Groups and Inverse Semigroups in Lambda Calculus

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Abstract

We study invertibility of λ-terms modulo λ-theories. Here a fundamental role is played by a class of λ-terms called finite hereditary permutations (FHP) and by their infinite generalisations (HP). More precisely, FHPs are the invertible elements in the least extensional λ-theory $\boldsymbol{\lambda\eta}$ and HPs are those in the greatest sensible λ-theory $\mathcal{H}^*$. Our approach is based on inverse semigroups, algebraic structures that generalise groups and semilattices. We show that FHP modulo a λ-theory $\mathcal{T}$ is always an inverse semigroup and that HP modulo $\mathcal{T}$ is an inverse semigroup whenever $\mathcal{T}$ contains the theory of Böhm trees. An inverse semigroup comes equipped with a natural order. We prove that the natural order corresponds to η-expansion in FHP/$\mathcal{T}$, and to infinite η-expansion in HP/$\mathcal{T}$. Building on these correspondences we obtain the two main contributions of this work: firstly, we recast in a broader framework the results cited at the beginning; secondly, we prove that the FHPs are the invertible λ-terms in all the λ-theories lying between $\boldsymbol{\lambda\eta}$ and $\mathcal{H}^+$. The latter is Morris' observational λ-theory, defined by using the β-normal forms as observables.

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1 Introduction

The λ-calculus, established by Alonzo Church in the 1930s, has played a prominent role in theoretical computer science for almost a century [6, 5, 7]. This stems from its position at the crossroads of programming language theory and various branches of mathematics, which draws the attention of a diverse community of researchers. Its denotational semantics has been studied using domain theory [2], category theory [40, 3] and bicategory theory [29, 37], game semantics [26, 9, 13], linear logic [1, 22], set theory [39, 21], and universal algebra [45]. Operational aspects of its semantics have been also mechanised in proof assistants [16, 41].

This article fits within the line of research that studies the λ-calculus from an algebraic perspective. In [12], Church formulated the λ-calculus as a semigroup where $M \circ N = \lambda x.M(Nx)$, whence its connection with algebra begins at the syntactic level, and this perspective also inspired subsequent works [11]. A problem that naturally arises is the characterisation of those λ-terms M that are *invertible* in the sense that $M \circ N = N \circ M = \mathbf{I}$ (the identity), for some N. The set of invertible λ-terms modulo a λ-theory is a group. When the notion of equality is $=_\beta$ the group is trivial, as the only invertible λ-term is $\mathbf{I}$. The problem becomes more interesting when considering extensional λ-theories: Dezani-Ciancaglini, Bergstra and Klop have characterised the λ-terms invertible modulo $=_{\beta\eta}$ as the *finite hereditary permutations* (FHPs) [20, 8]. Intuitively, FHPs are obtained from η-

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expansions of $\mathbf{I}$ by arbitrarily permuting the subtrees of their Böhm trees, and it can be proved that they are the only realisers of type isomorphisms in the simply typed λ-calculus [15]. The invertibility modulo $=_{\beta\eta}$ has also been studied by Folkerts [23], who proved that a λ-term is invertible if and only if it is both injective and surjective–a property that *seems* natural but fails in other λ-theories. More recently, Statman has shown that the semigroup of λ-terms modulo $=_\beta$ is Sq-universal [49], a profound result stating that every countable semigroup is embeddable in the semigroup of λ-terms modulo a suitable λ-theory.

There are other λ-theories that are of interest for computer scientists because they capture operational properties of programs. In particular, Morris defined observational equivalences where two λ-terms are equivalent if they display the same behaviour when they are plugged in any context [46]. Clearly, this notion depends on the behavioural property one decides to observe. The observational equivalence $\mathcal{H}^*$, where one observes head termination, is by far the most studied λ-theory: it arises as the theory of Scott's model $\mathcal{D}_\infty$ and captures the Böhm tree equality modulo an infinitary notion of extensionality [32]. In [8], Bergstra and Klop characterised the class of invertible λ-terms modulo $\mathcal{H}^*$ by generalising the finite hereditary permutations to permutations having a possibly infinite Böhm tree (HPs).

It is natural to consider the problem of characterising invertibility modulo λ-theories lying between $=_{\beta\eta}$ and $\mathcal{H}^*$; one of these is for instance $\mathcal{H}^+$, Morris's observational theory in which the observed behavioural property is β-normalisation [34]. In this paper, we aim to advance the study of this problem by extending the results presented in [6, Chapter 21]. Our novel approach consists in showing that HP, which is a monoid modulo $=_\beta$ and a group modulo $\mathcal{H}^*$, is an *inverse semigroup* (actually, an inverse monoid) modulo the λ-theory of Böhm trees. An inverse semigroup [42] is a semigroup in which every element admits a unique inverse in the generalised sense of semigroup theory. By virtue of their rich structure, inverse semigroups are connected with various areas of logic and theoretical computer science, including topos theory [24, 25], Stone duality [43], and automata theory [17]. They simultaneously and faithfully generalise groups–since they are equipped with a unary operation extending the group inverse–and semilattices, as the idempotent elements of an inverse semigroup always form a semilattice with respect to a natural order. An inverse semigroup is a semilattice if and only if every element is idempotent, and it is a group if and only if the identity is the only idempotent. The prototypical example of an inverse monoid is given by the injective partial endofunctions on a set; viewed in this way, inverse monoids provide a natural framework for describing partial symmetries, as opposed to the total ones described by groups.

We introduce a new monoid whose elements we call *permutation trees*, which can be put into correspondence with a monoid of Böhm-like trees. We show that the permutation trees form an inverse monoid, we characterise its idempotents and the associated order, and we prove that it is in fact F-inverse, a technical concept that will be appropriately explained. The analysis of this monoid has two important consequences: HP modulo Böhm tree equality and FHP modulo $=_\beta$ form inverse monoids; these sets remain so when considered modulo any theory extending Böhm tree equality and $=_\beta$, respectively. A noteworthy aspect is that the natural order on these inverse monoids coincides with the preorder induced by infinite η-expansion (on Böhm-like trees) and finite η-expansion (on λ-terms). Quotienting an inverse monoid by a suitable congruence σ, called minimum group congruence, yields a group in which all idempotents lie in the equivalence class of the identity. In this group, the generalised inverses become actual inverses in the usual group-theoretic sense. This naturally raises the question of what such a congruence corresponds to in the cases of HP modulo Böhm tree equality and FHP modulo $=_\beta$. The minimum group congruence is related to the natural order that corresponds to infinite η-expansion in the theory of Böhm trees and to

finite η-expansion in $=_\beta$. Therefore, we obtain the key result that quotienting HP by Böhm theory and then by σ gives precisely HP modulo $\mathcal{H}^*$; similarly, quotienting FHP by $=_\beta$ and then by σ gives precisely FHP modulo $=_{\beta\eta}$. This provides a new perspective on invertible λ-terms modulo these λ-theories.

The λ-theories $\mathcal{H}^*$ and $=_{\beta\eta}$ are special in that they arise precisely from the minimum group congruence, but this is not always the case. A counterexample is given by the theory $\mathcal{H}^+$, whose equality has been characterised in terms of the interaction between Böhm trees and *finite* η-expansions [14, 47, 34]. To characterise when two hereditary permutations are equal in $\mathcal{H}^+$, we therefore need to take a further step and study the interaction between infinite trees equipped with an order that nonetheless originates from *finite* idempotents. In other words, we are led to consider a congruence inspired by, but distinct from, the minimum group congruence. In the theory of Böhm trees, the order constructed in this way coincides with finite η-expansion. Building on the characterisation of the invertibles we gave for $\mathcal{H}^*$, we demonstrate that the λ-terms invertible modulo $\mathcal{H}^+$ are precisely FHPs. As a consequence, FHPs are the invertible λ-terms in all the λ-theories lying between $=_{\beta\eta}$ and $\mathcal{H}^+$. This result settles a conjecture formulated by Barendregt in his book [6, p. 547]. The conceptual reconstruction of invertibles modulo $\mathcal{H}^*$ and $=_{\beta\eta}$ together with the characterisation of invertibles modulo $\mathcal{H}^+$ constitute the main contributions of this work. Some directions for future research are outlined in the conclusion.

2 Preliminaries

We recall some basic notions and results concerning the λ-calculus and the theory of semigroups. We refer to [6, 7] for the former, and to [31, 42] for the latter.

We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}^+$ the set of positive natural numbers. For every $n \in \mathbb{N}$, let S_n be the symmetric group on n elements. Let $\mathrm{Sym}(\mathbb{N}^+)$ be the symmetric group on $\mathbb{N}^+$. For every $n \geq 1$, S_n can be embedded in $\mathrm{Sym}(\mathbb{N}^+)$ as the subgroup of all the $\rho \in \mathrm{Sym}(\mathbb{N}^+)$ such that $\rho i = i$ for all $i > n$. By abuse of notation, we shall write S_n for the image of this embedding. We denote by ι the identity permutation in $\mathrm{Sym}(\mathbb{N}^+)$.

2.1 Inverse semigroups

Given a semigroup S, we say that $u \in S$ is *regular* if there is $v \in S$ such that $uvu = u$. A semigroup is called *regular* if every element is regular. Moreover, we say that v is an *inverse* of u if $uvu = u$ and $vuv = v$. Every element in a regular semigroup admits at least on inverse. An extremely important class of regular semigroups is that of inverse semigroups, in which every element admits precisely one inverse.

▶ **Definition 1.** *An* inverse semigroup *is a semigroup S endowed with a unary operation $(-)^*$ that satisfies, for all $u, v \in S$:*

$$(u^*)^* = u, \qquad (uv)^* = v^*u^*, \qquad uu^*u = u, \qquad uu^*vv^* = vv^*uu^*. \tag{1}$$

An inverse monoid *is an inverse semigroup together with a unit for multiplication.*

In an inverse semigroup u^* is the unique inverse of u.

▶ **Example 2.** The prototypical example of an inverse monoid is the set $I(X)$ of partial injections on a given set X. An element u is a bijection between two subsets of X, the domain and the range of u, and u^* is its partial inverse.

An *idempotent* in a semigroup S is $u \in S$ such that $uu = u$. The idempotents of $I(X)$ are the partial identities. We denote by $E(S)$ the set of idempotents of S. In any inverse semigroup, every element of the form uu^* is idempotent. Conversely, since if u is idempotent, then $u = u^*$, idempotents are precisely those elements of the form uu^*. The last identity in (1) says that idempotents commute.

▶ Remark 3. If $f : S \to T$ is a surjective homomorphism of semigroups and S is an inverse semigroup, then T is also an inverse semigroup.

The natural order. Any inverse semigroup S comes equipped with a natural order that corresponds to inclusion between functions of $I(X)$. By definition, $u \leq v$ if $uu^*v = u$, or, equivalently, if $vu^*u = u$. Actually, one can prove that the following are equivalent:

1. $u \leq v$;
2. $u = ve$, for some idempotent e;
3. $u = dv$, for some idempotent d.

Therefore, if d is idempotent and $u \leq d$, then u is idempotent. Moreover, $(S, \leq)$ is a partial order compatible with the structure of inverse semigroup, i.e., if $s \leq t$ and $u \leq v$, then $s^* \leq t^*$ and $su \leq tv$. If S is an inverse monoid, then the poset $(E(S), \leq)$ has binary meets given by multiplication and a top element given by the unit of the monoid.

The minimum group congruence. Two important relations in the study of an inverse semigroup S are:

1. the relation of *compatibility* $\sim$, defined by $t \sim u$ if t^*u and tu^* are idempotents;
2. the relation σ_S, defined by $t \,\sigma_S\, u$ if there exists w such that $w \leq t$ and $w \leq u$.

When t and u are compatible, i.e., $t \sim u$, then $t \wedge u$ exists and $t \wedge u = tu^*u$. The relation σ_S is the smallest congruence such that S/σ_S is a group [42, Thm. 2.4.1] and will be called *minimum group congruence*.

Two classes of inverse semigroups that have been studied extensively are the class of E-unitary inverse semigroups and the class of F-inverse semigroups.

▶ **Definition 4.** *An inverse semigroup S is said to be*

1. E-unitary *if, whenever $e \in E(S)$ and $e \leq v$, it follows that $v \in E(S)$;*
2. F-inverse *if each equivalence class of σ_S has a unique maximal element.*

The inverse semigroup S is E-unitary iff $\sim = \sigma_S$ [42, Thm. 2.4.6]; if S is F-inverse, then it is E-unitary [42, Proposition 7.1.3]. The following result is [42, Thm. 2.4.6].

▶ **Lemma 5.** *The minimum group congruence σ_S of an E-unitary inverse monoid S is the least congruence equating all the idempotent elements of S. Moreover, for every $u \in S$, $u \,\sigma_S\, 1$ iff $u \in E(S)$.*

Green's relations. Certain equivalence relations, called Green's relations, have played a central role in the development of the theory. They can be defined on any semigroup, but assume a particularly simple form on inverse semigroups.

▶ **Definition 6.** *Let S be an inverse semigroup. We define two equivalence relations $\mathcal{L}$ and $\mathcal{R}$ on S by $u \,\mathcal{L}\, v$ if $u^*u = v^*v$ and $u \,\mathcal{R}\, v$ if $uu^* = vv^*$. Moreover, we define $\mathcal{H} := \mathcal{L} \cap \mathcal{R}$.*

Traditionally, given $s \in S$, L_s, R_s, and H_s, denote their equivalence classes under $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$, respectively. It is easy to check that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence, meaning that if $u \,\mathcal{L}\, v$, then $us \,\mathcal{L}\, vs$ for every $s \in S$, and similarly for $\mathcal{R}$.

▶ **Proposition 7.** [42, Prop. 3.2.1] *Let S be an inverse monoid and $e \in E(S)$. Then:*

1. *eSe is an inverse monoid, and is the largest monoid with identity e;*
2. *the $\mathcal{H}$-class H_e is a group with identity e and it coincides with $\{t \in eSe : tt^* = t^*t = e\}$;*
3. *every subgroup of S is contained in an $\mathcal{H}$-class.*

Thus, for any idempotent $e \in E(S)$, the $\mathcal{H}$-class H_e is the unique maximal subgroup of S with identity e. Since each $\mathcal{H}$-class contains at most one idempotent, the maximal subgroups of S are precisely the disjoint $\mathcal{H}$-classes containing an idempotent.

2.2 Trees

An *ordered tree* is a non-empty subset A of $(\mathbb{N}^+)^*$ such that:

(i) A is closed under initial segments;

(ii) A is *gapless*: for every $i, j \in \mathbb{N}^+$ and every $w \in A$, if $i < j$ and $wj \in A$, then $wi \in A$.

The elements of A are called *nodes*; a node is a *leaf* if it has no children (immediate successors). The empty word ϵ is the unique *root*, i.e., node without immediate predecessors. We will be exclusively interested in *finitely branching trees*, i.e., trees whose nodes have a finite number of children. We denote by $\mathrm{rd}(A)$ the *degree* of the root of A, i.e., the number of its children. For each $1 \leq i \leq \mathrm{rd}(A)$ we denote by $A_i := \{w \in (\mathbb{N}^+)^* : iw \in A\}$ the subtree of A of root i. Finally, we say that a *labelled* tree is a function $l : A \to L$ from a tree to a nonempty set L of labels. The tree A is called the *skeleton* of l, and is denoted by $\mathrm{sk}(l)$.

The set of finitely branching ordered trees can be defined coinductively. When we deal with infinite trees we will freely use coinductive arguments. We refer the reader to [38] for a gentle introduction to coinductive techniques.

2.3 Lambda calculus

The set Λ of *λ-terms* over a countably infinite set Var of variables is defined by the grammar:

$$M, N ::= x \mid \lambda x.M \mid MN, \text{ for } x \in \mathrm{Var}.$$

We assume that application associates on the left, and has higher precedence than abstraction. For instance, $\lambda xyz.xyz$ stands for $\lambda x.(\lambda y.(\lambda z.(xy)z))$. From now on, λ-terms are considered up to renaming of bound variables. A λ-term without free variables is called a *combinator*. The *β-reduction* $\to_\beta$ and *η-reduction* $\to_\eta$ are defined as the contextual closures of the rules:

$$(\lambda x.M)N \to_\beta M[N/x], \qquad \lambda x.Mx \to_\eta M, \text{ if } x \text{ does not occur free in } M$$

where we denote by $M[N/x]$ the *capture-free substitution* of N for all free occurrences of x in M. Given a notion of reduction $\to_r$ between λ-terms, we denote by $\twoheadrightarrow_r$ the reflexive, transitive closure of $\to_r$, and by $=_r$ the least equivalence relation containing $\to_r$. When $\to_r$ is confluent, $\mathrm{nf}_r(M)$ stands for the unique *r-normal form* of M (if any). We define *$\beta\eta$-reduction* as $\to_{\beta\eta} = \to_\beta \cup \to_\eta$. The combinators below are used throughout the paper:

$$\mathbf{I} = \lambda x.x, \qquad \mathbf{1} = \lambda xy.xy, \qquad \mathbf{B} = \lambda fgx.f(gx), \qquad \mathbf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)).$$

We recall that $\mathbf{I}$ is the identity, $\mathbf{1}$ the 1st Church numeral, $\mathbf{B}$ stands for the composition and $\mathbf{Y}$ is Curry's fixed point combinator satisfying $\mathbf{Y}M =_\beta M(\mathbf{Y}M)$, for every $M \in \Lambda$.

Böhm trees. A λ-term is in *head normal form* (*hnf*) if it is of the form $\lambda x_1 \dots x_n.yM_1 \cdots M_k$ for some $n, k \in \mathbb{N}$. If a λ-term has a hnf, this hnf can be reached by head reductions $\to_h$, i.e., by repeatedly β-reducing its head redex. A λ-term with a hnf is called *solvable* and a λ-term without a hnf is called *unsolvable*. The *Böhm tree* $\mathrm{BT}(M)$ of a λ-term M is a labelled ordered tree defined coinductively as follows: if M is unsolvable, then $\mathrm{BT}(M) = \bot$; otherwise $M \twoheadrightarrow_h \lambda x_1 \dots x_n.yM_1 \cdots M_k$, and

$$\begin{array}{l} \mathrm{BT}(M) := \lambda x_1 \dots x_n.y \\ \qquad\qquad\quad \diagup \;\; | \;\; \diagdown \\ \qquad\quad \mathrm{BT}(M_1) \cdots \mathrm{BT}(M_k) \end{array}$$

The set $\mathscr{B}$ of *Böhm-like trees* is coinductively defined as: $\bot \in \mathscr{B}$ and if $T \in \mathscr{B}$, then there are $k, n \in \mathbb{N}$, $x_1, \dots, x_n, y \in \mathrm{Var}$, and $T_1, \dots, T_k \in \mathscr{B}$ such that T is

$$\begin{array}{l} \lambda x_1 \dots x_n.y \\ \quad \diagup \;\; | \;\; \diagdown \\ T_1 \quad \cdots \quad T_k \end{array}$$

When $T \in \mathscr{B}$ is a finite tree without $\bot$, we identify T with the corresponding λ-term. Given $T \in \mathscr{B}$, there exists $M \in \Lambda$ such that $\mathrm{BT}(M) = T$ exactly when T is recursively enumerable and the number of free variables in T is finite [6, Thm. 10.1.23].

Theories. A λ-*theory* is an equivalence relation $\mathcal{T} \subseteq \Lambda^2$ on the set of λ-terms that contains $=_\beta$ and is compatible with abstraction and application. The minimal λ-theory $=_\beta$ is also denoted by $\boldsymbol{\lambda}$. Given a λ-theory $\mathcal{T}$, it is customary to write $\mathcal{T} \vdash M = N$ for $M \,\mathcal{T}\, N$.

A λ-theory $\mathcal{T}$ is called:

- *consistent* if it does not equate all λ-terms;
- *extensional* if it contains $=_{\beta\eta}$, or, equivalently, if $\mathcal{T} \vdash \mathbf{I} = \mathbf{1}$;
- *semisensible* if it does not equate a solvable and an unsolvable;
- *sensible* if it is consistent and equates all the unsolvables.

The set of λ-theories ordered by inclusion forms a complete lattice having a rich structure [44].

For every theory $\mathcal{T}$, we denote by $\mathcal{T}\boldsymbol{\eta}$ the least extensional λ-theory containing $\mathcal{T}$, and by $\mathcal{T}\boldsymbol{\omega}$ the closure of $\mathcal{T}$ under the so-called ω-*rule*: for all $M, N \in \Lambda$, if $\mathcal{T} \vdash MC = NC$ for every combinator C, then $\mathcal{T} \vdash M = N$. The following are the most important λ-theories:

- the least sensible λ-theory $\mathcal{H}$.
- the λ-theory $\mathcal{B}$ induced by the Böhm tree equality: $M \,\mathcal{B}\, N$ iff $\mathrm{BT}(M) = \mathrm{BT}(N)$.
- Morris' observational theory $\mathcal{H}^+$ defined by $\mathcal{H}^+ \vdash M = N$ if, for all contexts $C[\,]$,

  $C[M]$ has a β-nf iff $C[N]$ has a β-nf.

- the greatest sensible λ-theory $\mathcal{H}^*$ defined by $\mathcal{H}^* \vdash M = N$ if, for all contexts $C[\,]$,

  $C[M]$ has a hnf iff $C[N]$ has a hnf.

It can be proved that a λ-theory $\mathcal{T}$ is semisensible iff $\mathcal{T} \subseteq \mathcal{H}^*$. Since $\mathcal{B}$ and $\mathcal{H}^+$ are sensible, and $\mathcal{H}^+$ is moreover extensional, we have $\boldsymbol{\lambda} \subseteq \mathcal{H} \subseteq \mathcal{B} \subseteq \mathcal{H}^+ \subseteq \mathcal{H}^*$, and thus $\boldsymbol{\lambda\eta} \subseteq \mathcal{H}\boldsymbol{\eta} \subseteq \mathcal{B}\boldsymbol{\eta} \subseteq \mathcal{H}^+ \subseteq \mathcal{H}^*$. By [6, Thm. 17.4.16] and [33], these inclusions are *strict*.

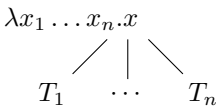
Hereditary permutations. A *finite η-expansion* ($\mathcal{I}^\eta$) of $\mathbf{I}$ is any combinator M such that $M \twoheadrightarrow_{\beta\eta} \mathbf{I}$. Every $M \in \mathcal{I}^\eta$ has a β-normal form, that can be taken as a representative of M. There is a bijection between β-normal forms of finite η-expansions of $\mathbf{I}$ and finite unlabelled ordered trees [35].

▶ **Example 8.** The following λ-terms are finite η-expansions of $\mathbf{I}$:

- $\mathbf{1}_n := \lambda x x_1 \dots x_n . x x_1 \cdots x_n$;
- $\mathbf{1}^0 := \mathbf{I}$, $\mathbf{1}^{n+1} := \lambda x z . x(\mathbf{1}^n z)$.

The skeleton of $\mathrm{BT}(\mathbf{1}^k)$ is $\{1^n : n \leq k\}$ and the union of these trees is the skeleton of the Böhm tree of Wadsworth's combinator $\mathbf{J} := \mathbf{Y}(\lambda f g x . g(fx))$: $\mathrm{sk}(\mathrm{BT}(\mathbf{J})) = \{1^n : n \in \mathbb{N}\}$; λ-terms like $\mathbf{J}$ are called "infinite η-expansions" of the identity. This concept will be defined shortly.

For every $n \in \mathbb{N}$, let S_n be the symmetric group on n elements. For all variables x, the set $\mathscr{HP}(x) \subseteq \mathscr{B}$ is defined coinductively as follows. If $T \in \mathscr{HP}(x)$, then there are $n \in \mathbb{N}$, $\pi \in S_n$ and $T_i \in \mathscr{HP}(x_{\pi i})$, $1 \leq i \leq n$ such that T is



(2)

▶ **Definition 9.** *We define:*

- $\mathscr{HP} := \{\lambda x.T : T \in \mathscr{HP}(x)\}$;
- $\mathrm{HP} := \{M \in \Lambda : \mathrm{BT}(M) \in \mathscr{HP}\}$;
- $\mathrm{FHP} := \{M \in \Lambda : \mathrm{BT}(M) \in \mathscr{HP}$ *and* $\mathrm{BT}(M)$ *is finite*$\}$.

An element of HP *is called a* hereditary permutation *and an element of* FHP *a* finite hereditary permutation.

The set $\mathscr{I}(x) \subseteq \mathscr{HP}(x)$ is defined coinductively as follows. If $T \in \mathscr{I}(x)$, then there are $n \in \mathbb{N}$, $T_i \in \mathscr{I}(x_i)$, $1 \leq i \leq n$ such that T is (2).

▶ **Definition 10.** *We define:*

- $\mathscr{I} := \{\lambda x.T : T \in \mathscr{I}(x)\}$;
- $\mathcal{I}^\eta_\omega := \{M \in \Lambda : \mathrm{BT}(M) \in \mathscr{I}\}$.

An element of $\mathcal{I}^\eta_\omega$ is called a possibly infinite η-expansion *of* $\mathbf{I}$.

When $M \in \mathcal{I}^\eta_\omega$ has a finite Böhm tree, we obtain a finite η-expansion of $\mathbf{I}$ as previously defined.

Invertibility. The operation $M \circ N := \mathbf{B}MN$ on $\Lambda/\boldsymbol{\lambda}$ is associative, hence $(\Lambda/\boldsymbol{\lambda}, \circ)$ is a semigroup. For a λ-theory $\mathcal{T}$, a term $M \in \Lambda$ is $\mathcal{T}$-*invertible* if $\mathcal{T} \vdash M \circ N = N \circ M = \mathbf{I}$ for some $N \in \Lambda$. This leads to the problem of characterising invertible λ-terms in a given theory; we summarise the known results, referring to [6, Chapter 21].

▶ **Theorem 11.** *Let $\mathcal{T}$ be a λ-theory.*

1. *If $\mathcal{T}$ is $\boldsymbol{\lambda}$, $\mathcal{H}$ or $\mathcal{B}$, then M is $\mathcal{T}$-invertible iff $\mathcal{T} \vdash M = \mathbf{I}$;*
2. *if $\mathcal{T}$ is $\boldsymbol{\lambda\eta}, \boldsymbol{\lambda\omega}, \mathcal{H}\boldsymbol{\eta}, \mathcal{H}\boldsymbol{\omega}$, then M is $\mathcal{T}$-invertible iff $M \in$ FHP;*
3. *if $\mathcal{T} = \mathcal{H}^*$, M is $\mathcal{T}$-invertible iff $M \in$ HP.*

3 Permutation trees

In this section we introduce and study a new inverse monoid PT of labelled ordered trees, that, as shown in Proposition 17, is isomorphic to the monoid of hereditary permutations $(\mathscr{HP}, \circ, \mathbf{I})$. Working with PT has the advantage of revealing the underlying structure in a more transparent way.

▶ **Definition 12.** *A* permutation tree *is a labelled ordered tree* $t : A \to \mathrm{Sym}(\mathbb{N}^+)$ *such that:*

- *if* $a \in A$ *has exactly* $n \geq 1$ *children, then* $t(a) \in S_n$*;*
- *if* $a \in A$ *is a leaf, then* $t(a) := \iota$*, where* ι *is the identity permutation in* $\mathrm{Sym}(\mathbb{N}^+)$*.*

We let PT *be the set of permutation trees.*

We denote by $\bar{\epsilon}$ the unique permutation tree with skeleton $\{\epsilon\}$, i.e. the tree with one node labelled ι.

We can conveniently write a permutation tree t as $\langle \pi; t_1, \ldots, t_n \rangle$ where $\pi \in S_n$ is the label of the root, $\mathrm{rd}(t) = n$ and $t_1, \ldots, t_n \in \mathrm{PT}$.

Observe that PT can be defined coinductively as the greatest set PT such that, if $t \in \mathrm{PT}$, then there is $n \in \mathbb{N}$, $\pi \in S_n$ and $t_1, \ldots, t_n \in \mathrm{PT}$ such that $t = \langle \pi; t_1, \ldots, t_n \rangle$.

▶ **Definition 13.** *Let* $t, u \in \mathrm{PT}$*. We define the product* tu *by coinduction as follows. First,* $t\bar{\epsilon} = t = \bar{\epsilon}t$*. If* $t = \langle \pi; t_1, \ldots, t_n \rangle$*,* $u = \langle \rho; u_1, \ldots, u_m \rangle$ *and* $k = \max\{n, m\}$*, then we define:*

$$tu := \langle \pi \circ \rho; u_1 t_{\rho 1}, \ldots, u_k t_{\rho k} \rangle$$

where, by convention, $t_j = \bar{\epsilon}$ *if* $j > n$*, and* $u_i = \bar{\epsilon}$ *if* $i > m$*.*

The product operation makes $(\mathrm{PT}, \cdot, \bar{\epsilon})$ into a monoid. We now show that this monoid is an inverse monoid.

▶ **Definition 14.** *Given* $t \in \mathrm{PT}$*, we define by coinduction* t^* *as follows. First,* $\bar{\epsilon}^* = \bar{\epsilon}$*. If* $t = \langle \pi; t_1, \ldots, t_n \rangle$ *with* $n \geq 1$*, then* $t^* = \langle \pi^{-1}; (t_{\pi^{-1}1})^*, \ldots, (t_{\pi^{-1}n})^* \rangle$*.*

We denote by E the set $\{t \in \mathrm{PT} : t(a) = \iota \text{ for every } a \in \mathrm{sk}(t)\}$. Every $t \in E$ is univocally determined by its skeleton and we identify $\mathrm{sk}(t)$ with the permutation tree t.

▶ **Proposition 15.** *The set* $E(\mathrm{PT})$ *of idempotent elements of* PT *coincides with* E*. Moreover, for every* $d, e \in E$*,* $de = d \cup e$*; consequently any two idempotents commute.*

Proof. Let $t = \langle \pi; t_1, \ldots, t_n \rangle \in E(\mathrm{PT})$ be idempotent. Then $\pi \circ \pi = \pi$, and this is possible only if $\pi = \iota$. The claim $t \in E$ follows from $t_i = (tt)_i = t_i t_i$. Conversely, if $d = \langle \iota; d_1, \ldots, d_n \rangle \in E$, then by definition of the product $dd = \langle \iota; d_1 d_1, \ldots, d_n d_n \rangle$. Since by coinduction $d_i d_i = d_i$, we have $dd = d$ and $E = E(\mathrm{PT})$. Finally, if $d, e \in E$, then, again by the coinductive definition of the product, $de = d \cup e$. ◀

▶ **Theorem 16.** *The monoid* $(\mathrm{PT}, \cdot, (-)^*, \bar{\epsilon})$ *is an* E*-unitary inverse monoid.*

Proof. We prove that PT is an inverse monoid by applying [31, Theorem 5.1.1], which states that S is an inverse monoid iff S is a regular monoid whose idempotents commute. For every $t = \langle \pi; t_1, \ldots, t_n \rangle \in \mathrm{PT}$, we prove that $tt^*t = t$: $\pi \circ \pi^{-1} \circ \pi = \pi$ and, coinductively,

$$(tt^*t)_i = ((tt^*)t)_i = t_i(tt^*)_{\pi i} = t_i(t^*)_{\pi i} t_{(\pi^{-1} \circ \pi) i} = t_i (t_i)^* t_i = t_i.$$

Since for every t there is t^* such that $tt^*t = t$, PT is regular. We conclude that PT is an inverse monoid applying Proposition 15.

We now prove that PT is E-unitary. Let $t = \langle \pi; t_1, \dots, t_n \rangle$, and $e = \langle \iota; e_1, \dots, e_m \rangle$ be an idempotent such that $e \leq t$. If $e = \bar{\epsilon}$, then necessarily $t = \bar{\epsilon}$, hence $t \in E$. If $e \neq \bar{\epsilon}$, then $e = td$ for some $d \in E$. We prove that $t \in E$. First, we immediately get that $\pi = \iota$ and $n \leq m$. Moreover, for every $1 \leq i \leq n$, we have $e_i = d_i t_i$. Since d_i is idempotent, we get that $e_i \leq t_i$, and coinductively, that $t_i \in E$. We conclude that $t \in E$. ◀

As a consequence of E-unitarity, if $t \leq u$ and one between t and u is idempotent, then both are idempotent.

We now show that PT is isomorphic to a monoid of Böhm-like trees. The set of Böhm-like trees is endowed with a composition that turns $(\mathscr{B}, \circ)$ into a semigroup [6, Thm. 18.3.10].

▶ **Proposition 17.** *There is an embedding $(-)^+$ of the inverse monoid* PT *of permutation trees into the semigroup $\mathscr{B}$ of Böhm-like trees whose image is $\mathscr{HP}$. In particular, $(\mathscr{HP}, \circ, \mathbf{I})$ is an inverse monoid.*

Proof. We coinductively define an injective map $(-)^+ : \mathrm{PT} \to \mathscr{B}$. We exploit the fact that $\mathscr{B}$ is a lambda abstraction algebra [48, Lemma 23], which entails that application and abstraction of Böhm-like trees are computed in the same way as for λ-terms. We define:

- $t^+ := \lambda x.x$ if $t = \bar{\epsilon}$;
- $t^+ := \lambda x x_1 \dots x_n . x((t_1)^+ x_{\pi 1}) \cdots ((t_n)^+ x_{\pi n})$ if $t = \langle \pi; t_1, \dots, t_n \rangle$.

We now prove that $(tu)^+ = t^+ \circ u^+$. Let $u = \langle \rho; u_1, \dots, u_m \rangle$. Without loss of generality, we assume $n = m$:

$$\begin{aligned}
(tu)^+ &= \lambda x x_1 \dots x_n . x((u_1 t_{\rho 1})^+ x_{(\pi\circ\rho)(1)}) \cdots ((u_n t_{\rho n})^+ x_{(\pi\circ\rho)(n)}) \\
&= \lambda x x_1 \dots x_n . x(((u_1)^+ \circ (t_{\rho 1})^+) x_{(\pi\circ\rho)(1)}) \cdots (((u_n)^+ \circ (t_{\rho n})^+) x_{(\pi\circ\rho)(n)}) \\
&= \lambda x x_1 \dots x_n . u^+ x((t_1)^+ x_{\pi 1}) \cdots ((t_n)^+ x_{\pi n}) \\
&= t^+ \circ u^+.
\end{aligned}$$

The last statement of the proposition follows from Remark 3. ◀

Note that, for $A \in \mathscr{HP}$, A^* coincides with the tree A^{-1} of [6, Def. 21.2.16]. The image of the idempotents of PT is the set $\mathscr{I}$ introduced in Definition 9.

3.1 The minimum group congruence

In this section we prove that the monoid PT is F-inverse. In the first lemma we characterise the natural partial order of the inverse monoid PT.

▶ **Lemma 18.** *Let $t = \langle \pi; t_1, \dots, t_n \rangle$ and $u = \langle \rho; u_1, \dots, u_m \rangle$. Then $t \leq u$ iff*

1. *$\pi = \rho$ and $m \leq n$;*
2. *$t_i \leq u_i$ for $1 \leq i \leq m$ and $t_i \in E$ for $m < i \leq n$.*

Proof. Assume that $t \leq u$, meaning that $t = ue$ for some $e \in E$. Then $\pi = \rho$ and $n = \max\{m, \mathrm{rd}(e)\} \geq m$. Moreover, since $t_i = e_i u_i$, for $1 \leq i \leq m$, and $t_i = e_i$, for $m < i \leq n$, the conclusion easily follows. Conversely, let $e = \langle \iota; e_1, \dots, e_m, t_{m+1}, \dots, t_n \rangle$, where e_i is such that $t_i = e_i u_i$ with $e_i \in E$ for $1 \leq i \leq m$. Then we have $t = ue$. ◀

Remark that the natural order is the greatest relation on PT that satisfies conditions (1) and (2) of Lemma 18.

▶ **Definition 19.** *Given $t = \langle \pi; t_1, \dots, t_n \rangle \in \mathrm{PT}$, we define coinductively $t^{\mathfrak{m}}$ as follows:*

$$t^{\mathfrak{m}} := \begin{cases} (\langle \pi; t_1, \dots, t_{n-1} \rangle)^{\mathfrak{m}}, & \text{if } t_n \in E \text{ and } \pi n = n, \\ \langle \pi; (t_1)^{\mathfrak{m}}, \dots, (t_n)^{\mathfrak{m}} \rangle, & \text{otherwise.} \end{cases}$$

Observe the similarity with the definition of Nakajima tree of a λ-term given in [7, Definition 2.60]. See also [6, Proposition 10.2.15]. Remark that if $e \in E$, then $e^{\mathfrak{m}} = \bar{\epsilon}$. For $t \in \mathrm{PT}$, let $t^{\uparrow} := \{u : t \leq u\}$.

▶ **Proposition 20.** *For every $t, u \in \mathrm{PT}$, the following hold:*
1. *$t \leq t^{\mathfrak{m}}$;*
2. *if $t \leq u$, then $u \leq t^{\mathfrak{m}}$.*

In particular, $t^{\mathfrak{m}}$ is the unique maximal element in $t^{\uparrow}$.

Proof. If t is idempotent, (1) is trivial and (2) follows from E-unitarity. Otherwise let $t = \langle \pi; t_1, \dots, t_n \rangle$ and let k be the greatest i such that $t_k \notin E$ or $\pi k \neq k$. Then $t^{\mathfrak{m}} = \langle \pi; (t_1)^{\mathfrak{m}}, \dots, (t_k)^{\mathfrak{m}} \rangle$. By coinductive hypothesis $t_i \leq (t_i)^{\mathfrak{m}}$ for $1 \leq i \leq k$, and $t_i \in E$ for $k < i \leq n$, so that by Lemma 18 $t \leq t^{\mathfrak{m}}$, showing (1). Concerning (2), let $u = \langle \rho; u_1, \dots, u_m \rangle$ such that $t \leq u$. Without loss of generality, we assume that $n = m$, so that $k \leq m$, where k is defined above. Moreover, we have $\pi = \rho$, so that by Lemma 18 and by coinductive hypothesis, $u_i \leq (t_i)^{\mathfrak{m}}$ for $1 \leq i \leq k$. As $t_i \leq u_i$ and $t_i \in E$ for $k < i \leq n$, we have $u_i \in E$ for $k < i \leq m$ by E-unitarity. By Lemma 18 we conclude that $u \leq t^{\mathfrak{m}}$. ◀

We now prove that PT is F-inverse. The proof relies on the following general lemma that will be used multiple times.

▶ **Lemma 21.** *Let S be a E-unitary inverse semigroup. Assume that for every $t \in S$, the set $t^{\uparrow}$ has a unique maximal element $t^{\mathfrak{m}}$. Then for all $t, u \in S$, $t \,\sigma_S\, u$ iff $t^{\mathfrak{m}} = u^{\mathfrak{m}}$.*

Proof. Assuming $t \,\sigma_S\, u$, there exists w such that $w \leq t, u$. Hence $w^{\mathfrak{m}} = t^{\mathfrak{m}} = u^{\mathfrak{m}}$. Conversely, let $w = t^{\mathfrak{m}} = u^{\mathfrak{m}}$. Then $u = we$ for some $e \in E(S)$ and $t = dw$ for some $d \in E(S)$. Moreover, $t^*u = t^*we \leq w^*we \in E(S)$ and $tu^* = dwu^* \leq dww^* \in E(S)$, so that t, u are compatible, i.e., $t \sim u$. As S is E-unitary, $\sim = \sigma_S$, so that $t \,\sigma_S\, u$. ◀

▶ **Theorem 22.** *The monoid* PT *is F-inverse.*

Proof. By Lemma 21 and Proposition 20. ◀

3.2 Finite permutation trees

We now focus on the subset of finite permutation trees (FPT). With the operations inherited from PT, this is an inverse monoid whose idempotent elements are the finite trees in E.

▶ **Definition 23.** *We denote by E_{fin} the set of finite permutation trees in E.*

▶ **Lemma 24.** *The monoid* FPT *is an F-inverse monoid.*

Proof. The class of F-inverse monoids is an equational class when adding $(-)^{\mathfrak{m}}$ as a unary operation to the signature (see [4]). As FPT is a submonoid of PT closed under $(-)^{\mathfrak{m}}$, FPT is F-inverse. ◀

The minimum group congruence σ_{FPT} is obtained restricting σ_{PT} to FPT.

▶ **Lemma 25.** *For all $t, u \in \mathrm{FPT}$, $t \,\sigma_{\mathrm{FPT}}\, u$ iff $t \,\sigma_{\mathrm{PT}}\, u$.*

Proof. One direction is trivial; the other follows from E-unitarity: if $t \,\sigma_{\mathrm{PT}}\, u$, then $t \sim u$, so that $t \wedge u$ exists and is given by tu^*u. Since $tu^*u \in \mathrm{FPT}$, we have $t \,\sigma_{\mathrm{FPT}}\, u$. ◀

Finite permutation trees have a rich structure in their own right. First, we show that FPT is residually finite. Second, we characterise the covering relation in the natural order, a result that will be useful later.

The monoid of finite permutation trees is residually finite. If $t \in \mathrm{FPT}$ with $\mathrm{sk}(t) = e$, we denote by $\nu(t)$ the number of nodes of e.

▶ **Lemma 26.** *For any $t, u \in \mathrm{FPT}$, the following hold:*
1. *if $t < u$, then $\nu(u) < \nu(t)$;*
2. *$\nu(t) = \nu(t^*)$;*
3. *$\nu(t), \nu(u) \leq \nu(tu) \leq \nu(t) + \nu(u)$;*
4. *$\nu(\mathrm{sk}(t)) = \nu(tt^*) = \nu(t^*t)$.*

As a consequence of (1) and (3) the sets $t^{\uparrow}$ and $\{(v, w) : t = vw\}$ are finite.

Two easy, yet remarkable, consequences of the previous lemma are the following facts.

▶ **Proposition 27.** *Every finitely generated submonoid of* FPT *is finite.*

▶ **Proposition 28.** *For every $t \in \mathrm{FPT}$, the equivalence classes L_t, R_t, H_t of Green's relations are all finite sets.*

Proof. By induction over the complexity of t. It $t = \bar{\epsilon}$, then $L_{\bar{\epsilon}} = R_{\bar{\epsilon}} = \{\bar{\epsilon}\}$. Otherwise, let $e = tt^*$. By definition, $t \mathcal{R} u$ iff $e = uu^*$. Since $\{u : e = uu^*\}$ is finite by Lemma 26, we get the conclusion. A similar proof works for $\mathcal{L}$. ◀

▶ **Definition 29.** *A monoid M is* residually finite *if for every $u, v \in M$ there is a finite monoid N and a homomorphism $\alpha : M \to N$ such that $\alpha(u) \neq \alpha(v)$. We say that N* separates *u and v.*

▶ **Theorem 30.** *The inverse monoid* FPT *is residually finite.*

Proof. The conclusion follows from the fact that every $\mathcal{R}$-class is finite by Proposition 28, using [36, Lemma 5.3]. ◀

Every residually finite monoid S can be equipped with a natural topology, the so-called *profinite* topology. This topology is defined by taking as a subbasis the equivalence classes of the congruences of finite index, that is, those congruences θ such that S/θ is finite. It is a metric topology with distance given by $d(u, v) = 2^{-r(u,v)}$ for all $u \neq v$, where $r(u, v)$ is the minimum cardinality of a monoid M separating u and v. With this topology, the monoid S embeds into its profinite completion $\hat{S}$, which is obtained as the inverse limit of all its finite quotients. The monoid FPT is residually finite and therefore admits such a profinite topology; this topology turns out to be discrete. The proof of this fact, however, relies on technical arguments that go beyond the scope of the present article. We leave the study of the completion of FPT for future work.

The covering relation. Given two distinct t and u in FPT, we write $t \prec u$ and say that u *covers* t if $t < u$ and there is no z such that $t < z < u$. The next two lemmas are proven by structural induction on FPTs. The covering relation is characterised as follows.

▶ **Lemma 31.** *Let $t = \langle \pi; t_1, \ldots, t_n \rangle$ and $u = \langle \rho; u_1, \ldots, u_m \rangle$ be finite permutation trees. Then $t \prec u$ iff one of the following conditions holds:*
1. *$\pi = \rho$, $n = m$, and there is $k \in \mathbb{N}$ such that $t_k \prec u_k$ and $t_i = u_i$ for every $1 \leq i \neq k \leq m$;*
2. *$\pi = \rho$, $n = m + 1$, $t_n = \bar{\epsilon}$ and $t_i = u_i$ for every $1 \leq i \leq m$.*

▶ **Lemma 32.** *Let $t, u \in \mathrm{FPT}$ such that $t < u$. Then there exist $n \in \mathbb{N}$ and $z_0, z_1, \ldots, z_{n+1} \in \mathrm{FPT}$ such that $z_0 = t \prec z_1 \prec \cdots \prec z_n \prec z_{n+1} = u$.*

Proof. The conclusion follows because by Lemma 26 $t^{\uparrow}$ is finite. ◀

We shall see that the covering relation, modulo the embedding of Proposition 17, corresponds to η-reduction.

3.3 Permutation trees and finite idempotents

In the next section, the interaction between infinite permutation trees and idempotent FPTs will be fundamental. This leads us to study a new order on PT inspired by the natural order and to investigate its maximal elements. Recall that $E_{\mathrm{fin}} = E(\mathrm{FPT})$.

▶ **Definition 33.** *We define $\leq_{\mathrm{fin}}$ as the greatest relation on PT such that if $t \leq_{\mathrm{fin}} u$ with $t = \langle \pi; t_1, \ldots, t_n \rangle$ and $u = \langle \rho; u_1, \ldots, u_m \rangle$, then:*
1. *$\pi = \rho$ and $m \leq n$;*
2. *$t_i \leq_{\mathrm{fin}} u_i$ for $1 \leq i \leq m$ and $t_i \in E_{\mathrm{fin}}$ for $m < i \leq n$.*

By coinduction, it is not difficult to see that this is a compatible partial order. Moreover, as $E_{\mathrm{fin}} \subseteq E$, if $t \leq_{\mathrm{fin}} u$ then $t \leq u$. In the next lemma we establish some properties of the order $\leq_{\mathrm{fin}}$. The last one asserts some kind of unitarity with respect to E_{fin}.

▶ **Lemma 34.** *Let $t, u \in \mathrm{PT}$. Then the following conditions hold:*
1. *there is $e \in E_{\mathrm{fin}}$ such that $t = eu$ iff there is $f \in E_{\mathrm{fin}}$ such that $t = uf$;*
2. *if there is $e \in E_{\mathrm{fin}}$ such that $t = ue$, then $t \leq_{\mathrm{fin}} u$;*
3. *if $f \leq_{\mathrm{fin}} t$ with $f \in E_{\mathrm{fin}}$, then $t \in E_{\mathrm{fin}}$.*

▶ **Definition 35.** *Let $t = \langle \pi; t_1, \ldots, t_n \rangle \in \mathrm{PT}$. We define coinductively $t^{\mathfrak{m}'}$ as:*

$$t^{\mathfrak{m}'} := \begin{cases} (\langle \pi; t_1, \ldots, t_{n-1} \rangle)^{\mathfrak{m}'}, & \text{if } t_n \in E_{\mathrm{fin}} \text{ and } \pi n = n, \\ \langle \pi; (t_1)^{\mathfrak{m}'}, \ldots, (t_n)^{\mathfrak{m}'} \rangle, & \text{otherwise.} \end{cases}$$

▶ **Proposition 36.** *For every $t, u \in \mathrm{PT}$, the following hold:*
1. *$t \leq_{\mathrm{fin}} t^{\mathfrak{m}'}$;*
2. *if $t \leq_{\mathrm{fin}} u$, then $u \leq_{\mathrm{fin}} t^{\mathfrak{m}'}$.*

In particular, $t^{\mathfrak{m}'}$ is the unique maximal element in $t^{\uparrow'} := \{u : t \leq_{\mathrm{fin}} u\}$.

▶ **Lemma 37.** *For every $t, u, v \in \mathrm{PT}$, if $t \leq_{\mathrm{fin}} v$ and $u \leq_{\mathrm{fin}} v$, then there is $w \in \mathrm{PT}$ such that $w \leq_{\mathrm{fin}} t$ and $w \leq_{\mathrm{fin}} u$.*

In analogy with the minimum group congruence, we define a relation σ'_{PT} by letting $t \, \sigma'_{\mathrm{PT}} \, u$ if there is $w \in \mathrm{PT}$ such that $w \leq_{\mathrm{fin}} t$ and $w \leq_{\mathrm{fin}} u$.

▶ **Lemma 38.** *The relation σ'_{PT} is a congruence on PT such that $\sigma'_{\mathrm{PT}} \subseteq \sigma_{\mathrm{PT}}$.*

Proof. Reflexivity and symmetry are immediate and transitivity follows from Lemma 37. Moreover, σ'_{PT} is a congruence because $\leq_{\mathrm{fin}}$ is compatible. ◀

▶ **Example 39.** The infinite permutation tree $e = \{1^n : n \in \mathbb{N}\}$ is idempotent. Now, $e^{\mathfrak{m}} = \bar{\epsilon}$ while $e^{\mathfrak{m}'} = e$. This implies that $e \not\leq_{\mathrm{fin}} \bar{\epsilon}$ and that σ'_{PT} is strictly contained in σ_{PT}.

▶ Remark 40. We observe that if $t \in \mathrm{PT}$ is infinite, then $t^{\mathfrak{m}'}$ is infinite, while $t^{\mathfrak{m}}$ may be finite. As a consequence, for an idempotent $e \in E$, $e \, \sigma'_{\mathrm{PT}} \, \bar{\epsilon}$ iff $e \in E_{\mathrm{fin}}$.

The congruences σ'_{PT} and σ_{PT} coincide on finite permutation trees. In particular, $\mathrm{FPT}/\sigma_{\mathrm{PT}} \simeq \mathrm{FPT}/\sigma'_{\mathrm{PT}}$. We characterise the congruence σ'_{PT} in a way analogous to σ_{PT}.

▶ **Proposition 41.** *For all $t, u \in \mathrm{PT}$, we have: $t\,\sigma'_{\mathrm{PT}}\,u$ iff $t^{\mathfrak{m}'} = u^{\mathfrak{m}'}$.*

Proof. ($\Rightarrow$) If $t\,\sigma'_{\mathrm{PT}}\,u$ then, by definition, there is $w \in \mathrm{PT}$ such that $w \leq_{\mathrm{fin}} t$ and $w \leq_{\mathrm{fin}} u$. Therefore $w^{\mathfrak{m}'} = t^{\mathfrak{m}'} = u^{\mathfrak{m}'}$. ($\Leftarrow$) If $t^{\mathfrak{m}'} = u^{\mathfrak{m}'}$ holds, then $t \leq_{\mathrm{fin}} t^{\mathfrak{m}'}$ and $u \leq_{\mathrm{fin}} u^{\mathfrak{m}'} = t^{\mathfrak{m}'}$. By Lemma 37, there exists w such that $w \leq_{\mathrm{fin}} t$ and $w \leq_{\mathrm{fin}} u$, whence $t\,\sigma'_{\mathrm{PT}}\,u$. ◀

By Remark 40, the group of units of $\mathrm{PT}/\sigma'_{\mathrm{PT}}$ is the group $\mathrm{FPT}/\sigma_{\mathrm{PT}}$.

4 Permutation trees and invertibility in λ-calculus

In this section we apply and adapt the previous results to the study of inverse monoids and groups in the λ-calculus. The natural order $\leq$ and the order $\leq_{\mathrm{fin}}$ on permutation trees are closely related to notions of η-expansion. The general construction extracting a canonical group from an inverse monoid recovers the invertible λ-terms in various λ-theories.

By Proposition 17, the image $\mathscr{HP}$ of the embedding $(-)^+ : \mathrm{PT} \to \mathscr{B}$ is an F-inverse monoid isomorphic to the monoid PT of permutation trees. Thus, we will transfer all the notions introduced for PT to $\mathscr{HP}$, including the natural order, the maximal elements, and the minimum group congruence.

Restricting the embedding $(-)^+$ to recursively enumerable permutation trees yields an E-unitary inverse monoid $\mathscr{HP}_{\mathrm{rec}}$, where $\mathscr{HP}_{\mathrm{rec}} = \{\mathrm{BT}(M) : M \in \mathrm{HP}\}$. This monoid is not F-inverse, since $\mathrm{BT}(M)^{\mathfrak{m}}$ is not in general recursively enumerable. We note that $\mathscr{HP}_{\mathrm{rec}} \simeq \mathrm{HP}/\mathcal{B}$. The set of idempotents of $\mathrm{HP}/\mathcal{B}$ is the set $\mathcal{I}^{\eta}_{\omega}/\mathcal{B}$ of possibly infinite η-expansions of $\mathbf{I}$ modulo $\mathcal{B}$ and it forms a $\wedge$-semilattice (see also [7, Thm. 11.22]).

Restricting $(-)^+$ to finite permutation trees yields an inverse monoid, $\mathrm{FHP}/\boldsymbol{\lambda}$, since $\mathcal{B}$ and $\boldsymbol{\lambda}$ coincide on β-normalisable λ-terms. We recover the fact that the idempotent elements $\mathcal{I}^{\eta}/\boldsymbol{\lambda}$ form a $\wedge$-semilattice [35].

▶ **Proposition 42.** *There exist minimal λ-theories $\mathcal{T}, \mathcal{T}'$ such that $\mathrm{HP}/\mathcal{T}$ and $\mathrm{FHP}/\mathcal{T}'$ are groups. Moreover, $\mathcal{T}' = \boldsymbol{\lambda\eta} \subsetneq \mathcal{T} \subseteq \mathcal{H}^*$.*

Proof. By Lemma 5 the λ-theory $\mathcal{T}$ is axiomatised by $\{M = \mathbf{I} : M \in \mathcal{I}^{\eta}_{\omega}\}$ and the λ-theory $\mathcal{T}'$ by $\{M = \mathbf{I} : M \in \mathcal{I}^{\eta}\}$. We have $\mathcal{T}' = \boldsymbol{\lambda\eta}$, because $\mathcal{T}' \vdash \mathbf{1} = \mathbf{I}$ and $\boldsymbol{\lambda\eta} \vdash M = \mathbf{I}$ for every $M \in \mathcal{I}^{\eta}$. Moreover, $\mathcal{T}'$ is strictly contained into $\mathcal{T}$, because every $M \in \mathcal{I}^{\eta}_{\omega} \setminus \mathcal{I}^{\eta}$ has no $\beta\eta$-normal form, and therefore $M = \mathbf{I}$ cannot be proven in $\mathcal{T}'$. ◀

We conjecture that the theory $\mathcal{T}$ of the previous proposition is strictly contained in $\mathcal{H}^*$.

Proposition 7 is central to determining invertible elements modulo λ-theories. In a F-inverse monoid S, the minimum group congruence σ_S collapses all idempotents in the σ_S-class of the unit 1, so every $t \in S$ becomes invertible modulo σ_S with inverse t^*. Since maximal elements are canonical representatives of σ_S-classes, they form a group under $t \cdot u = (tu)^{\mathfrak{m}}$, which is isomorphic to S/σ_S by Lemma 21. It is then natural to ask what σ_S corresponds to in the λ-calculus.

4.1 Hereditary permutations and the theory $\mathcal{H}^*$

Two λ-terms are equal in $\mathcal{H}^*$ whenever their Böhm trees coincide up to infinite η-expansions [6, Thm. 16.2.7]. We denote by $\leq^{\eta}_{\omega}$ the preorder on $\mathscr{B}$ such that $T \leq^{\eta}_{\omega} U$ holds precisely when U is obtained from T by performing countably many possibly infinite η-expansions [7, Def. 11.11]. Moreover, we denote by $\mathrm{nf}_{\eta!}(T)$ the Nakajima tree of a Böhm-like tree T [7, Def. 2.60]. One calculates $\mathrm{nf}_{\eta!}(T)$ by eliminating from T all finite and infinite η-expansions.

▶ **Theorem 43** ([7, Prop. 11.12]). *For every $M, N \in \Lambda$ the following are equivalent:*
1. $\mathcal{H}^* \vdash M = N$;
2. $\mathrm{nf}_{\eta!}(\mathrm{BT}(M)) = \mathrm{nf}_{\eta!}(\mathrm{BT}(N))$;
3. *there exists $T \in \mathscr{B}$ such that* $\mathrm{BT}(M) \leq^{\eta}_{\omega} T$ *and* $\mathrm{BT}(N) \leq^{\eta}_{\omega} T$.

▶ **Lemma 44.** *If $M, N \in \mathrm{HP}$, then we have:*
1. $\mathrm{BT}(M) \geq \mathrm{BT}(N)$ *iff* $\mathrm{BT}(M) \leq^{\eta}_{\omega} \mathrm{BT}(N)$;
2. $\mathrm{nf}_{\eta!}(\mathrm{BT}(M)) = \mathrm{BT}(M)^{\mathtt{m}}$.

Proof. (1) follows from the characterisation of the natural order, Lemma 18, and [7, Def. 11.11]. (2) follows from (1), Proposition 17, Definition 19 and [7, Def. 2.60]. ◀

Remark that the natural order $\leq$ in $\mathscr{HP}$ is the opposite of $\leq^{\eta}_{\omega}$.

▶ **Theorem 45.** *For any $M, N \in \mathrm{HP}$, the following are equivalent:*
1. $\mathcal{H}^* \vdash M = N$;
2. $\mathrm{BT}(M)\, \sigma_{\mathscr{HP}}\, \mathrm{BT}(N)$;
3. $\mathrm{BT}(M)^{\mathtt{m}} = \mathrm{BT}(N)^{\mathtt{m}}$.

Proof. (1) $\iff$ (3) by Theorem 43 and Lemma 44(2). (2) $\iff$ (3) by Lemma 21. ◀

The preceding analysis gives a characterisation of the invertible elements in $\mathcal{H}^*$. Recall that the monoid $\mathrm{HP}/\mathcal{B}$ is isomorphic to $\mathscr{HP}_{\mathrm{rec}}$.

▶ **Theorem 46.** *Let $S := \mathscr{HP}_{\mathrm{rec}}$. The following groups are isomorphic:*
1. *the group* $\mathrm{HP}/\mathcal{H}^*$ *of invertible elements of* $\mathcal{H}^*$;
2. *the group* S/σ_S;
3. *the group of maximal elements of S in the F-inverse monoid $\mathscr{HP}$.*

Proof. By Theorem 45 there is a bijective correspondence between the equivalence classes of σ_S and the equivalence classes of HP in $\mathcal{H}^*$. By Lemma 21 there is a bijective correspondence between maximal elements of $\mathscr{HP}$ and equivalence classes of $\sigma_{\mathscr{HP}}$. The maximal elements of $\mathscr{HP}_{\mathrm{rec}}$ in $\mathscr{HP}$ form a subgroup of the group of maximal elements of $\mathscr{HP}$. ◀

▶ **Theorem 47.** *Let $\mathcal{T}$ be a λ-theory with $\mathcal{B} \subseteq \mathcal{T}$. Then* $\mathrm{HP}/\mathcal{T}$ *is an E-unitary inverse submonoid of the semigroup* $\Lambda/\mathcal{T}$ *with composition.*

Proof. Since $\mathcal{H} \subseteq \mathcal{B} \subseteq \mathcal{T}$, $\mathcal{T}$ is sensible and thus $\mathcal{T} \subseteq \mathcal{H}^*$. Therefore, we have the following chain of surjective homomorphisms of monoids: $\mathrm{HP}/\mathcal{B} \xrightarrow{f} \mathrm{HP}/\mathcal{T} \xrightarrow{g} \mathrm{HP}/\mathcal{H}^*$. As $\mathrm{HP}/\mathcal{B}$ is an inverse monoid, $\mathrm{HP}/\mathcal{T}$ is such by Remark 3. We now prove that $\mathrm{HP}/\mathcal{T}$ is E-unitary. If $e \leq t$ in $\mathrm{HP}/\mathcal{T}$ with e idempotent, then $g(e) \leq g(t)$ in $\mathrm{HP}/\mathcal{H}^*$ with $g(e)$ idempotent. Since $\mathrm{HP}/\mathcal{H}^*$ is a group, $g(e) = g(t)$ and $g(e) = \mathbf{I}$, so that $g(t) = \mathbf{I}$. Let $t' \in \mathrm{HP}/\mathcal{B}$ be such that $f(t') = t$. Then $(g \circ f)(t') = \mathbf{I}$. By Theorem 46 $\ker(g \circ f) = \sigma_{\mathrm{HP}/\mathcal{B}}$, so that $t'\, \sigma_{\mathrm{HP}/\mathcal{B}}\, \mathbf{I}$. By Lemma 5, this implies that t' is idempotent in $\mathrm{HP}/\mathcal{B}$. Consequently, t is idempotent in $\mathrm{HP}/\mathcal{T}$ as desired. ◀

4.2 Finite hereditary permutations and the theory $\lambda\eta$

Finite permutation trees are in bijection with β-normal forms of finite hereditary permutations. We show that the natural order on the finite permutation trees perfectly captures η-reduction.

▶ **Proposition 48.** *Let $t, u \in \mathrm{FPT}$. Then:*
1. $t \prec u$ *iff* $t^+ \to_\eta u^+$;
2. $t \leq u$ *iff* $t^+ \twoheadrightarrow_\eta u^+$.

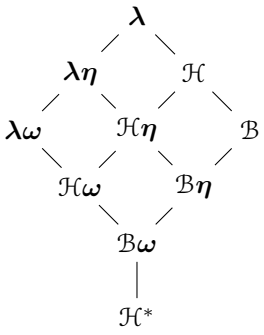


Figure 1 The kite.

Proof. (1) follows from Proposition 17 and Lemma 31; (2) from (1) and Lemma 32. ◀

By the above proposition, when $t \in \mathrm{FPT}$, the λ-term $(t^{\mathtt{m}})^+$ is the $\beta\eta$-normal form of t^+.

▶ **Proposition 49.** *Let $t, u \in \mathrm{FPT}$. The following are equivalent:*
1. $\boldsymbol{\lambda\eta} \vdash t^+ = u^+$;
2. $t\, \sigma_{\mathrm{FPT}}\, u$;
3. $t^{\mathtt{m}} = u^{\mathtt{m}}$.

Proof. The equivalence between (1) and (2) follows from Proposition 48 and the definition of σ_{FPT}; the equivalence between (2) and (3) follows from Lemma 21. ◀

▶ **Theorem 50.** *Let S be $\mathrm{FHP}/\boldsymbol{\lambda}$. The following groups are isomorphic:*
1. *the group $\mathrm{FHP}/\boldsymbol{\lambda\eta}$ of invertible elements of $\boldsymbol{\lambda\eta}$;*
2. *the group S/σ_S;*
3. *the group of maximal elements of S.*

Proof. By Proposition 49, $\mathrm{FHP}/\boldsymbol{\lambda\eta} \simeq S/\sigma_S$ and by Lemma 21 the group S/σ_S is isomorphic to the the group of maximal elements of S. ◀

▶ **Theorem 51.** *For every λ-theory $\mathcal{T}$, $\mathrm{FHP}/\mathcal{T}$ is an F-inverse submonoid of $(\Lambda/\mathcal{T}, \circ)$.*

Proof. Since FHPs are β-normal forms, and $\mathcal{T}$ cannot identifty β-normal forms that have distinct $\beta\eta$-normal forms, we have the following chain of surjective homomorphisms of monoids: $\mathrm{FHP}/\boldsymbol{\lambda} \xrightarrow{f} \mathrm{FHP}/\mathcal{T} \xrightarrow{g} \mathrm{FHP}/\boldsymbol{\lambda\eta}$. As $\mathrm{FHP}/\boldsymbol{\lambda}$ is an inverse monoid, $\mathrm{FHP}/\mathcal{T}$ is such. We prove that $\mathrm{FHP}/\mathcal{T}$ is F-inverse. Let $t \in \mathrm{FHP}/\boldsymbol{\lambda}$. We prove that $f(t^{\mathtt{m}})$ is maximal. If, towards a contradiction, there is $u \in \mathrm{FHP}/\boldsymbol{\lambda}$ such that $f(t^{\mathtt{m}}) < f(u)$ in $\mathrm{FHP}/\mathcal{T}$, then $(g \circ f)(t^{\mathtt{m}}) \leq (g \circ f)(u)$ in $\mathrm{FHP}/\boldsymbol{\lambda\eta}$, so that, as $\mathrm{FHP}/\boldsymbol{\lambda\eta}$ is a group, $(g \circ f)(t^{\mathtt{m}}) = (g \circ f)(u)$. By Theorem 50 $\ker(g \circ f) = \sigma_{\mathrm{FHP}/\boldsymbol{\lambda}}$, so that $t^{\mathtt{m}}\, \sigma_{\mathrm{FHP}/\boldsymbol{\lambda}}\, u$. By definition of $\sigma_{\mathrm{FHP}/\boldsymbol{\lambda}}$, there is $w \in \mathrm{FHP}/\boldsymbol{\lambda}$ such that $w \leq u, t^{\mathtt{m}}$. This implies that $u \leq t^{\mathtt{m}}$ in $\mathrm{FHP}/\boldsymbol{\lambda}$, so that $f(u) \leq f(t^{\mathtt{m}})$ in $\mathrm{FHP}/\mathcal{T}$. Contradiction. ◀

4.3 Invertible terms and the kite

A long-standing question concerned the inclusion relationship between $\mathcal{H}^+$ and $\mathcal{B}\boldsymbol{\omega}$. This question has now been settled: the inclusion $\mathcal{B}\boldsymbol{\omega} \subseteq \mathcal{H}^+$ was proved in [10], and the equality $\mathcal{B}\boldsymbol{\omega} = \mathcal{H}^+$ was subsequently established in [33]. Figure 1 depicts a subset of λ-theories ordered by reverse inclusion, commonly referred to as Barendregt's "kite". In this section we complete Theorem 11 by characterising invertible terms for all the λ-theories of the kite.

Similarly to $\mathcal{H}^*$, equality in $\mathcal{H}^+$ can also be characterised via a preorder on the set of Böhm-like trees. We denote by $\leq^\eta$ the preorder on $\mathscr{B}$ such that $T \leq^\eta U$ holds precisely when U is obtained from T by performing countably many finite η-expansions [7, Def. 11.8]. Moreover, we denote by $\mathrm{nf}_\eta(T)$ the η-normal form of a Böhm-like tree T (see [50, Def. 4] and [7, Def. 2.56]). Then, for every $W \in \mathscr{B}$, if $W \leq^\eta T$, then $\mathrm{nf}_\eta(T) \leq^\eta W$.

The next result was proved in [14]. See also [47, Prop. 11.2.20] and [7, Prop. 11.9].

▶ **Theorem 52.** *For every $M, N \in \Lambda$ the following are equivalent:*
1. $\mathcal{H}^+ \vdash M = N$;
2. $\mathrm{nf}_\eta(\mathrm{BT}(M)) = \mathrm{nf}_\eta(\mathrm{BT}(N))$;
3. *there exists $T \in \mathscr{B}$ such that* $\mathrm{BT}(M) \leq^\eta T$ *and* $\mathrm{BT}(N) \leq^\eta T$.

We now establish when two hereditary permutations are equal in $\mathcal{H}^+$.

▶ **Lemma 53.** *If $M, N \in \mathrm{HP}$, then we have:*
1. $\mathrm{BT}(N) \leq_{\mathrm{fin}} \mathrm{BT}(M)$ *iff* $\mathrm{BT}(M) \leq^\eta \mathrm{BT}(N)$;
2. $\mathrm{nf}_\eta(\mathrm{BT}(M)) = \mathrm{BT}(M)^{\mathfrak{m}'}$.

Proof. (1) is immediate from the definition of $\leq_{\mathrm{fin}}$ and [7, Def. 11.8]. (2) follows from (1), Proposition 17, Definition 35 and [50, Def. 4]. ◀

Remark that the order $\leq_{\mathrm{fin}}$ in $\mathscr{HP}$ is the opposite of $\leq^\eta$.

▶ **Theorem 54.** *For any $M, N \in \mathrm{HP}$, the following are equivalent:*
1. $\mathcal{H}^+ \vdash M = N$;
2. $\mathrm{BT}(M)\, \sigma'_{\mathscr{HP}}\, \mathrm{BT}(N)$;
3. $\mathrm{BT}(M)^{\mathfrak{m}'} = \mathrm{BT}(N)^{\mathfrak{m}'}$.

Proof. (1) $\iff$ (3) by Theorem 52 and Lemma 53(2). (2) $\iff$ (3) by Proposition 41. ◀

We now characterise the invertible elements in $\mathcal{H}^+$.

▶ **Theorem 55.** *Let $M \in \Lambda$. Then M is $\mathcal{H}^+$-invertible iff $M \in \mathrm{FHP}$.*

Proof. If $M \in \mathrm{FHP}$, then M is $\mathcal{H}^+$-invertible, since $\mathcal{H}^+$ is extensional. Conversely, let M be $\mathcal{H}^+$-invertible, i.e., there is N such that $\mathcal{H}^+ \vdash M \circ N = N \circ M = \mathbf{I}$. As $\mathcal{H}^+ \subseteq \mathcal{H}^*$, this implies that $\mathcal{H}^* \vdash M \circ N = N \circ M = \mathbf{I}$. By Theorem 11 $M, N \in \mathrm{HP}$. Let $P := M \circ N$ and $Q := N \circ M$. Let S be the inverse monoid $\mathscr{HP}_{\mathrm{rec}}$. By Theorem 46, $\mathrm{BT}(P)\, \sigma_S\, \mathrm{BT}(\mathbf{I})$ and $\mathrm{BT}(Q)\, \sigma_S\, \mathrm{BT}(\mathbf{I})$. By Lemma 5, this implies that $\mathrm{BT}(P)$ and $\mathrm{BT}(Q)$ are idempotent in $\mathscr{HP}_{\mathrm{rec}}$, i.e., P and Q are possibly infinite η-expansions of $\mathbf{I}$. Since $\mathcal{H}^+ \vdash \mathbf{I} = P$ and $\mathcal{H}^+ \vdash \mathbf{I} = Q$, then, by definition of $\mathcal{H}^+$, for every context $C[\,]$, $C[P]$ has a β-nf iff $C[Q]$ has a β-nf iff $C[\mathbf{I}]$ has a β-nf. If we choose the empty context, we derive that P, Q are finite η-expansions of $\mathbf{I}$. Therefore $M, N \in \mathrm{FHP}$. ◀

▶ **Lemma 56.** *Let $\mathcal{T}$ be non-extensional. If $\mathcal{T} \vdash D = \mathbf{I}$, for some $D \in \mathcal{I}^\eta_\omega$, then $D =_\beta \mathbf{I}$.*

Proof. For the sake of contradiction, if $D \neq_\beta \mathbf{I}$, then $D =_\beta \lambda uv.D'$ for some D'. Then $\mathcal{T} \vdash \mathbf{1} = \lambda xy.\mathbf{I}xy = \lambda xy.Dxy = \lambda xy.(\lambda uv.D')xy = D = \mathbf{I}$, so that $\mathcal{T}$ is extensional. ◀

▶ **Proposition 57.** *Let $\mathcal{T}$ be a λ-theory.*
1. *If $\mathcal{T}$ is semisensible and $\mathcal{T}$ is not extensional, then $M \in \Lambda$ is $\mathcal{T}$-invertible iff $M =_\beta \mathbf{I}$.*
2. *If $\boldsymbol{\lambda\eta} \subseteq \mathcal{T} \subseteq \mathcal{H}^+$, then $M \in \Lambda$ is $\mathcal{T}$-invertible iff $M \in \mathrm{FHP}$.*

Proof. (1) Let M be $\mathcal{T}$-invertible with inverse N. Since $\mathcal{T} \subseteq \mathcal{H}^*$, we have $M, N \in \mathrm{HP}$ by Theorem 11, and $\mathcal{H}^* \vdash M \circ N = N \circ M = \mathbf{I}$. Then, following the proof of Theorem 55, we can prove that $M \circ N \in \mathcal{I}_\omega^\eta$ and $N \circ M \in \mathcal{I}_\omega^\eta$. By Lemma 56 applied to $\mathcal{T}$, we get that $M \circ N =_\beta N \circ M =_\beta \mathbf{I}$. As the only invertible term modulo $\boldsymbol{\lambda}$ is $\mathbf{I}$, $M =_\beta N =_\beta \mathbf{I}$. (2) follows from Theorems 55 and 11. ◀

Thus we confirm Barendregt's conjecture [6, p. 547] stating that M is $\mathcal{T}$-invertible, for $\mathcal{T} = \mathcal{B}\boldsymbol{\eta}$ and $\mathcal{T} = \mathcal{B}\boldsymbol{\omega}$, exactly when $M \in \mathrm{FHP}$.

5 Related and further work

Inverse semigroups were first employed in the λ-calculus to study properties of β-reduction [18]. Subsequently, they played a role in Girard's Geometry of Interaction [27]. More recently, Goubault-Larrecq, inspired by the work [18, 19], has shown that a category constructed from an inverse semigroup models a fragment of linear logic [28]. Hofmann and Mislove employed the theory of inverse semigroups to prove that every compact Hausdorff topological model of λ-calculus is degenerate [30]. We have shown that the theory of inverse semigroups improves our understanding of the interaction between invertible and idempotent elements, which is essential for characterising λ-terms invertible modulo λ-theories. Looking ahead, we aim to characterise elements that are invertible modulo λ-theories lying strictly between $\mathcal{H}^+$ and $\mathcal{H}^*$. The role played by E and E_{fin} in $\mathcal{H}^*$ and $\mathcal{H}^+$, respectively, is taken on by other well-behaved semilattices of idempotents of PT. We conjecture that there exist λ-theories between $\mathcal{H}^+$ and $\mathcal{H}^*$ whose group of invertible elements is strictly between FHP and HP.

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A Proofs

We collect here all the missing proofs.

Proof of Lemma 26. Let $t = \langle \pi; t_1, \ldots, t_n \rangle$ and $u = \langle \rho; u_1, \ldots, u_m \rangle$.

1. Assume that $t < u$; then either $n > m$ or $n = m$. In the first case,

$$\nu(t) = 1 + \sum_{j=1}^{m} \nu(t_j) + \sum_{j=m+1}^{n} \nu(t_j) > 1 + \sum_{j=1}^{m} \nu(t_j) \geq 1 + \sum_{j=1}^{m} \nu(u_j).$$

   In the second case, there exists i such that $t_i < u_i$. Then

$$\nu(t) = 1 + \sum_{j=1}^{n} \nu(t_j) > 1 + \sum_{j=1}^{m} \nu(u_j),$$

   because $\nu(t_i) > \nu(u_i)$ and $\nu(t_j) \geq \nu(u_j)$ for every $j \neq i$.
2. The proof is by induction on t. Observe that

$$\nu(t^*) = 1 + \sum_{i=1}^{n} \nu((t^*)_i) = 1 + \sum_{i=1}^{n} \nu((t_{\pi^{-1}i})^*) = 1 + \sum_{i=1}^{n} \nu(t_{\pi^{-1}i}) = 1 + \sum_{i=1}^{n} \nu(t_i)$$

   which equals $\nu(t)$, because π^{-1} is a permutation of n.
3. We show (3) by induction on $\nu(t) + \nu(u)$. If $\nu(t) + \nu(u) = 2$, then $t = u = \bar{\epsilon}$, and the conclusion is trivial. If $\nu(t) + \nu(u) > 2$, then we have two cases: (a) $n \geq m$ and (b) $m \geq n$.
   a. If $m = 0$, then $u = \bar{\epsilon}$ and $tu = t$. Then the conclusion is trivial. Assume now $m > 0$. Since $(tu)_i = u_i t_{\rho i}$ for every $1 \leq i \leq m$ and $(tu)_i = t_i$ for every $\mathrm{rd}(u) < i \leq \mathrm{rd}(t)$, then by induction hypothesis $\nu(u_i), \nu(t_{\rho i}) \leq \nu((tu)_i)$ for every $1 \leq i \leq m$. Then we have:

$$\nu(tu) = 1 + \sum_{i=1}^{n} \nu(tu_i) = 1 + \sum_{i=1}^{m} \nu(tu_i) + \sum_{i=m+1}^{n} \nu(tu_i) \geq 1 + \sum_{i=1}^{m} \nu(u_i) = \nu(u)$$

   and

$$\nu(tu) = 1 + \sum_{i=1}^{m} \nu((tu)_i) + \sum_{i=m+1}^{n} \nu(t_i) \geq 1 + \sum_{i=1}^{m} \nu(u_{\rho i}) + \sum_{i=m+1}^{n} \nu(t_i) = \nu(t)$$

   because ρ is a permutation of m and $\sum_{i=1}^{m} \nu(t_{\rho i}) = \sum_{j=1}^{m} \nu(t_j)$. Finally

$$\begin{aligned}
\nu(tu) &= 1 + \sum_{i=1}^{m} \nu((tu)_i) + \sum_{i=m+1}^{n} \nu(t_i) \leq 2 + \sum_{i=1}^{m} (\nu(u_i) + \nu(t_{\rho i})) + \sum_{i=m+1}^{n} \nu(t_i) \\
&= \left(1 + \sum_{i=1}^{m} \nu(u_i)\right) + \left(1 + \sum_{i=1}^{m} \nu(t_{\rho i}) + \sum_{i=m+1}^{n} \nu(t)_i\right) \\
&= \nu(u) + \left(1 + \sum_{j=1}^{m} \nu(t_j) + \sum_{i=m+1}^{n} \nu(t_i)\right) = \nu(u) + \nu(t),
\end{aligned}$$

   because ρ is a permutation of m.

b. Assume $m > 0$. Since $(tu)_i = u_i t_{\rho i}$ for every $1 \leq i \leq m$ and $t_{\rho i} = \bar{\epsilon}$ for every $\rho i > n$, then $(tu)_i = u_i$ for every i such that $\rho i > n$. Let $A = \{i : \rho i > n, 1 \leq i \leq m\}$ and $B = \{i : \rho i \leq m, 1 \leq i \leq n\}$. Then we have:

$$\nu(tu) = 1 + \sum_{i \in A} \nu((tu)_i) + \sum_{i \in B} \nu((tu)_i) = 1 + \sum_{i \in A} \nu(u_i) + \sum_{i \in B} \nu((tu)_i),$$

so that

$$\nu(tu) \geq 1 + \sum_{i \in A} \nu(u_i) + \sum_{i \in B} \nu(u_i) = \nu(u)$$

and

$$\nu(tu) \geq 1 + \sum_{i \in B} \nu((tu)_i) = 1 + \sum_{i \in B} \nu(t_{\rho i}) = \nu(t),$$

because ρ is a permutation of m and $\sum_{i \in B} \nu(t_{\rho i}) = \sum_{j=1}^{n} \nu(t_j)$. Finally,

$$\nu(tu) \leq 2 + \sum_{i \in A} \nu(u_i) + \sum_{i \in B} \nu(u_i) + \sum_{i \in B} \nu(t_{\rho i}) = \nu(u) + 1 + \sum_{i \in B} \nu(t_{\rho i}) = \nu(u) + \nu(t)$$

because $\sum_{i \in B} \nu(t_{\rho i}) = \sum_{j=1}^{n} \nu(t_j)$.

4. The first equality is proved by induction on t. Since $(tt^*)_i = (t_{\pi^{-1}i})^* t_{\pi^{-1}i}$ and $(t^*t)_i = t_i (t_i)^*$, by induction hypothesis $\nu(\mathrm{sk}(t_i)) = \nu(t_i t_i^*)$. The second follows from (2). ◀

Proof of Lemma 31. First of all observe that, since $t \prec u$ implies $t \leq u$, then $\pi = \rho$. If $u = \bar{\epsilon}$, then t is idempotent. If t has more than two nodes, it can be easily seen that there is $z \in \mathrm{FPT}$ such that $t < z < \bar{\epsilon}$. Then t has two nodes and (ii) holds. If $u \neq \bar{\epsilon}$, then we have two cases: (a) $n = m$ and (b) $n > m$.

(a) If $n = m$, then $t_i \leq u_i$ for every $1 \leq i \leq n$ and it must exist k such that $t_k < u_k$. If there is $w \neq t_k, u_k$ such that $t_k < w < u_k$, then $t < \langle \pi; t_1, \ldots, t_{k-1}, w, t_{k+1}, \ldots, t_n \rangle < u$, leading to a contradiction. Therefore, $t_k \prec u_k$. If there is another $j \neq k$ such that $t_j \prec u_j$, then $t < \langle \pi; t_1, \ldots, t_j, \ldots, t_{k-1}, u_k, t_{k+1}, \ldots, t_n \rangle < u$, leading again to a contradiction. In conclusion, (i) holds.

(b) If $n > m$, similarly, (ii) has to hold, otherwise by Lemma 18 there would be $z \in \mathrm{FPT}$ such that $t < z < u$.

We prove the converse. Assume that (i) holds and that $t \leq w \leq u$ for some $w = \langle \pi; w_1, \ldots, w_p \rangle \in \mathrm{FPT}$. We immediately get that either $w = t$ or $w = u$. Assume that (ii) holds and that $t \leq w \leq u$. We have two cases: $n = p$ or $n = p + 1$. In the first case we have $t = w$, because $t_i \leq w_i \leq u_i = t_i$ for every $1 \leq i \leq m$ and $t_n = \bar{\epsilon} \leq w_n$. In the second case $w = u$, because $t_i = u_i \leq w_i \leq u_i$ for every $1 \leq i \leq m = p$. ◀

Proof of Lemma 34. Let $t := \langle \pi; t_1, \ldots, t_n \rangle$ and $u := \langle \rho; u_1, \ldots, u_m \rangle$.

1. We prove that there is $f \in E_{\mathrm{fin}}$ such that $uf = t$ by induction on $e \in E_{\mathrm{fin}}$. If $e = \bar{\epsilon}$, then $f = \bar{\epsilon}$. Let $e = \langle \iota; e_1, \ldots, e_k \rangle$ with $1 \leq k \leq n$. Assume that $m \geq k$; the other case is similar. If $t = eu$ then $\pi = \rho$ and $t_i = u_i e_{\rho i}$ for $1 \leq i \leq k$ and $t_i = u_i$ for $k < i \leq n$. As $e_{\rho i} \in E_{\mathrm{fin}}$ for every $1 \leq i \leq k$, there are $f_1, \ldots, f_k \in E_{\mathrm{fin}}$ such that $t_i = f_i u_i$. Let $f := \langle \iota; f_1, \ldots, f_k \rangle$; then $uf = \langle \pi; f_1 u_1, \ldots, f_k u_k, u_{k+1}, \ldots, u_m \rangle = t$.

2. The proof is by coinduction on $\leq_{\mathrm{fin}}$. Let $t = ue$ for some $e \in E_{\mathrm{fin}}$. Then $\pi = \rho$ and $n = \max\{m, \mathrm{rd}(e)\} \geq m$. Moreover, as $t_i = e_i u_i$, for $1 \leq i \leq m$, then by (1), there are $f_1, \ldots, f_m \in E_{\mathrm{fin}}$ such that $t_i = u_i f_i$ for $1 \leq i \leq m$. As $t_i = e_i$ for $m < i \leq n$ the conclusion follows from the coinductive hypothesis.

3. The proof is by induction on $f \in E_{\text{fin}}$. If $f = \bar{\epsilon}$, then $t = \bar{\epsilon} \in E_{\text{fin}}$. Let $f = \langle \iota; f_1, \ldots, f_m \rangle$, $m \geq 1$, such that $f \leq_{\text{fin}} t$. Then $t = \langle \iota; t_1, \ldots, t_n \rangle$ with $n \leq m$ and $f_i \leq_{\text{fin}} t_i$ for $1 \leq i \leq n$. As $f_i \in E_{\text{fin}}$, $t_i \in E_{\text{fin}}$ for every $1 \leq i \leq n$, so that $t \in E_{\text{fin}}$. ◀

Proof of Proposition 36. If t is idempotent, (1) is trivial and (2) follows from Lemma 34(3). Otherwise let $t = \langle \pi; t_1, \ldots, t_n \rangle$ and let k be the greatest i such that $t_k \notin E_{\text{fin}}$ or $\pi k \neq k$, and $t_i \in E_{\text{fin}}$, $\pi i = i$ for every $k < i \leq n$. Then $t^{\mathfrak{m}'} = \langle \pi; (t_1)^{\mathfrak{m}'}, \ldots, (t_k)^{\mathfrak{m}'} \rangle$. As to (1), by coinduction, $t_i \leq (t_i)^{\mathfrak{m}'}$ for $1 \leq i \leq k$, and $t_i \in E_{\text{fin}}$ for $k < i \leq n$, so that $t \leq t^{\mathfrak{m}'}$. Concerning (2), let $u = \langle \rho; u_1, \ldots, u_m \rangle$ such that $t \leq u$. Without loss of generality, we assume that $n = m$, so that $k \leq m$. Moreover, we have $\pi = \rho$, so that by coinductive hypothesis, $u_i \leq (t_i)^{\mathfrak{m}'}$ for $1 \leq i \leq k$. The fact that $t_i \leq u_i$ and $t_i \in E_{\text{fin}}$ for $k < i \leq n$, implies by Lemma 34(3) that $u_i \in E_{\text{fin}}$ for $k < i \leq m$. We conclude that $u \leq t^{\mathfrak{m}'}$. ◀

Proof of Lemma 37. Let $t = \langle \pi; t_1, \ldots, t_n \rangle$, $u = \langle \rho; u_1, \ldots, u_m \rangle$ and $v = \langle \tau; v_1, \ldots, v_k \rangle$. By definition of $\leq_{\text{fin}}$ we have $\pi = \rho = \tau$ and $k \leq n, m$. Without loss of generality we assume $n \leq m$. We define $w := \langle \pi; w_1, \ldots, w_m \rangle$ where w_i is

- for $1 \leq i \leq k$ a $\leq_{\text{fin}}$-lower bound of t_i and u_i obtained coinductively;
- for $k < i \leq n$, $w_i := t_i u_i$;
- for $n < i \leq m$, $w_i := u_i$.

Now we verify that $w \leq_{\text{fin}} t$:

- for $1 \leq i \leq k$ $w_i \leq t_i$ by coinductive hypothesis;
- for $k < i \leq n$, $w_i = t_i u_i \leq_{\text{fin}} t_i$ by Lemma 34.
- for $n < i \leq m$, $w_i = u_i \in E_{\text{fin}}$.

Similarly, $w \leq_{\text{fin}} u$, proving the claimed result. ◀