \begin{cases} \sum_{\substack{(a_1,\dots,a_n)\in(A/\cong_A)^n \text{ s.t.} \\ \mu=[a_1,\dots,a_n]}} \prod_{i=1}^{n} R_{a_i,b_i} & \text{if } n = m \\ 0 & \text{if } n \neq m, \end{cases}$$

where $[a_1, \dots, a_n]$ is the equivalence class of $\langle a_1, \dots, a_n\rangle$. It is easy to see that $\|-\|$ strictly preserves ! and the required diagrams commute.

4.5 Strong Functors from SProf_ω

Finally, we discuss functors from $\mathbf{SProf}_\omega$. To give a matrix representation, we must choose a factor for each groupoid. This corresponds to giving a functor $F\colon \mathbf{SProf}_\omega \longrightarrow \mathbf{SProf}^\star_\omega$ such that $U \circ F = \mathrm{id}$, where $U\colon \mathbf{SProf}^\star_\omega \longrightarrow \mathbf{SProf}_\omega$ is the forgetting map. We call such a functor F a *factor choice*.

The composite $\|F(-)\|\colon \mathbf{SProf}_\omega \longrightarrow \mathbf{WRel}_{[0,\infty]}$ is a functor. It maps $(\mathbb{A}, \mathcal{A})$ to $\|\mathbb{A}\|$ and $\|F(\mathcal{P})\|$ is given by $\|F(\mathcal{P})\|_{[a],[b]} = \#\mathcal{P}(a, b)/\Phi^\ominus(a)\Psi^\oplus(b)$ when $F(\mathbb{A}, \mathcal{A}) = (\mathbb{A}, \mathcal{A}, \Phi)$ and $F(\mathbb{B}, \mathcal{B}) = (\mathbb{B}, \mathcal{B}, \Psi)$. This functor is always strong, independent of the choice of F.

▶ **Proposition 22.** *For every factor choice $F\colon \mathbf{SProf}_\omega \longrightarrow \mathbf{SProf}^\star_\omega$, the functor $\|F(-)\|$ strongly preserves all the linear logic structures in the sense of Cockett et al. [12].*

Proof. It suffices to show that F is a strong functor, since strong functors compose [12]. Note that the forgetting map $U\colon \mathbf{SProf}^\star_\omega \longrightarrow \mathbf{SProf}_\omega$ is fully faithful and strictly preserves the linear logic structure, since a factor plays no role in the definitions of morphisms and composition. The natural transformation $F((\mathbb{A}, \mathcal{A})\otimes(\mathbb{B}, \mathcal{B})) \longrightarrow F(\mathbb{A}, \mathcal{A})\otimes F(\mathbb{B}, \mathcal{B})$ is defined as the unique morphism that is mapped to the identity by U. By faithfulness of U, it is enough to check commutativity of the required diagrams after applying U, but the application of U makes diagrams trivial. A similar argument applies to other structures. ◀

We remark that two factor choices F and G are isomorphic (in the 2-category in Cockett et al. [12]), so are $\|F(-)\|$ and $\|G(-)\|$. The functor(s) are "well-defined up to iso."

▶ **Example 23.** A factor choice $F_1\colon \mathbf{SProf}_\omega \longrightarrow \mathbf{SProf}^\star_\omega$ is given by $F_1((\mathbb{A}, \mathcal{A})) := (\mathbb{A}, \mathcal{A}, \Phi^\oplus)$ with $\Phi^\oplus(a) := 1$. Then $\|F_1(\mathcal{P})\| = \#\mathcal{P}(a, b)/\#\mathbb{A}(a, a)$. The functor $\|F_1(-)\|$ is strict monoidal, but it does not strictly preserve ! nor $(-)^\perp$. ◀

▶ **Example 24.** Another factor choice F_2 is given by $F_2((\mathbb{A}, \mathcal{A})) := (\mathbb{A}, \mathcal{A}, \Phi^\oplus)$ with $\Phi^\oplus(a) := \sqrt{\#\mathbb{A}(a, a)}$. Then $\|F_2(\mathcal{P})\|_{[a],[b]} = \#\mathcal{P}(a, b)/\sqrt{\#\mathbb{A}(a, a)\#\mathbb{B}(b, b)}$. The functor $\|F_2(-)\|$ strictly preserves $\otimes$ and $(-)^\perp$, but it preserves the exponential only strongly. ◀

5 Conclusion and Future Work

In this paper, we have established the relationship between profunctors and matrices. While the matrix representation is not functorial for general profunctors, it is known to be functorial when restricted to those definable by λ-terms. We have shown that stability is the key to the functoriality. Indeed, the matrix representation defines a functor from the (bi)category of stabilized profunctors to the category of matrices, preserving the structure of linear logic.

There are several possible directions for future work. The first is an extension to the weighted setting [35]. A direct application of our arguments should at least allow a proof of functoriality. Second, we may consider the relationship between the well-behavedness of the matrix representation and the correctness of proof structures. As we have seen in Section 4, the matrix representation preserves the structure of linear logic, and hence the interpretation of a proof net should be preserved by the matrix representation. On the other hand, as shown in Example 8, there exists an incorrect proof structure whose interpretation is not preserved by the matrix representation. We have a result suggesting that the preservation of interpretation under the matrix representation corresponds to the correctness of proof structures in a certain sense.

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