Simpler Presentations for Many Fragments of Quantum Circuits

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Abstract

Equational reasoning is central to quantum circuit optimisation and verification: one replaces subcircuits by provably equivalent ones using a fixed set of rewrite rules viewed as equations. A finite rule set is most informative when it separates the genuine algebra of a circuit fragment from the structural treatment of wires. This paper gives six near-Clifford fragments a common PROP treatment, where wire permutations are structural: qubit Clifford, real Clifford, Clifford+T (up to two qubits), Clifford+CS (up to three qubits), CNOT-dihedral, and qutrit Clifford. Starting from prior completeness theorems, we transfer completeness into this setting and remove redundant non-structural rules, then check minimality by separating interpretations tailored to individual axioms; the resulting presentations are minimal in all arities for qubit Clifford, real Clifford, and CNOT-dihedral, minimal in bounded ranges for the remaining fragments, and comparable by one transfer-and-separation pattern.

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1 Introduction

Optimisation and verification of quantum circuits are often carried out by *equational reasoning*: one rewrites a circuit by replacing a subcircuit with a provably equivalent one using a fixed set of identities [14, 12]. For a fixed gate signature, this is formalised as an equational theory, namely a set of well-typed equations between circuits, closed under sequential and parallel composition [13, 22, 16]. With respect to an intended interpretation by unitary operators, a rule set is *complete* if it derives every equality between circuits having the same denotation.

A finite complete presentation then gives a compact algebraic description of a circuit fragment: it records which identities are taken as primitive, which ones are consequences, and how that distinction depends on the ambient syntax. If wire permutations are represented by ordinary gates, then naturality and swap-generator interaction equations describe how those permutations interact with the generators of the fragment; if permutations are structural, those equations become part of the surrounding calculus. Reducing a presentation is therefore more than shortening a list of rules, because it can reveal which equalities belong to generic wiring and which belong to the algebra of the fragment itself.

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Alongside completeness, we study *axiom independence* and *minimality* [8, 2, 24]: an axiom is independent if it is not derivable from the others, and a finite presentation is minimal if all its axioms are independent. Minimal presentations isolate the irreducible algebraic content of a fragment and are also attractive for automated rewriting, where redundant rules enlarge the search space [15, 11]. For the fragments studied here, minimality also gives a useful check on the simplification process: after derivable rules have been removed, each remaining axiom is tested against the possibility that it was already forced by the others.

This work studies six near-Clifford fragments–qubit Clifford, real Clifford, Clifford+T up to two qubits, Clifford+CS up to three qubits, CNOT-dihedral, and qutrit Clifford–and places them in a common PROP setting, so that swaps are structural rather than fragment-specific. In this setting, objects are wire counts, morphisms are circuits, sequential composition is plugging, tensor is side-by-side juxtaposition, and the PROP structure accounts for wire crossings, thereby separating generic wiring from the fragment-specific equations whose size and independence we want to analyse.

Table 1 Gate signatures used for the six presented fragments; swaps are structural and therefore not listed as fragment generators.

Fragment	Dimension	Non-structural generators
Qubit Clifford	2	$\omega_8 : 0 \to 0$, H, S : $1 \to 1$, : $2 \to 2$
Real Clifford	2	$-1 : 0 \to 0$, H, Z : $1 \to 1$, : $2 \to 2$
Clifford+T	2	$\omega_8 : 0 \to 0$, H, T : $1 \to 1$, : $2 \to 2$
Clifford+CS	2	$\omega_8 : 0 \to 0$, H, S : $1 \to 1$, : $2 \to 2$
CNOT-dihedral	2	$\omega_8 : 0 \to 0$, X, T : $1 \to 1$, : $2 \to 2$
Qutrit Clifford	3	$\omega_{12} : 0 \to 0$, H, S : $1 \to 1$, : $2 \to 2$

The source presentations in the literature are therefore our point of departure: after moving to a common PROP setting with structural symmetries, we seek presentations with fewer non-structural axioms and separator data for the minimality bounds listed in Table 2, so simplification, completeness transfer, and minimality become one comparison rather than separate rule-count exercises.

1.1 Related work

Generators-and-relations completeness theorems already give presentations for all six fragments of Table 1: the n-qubit Clifford presentation [23], the real Clifford presentation [19], the CNOT-dihedral presentation [1], the 2-qubit Clifford+T presentation [3], the 3-qubit Clifford+CS presentation [4], and the n-qutrit Clifford presentation [17]. These theorems are the starting point for the completeness transfer in Section 4: explicit encodings and decodings carry the known complete presentations to the smaller PROP presentations of Section 3. Related Agda formalizations already check the 2-qubit Clifford+T and 3-qubit Clifford+CS presentations [3, 4].

Beyond the near-Clifford setting, recent completeness theorems cover more expressive circuit languages [10, 9], where the extra expressivity changes the shape of the axiomatisation: for such languages, complete equational theories require rules of unbounded arity [8]. The latter work also uses separating models to prove minimality for a complete theory of general quantum circuits, and Section 5 adapts that proof pattern to the six fragments of Table 1. This work preserves the strict unitary semantics of the earlier completeness papers and asks which axioms remain necessary after simplification.

1.2 Contributions

1. **Smaller presentations in a uniform PROP setting.** For each fragment we give a finite presentation with fewer non-structural axioms than the standard one while preserving the same strict unitary semantics.
2. **Completeness by transfer.** The transfer isolates a finite syntactic comparison: explicit encoding/decoding maps carry earlier complete presentations to the smaller ones after aligning the ambient PROP structure and scalar conventions where necessary.
3. **Minimality and independence results for the simplified rule sets.** By separating interpretations, we prove that the new presentations for qubit Clifford, real Clifford, and CNOT-dihedral are minimal in all arities, while the Clifford+T, Clifford+CS, and qutrit Clifford presentations are minimal in the bounded ranges listed in Table 2.

The arity bounds have two origins: for Clifford+T and Clifford+CS, the imported completeness theorems themselves are proved only up to 2 and 3 qubits, while for minimality the remaining bounds are the arities for which we construct explicit separating interpretations.

After the generic PROP structure has been factored out, Table 2 compares non-structural rule counts; the removed source rules are not lost, since they are derivable from the simplified presentations.

Table 2 Non-structural rule-count comparison after moving from source PRO presentations to PROP presentations with structural symmetries.

Fragment	Previous Presentation	Our Presentation	Figure
Qubit Clifford on n qubits	15 rules [23]	8 rules (minimal in all arities)	2
Real Clifford on n qubits	16 rules [19]	10 rules (minimal in all arities)	3
Qutrit Clifford on n qutrits	18 rules [17]	10 rules (minimal up to 2 qutrits; conjectured in all arities)	4
Clifford+T on 2 qubits	18 rules [3]	11 rules (minimal up to 1 qubit)	5
Clifford+CS on 3 qubits	17 rules [4]	14 rules (minimal up to 2 qubits)	6
CNOT-dihedral on n qubits	13 rules [1]	11 rules (minimal in all arities)	7

The rest of the paper follows the proof obligations behind Table 2: Section 2 fixes the PROP syntax used to lift PRO presentations, Section 3 defines the six presented circuit fragments and their unitary interpretations, Section 4 transfers completeness from the source presentations and handles scalar refinements, Section 5 checks independence by separating interpretations, and Appendix A records the finite shortcut, transfer, and scalar-refinement data used by the proofs.

2 Graphical languages for quantum circuits

The formal language is categorical, but its role is practical: it fixes the syntax for the usual boxes-and-wires calculus. In this setting, generic wiring is treated structurally, so the displayed rule sets can focus on fragment-specific generators and equations.

2.1 Graphical languages

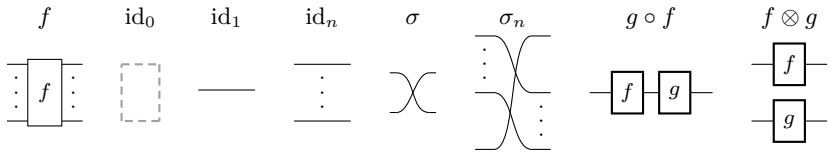
We work with the one-object-per-wire formalisms PROs and PROPs in the sense of Joyal–Street [13, 22]. We adopt strictness throughout: associativity and unit constraints are treated as literal equalities, so parentheses can be omitted.

▶ **Definition 1.** *A* PRO *is a strict monoidal category* $\mathbf{P}$ *whose objects are the natural numbers* $n \in \mathbb{N}$ *and whose tensor on objects is addition* $n \otimes m := n + m$ *with unit* 0*. We write* $\mathbf{P}(n, m)$ *for morphisms* $n \to m$*, with sequential composition* $\circ$ *and parallel composition* $\otimes$*, subject to the usual strict monoidal axioms (Figures 1a–1e).*

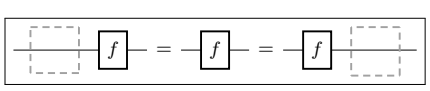
We write $f : n \to m$ when $f \in \mathbf{P}(n, m)$; n and m are the numbers of input and output wires of f. In particular $\mathrm{id}_n = \mathrm{id}_1^{\otimes n}$ for all $n \in \mathbb{N}$.

▶ **Definition 2.** *A* PROP *is a PRO* $\mathbf{P}$ *equipped with a symmetric monoidal structure, i.e. symmetry isomorphisms* $\sigma_{n,m} : n + m \to m + n$ $(n, m \in \mathbb{N})$ *generated by the basic swap* $\sigma := \sigma_{1,1}$ *and satisfying the axioms* (1f)-(1g) *in Figure 1. We write* $\sigma_n := \sigma_{n,1} : n+1 \to 1+n$ *for swapping one wire with a block of* n *wires.*

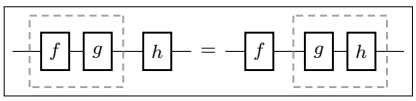
In a PROP, swaps and permutations are part of the ambient structure, and the resulting string-diagram representation draws a morphism $f : n \to m$ as a box with n incoming wires and m outgoing wires. Sequential composition is drawn by stacking boxes, tensor by placing them side by side, and $\sigma_{n,m}$ by crossing a block of n wires with a block of m wires. With these conventions, the coherence axioms are exactly the diagrammatic moves that make deformation sound. For example, if $g : 1 \to 1$ and $h : 2 \to 2$ are gates, then $g \otimes h : 3 \to 3$ is their side-by-side circuit, and composing with $\sigma_{1,2}$ just moves the first output wire past the other two. Such crossings are part of the syntax before any fragment-specific rule is added.



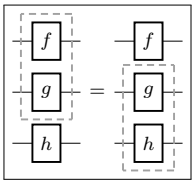
Equalities obtained solely by these coherence moves, including permutations, are *structural* and may be omitted from diagrammatic arguments.



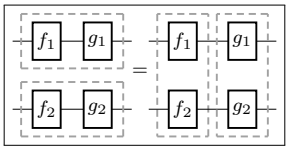
(a) $f \circ \mathrm{id}_n = f = \mathrm{id}_n \circ f$.



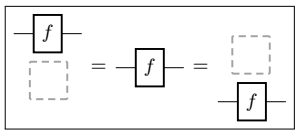
(b) $h \circ (g \circ f) = (h \circ g) \circ f$.



(c) $(f \otimes g) \otimes h = f \otimes (g \otimes h)$.

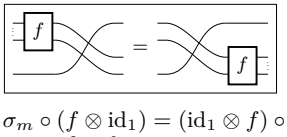


(d) $(g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2)$.



(e) $f \otimes \mathrm{id}_0 = f = \mathrm{id}_0 \otimes f$.

(f) $\sigma \circ \sigma = \mathrm{id}_2$.



(g) $\sigma_m \circ (f \otimes \mathrm{id}_1) = (\mathrm{id}_1 \otimes f) \circ \sigma_n$ for $f : n \to m$.

Figure 1 Coherence laws for PROs and PROPs.

2.2 Free PROPs and presentations

Circuit fragments are specified by *generators and relations*. Generators represent primitive gates with fixed arity, and relations are the rewrite rules used for equational reasoning.

▶ **Definition 3.** *A* (PROP) signature *is a set* Σ *equipped with arity functions* $\mathrm{dom}, \mathrm{cod} : \Sigma \to \mathbb{N}$. *An element* $g \in \Sigma$ *is written as a generating morphism* $g : \mathrm{dom}(g) \to \mathrm{cod}(g)$.

▶ **Definition 4.** *Let* Σ *be a signature. The* free PROP *on* Σ, *written* $\mathbf{P}_\Sigma$, *is the PROP whose morphisms are string diagrams generated from:*

- *the non-structural generators* $g \in \Sigma$ *(with their prescribed arities), and*
- *the structural morphisms of a PROP (identities and symmetries),*

quotiented by PROP coherence (i.e. the axioms of Figure 1).

▶ **Definition 5.** *Let* Σ *be a signature and let* $\mathcal{R}$ *be a set of well-typed equations* $L = R$ *between morphisms of the free PROP* $\mathbf{P}_\Sigma$. *The* presented PROP $\mathbf{P}_\Sigma/\mathcal{R}$ *is the quotient of* $\mathbf{P}_\Sigma$ *by the smallest congruence that contains* $\mathcal{R}$ *and is closed under* $\circ$, $\otimes$, *and symmetry.*

This quotient is the *graphical language* used in the rest of the paper: diagrams are equal exactly when their equality is derivable from PROP coherence together with the chosen relations.

2.3 Monoidal functors and completeness

An interpretation sends the presented syntax to its intended operations; completeness is the assertion that every semantic equality in that image can be recovered as a derivation in the quotient.

▶ **Definition 6.** *A* PROP morphism *is a strict symmetric monoidal functor* $F : \mathbf{P}_1 \to \mathbf{P}_2$ *between PROPs that is the identity on objects. Equivalently,* F *preserves identities, symmetries, sequential composition and tensor.*

▶ **Definition 7.** *Let* $\mathbf{P}$ *be a PROP. A* subPROP *of* $\mathbf{P}$ *is a wide strict symmetric monoidal subcategory* $\mathbf{Q} \subseteq \mathbf{P}$. *Since the objects of a PROP are fixed to be* $\mathbb{N}$, *this amounts to specifying subsets* $\mathbf{Q}(n,m) \subseteq \mathbf{P}(n,m)$ $(n,m \in \mathbb{N})$ *such that* $\mathrm{id}_n \in \mathbf{Q}(n,n)$ *for all* n; $\sigma_{n,m} \in \mathbf{Q}(n+m, m+n)$ *for all* n, m; *and* $\mathbf{Q}$ *is closed under* $\circ$ *and* $\otimes$.

In this situation, the inclusion functor $\mathbf{Q} \hookrightarrow \mathbf{P}$ *is a faithful PROP morphism.*

▶ **Definition 8.** *Let* $\mathbf{P}_\Sigma/\mathcal{R}$ *be a presented PROP and let* $\mathbf{C}$ *be a symmetric monoidal category. An* interpretation functor *is a strict symmetric monoidal functor* $[\![\cdot]\!] : \mathbf{P}_\Sigma \to \mathbf{C}$ *specified on generators* $g \in \Sigma$ *and extended monoidally to all diagrams.*

If every equation in $\mathcal{R}$ *holds in* $\mathbf{C}$, *then* $[\![\cdot]\!]$ *factors uniquely through the quotient, yielding a functor* $[\![\cdot]\!] : \mathbf{P}_\Sigma/\mathcal{R} \to \mathbf{C}$.

▶ **Definition 9.** *Let* $[\![\cdot]\!] : \mathbf{P}_\Sigma/\mathcal{R} \to \mathbf{C}$ *be such an interpretation. We say that the graphical language* $\mathbf{P}_\Sigma/\mathcal{R}$ *is* complete *for this semantics if, for all circuits* C_1, C_2, $[\![C_1]\!] = [\![C_2]\!]$ *implies* $\mathbf{P}_\Sigma/\mathcal{R} \vdash C_1 = C_2$. *Equivalently, the interpretation functor is faithful.*

We will also use a bounded variant. Given $k \in \mathbb{N}$, *we say that* $\mathbf{P}_\Sigma/\mathcal{R}$ *is* complete up to k wires *if the implication holds whenever* $C_1, C_2 : n \to m$ *with* $n, m \leq k$. *Equivalently, the interpretation functor is faithful on each hom-set* $\mathbf{P}_\Sigma/\mathcal{R}(n,m)$ *for* $n, m \leq k$.

For equational reasoning, completeness says that semantic equality can always be witnessed by an explicit diagrammatic derivation from the chosen relations; the bounded form restricts this requirement to circuits of limited arity.

When $\mathbf{C}$ is, by definition, the symmetric monoidal subcategory of an ambient category generated by the images of the generators, the interpretation functor is also full; in that case $\mathbf{P}_\Sigma/\mathcal{R}$ is actually isomorphic to the semantic subPROP $\mathbf{C}$.

Completeness theorems stated for PROs can still be used in this PROP setting through a single lifting principle.

▶ **Lemma 10** (PRO-to-PROP lifting). *Let $\mathbf{P}/\mathcal{R}$ be a complete presented PRO with an interpretation into a* symmetric *monoidal category $\mathbf{C}$. Assume there exists a PRO circuit $\tau : 2 \to 2$ such that $[\![\tau]\!]$ is the symmetry $\sigma_{1,1}$ of $\mathbf{C}$. Form the free PROP on the same generators, and quotient it by $\mathcal{R}$ together with the single equation $\sigma_{1,1} = \tau$. Then the resulting PROP presentation is complete for the same semantics.*

Proof. The equation $\sigma_{1,1} = \tau$ rewrites each structural swap between adjacent wires as a source PRO circuit. By the symmetric monoidal axioms of a PROP, every structural permutation is generated by such adjacent swaps, so each PROP diagram D has a translated PRO diagram $T(D)$ obtained by replacing structural swaps by copies of τ and leaving the ordinary generators unchanged. The interpretation satisfies $[\![D]\!] = [\![T(D)]\!]$ because $[\![\tau]\!] = \sigma_{1,1}$. If two PROP diagrams have the same semantics, their translations have the same semantics as PRO diagrams, hence are equal by PRO completeness. Replacing each occurrence of τ by $\sigma_{1,1}$ then gives the original PROP equality. ◀

3 Quantum circuits and their relations

The transfer and separation arguments use one Hilbert-space semantic setup: five qubit fragments share a signature, while qutrit Clifford uses its own presented subPROP.

Let **FdHilb** be the symmetric monoidal category of finite-dimensional complex Hilbert spaces and linear maps, with tensor product $\otimes$, tensor unit $\mathbb{C}$, and symmetry given by swapping tensor factors. By Mac Lane's coherence theorem [18] we may regard **FdHilb** as strict monoidal.

▶ **Definition 11.** *Let H, K be finite-dimensional Hilbert spaces. A linear map $U : H \to K$ is* unitary *if $U^\dagger U = \mathrm{id}_H$ and $UU^\dagger = \mathrm{id}_K$, where $U^\dagger$ denotes the Hilbert adjoint of U. We write $\mathrm{U}(H, K)$ for the set of unitary maps $H \to K$, and $\mathrm{U}(H) \coloneqq \mathrm{U}(H, H)$.*

▶ **Definition 12.** *For $d \geq 2$, let $\mathbf{Qudit}_d$ be the PROP with objects $\mathbb{N}$ and hom-sets $\mathbf{Qudit}_d(n, m) \coloneqq \mathrm{U}\big((\mathbb{C}^d)^{\otimes n}, (\mathbb{C}^d)^{\otimes m}\big)$, with composition, identities, tensor, and symmetry inherited from* **FdHilb** *viewed as strict monoidal. The qubit and qutrit PROPs are $\mathbf{Qubit} \coloneqq \mathbf{Qudit}_2$ and $\mathbf{Qutrit} \coloneqq \mathbf{Qudit}_3$.*

For $d \geq 2$, $\mathbf{Qudit}_d$ is endomorphism-only: there are no unitaries $n \to m$ for $n \neq m$. For a set G of morphisms in a PROP, $\langle G \rangle$ denotes the smallest subPROP containing G and all structural morphisms.

3.1 Qubit circuit fragments as presented subPROPs

The five qubit fragments share a common qubit signature and unitary interpretation.

▶ **Definition 13.** *The* common qubit signature $\Sigma^{(2)}$ *has non-structural generators* , : $0 \to 0$, , , , , : $1 \to 1$, *and* , : $2 \to 2$. *We write $\mathbf{P}^{(2)} \coloneqq \mathbf{P}_{\Sigma^{(2)}}$ for the free PROP on $\Sigma^{(2)}$. Structural morphisms id_n and $\sigma_{n,m}$ come from the ambient PROP structure.*

▶ Remark 14. All signatures used in this section are endomorphism-only: each generator has type $n \to n$. Hence every well-typed fragment circuit, and every equation between such circuits, preserves the wire count; this is the arity fact used in Section 5.

We use the standard Dirac notation for computational basis states: for $x \in \{0,1\}$ we write $|x\rangle \in \mathbb{C}^2$, and for $x, y \in \{0,1\}$ we abbreviate $|x,y\rangle := |x\rangle \otimes |y\rangle$ (similarly, $|x_1,\ldots,x_n\rangle := |x_1\rangle \otimes \cdots \otimes |x_n\rangle$).

By the universal property of the free PROP, there is a unique strict symmetric monoidal functor $[\![\cdot]\!]^{(2)} : \mathbf{P}^{(2)} \to \mathbf{Qubit}$ sending:

$[\![\omega_8]\!]^{(2)} : 1 \longmapsto e^{i\pi/4}$ $\quad$ $[\![\mathrm{H}]\!]^{(2)} : |x\rangle \longmapsto \frac{1}{\sqrt{2}}\sum_{k=0}^{1}(-1)^{xk}|k\rangle$

$[\![-1]\!]^{(2)} : 1 \longmapsto -1$ $\quad$ $[\![\mathrm{S}]\!]^{(2)} : |x\rangle \longmapsto i^x|x\rangle$

$[\![\mathrm{X}]\!]^{(2)} : |x\rangle \longmapsto |1-x\rangle$ $\quad$ $[\![\mathrm{Z}]\!]^{(2)} : |x\rangle \longmapsto (-1)^x|x\rangle$

$[\![\bullet\!-\!\oplus]\!]^{(2)} : |x,y\rangle \longmapsto |x, x\oplus y\rangle$ $\quad$ $[\![\mathrm{T}]\!]^{(2)} : |x\rangle \longmapsto e^{i\pi x/4}|x\rangle$

$[\![\bullet\!-\!\bullet]\!]^{(2)} : |x,y\rangle \longmapsto i^{xy}|x,y\rangle$

for $x, y \in \{0,1\}$ and addition $\oplus$ modulo 2.

Each qubit fragment is specified by a sub-signature of $\Sigma^{(2)}$ and equations on the free PROP it generates; the common unitary interpretation then restricts to this syntax. The index $\bullet$ records the chosen fragment, while $\Sigma_\bullet$, $\mathbf{P}_\bullet$, $\mathrm{C}_\bullet$, $\mathbf{QCirc}_\bullet$, $\mathbf{Qubit}_\bullet$, and $[\![\cdot]\!]_\bullet$ denote its gates, free PROP, equations, presented PROP, generated semantic subPROP, and quotient interpretation, respectively.

▶ **Definition 15.** *Let $\bullet \in \{Cliff, RCliff, CliffT, CliffCS, CNOTdihe\}$ range over the five qubit fragments considered in this paper. Given a sub-signature $\Sigma_\bullet \subseteq \Sigma^{(2)}$, let $\mathbf{P}_\bullet := \mathbf{P}_{\Sigma_\bullet}$ be the free PROP on $\Sigma_\bullet$. If $\mathrm{C}_\bullet$ is a finite set of well-typed equations in $\mathbf{P}_\bullet$, define the presented PROP $\mathbf{QCirc}_\bullet := \mathbf{P}_\bullet/\mathrm{C}_\bullet$. The semantic target is the subPROP $\mathbf{Qubit}_\bullet := \langle\, [\![g]\!]^{(2)} \mid g \in \Sigma_\bullet \,\rangle \subseteq \mathbf{Qubit}$, generated by the interpretations of the gates in $\Sigma_\bullet$. When $\mathrm{C}_\bullet$ is sound for the restricted common interpretation, the quotient carries the induced strict symmetric monoidal functor $[\![\cdot]\!]_\bullet : \mathbf{QCirc}_\bullet \to \mathbf{Qubit}_\bullet$. We call such a presentation complete when this functor is faithful.*

▶ **Definition 16.** *The five qubit fragments treated in this paper are specified by these sub-signatures and rule sets:*

Fragment	$\Sigma_\bullet \subseteq \Sigma^{(2)}$	***Rule set*** $\mathrm{C}_\bullet$
Clifford	$\Sigma_{Cliff} := \{\omega_8, \mathrm{H}, \mathrm{S}, \bullet\!-\!\oplus\}$	$\mathrm{C}_{Cliff} :=$ Cliff$_2$ *(Figure 2)*
Real Clifford	$\Sigma_{RCliff} := \{-1, \mathrm{H}, \mathrm{Z}, \bullet\!-\!\oplus\}$	$\mathrm{C}_{RCliff} :=$ RCliff$_2$ *(Figure 3)*
Clifford+T	$\Sigma_{CliffT} := \{\omega_8, \mathrm{H}, \mathrm{T}, \bullet\!-\!\oplus\}$	$\mathrm{C}_{CliffT} :=$ CliffT$_2$ *(Figure 5)*
Clifford+CS	$\Sigma_{CliffCS} := \{\omega_8, \mathrm{H}, \mathrm{S}, \bullet\!-\!\bullet\}$	$\mathrm{C}_{CliffCS} :=$ CliffCS$_2$ *(Figure 6)*
CNOT-dihedral	$\Sigma_{CNOTdihe} := \{\omega_8, \mathrm{X}, \mathrm{T}, \bullet\!-\!\oplus\}$	$\mathrm{C}_{CNOTdihe} :=$ CNOTdihe$_2$ *(Figure 7)*

3.2 The qutrit Clifford fragment

For qutrits, the paper uses only the Clifford fragment, denoted *Cliff3*, whose signature is separate from the common qubit signature; we use the computational basis $\{|0\rangle, |1\rangle, |2\rangle\}$ of $\mathbb{C}^3$ and the induced tensor-product basis.

▶ **Definition 17.** *The* qutrit Clifford *fragment is generated by the signature* $\Sigma_{Cliff3} := \{\omega_{12} : 0 \to 0, \mathrm{H}, \mathrm{S} : 1 \to 1, \bullet\!-\!\oplus : 2 \to 2\}$.

Let $\mathbf{P}_{Cliff3} \coloneqq \mathbf{P}_{\Sigma_{Cliff3}}$ *be the free PROP on* Σ_{Cliff3} *and let* $\mathrm{C}_{Cliff3} \coloneqq \mathrm{Cliff}_3$ *be the rule set given in Figure 4. The presented PROP is* $\mathbf{QCirc}_{Cliff3} \coloneqq \mathbf{P}_{Cliff3}/\mathrm{C}_{Cliff3}$.

By the universal property of the free PROP, the assignments

$$\llbracket \omega_{12} \rrbracket_{Cliff3} : 1 \longmapsto e^{i\pi/6} \qquad \llbracket \mathrm{H} \rrbracket_{Cliff3} : |x\rangle \longmapsto \tfrac{1}{\sqrt{3}} \sum_{k=0}^{2} e^{2\pi i\, xk/3} |k\rangle$$

$$\llbracket \text{CNOT} \rrbracket_{Cliff3} : |x, y\rangle \longmapsto |x, (x+y) \bmod 3\rangle \qquad \llbracket \mathrm{S} \rrbracket_{Cliff3} : |x\rangle \longmapsto e^{i\pi x(x-1)/3} |x\rangle$$

for $x, y \in \{0, 1, 2\}$ determine a strict symmetric monoidal functor $\mathbf{P}_{Cliff3} \to \mathbf{Qutrit}$. Let $\mathbf{Qutrit}_{Cliff3} \coloneqq \langle \llbracket g \rrbracket_{Cliff3} \mid g \in \Sigma_{Cliff3} \rangle$ be the semantic subPROP generated by the qutrit Clifford gates. Since the equations of C_{Cliff3} are sound for this interpretation, the free-PROP interpretation factors through the quotient and yields $\llbracket \cdot \rrbracket_{Cliff3} : \mathbf{QCirc}_{Cliff3} \longrightarrow \mathbf{Qutrit}_{Cliff3}$.

3.3 Rule-set presentations

Each displayed equation is an arity-preserving schema; gates outside the corresponding fragment signature are shortcuts expanded as in Figure 9.

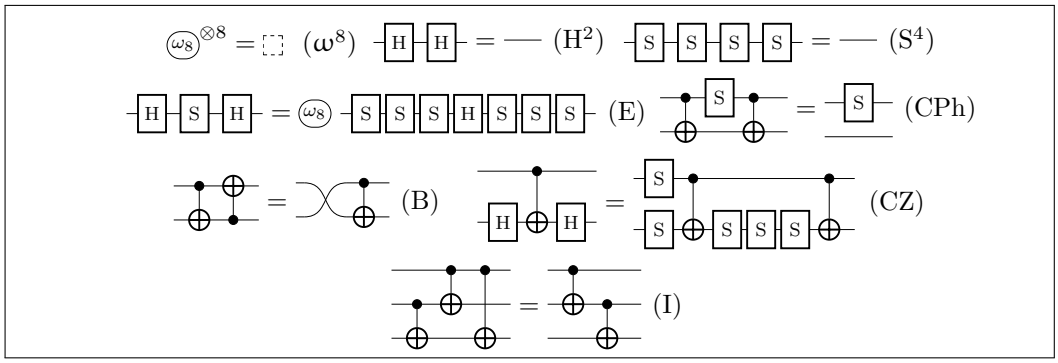


Figure 2 Qubit Clifford presentation Cliff_2, with the scalar, one-qubit, controlled-phase, swap-decomposition, controlled-Z, and three-wire interaction laws used in the completeness and minimality arguments.

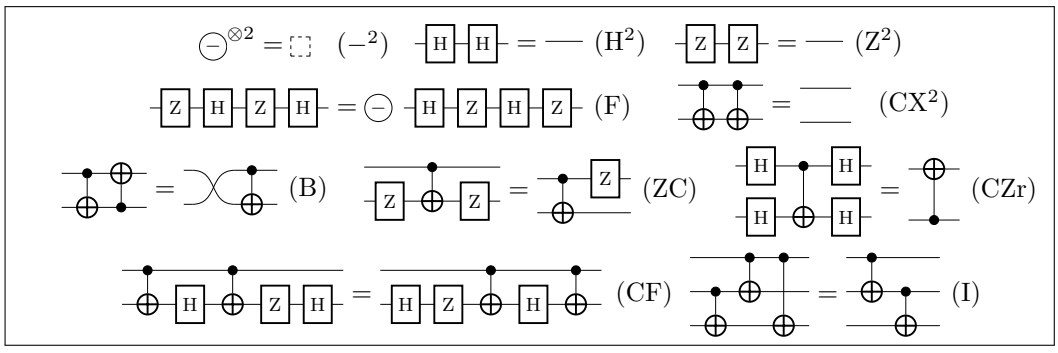


Figure 3 Real Clifford presentation RCliff_2, where the real phase, Hadamard-Z, controlled-X, controlled-Z, and three-wire interaction laws replace the corresponding Clifford rules in the real fragment.

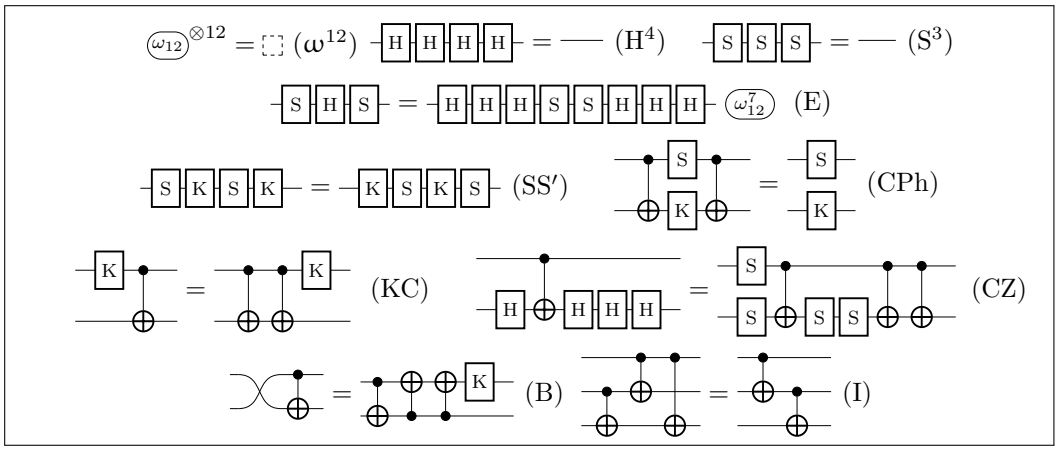


Figure 4 Qutrit Clifford presentation Cliff_3, using the qutrit scalar convention and the qutrit versions of the phase, controlled-addition, controlled-Z, swap-decomposition, and three-wire laws.

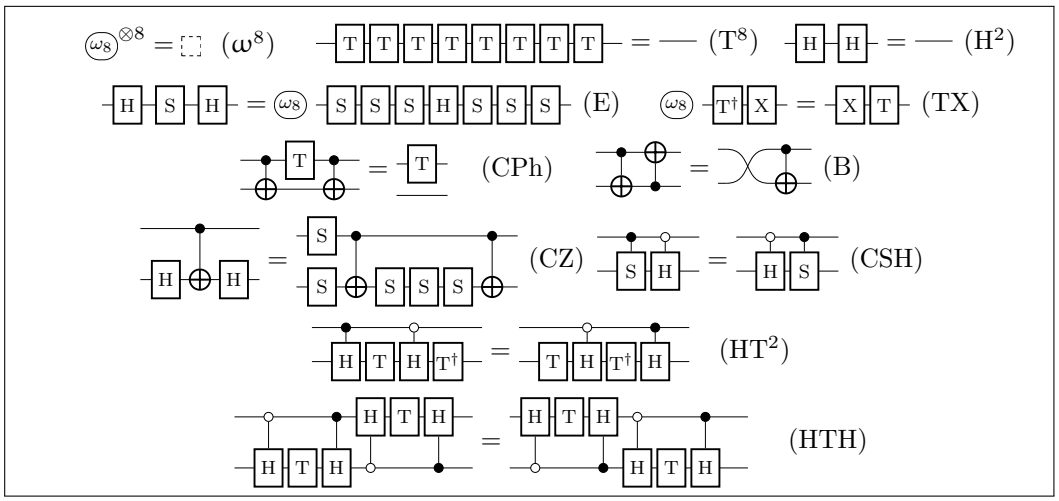


Figure 5 Qubit Clifford+T presentation CliffT_2, combining the Clifford core with the T-phase, controlled-phase, controlled-Hadamard, and HTH-type relations needed for the two-qubit completeness transfer.

4 Completeness: from PROs to PROPs

For each fragment $\bullet$ of Section 3 we have a presented PROP $\mathbf{QCirc}_\bullet = \mathbf{P}_\bullet/\mathrm{C}_\bullet$ and a strict symmetric monoidal interpretation $[\![\cdot]\!]_\bullet : \mathbf{QCirc}_\bullet \to \mathbf{Qubit}_\bullet$ (or $\mathbf{Qutrit}_{\mathit{Cliff3}}$ in the qutrit case). Completeness means faithfulness of this interpretation. Complete equational theories for these fragments already exist in the literature [23, 19, 17, 3, 4, 1]. Those theorems supply the source completeness results: their normal forms prove faithfulness for older source presentations, and the transfer argument in this section checks that the source syntax and our smaller figure-defined target syntax present the same semantic subPROP.

The transfer has to align three conventions before the cited normal forms can be used for our quotient PROPs. If a source theorem is stated for a PRO, Lemma 10 adds the structural symmetries needed to read it as a PROP theorem. For qutrit Clifford and Clifford+CS, the source and target scalar subgroups differ: the source presentations make fewer global phases explicit, which is harmless for the original proofs but does not match the standard strict matrices fixed in Section 3. The scalar-refinement step adjoins the missing roots of

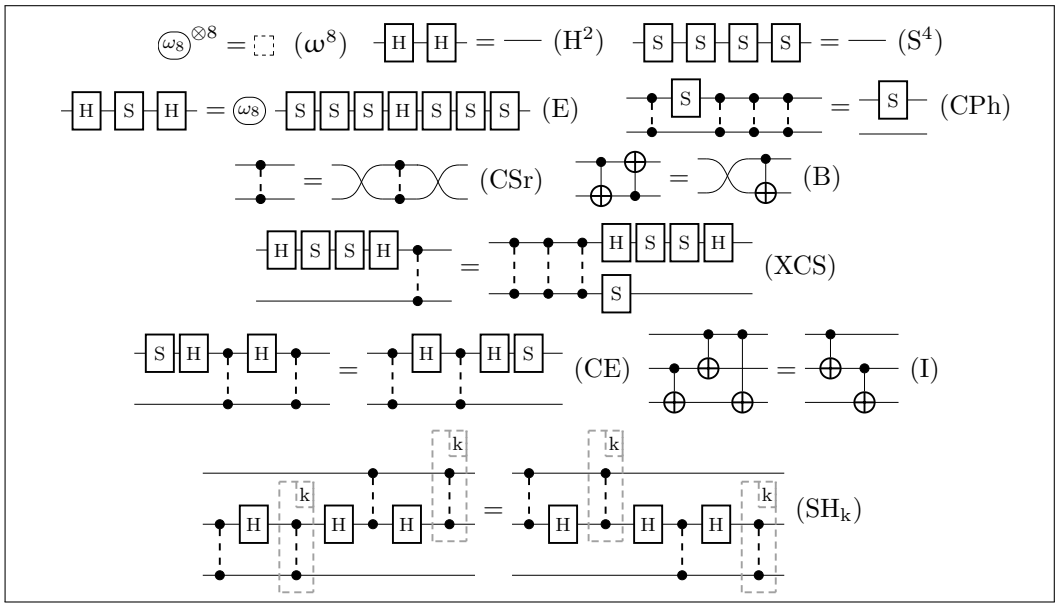


Figure 6 Qubit Clifford+CS presentation $\mathrm{CliffCS}_2$, combining the Clifford phase laws with controlled-S commutation and the two three-wire relations used for the three-qubit transfer; in Equation ($\mathrm{SH_k}$), the dashed box repeats the enclosed subcircuit k times for $k \in \{0, 1, 2, 3\}$.

unity, so source and target have the same strict semantics. Encoding and decoding are then identity-on-objects PROP morphisms, in the sense of Definition 6, that compare the source generators and relations with ours; once these maps preserve semantics and decode source axioms to derivable target equations, Theorem 21 transports faithfulness.

4.1 Scalar refinement

Because our semantics uses strict unitary equality rather than projective equality, global phases are visible. Equivalently, the scalar subgroup $S(\mathbf{C}) = \mathbf{C}(0, 0)$ of the semantic PROP is part of the data that must match between source and target presentations. For our purposes, a scalar mismatch is therefore a mismatch between presentation conventions: the source convention makes fewer global phases explicit, while our convention keeps the standard gate matrices and records the extra phases as scalars.

Only two source completeness results require scalar adjustment. The qutrit Clifford presentation of [17] uses an order-6 scalar subgroup whereas our semantics uses order 12, and the Clifford+CS presentation of [4] uses order 4 whereas our qubit semantics uses order 8. In both cases we appeal to the scalar-refinement construction of Appendix A.3, which adjoins the missing scalar generator and is conservative by Lemma 38. Source and target can then be compared inside the same semantic subPROP, while the refinement remains conservative on the old source signature.

4.2 Encoding, decoding, and transfer of completeness

The five fragments handled by encoding and decoding reduce to one generator-and-relation comparison.

Fix one encoding/decoding transfer instance. Here $\mathbf{P}$ and $\mathcal{R}$ denote the target free PROP and target relations, while $\mathbf{P}^{\mathrm{src}}$ and $\mathcal{R}^{\mathrm{src}}$ denote the corresponding source free PROP and source relations, after PRO-to-PROP lifting and scalar refinement where those steps apply.

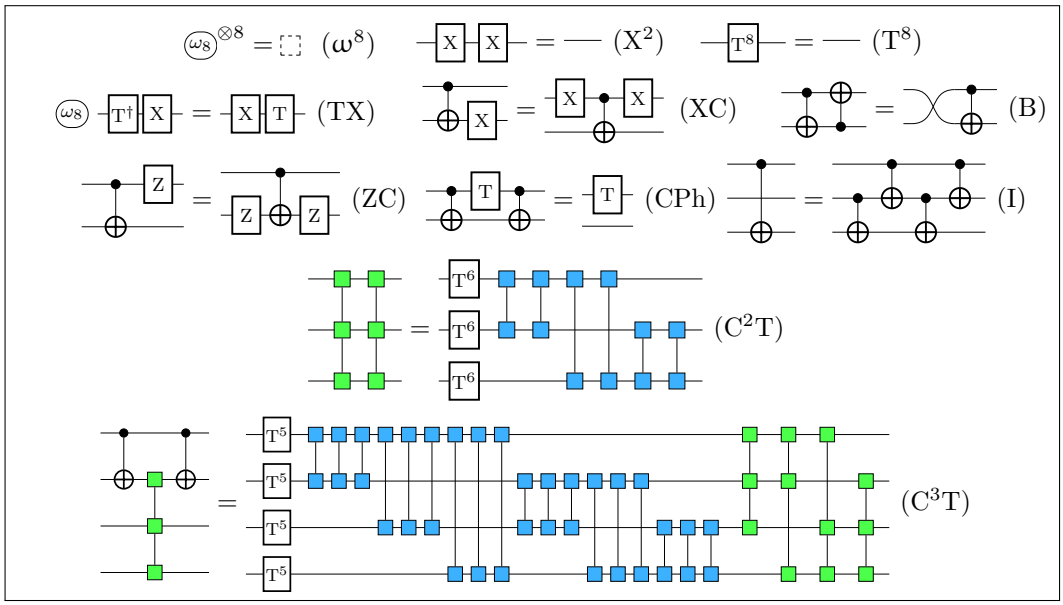


Figure 7 CNOT-dihedral presentation $\mathrm{CNOTdihe}_2$, with the X, T, CNOT, phase-commutation, C^2T, and C^3T relations used to recover the omitted source axioms.

▶ **Definition 18.** *An encoding/decoding pair consists of a target-to-source PROP morphism* $E : \mathbf{P} \to \mathbf{P}^{\mathrm{src}}$ *and a source-to-target PROP morphism* $D : \mathbf{P}^{\mathrm{src}} \to \mathbf{P}$ *such that* $[\![C]\!] = [\![E(C)]\!]_{\mathrm{src}}$ *for every target circuit* $C \in \mathbf{P}$ *and* $[\![C']\!]_{\mathrm{src}} = [\![D(C')]\!]$ *for every source circuit* $C' \in \mathbf{P}^{\mathrm{src}}$.

After importing a source completeness theorem, the comparison is local: it involves only the chosen generators and source relations. A typical mismatch is that a source presentation takes controlled-Z as primitive, whereas our target signature takes CNOT. The decoder replaces source controlled-Z by its fixed expansion through CNOT and Hadamards; the target-generator check then proves, for example, that decoding the encoded CNOT derives CNOT again, while the source-relation check verifies each source equation after the same replacement. The non-identity parts of the maps used here are recorded in Table 3.

The transfer therefore has a finite checklist: the round-trip equality $D(E(g)) = g$ for each target generator g, and the decoded equality $D(L) = D(R)$ for each source axiom $L = R$. This is the presentation-level analogue of Reidemeister–Schreier rewriting [20, 21]: instead of computing a subgroup presentation, we compare two chosen presentations by following the images of their generators and defining relations.

▶ **Lemma 19.** *Assume* $\mathbf{P}$ *is presented by generators* Σ *and relations* $\mathcal{R}$. *If for every non-structural generator* $g \in \Sigma$, $\mathbf{P}/\mathcal{R} \vdash D(E(g)) = g$, *then for every circuit* C *built from* Σ, $\mathbf{P}/\mathcal{R} \vdash D(E(C)) = C$.

Proof. By structural induction on C, using that E and D preserve $\circ$, $\otimes$, and the structural morphisms. ◀

▶ **Lemma 20.** *Assume that for every axiom* $L = R$ *in* $\mathcal{R}^{\mathrm{src}}$, $\mathbf{P}/\mathcal{R} \vdash D(L) = D(R)$. *Then* D *induces a PROP morphism* $\overline{D} : \mathbf{P}^{\mathrm{src}}/\mathcal{R}^{\mathrm{src}} \to \mathbf{P}/\mathcal{R}$, *and any derivable equality in the source quotient transports to a derivable equality between its decodings in the target theory.*

Proof. The hypothesis says exactly that D equalises all source relations, hence factors through the quotient. ◀

▶ **Theorem 21.** *Assume* $\mathbf{P}^{\mathrm{src}}/\mathcal{R}^{\mathrm{src}}$ *is complete for* $[\![\cdot]\!]_{\mathrm{src}}$*, and let* (E, D) *be an encoding/-decoding pair satisfying the hypotheses of Lemmas 19 and 20. Then* $\mathbf{P}/\mathcal{R}$ *is complete for* $[\![\cdot]\!]$*.*

Proof. If target circuits C_1, C_2 have the same semantics, then semantic preservation of E gives $[\![E(C_1)]\!]_{\mathrm{src}} = [\![E(C_2)]\!]_{\mathrm{src}}$. Source completeness yields $E(C_1) = E(C_2)$ in $\mathbf{P}^{\mathrm{src}}/\mathcal{R}^{\mathrm{src}}$; applying the quotient morphism $\overline{D}$ from Lemma 20 gives $D(E(C_1)) = D(E(C_2))$ in $\mathbf{P}/\mathcal{R}$, and Lemma 19 rewrites both sides to $C_1 = C_2$. ◀

The same proof gives the bounded form used in Theorem 22: if the source interpretation is faithful on all hom-sets $n \to m$ with $n, m \leq k$, then the target interpretation is faithful on the same range, because E and D are identity-on-objects PROP morphisms.

4.3 Fragment-wise completeness

Applying the transfer to the five encoding/decoding fragments and the direct CNOT-dihedral argument gives the completeness bounds used by the minimality theorem.

▶ **Theorem 22.**
1. $\mathbf{QCirc}_{Cliff}$ *is complete for* $\mathbf{Qubit}_{Cliff}$*.*
2. $\mathbf{QCirc}_{RCliff}$ *is complete for* $\mathbf{Qubit}_{RCliff}$*.*
3. $\mathbf{QCirc}_{Cliff3}$ *is complete for* $\mathbf{Qutrit}_{Cliff3}$*.*
4. $\mathbf{QCirc}_{CliffT}$ *is complete up to* 2 *qubits for* $\mathbf{Qubit}_{CliffT}$*.*
5. $\mathbf{QCirc}_{CliffCS}$ *is complete up to* 3 *qubits for* $\mathbf{Qubit}_{CliffCS}$*.*
6. $\mathbf{QCirc}_{CNOTdihe}$ *is complete for* $\mathbf{Qubit}_{CNOTdihe}$*.*

Proof. For the five fragments using encoding/decoding, start from the corresponding cited source completeness theorem; in the qutrit Clifford and Clifford+CS cases, first pass to the scalar-refined source presentation supplied by Lemma 38. The arity bounds in the Clifford+T and Clifford+CS items are the bounds inherited from those source results.

Appendix A records the finite transfer data needed to invoke Theorem 21: shortcut conventions, the non-identity encoding/decoding maps of Table 3, and the scalar-refinement lemma. With these ingredients fixed, the remaining derivability obligations are the target-generator checks for Lemma 19 and the decoded source-relation checks for Lemma 20; the target axioms used in these checks are exactly the labeled equations in the rule sets of Section 3. The full version of this work expands those generator round trips and decoded source-relation checks into diagrammatic derivations, with no further normal-form argument.

For CNOT-dihedral, use the completeness theorem of [1] directly: no encoding/decoding is needed, and the appendix of the full version derives, in $\mathrm{CNOTdihe}_2$, each source axiom removed from our simplified presentation. ◀

5 Minimality

Minimality asks which axioms of the six figure-defined presentations are genuinely necessary for completeness, and the full and bounded statements in Theorem 37 are certified by separating interpretations. Throughout, fix a fragment $\bullet$ with free PROP $\mathbf{P}_\bullet$ and finite rule set $\mathrm{C}_\bullet$. The arity bounds in the theorem are dictated by separator coverage: a bounded minimality claim is made only when the table contains a non-none separator for every axiom in that truncation. When this independence is read as necessity for completeness, the Clifford+T and Clifford+CS cases are understood relative to the bounded completeness results imported in Section 4.

5.1 Independence and minimality

To test whether an axiom ρ is needed, remove it from the rule set and ask whether its two sides are still equal in the quotient by the remaining rules. We write $\mathrm{C}_\bullet \setminus \{\rho\}$ for those remaining axioms, and $\mathbf{QCirc}_\bullet^{-\rho} \coloneqq \mathbf{P}_\bullet/(\mathrm{C}_\bullet \setminus \{\rho\})$ for the reduced presentation.

▶ **Definition 23.** *Let ρ be an axiom $C_1 = C_2$ in $\mathrm{C}_\bullet$ (with $C_1, C_2 : n \to n$). We say that ρ is independent of the other axioms of the same fragment if $\mathbf{QCirc}_\bullet^{-\rho} \not\vdash C_1 = C_2$. We say that $\mathrm{C}_\bullet$ is minimal if every $\rho \in \mathrm{C}_\bullet$ is independent.*

Because the presentations are sound for their intended semantics, an axiom derivable from the others cannot be necessary for completeness: removing it leaves the induced congruence unchanged. Conversely, in any arity range where the reduced presentation were still complete, soundness would force the removed equation to remain derivable. Thus independence is the syntactic test used here, and the separators in Figure 8 instantiate it by occurrence detectors, counting models, projective substitutions, and determinant phases.

Since derivations preserve arity by Remark 14, an n-wire axiom can be tested inside the subsystem generated by axioms of arity at most n. This gives the bounded notion used when the separator table contains separators only in a finite arity range.

▶ **Definition 24.** *Let $k \in \mathbb{N}$. Define $\mathrm{C}_{\bullet,\le k} \coloneqq \{\rho \in \mathrm{C}_\bullet \mid \rho \text{ has type } n \to n \text{ with } n \le k\}$. We say that $\mathrm{C}_\bullet$ is minimal up to k wires if every $\rho \in \mathrm{C}_{\bullet,\le k}$ is independent relative to $\mathrm{C}_{\bullet,\le k}$ (i.e. independent in the sense of Definition 23 after replacing $\mathrm{C}_\bullet$ by $\mathrm{C}_{\bullet,\le k}$).*

5.2 Separation by alternative interpretations

Our independence proofs are semantic: to show that an axiom ρ is not derivable from the others, we construct a PROP-valued model that satisfies all remaining axioms but not ρ. The underlying countermodel principle for equational logic is that derivable equations hold in every model of the theory, so one model of the reduced theory that violates ρ rules out a derivation. This principle goes back at least to Birkhoff's correspondence between equational theories and varieties of algebras [5] and is explicit in modern accounts of equational completeness [6, Thm. 14.19]. In the quantum-circuit literature, the same proof pattern is used in [8], which proves a minimal and complete equational theory for general quantum circuits.

▶ **Lemma 25.** *Let $\rho : C_1 = C_2$ be an axiom in $\mathrm{C}_\bullet$. Assume there exist a PROP $\mathbf{P}_{\mathrm{alt}}$ and a PROP morphism $F : \mathbf{P}_\bullet \to \mathbf{P}_{\mathrm{alt}}$ such that $F(L) = F(R)$ for every $(L = R) \in \mathrm{C}_\bullet \setminus \{\rho\}$, but $F(C_1) \neq F(C_2)$. Then ρ is independent.*

Proof. Since F equalises every axiom in $\mathrm{C}_\bullet \setminus \{\rho\}$, it factors through the quotient $\mathbf{QCirc}_\bullet^{-\rho}$, yielding a PROP morphism $\overline{F} : \mathbf{QCirc}_\bullet^{-\rho} \to \mathbf{P}_{\mathrm{alt}}$. If $\mathbf{QCirc}_\bullet^{-\rho} \vdash C_1 = C_2$ then applying $\overline{F}$ gives $F(C_1) = F(C_2)$, a contradiction. ◀

Because morphisms of the free PROP are already quotiented by coherence, once F is a PROP morphism it automatically respects all structural equalities, so the separator check is purely fragment-specific.

5.3 Families of separating interpretations

Figure 8 is the finite certificate for the independence proofs: each non-none row supplies one separator to be checked against the other axioms in the relevant arity range, while the none rows mark exactly the gaps that prevent full minimality claims for qutrit Clifford, Clifford+T,

and Clifford+CS. The separators fall into four families–counting models, occurrence detectors, projective substitutions, and determinant-phase interpretations–so the proof remains uniform even though the fragments differ. In displayed rule sets and tables, abbreviated gates stand for fixed morphisms of the free PROP, and every separating interpretation is applied to the expanded morphisms.

Counting and occurrence detectors. Counting models record selected generator multiplicities or swap parity, while occurrence detectors record whether a chosen generator appears; both kinds of separator land in endomorphism-only PROPs, so there are no morphisms between different arities.

▶ **Definition 26.** *Let $(M, \oplus, 0)$ be a commutative monoid and let $\epsilon \in M$ satisfy $\epsilon \oplus \epsilon = 0$. Define $\mathbf{End}_\epsilon(M)$ to be the PROP with objects $\mathbb{N}$ and hom-sets $\mathbf{End}_\epsilon(M)(n,n) \coloneqq M$ $(n \in \mathbb{N})$, $\mathbf{End}_\epsilon(M)(n,m) \coloneqq \varnothing$ $(n \neq m)$. Composition and tensor are given by $\oplus$, identities by 0, and the basic symmetry $\sigma_{1,1}$ is interpreted as ϵ.*

▶ Remark 27. The condition $\epsilon \oplus \epsilon = 0$ implies $\sigma_{m,n} \circ \sigma_{n,m} = \mathrm{id}_{n+m}$. All other PROP coherence axioms hold because every structural map is interpreted as an element of the commutative monoid M, and both $\circ$ and $\otimes$ are interpreted as $\oplus$.

▶ **Definition 28.** *For $m \geq 2$ define $\mathbf{Count}_m \coloneqq \mathbf{End}_0(\mathbb{Z}_m, +, 0)$ and $\mathbf{Has} \coloneqq \mathbf{End}_0(\mathbb{B}, \vee, 0)$, with $\mathbb{B} = \{0,1\}$. When we want to record the parity of the structural swap, we also use $\mathbf{Count}_2^{\mathrm{swap}} \coloneqq \mathbf{End}_1(\mathbb{Z}_2, +, 0)$, so that the basic symmetry $\sigma_{1,1}$ is interpreted as $1 \in \mathbb{Z}_2$.*

▶ **Definition 29.** *Let $m \geq 2$. Given a finite list $(g_1, \ldots, g_r)$ of generators in $\Sigma_\bullet$, with repetitions counted with multiplicity, write $\#\{g_1, \ldots, g_r\}_{[m]} : \mathbf{P}_\bullet \to \mathbf{Count}_m$ for the unique PROP morphism sending each generator g to the number of its occurrences in the list modulo m.*

When $m = 2$, we use the symbol $\times$ inside the braces only as a mnemonic that swaps are counted: $\#\{g_1, \ldots, g_r, \times\}_{[2]} : \mathbf{P}_\bullet \to \mathbf{Count}_2^{\mathrm{swap}}$. In particular, $\#\{\times\}_{[2]}$ denotes the morphism that sends every non-structural generator to 0 and counts each structural swap as 1.

▶ **Definition 30.** *For a generator $g \in \Sigma_\bullet$, write $?g : \mathbf{P}_\bullet \to \mathbf{Has}$ for the unique PROP morphism with $?g(g) = 1$ and $?g(h) = 0$ for every generator $h \neq g$. Equivalently, for any circuit C, one has $?g(C) = 1$ if and only if g occurs at least once in C.*

▶ **Example 31.** In the one-wire Clifford+T truncation, the detector $?\text{-}\boxed{\text{H}}\text{-}$ separates the axiom (H^2) from the remaining rules. The equations (ω^8) and (T^8) contain no $\text{-}\boxed{\text{H}}\text{-}$ on either side, while (E) contains an occurrence of $\text{-}\boxed{\text{H}}\text{-}$ on both sides. The same is true of (TX) after expanding the displayed shortcut $\text{-}\boxed{\text{X}}\text{-}$. Thus $?\text{-}\boxed{\text{H}}\text{-}$ sends each remaining equation in $\mathrm{C}_{\mathit{CliffT},\leq 1}$ to either $0 = 0$ or $1 = 1$. On the removed axiom, however, $?\text{-}\boxed{\text{H}}\text{-}(\text{-}\boxed{\text{H}}\text{-}\boxed{\text{H}}\text{-}) = 1$ and $?\text{-}\boxed{\text{H}}\text{-}(\mathrm{id}_1) = 0$, so (H^2) is independent by Lemma 25.

Projective substitutions. These separators alter one generator while quotienting only by global phase, which separates rows whose two sides have the same generator multiplicities but differ after a projective change of one generator.

▶ **Definition 32.** *For $d \in \{2,3\}$, let $\mathbf{Qudit}_d^{\sim}$ be the projective quotient of $\mathbf{Qudit}_d$ from Definition 12: its endomorphism hom-set is $\mathbf{Qudit}_d(n,n)/\mathrm{U}(1)$, where $\mathrm{U}(1)$ acts by global phase, and its non-endomorphism hom-sets remain empty. Write $\Pi_d : \mathbf{Qudit}_d \to \mathbf{Qudit}_d^{\sim}$ for the quotient PROP morphism.*

▶ **Definition 33.** *For a fragment* $\bullet$*, set* $d(\bullet) = 3$ *for the qutrit Clifford fragment and* $d(\bullet) = 2$ *for the qubit fragments. Its usual projective interpretation on the free presentation is* $[\![\cdot]\!]_{\bullet,\sim} := \Pi_{d(\bullet)} \circ [\![\cdot]\!]^{\mathrm{free}}_{\bullet}$*, regarded as a PROP morphism* $\mathbf{P}_\bullet \to \mathbf{Qudit}^{\sim}_{d(\bullet)}$*, where* $[\![\cdot]\!]^{\mathrm{free}}_{\bullet} : \mathbf{P}_\bullet \to \mathbf{Qudit}_{d(\bullet)}$ *is the standard unitary interpretation before quotienting by the fragment equations.*

▶ **Definition 34.** *Given a fragment* $\bullet$*, a generator* $g : n \to n$*, and a circuit* $C : n \to n$ *of that fragment, write* $[\![\cdot]\!]^{g:=C}_{\bullet,\sim}$ *for the unique PROP morphism* $\mathbf{P}_\bullet \to \mathbf{Qudit}^{\sim}_{d(\bullet)}$ *sending* g *to the already-evaluated class* $[\![C]\!]_{\bullet,\sim}$ *and every generator* $h \neq g$ *to* $[\![h]\!]_{\bullet,\sim}$*. Displayed abbreviations in* C *are expanded before this interpretation.*

Thus a table entry $[g := C]_\sim$ means that every generator is interpreted projectively as usual except g, whose value is replaced by the projective value of C.

Determinant phases. These separators record a scaled determinant phase for the qubit rows, chosen so as to be strictly monoidal on a bounded number of wires.

▶ **Definition 35.** *Fix a qubit fragment, let* $k \geq 2$*, and write* $\arg\det(U) \in \mathbb{R}/2\pi\mathbb{Z}$ *for the phase of* $\det(U)$*. Set* $\epsilon_k := 2^{k-2}\pi \bmod 2\pi$*, so* $\epsilon_2 = \pi$ *and* $\epsilon_k = 0$ *for* $k > 2$*.*

Define a PROP morphism $\arg\det_k : \mathbf{P}_\bullet \to \mathbf{End}_{\epsilon_k}(\mathbb{R}/2\pi\mathbb{Z}, +, 0)$ *on generators* $g : n \to n$ *by* $(\arg\det_k)(g) := 2^{k-n} \arg\det([\![g]\!]) \bmod 2\pi$*, and extend it to all circuits using preservation of* $\circ$*,* $\otimes$*, and symmetries.*

▶ Remark 36. The scaling factor 2^{k-n} is chosen so that $\arg\det_k$ is compatible with tensoring on arities $\leq k$, using $\det(U \otimes V) = \det(U)^{2^m} \det(V)^{2^n}$ for U on n qubits and V on m qubits. The choice of ϵ_k makes the image of the structural swap $\sigma_{1,1}$ coincide with its determinant phase after scaling: it contributes π when $k = 2$ and becomes 0 for $k > 2$. For equations on more than k wires, $\arg\det_k$ is still used only as the generator-defined PROP morphism above; the row-wise separator check is a check in $\mathbf{End}_{\epsilon_k}(\mathbb{R}/2\pi\mathbb{Z}, +, 0)$, not an assertion that the displayed value is the scaled determinant of the whole higher-arity unitary.

For each non-none row of Figure 8, the relation named on the left is removed and the separator on the right equalises the other axioms in the relevant arity range while distinguishing the removed relation. The table records the separator rather than every value in the finite check. Detector and counting rows use the relevant commutative monoid, projective substitution rows use the projective quotient from Definition 33, and determinant rows use the scaled determinant phase from Definition 35. When a higher-arity separator is used in Theorem 37, its check is against all axioms in the relevant arity truncation, including same-arity rows marked none in Figure 8.

5.4 Minimality statements

The none rows lie outside the bounded ranges claimed in Theorem 37. When the table contains a separator beyond such a range, the corresponding higher-arity axiom is stated separately.

▶ **Theorem 37.** *The indicated presented circuit fragments have minimal rule sets as listed:*

1. Cliff_2 *is minimal for* $\mathbf{P}_{Cliff}$*-circuits (all arities).*
2. RCliff_2 *is minimal for* $\mathbf{P}_{RCliff}$*-circuits (all arities).*
3. Cliff_3 *is minimal for* $\mathbf{P}_{Cliff3}$*-circuits up to 2 qutrit wires.*[1]

[1] Full minimality would follow from a separator for the 3-qutrit axiom Equation (I).

(a) Clifford.

Rel.	Separator
(ω^8)	$?\,\omega_8$
(H^2)	$?\,H$
(S^4)	$?\,S$
(E)	$\#\{H\}_{[2]}$
(CPh)	$?$
(B)	$\#\{\times\}_{[2]}$
(CZ)	$\#\{\bullet,\times\}_{[2]}$
(I)	$\arg\det_2$

(b) Real Clifford.

Rel.	Separator
$(-^2)$	$?\,-$
(H^2)	$?\,H$
(Z^2)	$?\,Z$
(F)	$\#\{-\}_{[2]}$
(CX^2)	$?$
(B)	$\#\{\times\}_{[2]}$
(ZC)	$\#\{Z\}_{[2]}$
(CF)	$[Z := —]_{\sim}$
(CZr)	$[H := —]_{\sim}$
(I)	$\arg\det_2$

(c) Qutrit Clifford.

Rel.	Separator
(ω^{12})	$?\,\omega_{12}$
(H^4)	$?\,H$
(S^3)	$?\,S$
(E)	$\#\{H\}_{[2]}$
(SS′)	$[S := SX]_{\sim}$
(CPh)	$?$
(B)	$\#\{\times\}_{[2]}$
(CZ)	$\#\{S\}_{[3]}$
(KC)	$\#\{\bullet\}_{[2]}$
(I)	none

(d) Clifford+T.

Rel.	Separator
(ω^8)	$?\,\omega_8$
(H^2)	$?\,H$
(T^8)	$?\,T$
(E)	$\#\{H\}_{[2]}$
(TX)	$\#\{H,T,\omega_8\}_{[2]}$
(CPh)	$?$
(B)	$\#\{\times\}_{[2]}$
(CZ)	$\#\{\bullet,\times\}_{[2]}$
(CSH)	none
(HT^2)	none
(HTH)	none

(e) Clifford+CS.

Rel.	Separator
(ω^8)	$?\,\omega_8$
(H^2)	$?\,H$
(S^4)	$?\,S$
(E)	$\#\{H\}_{[2]}$
(CPh)	$?$
(B)	$\#\{\times\}_{[2]}$
(XCS)	$\#\{S,H\}_{[2]}$
(CSr)	$[\bullet := \bullet Z]_{\sim}$
(CE)	$[\bullet := \bullet Z\,Z]_{\sim}$
(I)	$\arg\det_2$
(SH_k)	none

(f) CNOT-dihedral.

Rel.	Separator
(ω^8)	$?\,\omega_8$
(T^8)	$?\,T$
(X^2)	$?\,X$
(TX)	$\#\{\omega_8\}_{[2]}$
(XC)	$\#\{X\}_{[2]}$
(CPh)	$?$
(B)	$\#\{\times\}_{[2]}$
(ZC)	$\#\{T,\omega_8,\omega_8\}_{[8]}$
(I)	$\#\{\bullet,\times\}_{[2]}$
(C^2T)	$\#\{T,\omega_8,\omega_8\}_{[4]}$
(C^3T)	$\arg\det_3$

Figure 8 Separators witnessing independence, indexed by the equation labels from Section 3; entries marked none are the cases for which no separator is currently provided.

4. CliffT_2 *is minimal for* $\mathbf{P}_{CliffT}$*-circuits up to* 1 *qubit wire.*
5. $\mathrm{CliffCS}_2$ *is minimal for* $\mathbf{P}_{CliffCS}$*-circuits up to* 2 *qubit wires.*
6. $\mathrm{CNOTdihe}_2$ *is minimal for* $\mathbf{P}_{CNOTdihe}$*-circuits (all arities).*

For the bounded fragments, the constructed higher-arity separators give:

- *In* CliffT_2*, the* 2*-qubit axioms* (CPh), (B) *and* (CZ) *are independent.*
- *In* $\mathrm{CliffCS}_2$*, the* 3*-qubit axiom* (I) *is independent.*

Proof. For each axiom ρ covered by the statement, the corresponding non-none row of Figure 8 gives a separator F_ρ that equalises the remaining axioms of the relevant truncated system while distinguishing the two sides of ρ. For the additional higher-arity statements, the relevant truncated system is the one at the arity of the displayed axiom. Apply Lemma 25 and Definition 24. ◀

6 Conclusion

By placing the six fragments in a common PROP syntax with structural symmetries, the paper isolates fewer non-structural axioms while preserving the strict unitary semantics inherited from the source presentations. The completeness transfer and separator table

show that this simplification often preserves irredundancy: qubit Clifford, real Clifford, and CNOT-dihedral are minimal in all arities, while the other fragments are minimal in the bounded ranges certified in Section 5. The none rows of Figure 8 identify the remaining separator-construction problem, and the reduced presentations give mechanised rewriting a smaller algebraic core: related work on 2-qubit Clifford+T and 3-qubit Clifford+CS already gives Agda checks for this style of argument [3, 4], while recent Isabelle/HOL work on verified rewriting infrastructure for diagrammatic equivalence [7] offers another benchmark for future formalisation.

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A Shortcut and transfer data

A.1 Shortcut conventions

The rule sets use a few gates, listed in Figure 9, that are not primitive generators of the corresponding fragment. These are fixed abbreviations, not extra axioms: before an equation is counted as a fragment-specific rule, every shortcut is expanded into the fragment signature.

A.2 Completeness-transfer data

After the source theorem has been lifted to a PROP where necessary and scalar conventions have been aligned, the transfer reduces to the non-identity generator images in Table 3; omitted generators are fixed by the corresponding map. For CNOT-dihedral, no encoding/decoding map is needed, and the full-version appendix titled "CNOT-dihedral circuits" derives the source axioms removed from the simplified presentation directly in $\mathrm{CNOTdihe}_2$.

A.3 Scalar refinement

▶ **Lemma 38** (Scalar refinement). *Let $\mathbf{P}^{\mathrm{src}}/\mathcal{R}^{\mathrm{src}}$ be a complete presented PROP with faithful strict symmetric monoidal interpretation into a unitary semantic subPROP $\mathbf{C}^{\mathrm{src}} \subseteq \mathbf{FdHilb}$. Assume its visible scalar group is finite cyclic, generated by a source scalar $s : 0 \to 0$, and that the semantic subPROP has no hidden global phases: whenever $\lambda \mathrm{id}_n \in \mathbf{C}^{\mathrm{src}}(n,n)$ for some $n \geq 1$, the scalar λ already lies in $\mathbf{C}^{\mathrm{src}}(0,0)$. Assume also that the source semantic subPROP*

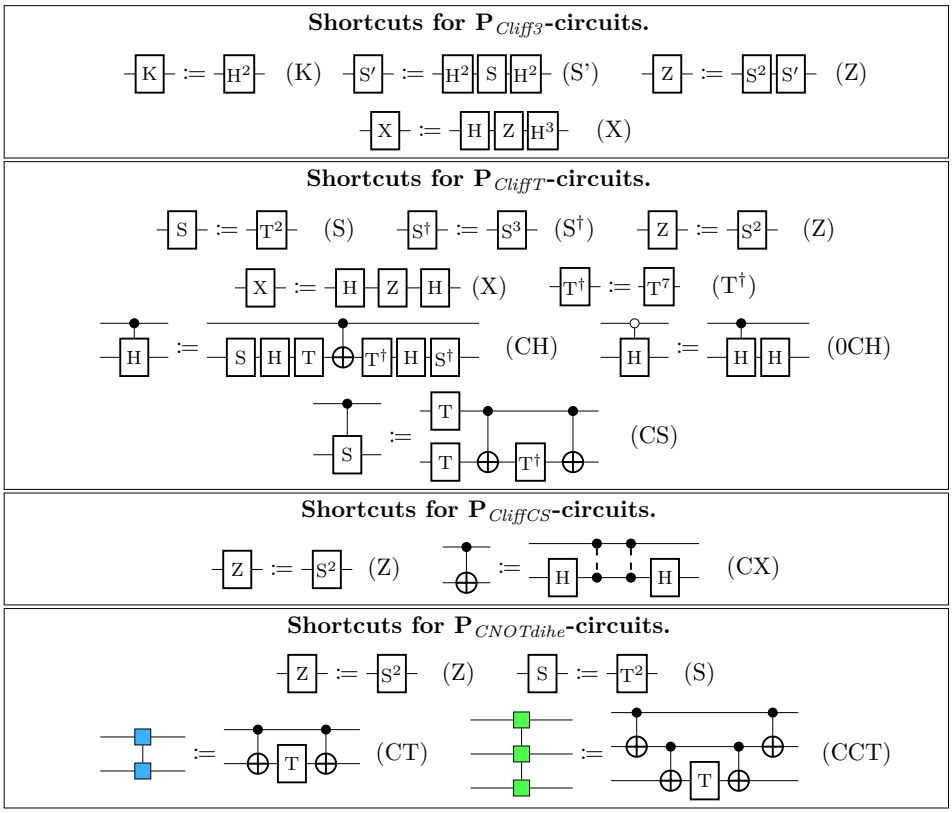


Figure 9 Shortcut definitions expanded before applying the fragment rule sets.

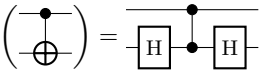
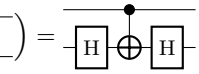
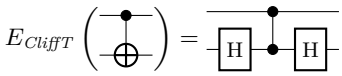
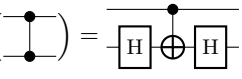
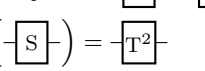
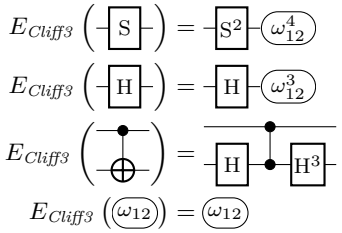
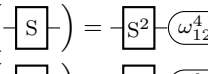
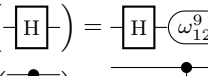
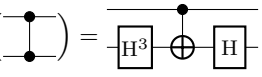
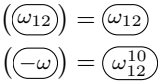
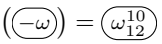
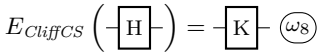
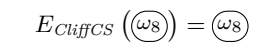
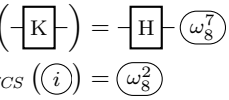
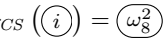
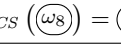
is closed under inverses, so each endomorphism hom-set is a group under composition. If one adjoins a new scalar $\omega : 0 \to 0$ *together with relations* $\omega^{m\ell} = \mathrm{id}_0$ *and* $\omega^r = s$*, interpreting* ω *as a chosen primitive* $(m\ell)$*-th root of unity satisfying* $\omega^r = [\![s]\!]_{\mathrm{src}}$*, then the refined presentation is complete for the semantic subPROP obtained by adjoining that scalar. The refinement is conservative on the old signature.*

Proof. *Soundness.* Write ζ for the semantic value of the new scalar ω. The refined interpretation agrees with the source interpretation on the old generators, sends ω to ζ, and satisfies the two added scalar equations by construction.

Scalar extraction. In a strict monoidal category, scalars are central: for a scalar $a : 0 \to 0$ and an endomorphism $f : n \to n$, $(a \otimes \mathrm{id}_n) \circ f = (a \circ \mathrm{id}_0) \otimes (\mathrm{id}_n \circ f) = (\mathrm{id}_0 \circ a) \otimes (f \circ \mathrm{id}_n) = f \circ (a \otimes \mathrm{id}_n)$, by interchange. Thus every occurrence of ω can be moved past the non-scalar part of a diagram and collected into one scalar factor. Using $\omega^{m\ell} = \mathrm{id}_0$, every refined circuit $C : n \to n$ is derivably equal to $\omega^a \otimes C_0$, with $0 \le a < m\ell$ and C_0 a source circuit.

Faithfulness. Now suppose two refined circuits have the same semantics. Put them in scalar-extraction form $\omega^a \otimes C_0$ and $\omega^b \otimes D_0$, and write $U = [\![C_0]\!]_{\mathrm{src}}$ and $V = [\![D_0]\!]_{\mathrm{src}}$. Semantic equality gives $\zeta^a U = \zeta^b V$. Since the source semantic category is unitary, V is invertible, so $UV^{-1} = \zeta^{b-a}\mathrm{id}_n$ lies in the source semantic subPROP. By the no-hidden-phases hypothesis, the phase ζ^{b-a} is already a visible source scalar. Since the visible scalar group is generated by $[\![s]\!]_{\mathrm{src}} = \zeta^r$, there is some t with $\zeta^{b-a} = [\![s^t]\!]_{\mathrm{src}}$, hence $[\![C_0]\!]_{\mathrm{src}} = [\![s^t \otimes D_0]\!]_{\mathrm{src}}$. Source completeness gives $C_0 = s^t \otimes D_0$ in the source theory, and therefore in the refined theory. The added relation $s = \omega^r$ gives $\omega^a \otimes C_0 = \omega^{a+rt} \otimes D_0$. Since $\zeta^{b-a} = \zeta^{rt}$ and ζ has order $m\ell$, we have $a + rt \equiv b \pmod{m\ell}$; using $\omega^{m\ell} = \mathrm{id}_0$, the last term is $\omega^b \otimes D_0$. Thus equal semantics imply derivable equality, so the refined interpretation is faithful.

Table 3 Encoding/decoding pairs used in the completeness-transfer arguments. For the qutrit Clifford and Clifford+CS rows, the source presentation is the scalar-refined one from Lemma 38.

Fragment	Encoding $E_\bullet$ (new → source)	Decoding $D_\bullet$ (source → new)
Cliff, *RCliff*	$E_\bullet$ 	$D_\bullet$ 
CliffT	E_{CliffT} 	D_{CliffT}  D_{CliffT} 
Cliff3	E_{Cliff3} 	D_{Cliff3}  D_{Cliff3}  D_{Cliff3}  D_{Cliff3}  D_{Cliff3} 
CliffCS	$E_{CliffCS}$  $E_{CliffCS}$ 	$D_{CliffCS}$  $D_{CliffCS}$  $D_{CliffCS}$ 

Conservativity. If two source circuits become equal in the refined theory, their source semantics are equal; faithfulness of the source interpretation then already derives their equality in the original source presentation. ◀

The two applications are the scalar refinements recorded in Section 4.1: qutrit Clifford is refined from the order-6 source scalar convention to the order-12 strict semantics, and Clifford+CS is refined from order 4 to order 8. In both applications, the imported source semantics are already group-valued on each arity: the gates are unitary, and the source normal forms contain representatives for inverses. The visible scalar groups are the finite cyclic groups displayed in the source presentations, and the source normal forms expose every scalar multiple of an identity as one of those visible scalars. Thus the inverse-closure and no-hidden-phases hypotheses hold before the refinement step.