Classification of Non-Redundancy of Boolean Predicates of Arity 4

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Abstract

Given a constraint satisfaction problem (CSP) predicate $P \subseteq D^r$, the non-redundancy (NRD) of P is the maximum-sized instance on n variables such that for every clause of the instance, there is an assignment which satisfies all clauses but that one. The study of NRD for various CSPs is an active area of research which combines ideas from extremal combinatorics, logic, lattice theory, and other techniques. Complete classifications are known in the cases $r = 2$ and $(|D| = 2, r = 3)$.

In this paper, we give a near-complete classification of the case $(|D| = 2, r = 4)$. Of the 400 distinct non-trivial Boolean predicates of arity 4, we implement an algorithmic procedure which perfectly classifies 397 of them. Of the remaining three, we solve two by reducing to extremal combinatorics problems – leaving the last one as an open question. Along the way, we identify the first Boolean predicate whose non-redundancy asymptotics are non-polynomial.

2012 ACM Subject Classification Theory of computation → Constraint and logic programming

Keywords and phrases constraint satisfaction problem, redundancy

Digital Object Identifier 10.4230/LIPIcs.CP.2026.8

Related Version *Full Version*: `https://arxiv.org/abs/2603.21353`

Supplementary Material *Software*: `https://github.com/jbrakensiek/Arity-4-NRD`
archived at `swh:1:dir:748465b0d7f28aa1b0e7c1e37fd74d69487fe301`

Funding *Joshua Brakensiek*: Supported in part by the Simons Investigator award of Venkatesan Guruswami and NSF grants CCF-2211972 and DMS-2503280.
Venkatesan Guruswami: Supported in part by a Simons Investigator award and NSF award CCF-2211972.
Aaron Putterman: Supported in part by the Simons Investigator Awards of Madhu Sudan and Salil Vadhan and AFOSR award FA9550-25-1-0112.

Acknowledgements We thank Santhoshini Velusamy and Amatya Sharma for valuable conversations which indirectly led to fixing some typos in Appendix A.

1 Introduction

In the study of constraint satisfaction problems (CSPs), an often-studied question is to find succinct representations of an input CSP instance such that some essential property of the original instance is preserved. The most common such name for this task is that of *sparsification*, although many of these formulations have subtly different objectives. For example, one can seek efficient transformations of an instance such that the transformed instance is satisfiable if and only if the input instance is (even if one cannot efficiently

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determine if the instance is satisfiable in the first place) [14, 23, 13, 31, 11]. A seemingly unrelated line of work seeks to (sparsely) reweight the clauses of an input CSP instance such that the fraction of clauses satisfied by every assignment is approximately preserved [26, 4, 2, 37, 20, 30, 16, 36, 10, 12, 25, 22, 24, 32, 29, 27, 28, 9].

A common thread between these two lines of work is the core of *non-redundancy* (NRD). The systematic study of NRD (and a related concept called *chain length*) was initiated by Bessiere, Carbonnel, and Katsirelos [5] towards understanding the query complexity of CSPs in the partial membership query model [6]. The essence of non-redundancy is that it measures the maximal amount of "logical independence" the clauses a given CSP can have (see Definition 2). Later, Carbonnel [11] observed the utility of NRD in the study of CSP kernelization and further conjectured that the non-redundancy of any CSP is (approximately) an upper bound[1] on the kernelization of any CSP. Very recently, Brakensiek and Guruswami [7] gave a direct link between combinatorial non-redundancy and the construction of (weighted) CSP sparsifiers, showing that the approaches of these disparate communities are (for the most part) working towards the same fundamental questions, often with overlapping results.

An example of these overlapping results is in the systematic classification of the non-redundancy of certain families of CSP predicates. For example, both Bessiere, Carbonnel, and Katsirelos [5] and Butti and Živný [10] (see also [16]) classified all predicates of arity 2, that is predicates of the form $P \subseteq D^2$ for some finite domain D. Although the classifications by these two works were in rather different settings, the dichotomies were identical:[2] if P can define via a gadget reduction (more precisely an fgpp-reduction [11]) the predicate $\mathrm{OR}_2 := \{00, 01, 10\}$, then P has quadratic non-redundancy (where the growth rate is measured by the number of variables n in the instance). Otherwise, P has linear non-redundancy.

In a similar vein, both Chen, Jansen, and Pieterse [13] and Khanna, Putterman, and Sudan [28] (in effect) fully classified the non-redundancy of every non-trivial[3] ternary Boolean predicate $P \subseteq \{0,1\}^3$. The gadget reduction is very simple. If P lacks a gadget reduction to OR_2, then P has linear non-redundancy. Otherwise, if P has a gadget reduction to OR_2 but not to $\mathrm{OR}_3 := \{0,1\}^3 \backslash \{000\}$, then P has quadratic non-redundancy. Otherwise, P has cubic non-redundancy. In particular, every such predicate has as its non-redundancy a polynomial function of the number of variables. This led Khanna, Putterman, and Sudan [28] to conjecture that for their notion of CSP sparsifiability, the dependence on the number of variables is always a polynomial with an integral exponent.

However, such "integral optimism" turns out to be far from the truth. Both Bessiere, Carbonnel, and Katsirelos [5] and Carbonnel [11] gave examples of predicates with natural non-redundancy upper bounds which were not integral polynomial functions of the number of variables, although they could not give lower bounds which proved the non-polynomial dependence is necessary. However, a few years later, Brakensiek and Guruswami [7] identified an explicit predicate $P \subseteq \{0,1,2\}^3$ whose non-redundancy as a function of the number of variables n lies between $\Omega(n^{1.5})$ and $O(n^{1.6} \log n)$. Other predicates with fractional exponents were identified by Brakensiek, Guruswami, Jansen, Lagerkvist, and Wahlström [8], including showing for every rational number $q \geqslant 1$ there exists a finite arity r and a finite domain D such that some $P \subseteq D^r$ has non-redundancy growing as $\Theta(n^q)$.

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[1] The reason non-redundancy can only be an upper bound is that tractable CSPs (such as 2-SAT) have trivial kernels, although one can still study their NRD as a combinatorial quantity.

[2] We present the classification as it appears in [5], however due to a different "satisfiability convention" the classification in [10] is subtly different. In particular, Butti and Živný [10] show if P has a gadget reduction to $\mathbf{AND}_2 := \{11\}$, then P has quadratic sparsifiability; otherwise P has near-linear sparsifiability. These differing conventions each appear in various works, but for simplicity of exposition, we stick with the convention used by Bessiere, Carbonnel, and Katsirelos [5] which is consistent with the CSP kernelization literature. See Brakensiek and Guruswami [7] for further discussion.

[3] We ignore the cases $P = \varnothing$ and $P = \{0,1\}^3$ as these trivially have constant non-redundancy.

However, all such examples involve domains D of size at least three. In particular, there are *no known examples* of Boolean predicates $P \subseteq \{0,1\}^r$ whose non-redundancy fails to be an integral polynomial, thus leaving the conjecture of Khanna, Putterman, and Sudan [28] open in this setting. That said, this conjecture is still generally believed to be false in the Boolean setting due to the existence of a notorious predicate[4] $\mathsf{BoolBCK} \subseteq \{0,1\}^9$ independently identified by many different research groups [13, 31, 28], all of which derive from a non-Boolean ternary predicate identified by Bessiere, Carbonnel, and Katsirelos [5] – see [7, 8] for further discussion and analysis.

1.1 Our Contributions

Note, however, that there is a large gap between the fully classified arity-3 predicates and this suspected-fractional predicate of arity 9. The focus of this paper is to systematically understand Boolean predicates of arity 4. In particular, using a computer search, we enumerate all 400 non-trivial[5] Boolean predicates $P \subseteq \{0,1\}^4$. To analyze these predicates, we implemented various techniques for both lower-bounding and upper-bounding the non-redundancy of P. In particular, for the lower bound, we checked the largest r such that P has a gadget reduction to OR_r. For the upper bound, we used lattice-based techniques [13, 28] to identify low-degree polynomial representations of P. With only these two techniques, we were able to perfectly classify the non-redundancy of 397 of the 400 predicates, giving an integral exponent for each of them. The other three identified predicates are as follows (numbers derive from the table in Appendix A).

$$R_{181} = \{0000, 0001, 0010, 0100, 0111, 1000, 1011, 1101, 1111\}$$
$$R_{299} = \{0000, 0001, 0010, 0011, 0100, 0101, 1000, 1110, 1111\}$$
$$R_{317} = \{0000, 0001, 0010, 0011, 0100, 0101, 1000, 1001, 1110\}.$$

For each $R \in \{R_{181}, R_{299}, R_{317}\}$, our initial techniques show that $\mathrm{NRD}(R,n) \in [\Omega(n^2), O(n^3)]$, so further techniques are needed to resolve the non-redundancy of these predicates. For R_{317}, we prove that $\Theta(n^3)$ is the correct answer. However, the lower bound does not follow from a gadget reduction to OR_3 but rather from the structure of *linear 3-uniform hypergraphs*; see Section 4.1.

For R_{299}, we prove that $\mathrm{NRD}(R_{299}, n)$ does *not* grow as a polynomial function of n, the first Boolean example of this phenomenon. This is also the first example of a predicate whose non-redundancy is asymptotically $n^{s-o(1)}$, where s is a rational number. More precisely, we prove the following.

▶ **Theorem 1.** $\mathrm{NRD}(R_{299}, n) \in \left[\frac{n^3}{2^{O(\sqrt{\log(n)})}}, \frac{n^3}{2^{\Omega(\log^*(n))}}\right]$, *where* $\log*(n)$ *denotes the iterated logarithm.*

The proof proceeds by showing that the non-redundancy of R_{299} is closely linked to the optimal answer to the Rusza-Szemeredi problem of packing induced matchings in a graph. Such a method was used in [5] to upper bound the non-redundancy of a predicate $\mathsf{BCK} \subseteq \{0,1,2\}^3$, but this upper bound approach was later established to not be tight [7, 8]. In contrast, we identify a predicate in which the Rusza-Szemeredi analysis is (essentially) tight. See Section 4.2 and Theorem 12 for further details.

[4] A succinct description of this predicate is the representation of the symmetric group S_3 over GL(3) with the identity matrix removed [7].

[5] Note that non-redundancy is unchanged under certain operations of the predicate, such as negating a bit or swapping two coordinates. This is why we are able to work with 400 predicates instead of $2^{2^4} = 65536$.

For R_{181}, we could not get an improved bound on its non-redundancy. However, we show that R_{181} is closely related to a *symmetric* Boolean predicate of arity 5 which also remains unresolved by known methods, see Section 4.3 for further discussion.

1.2 Outline

In Section 2, we formally define non-redundancy as well as give many known properties of non-redundancy (and related notions such as conditional non-redundancy). In Section 3, we describe in detail the procedure we used to systematically classify the 400 Boolean predicates of arity 4. In Section 4, we discuss in detail the three exceptional predicates. In Section 5, we conclude the paper with some thoughts and open questions. In Appendix A, we provide a table with the full 400-predicate classification.

2 Preliminaries

Our notation is adapted from Brakensiek and Guruswami [7]. We define a *predicate* (or *relation*) to be a subset $P \subseteq D^r$, where D is the *domain* of the predicate and r is the *arity* of the predicate. In most situations, we assume that $D = \{0, 1\}$ – that is, P is a *Boolean* predicate. We say that a predicate is *trivial* if $P = \varnothing$ or $P = D^r$, otherwise the predicate is *non-trivial*.

Given an arity r predicate P, an *instance* of $\mathrm{CSP}(P)$ is an r-uniform hypergraph $H := (V, E)$ where V is the set of *variables* of the instance and $E \subseteq V^r$ is the set of *constraints*. In general, a constraint may use the same variable multiple times, however we usually assume that the instance is r-partite that the variables V have a partitioning $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_r$ such that $E \subseteq V_1 \times V_2 \times \cdots \times V_r$. We can even assume that $|V_1| = \cdots = |V_r|$ (assuming $|V|$ is a multiple of r).

An *assignment* to H is a map $\psi : V \to D$, where D is the domain of the predicate P. A constraint $(x_1, \ldots, x_r) \in E$ is *P-satisfied* (or just *satisfied*) by ψ if $(\psi(x_1), \ldots, \psi(x_r)) \in P$. Otherwise, the constraint is *P-unsatisfied* (or just *unsatisfied*). We now define the key notion of non-redundancy.

▶ **Definition 2.** *An instance $H := (V, E)$ is a* non-redundant *instance of $\mathrm{CSP}(P)$ if for every $e \in E$ there is an assignment $\psi_e : V \to D$ such that $e' \in E$ is satisfied by ψ_e if and only if $e \neq e'$. We let $\mathrm{NRD}(P, n)$ denote the maximum value of $|E|$ among all non-redundant instances (V, E) of $\mathrm{CSP}(P)$ with $|V| = n$.*

As an example, if we let $\mathrm{OR}_r := \{0, 1\}^r \backslash \{0^r\}$, then $\mathrm{NRD}(\mathrm{OR}_r, n) = \binom{n}{r}$ [16, 11, 28] as the complete r-uniform hypergraph on n vertices is non-redundant: for each $S \in \binom{[n]}{r}$, assign $\psi_S(v) = \mathbf{1}[v \notin S]$. Note that our assumption that H is r-partite only affects quantities like the non-redundancy of the predicate up to a factor of $O_r(1)$ [7].

2.1 Conditional Non-redundancy

Another crucial notion formally defined by Brakensiek and Guruswami [7] (although implicit in the work of Bessiere, Carbonnel, and Katsirelos [5]) is the notion of *conditional* non-redundancy.

▶ **Definition 3.** *Given predicates $P \subsetneq Q \subseteq D^r$, we say that an r-partite hypergraph $H := (V, E)$ is a* non-redundant *instance of $\mathrm{CSP}(P \mid Q)$ if for every $e \in E$ there is an assignment $\psi_e : V \to D$ (called a witness) such that $e' \in E$ is P-satisfied by ψ_e if and only if $e \neq e'$*

and further e is $Q\backslash P$-satisfied by ψ_e. We let $\mathrm{NRD}(P \mid Q, n)$ *denote the maximum value of* $|E|$ *among all non-redundant instances* (V, E) *of* $\mathrm{CSP}(P \mid Q)$ *with* $|V| = n$*. Note that* $\mathrm{NRD}(P \mid Q, n) \leqslant \mathrm{NRD}(P, n)$*, with equality when* $Q = D^r$*.*

We make use of the following fact about conditional non-redundancy.

▶ **Proposition 4** (Triangle Inequality [7])**.** *Let* $P \subsetneq Q \subsetneq R \subseteq D^r$ *be predicates, and let* n *be a positive integer. Then,* $\mathrm{NRD}(P \mid R, n) \leqslant \mathrm{NRD}(P \mid Q, n) + \mathrm{NRD}(Q \mid R, n)$.

We also make use of the following conditional non-redundancy trick due to Brakensiek et al. [8], where in some circumstances we can reduce the non-redundancy of a predicate of arity r to a conditional NRD problem of arity $r - 1$. We need the trick in more generality than their use case, so we give a full proof.

▶ **Proposition 5.** *Assume* $\{0, 1\} \subseteq D$*. Let* $R \subseteq D^4$ *be a predicate such that* $R = (Q \times \{0\}) \cup (P \times \{1\})$ *with* $P \subseteq Q \subseteq D^3$*. Then,*

$$\mathrm{NRD}(P \mid Q, n/2) \cdot \Omega(n) \leqslant \mathrm{NRD}(R, n) \leqslant \mathrm{NRD}(P \mid Q, n) \cdot O(n) + \mathrm{NRD}(Q \times \{0, 1\}).$$

Proof. We first prove the lower bound on $\mathrm{NRD}(R, n)$. Assume n is even, and let (V, E) be a non-redundant instance of $\mathrm{CSP}(P \mid Q)$ with $V = [n/2]$ and $E \subseteq V^3$, with witnessing assignments $\psi_e : [n/2] \to D$ for all $e \in E$. Let (V', E') be an instance of $\mathrm{CSP}(R)$ with $V' = [n]$ and $E' = E \times \{n/2 + 1, \ldots, n\}$.

We claim that (V', E') is a non-redundant instance of $\mathrm{CSP}(R)$. To see why, fix a constraint $e := (x_1, x_2, x_3) \in E$ of (V, E) and variable $x_4 \in \{n/2 + 1, \ldots, n\}$ and consider $f := (x_1, x_2, x_3, x_4) \in E'$. Our witness $\psi_f : [n] \to D$ is defined by $\psi_f \restriction [n/2] = \psi_e$, $\psi_f(x_4) = 1$ and $\psi_f(y) = 0$ for all $y \in \{n/2 + 1, \ldots, n\}\backslash\{y\}$. Since $R = (P \times \{1\}) \cup (Q \times \{0\})$, we have that ψ_f satisfies all constraints of Y except f.

For the upper bound, first note that by Proposition 4, we have that $\mathrm{NRD}(R, n) \leqslant \mathrm{NRD}(R \mid Q \times \{0, 1\}, n) + \mathrm{NRD}(Q \times \{0, 1\}, n)$. Thus, it suffices to prove that $\mathrm{NRD}(R \mid Q \times \{0, 1\}, n) \leqslant \mathrm{NRD}(P \mid Q, n) \cdot O(n)$. To see why, let $(V = [n], E)$ be a non-redundant instance of $\mathrm{CSP}(R \mid Q \times \{0, 1\})$, and let $E' \subseteq E$ be a subset of edges such that $|E'| \geqslant |E|/n$ and all clauses $e \in E'$ have the same last coordinate $x \in [n]$. Let $E'' := \{(x_1, x_2, x_3) : (x_1, x_2, x_3, x) \in E\}$, so $|E''| = |E'| \geqslant |E|/n$. We claim that (V, E'') is a non-redundant instance of $\mathrm{CSP}(P \mid Q)$, proving $|E| \leqslant E''|n \leqslant \mathrm{NRD}(P \mid Q, n) \cdot n$, as desired.

For each $f := (x_1, x_2, x_3, x) \in E$, we have a witness $\psi_f : [n] \to D$ such that $\psi_f(f) \in (Q\backslash P) \times \{1\}$, so $\psi_f(x) = 1$. Thus, the condition that $\psi_f(f') \in R$ for all other $f' \in E$ implies $\psi_f(f') \in P \times \{0\}$. Therefore, if we use ψ_f as the witness for $(x_1, x_2, x_3) \in E''$, we have that $\psi_f(e) \in P$ unless $e = (x_1, x_2, x_3)$. Thus, (V, E'') is a non-redundant instance of $\mathrm{CSP}(P \mid Q)$. ◀

2.2 Balanced Predicates and Polynomial Representations

In general, the best known[6] method for establishing that $\mathrm{NRD}(P, n) = O(n)$ for some predicate $P \subseteq D^r$ is checking if P has what is known as a *Malt'sev embedding* [13, 31, 5]. However, due to a recent result of Brakensiek et al. [8], for Boolean predicates P, such criteria is equivalent to the much simpler criteria of checking whether P is *balanced* (e.g., [13]).

[6] It is a major open question to establish if a Mal'tsev embedding is indeed the only reason a predicate can have linear (or even near-linear) non-redundancy [13, 5, 31, 11, 7, 8].

▶ **Definition 6** (e.g., [13]). *A Boolean predicate $P \subseteq \{0,1\}^r$ is* balanced *if for all odd integers $m \geqslant 1$ and all $t_1, \ldots, t_m \in P$, we have that*

$$t_1 - t_2 + t_3 - \cdots + t_m \in \{0,1\}^r \implies t_1 - t_2 + t_3 - \cdots + t_m \in P,$$

where addition and subtraction is done over $\mathbb{Z}^r$.

Given a Boolean predicate P, determining whether P is balanced can be done using a lattice-based algorithm (e.g., [13, 28]), see Section 3.3 for more details. However, *verifying* whether P is balanced can be done much more efficiently using a linear polynomial certificate. In particular, Chen, Jansen, and Pieterse (Theorems 4.10 and 4.11 of [13]) show that $P \in \{0,1\}^r$ is balanced if and only if there exists integers $c_0, c_1, \ldots, c_r, q$ such that

$$(x_1, \ldots, x_r) \in P \iff c_1 x_1 + \cdots + c_r x_r \equiv c_0 \mod q.$$

This is equivalent to a Mal'tsev embedding of P in $\mathbb{Z}/q\mathbb{Z}$ (see Theorem 7.6 of [13]).

More generally, a method of establishing $\mathrm{NRD}(P, n) = O(n^d)$ for some $d \geqslant 1$ is to find a modulus q and coefficients $\{c_S \in \mathbb{Z} : S \subseteq [r], |S| \leqslant d\}$ such that

$$(x_1, \ldots, x_r) \in P \iff \sum_{\substack{S \subseteq [r] \\ |S| \leqslant d}} c_S \prod_{i \in S} x_i \equiv 0 \mod q.$$

See [23, 13, 31] for a proof of this fact[7] and further discussion.

3 Experiment Methodology

The primary contribution of this paper is a comprehensive classification of the non-redundancy of Boolean predicates of arity 4. In general, the techniques build on the previous classifications of Boolean predicates of arity 3 [13, 28]; however due to the sheer number of predicates of arity 4 (400 total), our entire verification pipeline needs to be automated. In this section, we describe the high-level technical details of implementing this classification algorithm. The full classification is available in Appendix A with sufficient detail that hand-verification is possible. At the highest level, our classification algorithm (implemented[8] in Python and SageMath [38, 33]) consists of four components.

1. Enumeration of Boolean predicates of arity 4 with symmetry-breaking.
2. Calculation of NRD lower bounds using OR projections.
3. Calculation of NRD upper bounds using lattice-based methods.
4. Certifying NRD upper bound by extracting a polynomial equation.

We now discuss the primary technical ideas behind each of these steps.

3.1 Enumeration of Boolean Predicates of Arity 4

Enumerating all Boolean predicates of arity 4 is the most straightforward (and standard) component of our algorithm. Given that there are only $2^{2^4} = 65536$ possible predicates, a brute-force enumeration is practical. However, it is easy to observe that the non-redundancy

[7] To be truly pedantic, the notion of sparsification considered in [23, 13, 31] is not formally equivalent to non-redundancy; however as noted by (e.g., [8]) the proof methods extend to the non-redundancy setting with no modification.

[8] A copy of our code is available at `https://github.com/jbrakensiek/Arity-4-NRD`. In terms of verifying correctness, Appendix A includes sufficient details to make that possible by hand.

of a predicate $P \subseteq \{0,1\}^4$ is invariant under swaps of the coordinates as well as bit-flipping individual coordinates. For example, if $P = \{1000, 0100\}$, bit-flipping the first coordinate and swapping the middle two coordinates yields the predicate $P' = \{0000, 1010\}$ with the same non-redundancy. Thus, we can substantially reduce the number of predicates by picking the lexicographically first predicate in each equivalence class.

Although we implemented our symmetry-breaking entirely by hand, the total number of distinct nontrivial Boolean predicates we found, 400, precisely matches previous enumerations of such predicates in the literature by Hast [21] in the context of the Max CSP problem and Austrin, Håstad and Martinsson [1] in the context of the Promise CSP problem. These 400 predicates are enumerated from 0 to 399 in Appendix A. For example, in our encoding of the predicates, the lexicographically earliest representation of $P = \{1000, 0100\}$ is $P'' = \{0000, 0011\}$ which is predicate number 3 in the table.

3.2 NRD Lower Bound: OR Projections

A well-known non-redundancy bound is that the predicate $\mathrm{OR}_k := \{0,1\}^k \backslash \{0^k\}$ satisfies $\mathrm{NRD}(\mathrm{OR}_k, n) = \Theta_k(n^k)$ [14, 16, 10, 13, 31, 11, 28]. Such an NRD lower bound can be transferred to an arbitrary Boolean predicate $P \subseteq \{0,1\}^r$ using a suitable gadget reduction. Following the notation of Khanna, Putterman, and Sudan [28], we say that OR_k is a *projection* of $P \subseteq \{0,1\}^r$ if there exists literals $\ell_1, \ldots, \ell_r \in \{0, 1, x_1, \ldots, x_k, \bar{x}_1, \ldots, \bar{x}_k\}$ such that

$$\mathrm{OR}_k(x_1, \ldots, x_k) = P(\ell_1, \ldots, \ell_r), \tag{1}$$

where we interpret the predicates as Boolean functions $\mathrm{OR}_k : \{0,1\}^k \to \{0,1\}$ and $P : \{0,1\}^r \to \{0,1\}$ in the standard manner. In this case, we can deduce that $\mathrm{NRD}(P, n) = \Omega(\mathrm{NRD}(\mathrm{OR}_k, n)) = \Omega_k(n^k)$. These "OR projections" are a special case of a much more general family of methods known as functionally guarded primitive positive (fgpp) reductions [11, 8].

In our setting, we have $r = 4$ and $k \in \{1, 2, 3, 4\}$. Thus, for our 400 predicates, we can efficiently enumerate over all choices of literals $\ell_1, \ldots, \ell_4$ and check if Equation (1) is satisfied. In Appendix A, we provide an explicit choice of literals certifying to our claimed lower bound; note that such a certificate need not be unique in general.

3.3 NRD Upper Bound: Lattice Methods

Next, we discuss our algorithmic techniques for upper-bounding the non-redundancy of a given predicate $P \subseteq \{0,1\}^4$. As a warm up, we discuss testing whether $\mathrm{NRD}(P, n) = O(n)$. In practice, this balanced condition is easy to check using a Lattice algorithm. First, consider map $e_1 : \{0,1\}^r \to \mathbb{Z}^{r+1}$ defined by $e_1(x_1, \ldots, x_r) = (1, x_1, \ldots, x_r)$. We can then define a lattice $\Lambda_{P,1}$ corresponding to a predicate $P \subseteq \{0,1\}^r$ as follows.

$$\Lambda_{P,1} := \left\{ \sum_{t \in P} z_t e_1(t) \mid \forall t \in P, z_t \in \mathbb{Z} \right\}.$$

We can then observe the following fact (implicit in previous works [13, 31, 28, 8]).

▶ **Proposition 7.** *P is balanced iff for all $t \in \{0,1\}^r$, $e_1(t) \in \Lambda_{P,1}$ implies $t \in P$.*

Proof. For all $t' \in \{0,1\}^r$, the condition $e_1(t') \in \Lambda_{P,1}$ implies there exists integers $z_t^{t'} \in \mathbb{Z}$ for all $t \in P$ such that $e_1(t') = \sum_{t \in P} z_t^{t'} e_1(t)$. Since the first coordinate of $e_1(t')$ is always 1, we have that $\sum_{t \in P} z_t^{t'} = 1$. Let $m = \sum_{t \in P} |z_t^{t'}|$, which we observe is odd. Then, there exists a

choice of $t_1, \ldots, t_m \in P$ such that each $t \in P$ appears exactly $|z_t^{t'}|$ times. Furthermore, if $z_t^{t'} \geqslant 1$, then t appears only in odd positions and if $z_t^{t'} \leqslant -1$, then t appears only in even positions. In other words, we have that $e_1(t') = e_1(t_1) - e_1(t_2) + e_1(t_3) - \cdots + e_1(t_m)$. Hence, $t' = t_1 - t_2 + t_3 - \cdots + t_m$. Therefore, if P is balanced, we have that $t' \in P$, so $e_1(t') \in \Lambda_P$ implies $t' \in P$.

Conversely, if P is not balanced, then there are $t_1, \ldots, t_m \in P$ such that $t' := t_1 - t_2 + \cdots + t_m \in \{0,1\}^r \backslash P$. As such, it is clear that $e_1(t') = e_1(t_1) - e_1(t_2) + \cdots + e_1(t_m) \in \Lambda_P$. Thus, $e_1(t') \in \Lambda_P$ does not imply $t' \in P$. ◀

Using Proposition 7, we have a straightforward algorithm for checking if our predicate P is balanced. We use the IntegralLattice library of SageMath [38, 33] to build the lattice Λ_P. More precisely, we build a matrix M whose rows are $e_1(t)$ for all $t \in P$. We compute the Hermite Normal Form of this matrix and extract the nonzero rows to get the basis B of Λ_P. Using B, we build an IntegralLattice library which support membership queries. Thus, we can test for $t \in \{0,1\}^r$ whether $e_1(t) \in \Lambda_P$. If the test succeeds iff $t \in P$, then we know that P is balanced and so $\mathrm{NRD}(P, n) = O(n)$.

Higher Degrees

So far, we have discussed a method of checking if $\mathrm{NRD}(P, n) = O(n)$. However, as discussed in Section 2.2 we can certify that $\mathrm{NRD}(P, n) = O(n^d)$ using a degree d polynomial. More precisely, for any $d \geqslant 1$, let $\binom{r}{\leqslant d} := \sum_{i=0}^{d} \binom{r}{i}$ and consider the mapping $e_d : \{0,1\}^r \to \{0,1\}^{\binom{r}{\leqslant d}}$ defined by

$$e_d(x_1, \ldots, x_r) = \left(\prod_{i \in S} x_i : \forall S \subseteq [r], |S| \leqslant d \right).$$

Note that e_1 coincides with our previous definition. For any $d \geqslant 1$ and predicate $P \subseteq \{0,1\}^d$, we can then build a lattice $\Lambda_{P,d}$ generated by $\{e_d(t) : t \in P\}$. We then have the following analogue of Proposition 7

▶ **Proposition 8.** *If for all $t \in \{0,1\}^r$, $e_d(t) \in \Lambda_{P,d}$ implies $t \in P$, then* $\mathrm{NRD}(P, n) = O(n^d)$.

Proof. Let $R := \Lambda_{P,d} \cap \{0,1\}^{\binom{r}{\leqslant d}}$. Since $e_d(t) \in R$ for all $t \in P$, we have that $\Lambda_{P,d} \subseteq \Lambda_{R,1}$. Since $R \subseteq \Lambda_{P,d}$, we have that $\Lambda_{R,1} \subseteq \Lambda_{P,d}$. Thus, $\Lambda_{R,1} = \Lambda_{P,d}$. Thus, $R = \Lambda_{R,1} \cap \{0,1\}^{\binom{r}{\leqslant d}}$, so R is balanced by Proposition 7. Therefore, there exists a finite q and coefficients $c_S \in \mathbb{Z}/q\mathbb{Z}$ for all $S \subseteq [r], |S| \leqslant d$ such that

$$x \in R \iff \sum_{\substack{S \subseteq [r] \\ |S| \leqslant d}} c_S x_S \equiv 0 \mod q.$$

In particular, for all $t \in \{0,1\}^r$, we have that $e_d(t) \in R$ (that is, $t \in P$) iff

$$\sum_{\substack{S \subseteq [r] \\ |S| \leqslant d}} c_S \prod_{i \in S} t_i \equiv 0 \mod q.$$

In other words, P is the zeros of a degree d polynomial over $\mathbb{Z}/q\mathbb{Z}$. Therefore, by [23, 13, 31], we have that $\mathrm{NRD}(P, n) = O(n^d)$. ◀

Using Proposition 8, we may test for arbitrary bounds of the form $O(n^d)$ by testing when Proposition 8 holds for all $d \in [r]$. These degree bounds for all 400 predicates appear in Appendix A.

3.4 Extracting a Polynomial Certificate

Unlike the OR projections, the lattice algorithm does not immediately yield a succinct certificate of the claimed non-redundancy upper bound. However, by adapting the methods of Chen, Jansen, and Pieterse [13] and Khanna, Putterman, and Sudan [28], we can extract a suitable polynomial for which hand-verification becomes feasible. We concern ourselves with the task of verifying whether P is balanced – the extension of this technique to higher degrees follows with minimal adaptation.

Recall that $\Lambda_{P,1} \subseteq \mathbb{Z}^{r+1}$ is the lattice generated by $\{e_1(t) : t \in P\}$. We let $\Lambda_{P,1} \otimes \mathbb{Q} \subseteq \mathbb{Q}^{r+1}$ denote the vector space over $\mathbb{Q}$ generated by $\Lambda_{P,1}$. Recall that to verify if P is balanced, we must check for every $t \in \{0,1\}^d \backslash P$ that $e_1(t) \notin \Lambda_{P,1}$. This can happen for two reasons: either (1) $e_1(t) \notin \Lambda_{P,1} \otimes \mathbb{Q}$, or (2) $e_1(t) \in \Lambda_{P,1} \otimes \mathbb{Q}$ but $e_1(t) \notin \Lambda_{P,1}$.

The first case is equivalent to the existence of a vector $y \in \mathbb{Q}^{d+1}$ such that $\langle y, \Lambda_{P,1} \rangle = 0$ but $\langle y, e_1(t)) \rangle \neq 0$. By picking a suitable positive integer N such that Ny has integral entries as well as a positive integer q which does not divide $\langle Ny, e_1(t) \rangle$, then the equation $\langle Ny, e_1(t') \rangle \equiv 0 \mod q$ includes all $t' \in P$ but excludes $t' = t$.

In the second case, we can construct a similar map, by looking at the dual lattice

$$\Lambda_{P,1}^* := \{y \in \mathbb{Q}^{d+1} \mid \forall x \in \Lambda_{P,1}, \langle y, x \rangle \in \mathbb{Z}\},$$

and finding a map $y \in \Lambda_{P,1}^*$ such that $\langle y, e_1(t) \rangle \notin \mathbb{Z}$. If $\langle y, e_1(t) \rangle = p/q$ with $p, q \in \mathbb{Z}$ and pick positive N such that Ny has integral entries then $Nq\langle y, e_1(t') \rangle \equiv 0 \mod Nq$ for all $t' \in P$ but $Nq\langle y, e_1(t) \rangle \not\equiv 0 \mod Nq$.

In both cases, we have found a linear polynomial which has all elements of P as zeros but not t. In Khanna, Putterman, and Sudan [28], the authors aggregate these linear polynomials for all $t \in \{0,1\}^r \backslash P$ to get a system[9] of linear equations witnessing P is balanced. However, as predicated by [13], we found (with a suitable brute-force search over small moduli) a single linear equation excluding all $t \in \{0,1\}^r \backslash P$ lying in Case (1) as well as a single linear equation excluding all $t \in \{0,1\}^r \backslash P$ lying in Case (2). Given the freedom in Case (1) of choosing the modulus q, we could pick the two equations to be over relatively prime moduli, ensuring by the Chinese Remainder Theorem can be combined into a single equation. This extracted linear equation for each of the 400 predicates is stated in Appendix A.

4 The Three Exceptional Predicates

As previously mentioned, the algorithmic techniques in Section 3 perfectly classified the non-redundancy of 397 of the 400 predicates. In this section, we study the three exceptional predicates, namely Predicates 181, 299, and 317, providing tighter theoretical analyses of their NRD for two of them than would otherwise be guaranteed by our classification algorithm. We start with the analysis of Predicates 317 and 299 which we provide tight bounds for, and conclude with the discussion of Predicate 181, which we leave as an open problem.

4.1 Predicate 317

Predicate 317 has satisfying assignments

$$R_{317} = \{0000, 0001, 0010, 0011, 0100, 0101, 1000, 1001, 1110\}.$$

[9] Equivalently, this system can be represented as a single linear equation over an Abelian group.

From Appendix A, we see that $x \in \{0,1\}^4$ satisfies $x \in R_{317}$ if and only if $x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_4 \equiv 0 \mod 3$. Therefore, $\mathrm{NRD}(R_{317}, n) = O(n^3)$. We thus seek to prove that $\mathrm{NRD}(R_{317}, n) = \Omega(n^3)$. First, observe that

$$R_{317} = \{000, 001, 010, 100, 111\} \times \{0,1\} - \{1111\}.$$

Thus, setting $P_{317} = \{000, 001, 010, 100\}, Q_{317} = P_{317} \cup \{111\}$, we have $R_{317} = (Q_{317} \times \{0\}) \cup (P_{317} \times \{1\})$. By Proposition 5, we know that $\mathrm{NRD}(P_{317} \mid Q_{317}, n/2) \cdot \Omega(n) \leqslant \mathrm{NRD}(R_{317}, n)$.

We now proceed to a *lower bound* on $\mathrm{NRD}(P_{317} \mid Q_{317}, n)$. For this, we use the following constructive property of linear hypergraphs:

▶ **Theorem 9** (See, for instance [15]). *There exist tripartite linear hypergraphs on vertex set $[n] \times [n] \times [n]$ with $\Omega(n^2)$ edges.*

With this, we show:

▶ **Lemma 10.** $\mathrm{NRD}(P_{317} \mid Q_{317}, n) = \Omega(n^2)$.

Proof. We consider a tripartite linear hypergraph $H = (V, E)$ as guaranteed by Theorem 9, with $|E| = \Omega(n^2)$. We claim that this hypergraph is non-redundant, thus yielding the lemma. To see why this hypergraph is non-redundant, we consider any hyperedge $e \in E$, along with the assignment ψ_e which maps $\psi_e(x) = 1$ if $x \in e$, and otherwise maps variables to 0. Note then that $\psi_e(e) = 111$, as we desire. It remains only to show that $\psi_e(e') \in \{000, 001, 010, 100\}$ for any $e' \neq e$. This follows because $|e' \cap e| \leqslant 1$, and so at least two variables in e' are mapped to 0 by ψ_e. This concludes the proof. ◀

Thus, we are now able to prove the following:

▶ **Theorem 11.** $\mathrm{NRD}(R_{317}, n) = \Theta\left(n^3\right)$.

Proof. By Proposition 5, we know that $\mathrm{NRD}(P_{317} \mid Q_{317}, n/2) \cdot \Omega(n) \leqslant \mathrm{NRD}(R_{317}, n)$. Plugging in Lemma 10 then yields $\mathrm{NRD}(R_{317}, n) = \Omega(n^3)$. Since R_{317} has a cubic polynomial representation, we know that $\mathrm{NRD}(R_{317}, n) = O(n^3)$, completing the proof. ◀

4.2 Predicate 299

Predicate 299 has satisfying assignments

$$R_{299} = \{0000, 0001, 0010, 0011, 0100, 0101, 1000, 1110, 1111\}.$$

In this section, we show the following theorem:

▶ **Theorem 12.** $\mathrm{NRD}(R_{299}, n) \geqslant \frac{n^3}{2^{O(\sqrt{\log(n)})}}$ *and* $\mathrm{NRD}(R_{299}, n) \leqslant \frac{n^3}{2^{\Omega(\log^* (n))}}$.

Towards proving this theorem, observe that we can write

$$R_{299} = \{000, 001, 010, 100, 111\} \times \{0,1\} - \{1001\}.$$

Now, let us set $P_{299} = \{000, 001, 010, 111\}$ and $Q_{299} = P_{299} \cup \{100\}$. Immediately, we can observe that $R_{299} = (Q_{299} \times \{0\}) \cup (P_{299} \times \{1\})$. Thus, by Proposition 5, we know that

$$\begin{aligned} \mathrm{NRD}(P_{299} \mid Q_{299}, n/2) \cdot \Omega(n) &\leqslant \mathrm{NRD}(R_{299}, n) \\ &\leqslant \mathrm{NRD}(P_{299} \mid Q_{299}, n) \cdot O(n) + \mathrm{NRD}(Q_{299} \times \{0,1\}). \end{aligned}$$

Next, we show that $Q_{299} \times \{0,1\}$ has non-redundancy $O(n^2)$:

▷ Claim 13. $\mathrm{NRD}(Q_{299} \times \{0,1\}, n) = O(n^2)$.

Proof. $Q_{299} \times \{0,1\}$ corresponds to R_{318} in Appendix A. The upper bound of $O(n^2)$ follows from the fact that for any $x \in \{0,1\}^4$, we have that $x \in Q_{299} \times \{0,1\}$ if and only if $x_1x_2 + x_1x_3 + x_2x_3 \equiv 0 \mod 3$. ◁

We make an additional observation that simplifies our notation going forward: because NRD is unchanged when permuting and negating variables in a predicate, we are free to re-define $P'_{299} = \{001, 011, 101, 110\}$ and $Q'_{299} = P'_{299} \cup \{000\}$, while ensuring that $\mathrm{NRD}(P'_{299} \mid Q'_{299}, n) = \mathrm{NRD}(P_{299} \mid Q_{299}, n)$. Note that here we have simply negated the first bit, and then swapped the first bit with the third bit. Now, our goal is to bound $\mathrm{NRD}(P'_{299} \mid Q'_{299}, n)$.

4.2.1 Upper Bounding the Conditional Non-Redundancy

Before bounding the conditional non-redundancy of these predicates, we first recall the statement of Fox's triangle removal lemma [18]:

▶ **Theorem 14** (Fox's Triangle Removal Lemma, simplified). *Let G be a graph on n vertices with at most cn^2 many triangles, where c is an absolute constant. Then, there exists a set of $\frac{n^2}{2^{\Omega_c(\log^*(n))}}$ many edges that can be removed such that G becomes triangle-free.*

▶ **Lemma 15.** $\mathrm{NRD}(P'_{299} \mid Q'_{299}, n) \leqslant \frac{n^2}{2^{\Omega(\log^*(n))}}$.

Proof. We show that $\mathrm{NRD}(P'_{299} \mid Q'_{299}, 3n) \leqslant \frac{n^2}{2^{\Omega(\log^*(n))}}$, from which the stated lemma follows.

Let $E = \{e_1 \dots e_m\} \subseteq [n] \times [n] \times [n]$ denote the constraints which are a non-redundant instance of $P'_{299} \mid Q'_{299}$ (at a constant factor loss in size, we may assume that that non-redundant instances are tripartite). That is, for each constraint $e_i \in E$, there exists a corresponding assignment $\psi_{e_i} \in \{0,1\}^n$ such that $\psi_{e_i}(e_i) = 000$, and for all other $e_j \neq e_i$, we have that $\psi_{e_i}(e_j) \in \{001, 011, 101, 110\}$.

Now, we consider the following construction: we create an auxiliary graph G with vertex set $A = [n]$, $B = [n]$ and $C = [n]$, and for every edge $e_i = (u, v, w)$, we add an edge from $u \in A$ to $v \in B$, from $u \in A$ to $w \in C$, and from $v \in B$ to $w \in C$. We define these triangles (u, v, w) to be **canonical triangles** and note that their number is $|E|$. As our goal is to bound $|E|$, we will show that one can remove all triangles in G (and in particular all the canonical triangles) by deleting few edges, and that each edge on average cannot remove too many canonical triangles.

For ease of analysis later, we will also use the following subgraphs: for a choice of $c \in [n]$, we let $E[c]$ denote the graph which contains edges $\{e - c : e \in \{e_1 \dots e_m\}, e[3] = c\}$ (in words, we are taking all constraints which have variable c in the third position, and considering the graph they induce on $A \cup B$). We only include those vertices in this graph which touch one of these edges. Importantly, we have the following two claims about these induced graphs:

▷ Claim 16. Let $E = \{e_1 \dots e_m\} \subseteq [n] \times [n] \times [n]$ denote the (ordered) constraints which form a witness to the non-redundancy of $P'_{299} \mid Q'_{299}$, and let $E[c]$ be defined as above for $c \in [n]$. Then, $E[c]$ is a matching.

▷ Claim 17. Let $E = \{e_1 \dots e_m\} \subseteq [n] \times [n] \times [n]$ denote the (ordered) constraints which form a witness to the non-redundancy of $P'_{299} \mid Q'_{299}$, and let $E[c]$ be defined as above for $c \in [n]$. Then, for any $c \neq c' \in [n]$, $V(E[c])$ induces at most 2 edges of $E[c']$.

We defer the proofs of these claims for now, but, as we shall see, they suffice for deriving our ultimate bound on the size of our starting non-redundant instance. Towards this end, we first use the above claims to show that the graph G has only $O(n^2)$ many triangles. To see why, consider any triple of vertices $a \in A, b \in B, c \in C$ which form a triangle (note that no triangle can have two vertices in any of A, B, C, as there are no internal edges in these parts). If a, b, c form a triangle, this means that there must be edges $(a, b), (b, c), (a, c)$ in the graph G. There are two cases for us to consider: Either (a) The edge $(a, b) \in E[c]$. Then, this trivially means that $(a, b), (b, c), (a, c) \in G$ which form the aforementioned canonical triangles. Otherwise (b) the edge $(a, b) \notin E[c]$. Then, it must be the case that there is some edge $(a, y) \in E[c]$, some edge $(x, b) \in E[c]$, and some c' for which $(a, b) \in E[c']$. It may even be the case that $x = a$ and $y = b$ (i.e., $E[c']$ and $E[c]$ may have shared edges). Importantly, this means that the edge (a, b) from $E[c']$ is *induced* by $V(E[c])$. We call these **non-canonical triangles**.

Now we bound the number of these triangles. First, we see that the number of canonical triangles in G is bounded by n^2 because $\sum_{c\in[n]} |E[c]| \leqslant n \cdot \max_{c\in[n]} |E[c]| \leqslant n^2$, as by Claim 16, we know that each $E[c]$ is a matching (and so has at most n edges).

Next, we bound the number of non-canonical triangles. The key observation here, as remarked above, is that any non-canonical triangle contains an edge $(a, b) \in E[c']$ which is induced by $V(E[c])$. To bound the number of non-canonical triangles, we see (by Claim 17):

$$\sum_{c\in[n]} \sum_{c'\neq c\in[n]} \left|\{e \in E[c'] : e \text{ induced by } V(E[c])\}\right| \leqslant \sum_{c\in[n]} \sum_{c'\neq c\in[n]} 2 \leqslant 2n^2.$$

So, for this graph G, we know that G has at most $3n^2$ triangles. By Fox's triangle removal lemma (Theorem 14), we then know that there exists a set of $\frac{n^2}{2^{\Omega(\log^*(n))}}$ many edges that can be removed from G such that the resulting graph is triangle-free.

Importantly, this implies that removing $\frac{n^2}{2^{\Omega(\log^*(n))}}$ many edges from G removes *all* of the canonical triangles from the graph G. Let E' denote this set of edges which are deleted from G. We now analyze how many *canonical triangles* are removed by the removal of each edge $e \in E'$, call it ℓ_e.

1. Suppose that a deleted edge in E' is of the form (a, c) or (b, c) (WLOG we consider (a, c)). In this case, the only canonical triangles which are lost are those which correspond to edges in $E[c]$ which contain the vertex a. But, because $E[c]$ is a matching (by Claim 16), we know that there is at most one edge in $E[c]$ which contains a. Thus, there is at most one canonical triangle which is lost. Thus, we set $\ell_{(a,c)} = 1$.
2. Suppose a deleted edge in E' is of the form (a, b). In this case, a canonical triangle (a, b, c) is lost if and only if $(a, b) \in E[c]$. Thus, the number of canonical triangles which is removed is exactly $\ell_{(a,b)}$: the number of choices of $c \in [n]$ such that $(a, b) \in E[c]$.

Now, by Claim 17, we know that every pair of labels $c, c' \in [n]$, the edge sets $E[c], E[c']$ have at most two edges in common. Hence,

$$2 \cdot \binom{n}{2} \geqslant \sum_{e\in E'} \binom{\ell_e}{2} \geqslant |E'| \cdot \binom{\frac{1}{|E'|} \cdot \sum_{e\in E'} \ell_e}{2},$$

where the second inequality above is simply using convexity. The above then implies that $|E'| \cdot \frac{\left(\sum_{e\in E'} \ell_e\right)^2}{|E'|^2} \leqslant 2 \cdot \binom{n}{2} \leqslant n^2$, or equivalently, that $\sum_{e\in E'} \ell_e \leqslant n \cdot \sqrt{|E'|}$. By the above, we know that $|E'| = \frac{n^2}{2^{\Omega(\log^*(n))}}$, and so this immediately implies that $\sum_{e\in E'} \ell_e \leqslant \frac{n^2}{2^{\Omega(\log^*(n))}}$. To conclude, we have shown that deleting the set of edges E' removes all triangles from the graph G. At the same time, this process removes at most $\frac{n^2}{2^{\Omega(\log^*(n))}}$ canonical triangles from the graph. Together, this means that G has at most $\frac{n^2}{2^{\Omega(\log^*(n))}}$ many canonical triangles, and so the number of constraints in our starting non-redundant CSP is at most $\frac{n^2}{2^{\Omega(\log^*(n))}}$. ◀

Now we prove our auxiliary claims. We start by proving Claim 16.

Proof of Claim 16. As before, let the constraints in the non-redundant CSP instance be $E = \{e_1, \dots e_m\}$, and consider $E[c]$ for any $c \in [n]$. Now, for any edge $(a, b) \in E[c]$, we know that (a, b, c) is a constraint in our CSP instance. Importantly, because our CSP instance is non-redundant, we know there must exist an assignment $\psi \in \{0, 1\}^n$ such that $\psi(a, b, c) = 000$, while for every other constraint e', $\psi(e') \in \{001, 011, 101, 110\}$. Now, in order for $\psi(a, b, c) = 000$, this means that all of $\psi(a), \psi(b), \psi(c) = 0$. Now, suppose for the sake of contradiction that there were any other edge in $E[c]$ which shared a vertex with (a, b) (WLOG assume a), and denote this edge by (a, b'). Then, under the assignment ψ, we see that $\psi(a, b', c) = 0 \circ \psi(b') \circ 0 \notin \{001, 011, 101, 110\}$ for any choice of $\psi(b')$. Thus, this instance could not be a witness to the non-redundancy of $P'_{299} \mid Q'_{299}$. So, in any non-redundant instance of $P'_{299} \mid Q'_{299}$, it must be the case that each vertex in $E[c]$ has degree $\leqslant 1$ (i.e., a matching). ◁

Proof of Claim 17. Consider any $c, c' \in [n]$, and let us look at $E[c]$ and $E[c']$. Now, suppose for the sake of contradiction that there are three edges $e_1 = (a_1, b_1), e_2 = (a_2, b_2), e_3 = (a_3, b_3)$ from $E[c']$ which are induced by $V(E[c])$. Because there are three distinct edges, and $E[c']$ is a matching, it must be the case that there are three distinct left vertices a_1, a_2, a_3 and likewise three distinct right vertices b_1, b_2, b_3.

Now, let us consider the edge $e \in E[c]$ such that $a_1 \in e$, along with the assignment ψ which is the non-redundant certificate for this edge e. By definition, it must then be the case that $\psi(a_1) = 0$. Importantly, this now forces the value of $\psi(c')$: indeed, conditioned on the first variable being 0, the only value of the third variable which is in P'_{299} is 1. Thus, we know that $\psi(c') = 1$.

At the same time, we know that for every variable x in $V(E[c]) - e$, it must be the case that $\psi(x) = 1$. This is because the only tuple which is in P'_{299} with final variable 0 is 110 (and by the above, we know that $\psi(c) = 0$, effectively forcing these other assignments). To reach a contradiction then, we know that because a_1, a_2, a_3 are distinct, and b_1, b_2, b_3 are distinct, one of (a_2, b_2) or (a_3, b_3) must *not* intersect e (WLOG let us say that it is (a_3, b_3)). But, because (a_3, b_3) is induced by $V(E[c])$, this means that $a_3, b_3 \in V(E[c])$, and so $\psi(a_3) = \psi(b_3) = 1$. At the same time, know that $\psi(c') = 1$, which means that $\psi(a_3, b_3, c') = 111$ which is *not* in P'_{299}. Thus, this violates the fact that we started with an NRD instance.

So, it must be that $V(E[c])$ induces at most 2 edges from $E[c']$. ◁

4.2.2 Lower Bounding the Conditional Non-Redundancy

We now provide a lower bound on the conditional non-redundancy $\mathrm{NRD}(P'_{299} \mid Q'_{299})$. For our construction, we will make use of the fascinating *Ruzsa-Szemeredi* graphs [35].

▶ **Definition 18.** *We say that a graph $G = (V, E)$ on n vertices is an (r, t)-RS graph if E can be partitioned in a set of t matchings $M_1, \dots M_t$, each of size r such that for every $i \in [t]$, M_i is an* induced *matching in G.*

The key fact we will use is the following.

▶ **Theorem 19** ([3, 17])**.** *There exist bipartite (r, t)-RS graphs on vertex set $|L| = |R| = n$ with $t = \Omega(n)$ and $r \geqslant \frac{n}{2^{O(\sqrt{\log n})}}$.*

With this in hand, we now show the following.

▶ **Lemma 20.** $\mathrm{NRD}(P'_{299} \mid Q'_{299}, n) \geqslant \frac{n^2}{2^{O(\sqrt{\log n})}}$.

Proof. First, we fix a (bipartite) (r,t)-RS graph on $2n$ vertices (vertex set L, R) with $t = \Omega(n)$ and $r = \frac{n}{2^{O(\sqrt{\log(n)})}}$ as guaranteed by Theorem 19. To build our NRD instance, we now add t distinguished variables $z_1, \dots z_t$, along with $2n$ variables $x_1, \dots x_n, y_1, \dots y_n$ for the vertices in the graph. Now, we add a constraint of $\{x_u, y_v, z_i\}$ for each matching $M_i : i \in [t]$, $e = (u,v) \in M_i$ (where we understand that P'_{299} operates on x_u, y_v, z_i, where $e = (u,v)$). Thus, the total number of constraints in the resulting CSP (denote this CSP by C) is exactly $|E| = t \cdot r = \frac{n^2}{2^{O(\sqrt{\log n})}}$.

It remains only to show that C is a (conditionally) non-redundant instance. To see this, let us fix a single constraint $P'_{299}(e \cup \{z_i\})$ for $e = (u,v) \in M_i$, with the understanding that $u \in L$, $v \in R$. We must show that there is an assignment $\psi : \{x_1, \dots x_n, y_1, \dots y_n, z_1, \dots z_t\} \to \{0,1\}$ such that $\psi(e \cup \{z_i\}) = 000$, while for every other constraint $\psi(e' \cup \{z_j\}) \in \{001, 011, 101, 110\}$.

For this, we consider the assignment ψ such that

$$\psi(z_i) = 0 \qquad\qquad \forall j \neq i \in [t] : \psi(z_j) = 1,$$

$$\psi(x_a) = \begin{cases} 0 \text{ if } a = u \\ 1 \text{ if } a \in L \cap V(M_i) - \{u\} \\ 0 \text{ if } a \in L - V(M_i) \end{cases}, \quad \psi(y_b) = \begin{cases} 0 \text{ if } b = v \\ 1 \text{ if } b \in R \cap V(M_i) - \{v\} \\ 0 \text{ if } b \in R - V(M_i) \end{cases}.$$

By construction, we know that $\psi(e \cup \{z_i\}) = 000$. So, it remains only to show that $\psi(e' \cup \{z_j\}) \in \{001, 011, 101, 110\}$ for every $e' \in M_j$. We do this by cases:

1. Suppose $e' \in M_i$, where M_i is the same matching that contains e. Observe then that $e' \cap e = \varnothing$ as M_i is a matching. This means that both vertices in e' are given value 1 by the assignment ψ, i.e., $\psi(e') = 11$. By construction, we know that $\psi(z_i) = 0$, so $\psi(e' \cup \{z_j\}) = 110$.
2. Suppose $e' \in M_j$ for $j \neq i$. Then, we know that $\psi(z_j) = 1$ by construction. Likewise, we know that $|e' \cap V(M_i)| \leqslant 1$, as M_i was an *induced* matching in the graph G (which implies that every edge $e' \in G - M_i$ has at least one vertex outside of $V(M_i)$). Thus, $\psi(e') \in \{00, 01, 10\}$. Thus, $\psi(e' \cup \{z_j\}) \in \{001, 011, 101\}$, as we desire. ◀

Thus, we are now able to prove Theorem 12.

Proof of Theorem 12. By Proposition 5, we know that

$$\mathrm{NRD}(P_{299} \mid Q_{299}, n/2) \cdot \Omega(n)$$

$$\leqslant \mathrm{NRD}(R_{299}, n) \leqslant \mathrm{NRD}(P_{299} \mid Q_{299}, n) \cdot O(n) + \mathrm{NRD}(Q_{299} \times \{0,1\}).$$

Plugging in Lemma 15, Lemma 20 and Claim 13, then yields the desired claim. ◀

4.3 Predicate 181 and Connections to Hypergraph Intersection Problems

Predicate 181 has satisfying assignments

$$R_{181} = \{0000, 0001, 0010, 0100, 0111, 1000, 1011, 1101, 1111\}.$$

From Appendix A we know that $\mathrm{NRD}(R_{181}, n) \in [\Omega(n^2), O(n^3)]$. More precisely, for the lower bound, we can observe that $R_{181}(0, 0, \bar{x}_1, \bar{x}_2) = \mathrm{OR}_2(x_1, x_2)$. For the upper bound, R_{181} has a polynomial representation of

$$2x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 + 2x_1x_2x_4 \mod 3.$$

However, we are unable to further classify the non-redundancy of R_{181}.

Consider the predicate $P_{181} = \{x \in \{0,1\}^5 : |x| \in \{0,2,3\}\}$. Immediately, we can observe that predicate $Q_{181} = \{x \in \{0,1\}^5 : |x| \in \{0,2,3,5\}\}$ satisfies $\mathrm{NRD}(Q_{181}, n) = O(n^2)$, as Q_{181} is exactly the zeros of the degree polynomial $f(x) = |x| \cdot (|x| - 2) \mod 3$.

Thus, by Proposition 4, we know that

$$\mathrm{NRD}(P_{181}, n) \leqslant \mathrm{NRD}(P_{181} \mid Q_{181}, n) + \mathrm{NRD}(Q_{181}, n) \leqslant \mathrm{NRD}(P_{181} \mid Q_{181}, n) + O(n^2).$$

So understanding the NRD of P_{181} reduces to understanding the NRD of $P_{181} \mid Q_{181}$ (up to an additive error of $O(n^2)$).

A priori, it may seem mysterious that we mention this arity 5 predicate P_{181} in the discussion of Predicate 181. However, we can consider the restriction of P_{181} that is achieved by forcing $x_4 = x_5$: this then yields satisfying assignments

$$(P_{181})_{x_4=x_5} = \{0000, 0001, 0110, 1100, 1010, 1001, 0101, 0011, 1110\}.$$

Even further, if we now negate the last variable, we see that:

$$(P_{181})_{x_4=x_5,\neg x_4} = \{0001, 0000, 0111, 1101, 1011, 1000, 0100, 0010, 1111\}$$

$$= \{0000, 0001, 0010, 0100, 0111, 1000, 1011, 1101, 1111\} = R_{181}$$

Thus, by using Proposition 4, we have:

▷ Claim 21. $\mathrm{NRD}(R_{181}, n) \leqslant \mathrm{NRD}(P_{181}, n) \leqslant \mathrm{NRD}(P_{181} \mid Q_{181}, n) + \mathrm{NRD}(Q_{181}, n) \leqslant \mathrm{NRD}(P_{181} \mid Q_{181}, n) + O(n^2)$.

Therefore, one promising approach towards bounding $\mathrm{NRD}(R_{181}, n)$ is via an upper bound on $\mathrm{NRD}(P_{181} \mid Q_{181}, n)$.

To better understand $\mathrm{NRD}(P_{181} \mid Q_{181}, n)$, we consider the hypergraph interpretation of this quantity: any hypergraph $H = (V, E)$ which constitutes a non-redundant instance for $\mathrm{NRD}(P_{181} \mid Q_{181}, n)$ must satisfy the following property: for each $e \in E$ (each e is of size 5) there is an assignment $\psi_e : V \to \{0,1\}$ such that $(\psi_e)(e) = 11111$, but for every other $e' \neq e \in E$, that $\mathrm{wt}(\psi_e(e')) \in \{0,2,3\}$.

A natural approach towards building such non-redundant hypergraphs is to consider the set of assignments $\psi_e(x_i) = \mathbf{1}[i \in e]$, i.e., where ψ_e only maps variables in e to 1, and all other variables are mapped to 0. In this setting, we can observe that a hypergraph is non-redundant *if and only if* for every pair of hyperedges e, e', $|e \cap e'| \in \{0,2,3\}$. This motivates a natural *extremal hypergraph intersection problem*: *What is the maximum number of hyperedges in a hypergraph $H = (V, E)$ on n vertices of arity 5 such that for every pair of hyperedges e, e', $|e \cap e'| \in \{0,2,3\}$?*

This style of question is studied exactly in the work of Deza, Erdös, and Frankl [15], where they showed that any such hypergraph must *necessarily have* $O(n^2)$ many hyperedges. In turn, this immediately implies that the "canonical" set of assignments ψ_e defined above cannot create non-redundant instances of size larger than n^2. Note that an upper bound of $O(n^3)$ is standard by the linear algebra/polynomial method of Ray-Chaudhuri, Wilson [34] (and prime modulus versions in [19]). Deza *et al* [15] provide a powerful refinement of this approach to derive a tighter $O(n^2)$ bound, which we can hope to emulate in our CSP setting.

5 Conclusion and Open Questions

In this paper, we classified the non-redundancy of 399 of 400 Boolean predicates of arity 4. We leave the remaining case R_{181} as our main open question. The most notable case we classified, R_{299}, is the first example of a predicate Boolean or otherwise whose non-redundancy is

provably not of the form $\Theta(n^\beta)$ for some rational number β. Does there exist a Boolean predicate (of arity $r \geqslant 4$) whose non-redundancy is $\Theta(n^\beta)$ for some $\beta \in \mathbb{Q}\backslash\mathbb{N}$? We know that β cannot lie in the range $(r-1, r)$ [13, 11, 28]. Does there exist any Boolean predicate whose non-redundancy lies in the range $[\omega(n), o(n^2)]$?

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A Full table of results

In this appendix, we present the complete classification of Boolean predicates of arity 4. We give a brief explanation of each column of the table.

- **Predicate.** The specific predicate $P \subseteq \{0,1\}^4$ written in binary in the following ordering:

$$(0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111).$$

- **OR.** The largest $k \in [4]$ such that some projection of P is equal to OR_k.
- **OR Certificate.** A specific projection of P certifying the OR degree.
- **Deg.** TA minimal degree of a polynomial representation of P.
- **Polynomial.** A polynomial representation of P. We use x_S as shorthand for $\prod_{i \in S} x_i$.

Num.	Predicate	OR	OR Cert.	Deg	Polynomial
0	1000000000000000	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + x_3 + x_2 + x_1 \mod 5$
1	1000000000000001	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + x_3 + x_2 + 2x_1 \mod 5$
2	1000000100000000	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + x_3 + 5x_2 + 3x_1 \mod 7$
3	1001000000000000	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + 6x_3 + 2x_2 + 2x_1 \mod 7$
4	1001000000000100	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + 4x_3 + 2x_2 + 2x_1 \mod 5$
5	1001000000000110	1	$(0,0,0,\bar{x}_1)$	1	$6x_4 + 6x_3 + x_2 + 5x_1 \mod 12$
6	1001000000001000	2	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,\bar{x}_2)$	2	$x_4 + x_3 + x_{34} + x_2 + 2x_1 + x_{14} + x_{13} \mod 3$
7	1001000000001001	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + 4x_3 + 2x_2 + 3x_1 \mod 5$
8	1001010000000000	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + 4x_3 + 4x_2 + 3x_1 \mod 5$
9	1001010000000010	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + 4x_3 + 4x_2 + 2x_1 \mod 5$
10	1001010000100000	2	$(\bar{x}_1,\bar{x}_2,\bar{x}_1,\bar{x}_2)$	2	$x_4 + x_3 + x_{34} + x_2 + x_{24} + x_1 + x_{13} \mod 3$
11	1001010000101000	2	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,\bar{x}_2)$	2	$x_4 + x_3 + x_{34} + x_2 + x_{24} + x_1 + x_{13} + x_{12} \mod 3$
12	1001010000101001	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + 2x_3 + 2x_2 + x_1 \mod 3$
13	1001010001000000	1	$(0,0,0,\bar{x}_1)$	1	$x_4 + 4x_3 + 4x_2 + 4x_1 \mod 5$
14	1001010001000010	1	$(0,0,0,\bar{x}_1)$	1	$4x_4 + 2x_3 + 2x_2 + 2x_1 \mod 6$
15	1001011000000000	1	$(0,0,0,\bar{x}_1)$	1	$3x_4 + 3x_3 + 3x_2 + 4x_1 \mod 6$

Num.	Predicate	OR	OR Cert.	Deg	Polynomial
16	1001011001000000	2	$(\bar{x}_1, x_1, \bar{x}_2, x_2)$	2	$x_4 + x_3 + x_{34} + 2x_2 + x_1 + x_{14} + 2x_{13} + 2x_{12} \mod 3$
17	1001011001100000	2	$(\bar{x}_1, \bar{x}_2, x_1, x_2)$	2	$x_4 + x_3 + x_{34} + 2x_2 + 2x_1 \mod 3$
18	1001011001101000	2	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_2)$	2	$x_4 + x_3 + 3x_{34} + x_2 + 3x_{24} + 3x_{23} + x_1 + 3x_{14} + 3x_{13} + 3x_{12} \mod 5$
19	1001011001101001	1	$(0, 0, 0, \bar{x}_1)$	1	$3x_4 + 3x_3 + 3x_2 + 3x_1 \mod 6$
20	1100000000000000	1	$(0, 0, \bar{x}_1, 0)$	1	$x_3 + x_2 + x_1 \mod 5$
21	1100000000000010	2	$(\bar{x}_1, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + 2x_{23} + x_1 + x_{14} + 2x_{13} + 2x_{12} \mod 3$
22	1100000000000011	1	$(0, 0, \bar{x}_1, 0)$	1	$x_3 + x_2 + x_1 \mod 3$
23	1100001000000000	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{24} + x_{23} + 2x_1 + x_{13} + x_{12} \mod 3$
24	1100001000000001	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{24} + x_{23} + x_1 + 2x_{13} + 2x_{12} \mod 3$
25	1100001000000010	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{24} + x_{23} + x_1 + 2x_{12} \mod 3$
26	1100001000010000	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{24} + x_{23} + 2x_1 + 2x_{14} + x_{13} + x_{12} \mod 3$
27	1100001000010010	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + 2x_2 + 2x_{24} + 2x_1 + 2x_{14} + x_{13} \mod 3$
28	1100001000100000	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{24} + x_{23} + x_1 + x_{14} + x_{13} \mod 3$
29	1100001000100001	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + 2x_2 + 2x_{24} + 2x_1 + 2x_{14} \mod 3$
30	1100001000100100	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_{34} + x_2 + x_{23} + x_1 + x_{14} + x_{13} \mod 3$
31	1100001000100101	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{24} + x_{23} + x_1 + x_{14} + x_{13} + 2x_{12} \mod 3$
32	1100001000101000	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + 2x_2 + 2x_{24} + 2x_1 + 2x_{14} + 2x_{12} \mod 3$
33	1100001000101001	2	$(0, \bar{x}_1, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{24} + 3x_{23} + x_1 + 2x_{14} + 3x_{13} + 3x_{12} \mod 5$
34	1100001100000000	1	$(0, 0, \bar{x}_1, 0)$	1	$x_3 + 4x_2 + 2x_1 \mod 5$
35	1100001100100000	2	$(\bar{x}_1, 0, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{23} + x_1 + x_{14} + x_{13} + 2x_{12} \mod 3$
36	1100001100100100	2	$(\bar{x}_1, 0, \bar{x}_1, \bar{x}_2)$	2	$x_3 + 2x_2 + x_1 + x_{14} + x_{13} + 2x_{12} \mod 3$
37	1100001100101000	2	$(\bar{x}_1, 0, \bar{x}_1, \bar{x}_2)$	2	$x_3 + x_2 + x_{23} + 2x_1 + 2x_{14} \mod 3$
38	1100001100110000	1	$(0, 0, \bar{x}_1, 0)$	1	$x_3 + 2x_2 + 2x_1 \mod 3$
39	1100001100111000	2	$(1, \bar{x}_1, x_1, \bar{x}_2)$	2	$x_3 + x_{34} + 2x_2 + 4x_{24} + 2x_{23} + 3x_1 + 4x_{14} + x_{13} \mod 5$
40	1100001100111100	1	$(0, 0, \bar{x}_1, 0)$	1	$3x_3 + 3x_2 + 3x_1 \mod 6$
41	1110000000000000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_1 + 2x_{12} \mod 3$
42	1110000000000001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_1 \mod 3$
43	1110000000000100	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_1 + 2x_{14} + x_{12} \mod 3$
44	1110000000000101	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + 2x_2 + 2x_{24} + 2x_1 + 2x_{14} + 2x_{13} + x_{12} \mod 3$
45	1110000000000110	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_1 + x_{14} + x_{13} \mod 3$
46	1110000000000111	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_1 + 2x_{14} + 2x_{13} + x_{12} \mod 3$
47	1110000000001000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_1 + 2x_{14} + 2x_{13} \mod 3$
48	1110000000001001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_1 + x_{14} + x_{13} + x_{12} \mod 3$
49	1110000000001100	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{24} + x_{23} + 2x_1 + 2x_{14} \mod 3$
50	1110000000001101	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_1 + 2x_{13} \mod 3$
51	1110000000001110	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_1 + x_{12} \mod 3$
52	1110000100000000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + 2x_1 + 2x_{13} + x_{12} \mod 3$
53	1110000100000001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + x_1 + 2x_{12} \mod 3$
54	1110000100000100	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + 2x_1 + 2x_{14} + x_{12} \mod 3$
55	1110000100000101	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + x_1 + x_{14} + x_{13} \mod 3$
56	1110000100000110	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + 2x_2 + x_1 + x_{14} + x_{13} + 2x_{12} \mod 3$
57	1110000100001000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + x_1 + x_{14} + x_{13} + x_{12} \mod 3$
58	1110000100001001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + 2x_2 + x_1 + x_{14} + x_{13} \mod 3$
59	1110000100001100	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + x_1 + x_{12} \mod 3$
60	1110000100001110	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + 2x_1 + 2x_{13} \mod 3$
61	1110000100010000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + x_1 + x_{13} \mod 3$
62	1110000100010001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_1 + x_{12} \mod 2$
63	1110000100010100	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{24} + x_1 + x_{14} + 2x_{12} \mod 3$
64	1110000100010110	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + x_1 + x_{14} \mod 3$
65	1110000100011000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 3x_{23} + x_1 + 2x_{14} + x_{13} + 3x_{12} \mod 5$
66	1110000100011001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + x_1 + x_{14} + x_{12} \mod 3$
67	1110000100011100	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{23} + 2x_1 \mod 3$
68	1110000100011110	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_1 \mod 2$
69	1110010000000000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_1 \mod 3$
70	1110010000000010	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_1 + x_{14} + x_{13} \mod 3$
71	1110010000000011	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_{23} + x_1 \mod 3$
72	1110010000010000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_1 + x_{14} + 2x_{12} \mod 3$
73	1110010000010001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_1 + x_{14} \mod 3$
74	1110010000010010	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_{23} + x_1 + x_{14} \mod 3$
75	1110010000010011	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_1 + x_{13} \mod 3$
76	1110010000011000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_1 + x_{13} + x_{12} \mod 3$
77	1110010000011010	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + 2x_2 + x_{24} + 2x_1 + 2x_{12} \mod 3$
78	1110010000011011	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{24} + x_1 \mod 2$
79	1110010000100000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_1 + 2x_{13} \mod 3$
80	1110010000100001	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + x_{23} + x_1 + x_{14} + 2x_{13} \mod 3$
81	1110010000100100	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + 2x_{24} + 2x_1 + x_{13} + x_{12} \mod 3$
82	1110010000100110	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + 2x_2 + x_{24} + 2x_1 + x_{13} + x_{12} \mod 3$
83	1110010000100111	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{24} + x_1 + x_{13} + x_{12} \mod 2$
84	1110010000101000	2	$(0, 0, \bar{x}_1, \bar{x}_2)$	2	$x_{34} + x_2 + x_{24} + x_1 + x_{13} \mod 2$

Num.	Predicate	OR	OR Cert.	Deg	Polynomial
85	1110010000101001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+2x_1+x_{13}+2x_{12} \mod 3$
86	1110011000000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1+x_{12} \mod 3$
87	1110011000000001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+2x_1+2x_{13} \mod 3$
88	1110011000000100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+2x_1+2x_{13}+x_{12} \mod 3$
89	1110011000000101	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+3x_{24}+3x_{23}+x_1+x_{13}+4x_{12} \mod 5$
90	1110011000000110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1+2x_{12} \mod 3$
91	1110011000000111	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1+x_{14}+x_{13}+x_{12} \mod 3$
92	1110011000001000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1+x_{14}+x_{13} \mod 3$
93	1110011000001001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1 \mod 3$
94	1110011000001100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+2x_1+2x_{14}+2x_{12} \mod 3$
95	1110011000001110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+2x_1+2x_{14}+2x_{13}+2x_{12} \mod 3$
96	1110011000010000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+2x_1 \mod 3$
97	1110011000010001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+3x_{24}+3x_{23}+2x_1+2x_{13}+2x_{12} \mod 5$
98	1110011000010100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1+x_{14}+x_{12} \mod 3$
99	1110011000010101	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1+x_{13}+2x_{12} \mod 3$
100	1110011000010110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+2x_1+x_{12} \mod 3$
101	1110011000011000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+x_1+x_{13} \mod 3$
102	1110011000011001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+x_{24}+x_{23}+2x_1+2x_{12} \mod 3$
103	1110011000011100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_2+3x_{24}+3x_{23}+2x_1+2x_{14}+x_{12} \mod 5$
104	1110011000011110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+3x_2+2x_{24}+2x_{23}+3x_1+3x_{14}+3x_{13}+4x_{12} \mod 5$
105	1110011100000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_2+2x_{24}+2x_{23}+x_1 \mod 3$
106	1110011100010000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_2+2x_{24}+2x_{23}+x_1+x_{13} \mod 3$
107	1110011100011000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_2+x_{24}+x_{23}+x_1 \mod 2$
108	1110100000000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+x_1+x_{14}+x_{13}+x_{12} \mod 3$
109	1110100000000001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+2x_1+2x_{14}+2x_{13}+2x_{12} \mod 3$
110	1110100000010000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+x_1+x_{14}+x_{12} \mod 3$
111	1110100000010001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+x_1+x_{13} \mod 3$
112	1110100000010100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+x_1+x_{13}+x_{12} \mod 3$
113	1110100000010101	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+2x_1 \mod 3$
114	1110100000010110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+x_{14}+x_{13}+x_{12} \mod 5$
115	1110100000010111	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1 \mod 2$
116	1110100001000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+x_1+2x_{14}+x_{12} \mod 3$
117	1110100001000001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+x_1+2x_{14}+x_{13}+x_{12} \mod 3$
118	1110100001000010	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+x_{14} \mod 2$
119	1110100001000011	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+3x_{14}+2x_{12} \mod 5$
120	1110100001010000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+x_{23}+2x_1+x_{14}+2x_{13}+2x_{12} \mod 3$
121	1110100001010001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+x_{23}+2x_1+x_{14}+2x_{13} \mod 3$
122	1110100001010010	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+2x_{23}+2x_1+x_{14}+2x_{13} \mod 3$
123	1110100001010011	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+2x_1+x_{14}+2x_{13} \mod 3$
124	1110100001010100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+2x_1+x_{14}+2x_{13}+2x_{12} \mod 3$
125	1110100001010110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+4x_{23}+3x_1+2x_{14}+4x_{13}+4x_{12} \mod 5$
126	1110100001100000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+2x_1+x_{14}+x_{13} \mod 3$
127	1110100001100001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+2x_{23}+2x_1+x_{14}+x_{13} \mod 3$
128	1110100001100100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+2x_1+x_{14}+x_{13}+2x_{12} \mod 3$
129	1110100001100101	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+3x_1+2x_{14}+2x_{13}+4x_{12} \mod 5$
130	1110100001100110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+3x_1+2x_{14}+2x_{13}+4x_{12} \mod 5$
131	1110100001100111	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+3x_{14}+3x_{13}+4x_{12} \mod 5$
132	1110100001101000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+3x_{14}+3x_{13}+3x_{12} \mod 5$
133	1110100001101001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+4x_{23}+3x_1+2x_{14}+2x_{13}+2x_{12} \mod 5$
134	1110100001110000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+2x_{23}+x_1+2x_{14}+2x_{13} \mod 3$
135	1110100001110001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+x_{14}+x_{13} \mod 2$
136	1110100001110100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+x_{23}+x_1+2x_{14}+2x_{13}+x_{12} \mod 3$
137	1110100001110110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+2x_{23}+x_1+2x_{14}+2x_{13}+x_{12} \mod 3$
138	1110100001111000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+3x_{23}+x_1+4x_{14}+4x_{13}+4x_{12} \mod 5$
139	1110100001111100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+x_1+2x_{14}+2x_{13}+2x_{12} \mod 3$
140	1110100001111110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+x_{14}+x_{13}+x_{12} \mod 2$
141	1110100010000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+2x_{23}+2x_{14}+x_{13}+x_{12} \mod 3$
142	1110100010000001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_{14}+x_{13}+x_{12} \mod 2$
143	1110100100000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1 \mod 3$
144	1110100100000001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+2x_{13}+2x_{12} \mod 3$
145	1110100100010000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+2x_{12} \mod 3$
146	1110100100010001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+x_1+3x_{13}+x_{12} \mod 5$
147	1110100100010100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+x_{14} \mod 3$
148	1110100100010101	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+x_{13}+x_{12} \mod 3$
149	1110100100010110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1 \mod 3$
150	1110100101000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+2x_{14}+x_{13}+x_{12} \mod 3$
151	1110100101000001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+2x_{14} \mod 3$
152	1110100101000010	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+2x_{14}+x_{12} \mod 3$
153	1110100101000011	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+x_{14} \mod 3$

Num.	Predicate	OR	OR Cert.	Deg	Polynomial
154	1110100101010000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+x_{14}+2x_{13} \mod 3$
155	1110100101010001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+3x_1+2x_{14}+4x_{13}+x_{12} \mod 5$
156	1110100101010010	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+3x_1+2x_{14}+4x_{13} \mod 5$
157	1110100101010011	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+2x_1+3x_{14}+4x_{13}+x_{12} \mod 5$
158	1110100101010100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+x_{14}+2x_{13}+2x_{12} \mod 3$
159	1110100101010110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+4x_1+x_{14}+4x_{13}+4x_{12} \mod 5$
160	1110100101100000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+x_{14}+x_{13} \mod 3$
161	1110100101100001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+3x_1+2x_{14}+2x_{13}+3x_{12} \mod 5$
162	1110100101100100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+2x_1+3x_{14}+3x_{13}+4x_{12} \mod 5$
163	1110100101100101	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+4x_1+x_{14}+x_{13}+4x_{12} \mod 5$
164	1110100101100110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$2x_{34}+4x_{24}+4x_{23}+3x_1+2x_{14}+2x_{13}+x_{12} \mod 5$
165	1110100101100111	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$2x_{34}+2x_{24}+2x_{23}+x_1+2x_{14}+2x_{13}+x_{12} \mod 3$
166	1110100101101000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_1+x_{14}+x_{13}+x_{12} \mod 3$
167	1110100101101001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$6x_{34}+6x_{24}+18x_{23}+15x_1+15x_{14}+15x_{13}+15x_{12} \mod 30$
168	1110100101110000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+2x_{14}+2x_{13} \mod 3$
169	1110100101110001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_1+2x_{14}+2x_{13}+x_{12} \mod 3$
170	1110100101110100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+3x_{24}+x_{23}+x_1+4x_{14}+4x_{13}+2x_{12} \mod 5$
171	1110100101110110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$2x_{34}+4x_{24}+4x_{23}+2x_1+3x_{14}+3x_{13}+x_{12} \mod 5$
172	1110100101111000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+2x_{24}+2x_{23}+x_1+4x_{14}+4x_{13}+4x_{12} \mod 5$
173	1110100101111001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$6x_{34}+x_{24}+5x_{23}+6x_1+6x_{14}+6x_{13}+6x_{12} \mod 12$
174	1110100101111100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+x_1+4x_{14}+4x_{13}+4x_{12} \mod 5$
175	1110100101111110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$4x_{34}+4x_{24}+4x_{23}+4x_1+2x_{14}+2x_{13}+2x_{12} \mod 6$
176	1110100110000000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_{14}+2x_{13}+2x_{12} \mod 3$
177	1110100110000001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+2x_{14}+2x_{13}+2x_{12} \mod 3$
178	1110100110010000	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_{14}+x_{13}+2x_{12} \mod 3$
179	1110100110010001	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+x_{14}+3x_{13}+x_{12} \mod 5$
180	1110100110010100	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+3x_{23}+x_{14}+3x_{13}+3x_{12} \mod 5$
181	1110100110010101	2	$(0,0,\bar{x}_1,\bar{x}_2)$	3	$x_{34}+x_{24}+x_{23}+x_{14}+x_{13}+2x_{12}+2x_{124} \mod 3$
182	1110100110010110	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+28x_{23}+13x_{14}+16x_{13}+16x_{12} \mod 30$
183	1110100110010111	2	$(0,0,\bar{x}_1,\bar{x}_2)$	2	$x_{34}+x_{24}+x_{23}+x_{14}+x_{13}+x_{12} \mod 3$
184	1111000000000000	1	$(0,\bar{x}_1,0,0)$	1	$x_2+x_1 \mod 3$
185	1111000000001000	2	$(\bar{x}_1,\bar{x}_1,0,\bar{x}_2)$	2	$x_2+x_{23}+x_1+x_{14}+x_{12} \mod 3$
186	1111000000001001	2	$(\bar{x}_1,\bar{x}_1,0,\bar{x}_2)$	2	$x_2+x_{23}+2x_1+2x_{14} \mod 3$
187	1111000000001100	2	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,0)$	2	$x_2+x_1+x_{13}+x_{12} \mod 3$
188	1111000000001110	3	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_2+x_1+x_{134}+x_{12} \mod 3$
189	1111000000001111	1	$(0,\bar{x}_1,0,0)$	1	$x_2+x_1 \mod 2$
190	1111100000000000	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+2x_{13}+2x_{12} \mod 3$
191	1111100000000001	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1 \mod 3$
192	1111100000000100	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+x_{13}+x_{12} \mod 3$
193	1111100000000101	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+2x_{13} \mod 3$
194	1111100000000110	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+x_{12} \mod 3$
195	1111100000000111	3	$(\bar{x}_1,\bar{x}_1,x_2,x_3)$	3	$x_{24}+x_{23}+x_{234}+x_1 \mod 2$
196	1111100000001000	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+2x_{12} \mod 3$
197	1111100000001001	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+x_{13}+2x_{12} \mod 3$
198	1111100000001100	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+2x_{14}+x_{12} \mod 3$
199	1111100000001101	3	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,x_3)$	3	$x_{24}+x_{23}+x_{234}+x_1+x_{12}+x_{124} \mod 2$
200	1111100000001110	3	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{24}+x_{23}+x_{234}+x_1+x_{12}+x_{124}+x_{123} \mod 2$
201	1111100000001111	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+3x_1+4x_{14}+4x_{13}+2x_{12} \mod 5$
202	1111100000010000	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+2x_{14}+2x_{13} \mod 3$
203	1111100000010100	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+x_{14}+3x_{13}+2x_{12} \mod 5$
204	1111100000010110	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+x_{14}+x_{13} \mod 3$
205	1111100000011000	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+x_{14}+x_{13}+2x_{12} \mod 3$
206	1111100000011001	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+x_{14}+2x_{13}+3x_{12} \mod 5$
207	1111100000011100	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+3x_1+4x_{14}+3x_{13}+2x_{12} \mod 5$
208	1111100000011101	3	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,x_3)$	3	$x_{24}+x_{23}+x_1+2x_{134}+2x_{12}+2x_{124} \mod 3$
209	1111100000011110	3	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{24}+x_{23}+x_1+2x_{134}+2x_{12}+2x_{124}+2x_{123} \mod 3$
210	1111100000011111	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+2x_{14}+2x_{13}+x_{12} \mod 3$
211	1111100001000000	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+2x_{14}+x_{13}+x_{12} \mod 3$
212	1111100001000010	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+3x_{14}+x_{13}+x_{12} \mod 5$
213	1111100001000011	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+2x_{14}+x_{13} \mod 3$
214	1111100001001000	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+3x_{14}+x_{13}+3x_{12} \mod 5$
215	1111100001001001	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+3x_1+2x_{14}+x_{13}+2x_{12} \mod 5$
216	1111100001001010	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_1+x_{14}+2x_{13}+x_{12} \mod 3$
217	1111100001001011	3	$(\bar{x}_1,\bar{x}_1,x_2,\bar{x}_3)$	3	$x_{24}+x_{23}+x_1+2x_{14}+x_{134}+2x_{12}+2x_{124}+2x_{123} \mod 3$
218	1111100001001100	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_1+2x_{14}+x_{13}+2x_{12} \mod 3$
219	1111100001001110	3	$(\bar{x}_1,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{24}+x_{23}+x_1+2x_{14}+x_{134}+2x_{12}+2x_{123} \mod 3$
220	1111100001001111	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+2x_{23}+x_1+4x_{14}+3x_{13}+4x_{12} \mod 5$
221	1111100010000000	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+2x_{14}+2x_{13}+x_{12} \mod 3$
222	1111100010000001	2	$(0,\bar{x}_1,0,\bar{x}_2)$	2	$x_{24}+x_{23}+x_{14}+x_{13}+2x_{12} \mod 3$

Num.	Predicate	OR	OR Cert.	Deg	Polynomial
223	1111100010000100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_{14} + 2x_{13} + 3x_{12} \mod 5$
224	1111100010000101	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + 2x_{14} + 4x_{13} + 2x_{12} \mod 5$
225	1111100010000110	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_{14} + x_{13} + x_{12} \mod 3$
226	1111100010000111	3	$(\bar{x}_1, \bar{x}_1, x_2, x_3)$	3	$x_{24} + x_{23} + x_{14} + x_{13} + 2x_{134} + 2x_{12} + 2x_{124} + 2x_{123} \mod 3$
227	1111100010001000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_{14} + x_{13} \mod 3$
228	1111100010001001	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_{14} + 2x_{13} \mod 5$
229	1111100100000000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_1 + x_{13} \mod 3$
230	1111100100000100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_1 + x_{12} \mod 3$
231	1111100100000110	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_1 \mod 2$
232	1111100100001000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_1 + x_{14} + 2x_{12} \mod 3$
233	1111100100001001	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_1 + x_{12} \mod 2$
234	1111100100001100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + 2x_1 + 2x_{14} + x_{12} \mod 3$
235	1111100100001101	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, x_3)$	3	$x_{24} + x_{23} + x_{234} + x_1 + 2x_{12} + 2x_{124} + x_{123} \mod 3$
236	1111100100001110	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{24} + x_{23} + x_{234} + x_1 + 2x_{12} + 2x_{124} + 2x_{123} \mod 3$
237	1111100100001111	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + 2x_1 + 4x_{14} + x_{13} + 3x_{12} \mod 5$
238	1111100101000000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_1 + 2x_{14} + x_{13} \mod 3$
239	1111100101000010	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + 2x_1 + x_{14} + 2x_{13} \mod 3$
240	1111100101000100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + 2x_1 + x_{14} + 2x_{13} + 2x_{12} \mod 3$
241	1111100101000110	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + 3x_1 + 2x_{14} + 4x_{13} + 4x_{12} \mod 5$
242	1111100101001000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + 2x_1 + 3x_{14} + 4x_{13} + 3x_{12} \mod 5$
243	1111100101001001	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + 2x_1 + x_{14} + 2x_{13} + x_{12} \mod 3$
244	1111100101001010	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + 2x_1 + 3x_{14} + x_{13} + 3x_{12} \mod 5$
245	1111100101001011	3	$(\bar{x}_1, \bar{x}_1, x_2, \bar{x}_3)$	3	$x_{24} + x_{23} + x_1 + x_{14} + x_{134} + x_{12} + x_{124} + x_{123} \mod 2$
246	1111100101001100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + x_1 + 4x_{14} + 2x_{13} + 4x_{12} \mod 5$
247	1111100101001101	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, x_3)$	3	$x_{24} + x_{23} + x_1 + x_{14} + x_{134} + x_{12} \mod 2$
248	1111100101001110	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{24} + x_{23} + x_1 + x_{14} + x_{134} + x_{12} + x_{123} \mod 2$
249	1111100101001111	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_1 + 2x_{14} + x_{13} + 2x_{12} \mod 3$
250	1111100101100000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_1 + x_{14} + x_{13} \mod 2$
251	1111100101101000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + 2x_1 + 3x_{14} + 3x_{13} + 3x_{12} \mod 5$
252	1111100101101001	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$4x_{24} + 2x_{23} + 3x_1 + 3x_{14} + 3x_{13} + 3x_{12} \mod 6$
253	1111100101101100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_1 + 2x_{14} + 2x_{13} + 2x_{12} \mod 3$
254	1111100101101110	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{24} + x_{23} + x_{234} + x_1 + 2x_{14} + 2x_{13} + 2x_{12} \mod 3$
255	1111100101101111	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$3x_{24} + 3x_{23} + 3x_1 + 3x_{14} + 3x_{13} + 3x_{12} \mod 6$
256	1111100110000000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_{14} + x_{13} + 2x_{12} \mod 3$
257	1111100110000001	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + x_{14} + 2x_{13} + 2x_{12} \mod 5$
258	1111100110000100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + x_{14} + 2x_{13} + 3x_{12} \mod 5$
259	1111100110000101	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_{14} + x_{13} + x_{12} \mod 3$
260	1111100110000110	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + x_{14} + 3x_{13} + 3x_{12} \mod 5$
261	1111100110000111	3	$(\bar{x}_1, \bar{x}_1, x_2, x_3)$	3	$x_{24} + x_{23} + x_{14} + x_{13} + x_{134} + x_{12} + x_{124} + x_{123} \mod 2$
262	1111100110001000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 4x_{23} + x_{14} + 2x_{13} \mod 5$
263	1111100110001001	3	$(\bar{x}_1, x_2, \bar{x}_3, \bar{x}_3)$	3	$x_{24} + x_{23} + x_{234} + x_{14} + x_{13} + 2x_{124} + 2x_{123} \mod 3$
264	1111100110001100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + 2x_{14} + 2x_{13} \mod 3$
265	1111100110001101	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, x_3)$	3	$x_{24} + x_{23} + x_{14} + x_{13} + x_{134} + x_{123} \mod 2$
266	1111100110010000	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_{14} + x_{13} + x_{12} \mod 2$
267	1111100110010100	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_{14} + 2x_{13} + x_{12} \mod 3$
268	1111100110010110	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 11x_{23} + 5x_{14} + 7x_{13} + 6x_{12} \mod 12$
269	1111100110011000	3	$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_3)$	3	$x_{24} + x_{23} + x_{234} + x_{14} + x_{13} + x_{134} + 2x_{123} \mod 3$
270	1111100110011001	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + 2x_{23} + x_{14} + 2x_{13} \mod 3$
271	1111100110011100	3	$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_3)$	3	$x_{24} + x_{23} + x_{14} + x_{13} + x_{123} \mod 2$
272	1111100110011101	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, x_3)$	3	$x_{24} + x_{23} + x_{234} + x_{14} + x_{13} + x_{134} + x_{124} + 2x_{123} \mod 3$
273	1111100110011110	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{24} + x_{23} + x_{234} + x_{14} + x_{13} + x_{134} + x_{124} + x_{123} \mod 3$
274	1111100110011111	2	$(0, \bar{x}_1, 0, \bar{x}_2)$	2	$x_{24} + x_{23} + x_{14} + x_{13} \mod 2$
275	1111110000000000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_1 \mod 3$
276	1111110000000010	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + 2x_{14} \mod 3$
277	1111110000000011	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_1 \mod 2$
278	1111110000100000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_1 + x_{14} + 2x_{13} \mod 3$
279	1111110000100001	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + 2x_{14} + x_{13} \mod 3$
280	1111110000100010	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + x_{14} + 3x_{13} + 4x_{12} \mod 5$
281	1111110000100011	3	$(\bar{x}_1, x_2, \bar{x}_1, \bar{x}_3)$	3	$x_{23} + x_1 + 2x_{13} + x_{134} + x_{12} + 2x_{124} + x_{123} \mod 3$
282	1111110000100100	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + x_{14} + 3x_{13} + 2x_{12} \mod 5$
283	1111110000100110	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + 2x_{14} + x_{13} + 2x_{12} \mod 3$
284	1111110000100111	3	$(\bar{x}_1, \bar{x}_1, x_2, x_3)$	3	$x_{23} + x_1 + x_{13} + x_{134} + x_{124} + x_{123} \mod 2$
285	1111110000101000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + 2x_{14} + x_{13} + x_{12} \mod 3$
286	1111110000101001	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + x_{14} + 3x_{13} + 3x_{12} \mod 5$
287	1111110000101010	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_1 + x_{14} + 2x_{13} + 2x_{12} \mod 3$
288	1111110000101011	3	$(\bar{x}_1, \bar{x}_1, x_2, \bar{x}_3)$	3	$x_{23} + x_1 + x_{13} + x_{134} + x_{12} + x_{124} \mod 2$
289	1111110000110000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_1 + x_{13} \mod 2$
290	1111110000111000	3	$(\bar{x}_1, \bar{x}_2, x_2, \bar{x}_3)$	3	$x_{23} + x_1 + 2x_{13} + 2x_{12} + x_{124} + x_{123} \mod 3$
291	1111110000111001	3	$(\bar{x}_1, \bar{x}_2, \bar{x}_1, x_3)$	3	$x_{23} + x_1 + x_{13} + x_{12} + x_{124} + x_{123} \mod 2$

Num.	Predicate	OR	OR Cert.	Deg	Polynomial
292	1111110000111010	3	$(\bar{x}_1, \bar{x}_2, \bar{x}_1, \bar{x}_3)$	3	$x_{23} + x_1 + x_{13} + x_{12} + x_{124} \mod 2$
293	1111110000111100	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_1 + x_{13} + x_{12} \mod 3$
294	1111110000111110	3	$(\bar{x}_1, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{23} + x_1 + x_{14} + 2x_{13} + 2x_{134} + 2x_{12} + 2x_{124} \mod 3$
295	1111110000111111	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_1 + x_{13} + x_{12} \mod 2$
296	1111110010000000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_{14} + 2x_{13} + 2x_{12} \mod 3$
297	1111110010000001	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} + x_{13} + 2x_{12} \mod 5$
298	1111110010000010	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} + x_{13} + x_{12} \mod 3$
299	1111110010000011	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	3	$x_{23} + x_{14} + x_{13} + x_{12} + 2x_{124} \mod 3$
300	1111110010010000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} + 2x_{13} + x_{12} \mod 3$
301	1111110010010001	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_{14} + x_{13} + 2x_{12} \mod 3$
302	1111110010010010	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_{14} + 3x_{13} + x_{12} \mod 5$
303	1111110010010011	3	$(\bar{x}_1, x_2, \bar{x}_1, x_3)$	3	$x_{23} + x_{14} + x_{13} + x_{12} + x_{124} + x_{123} \mod 2$
304	1111110010010100	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_{14} + 3x_{13} + 3x_{12} \mod 5$
305	1111110010010101	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} + x_{13} + x_{12} \mod 2$
306	1111110010010110	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$2x_{23} + x_{14} + 2x_{13} + 2x_{12} \mod 3$
307	1111110010010111	3	$(\bar{x}_1, \bar{x}_1, x_2, x_3)$	3	$x_{23} + x_{14} + x_{13} + x_{134} + x_{12} + x_{124} \mod 3$
308	1111110010100000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} + x_{12} \mod 5$
309	1111110010100001	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} + x_{12} \mod 3$
310	1111110010100100	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + 2x_{14} + x_{12} \mod 3$
311	1111110010100101	3	$(x_1, \bar{x}_2, \bar{x}_3, \bar{x}_2)$	3	$x_{23} + x_{14} + x_{12} + x_{123} \mod 2$
312	1111110010100110	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} + x_{12} \mod 2$
313	1111110010100111	3	$(x_1, \bar{x}_2, \bar{x}_3, \bar{x}_2)$	3	$x_{23} + x_{14} + x_{134} + x_{12} + x_{124} + x_{123} \mod 3$
314	1111110010101000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} \mod 3$
315	1111110010101001	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{14} \mod 2$
316	1111110011000000	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{13} + x_{12} \mod 2$
317	1111110011000010	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	3	$x_{23} + x_{13} + x_{12} + x_{124} \mod 3$
318	1111110011000011	2	$(0, \bar{x}_1, \bar{x}_2, 0)$	2	$x_{23} + x_{13} + x_{12} \mod 3$
319	1111111000000000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 \mod 3$
320	1111111000000001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 \mod 2$
321	1111111000010000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{134} \mod 2$
322	1111111000010001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{134} + x_{124} + x_{123} \mod 3$
323	1111111000010100	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{134} + 2x_{124} + x_{123} \mod 3$
324	1111111000010101	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{134} + x_{124} \mod 2$
325	1111111000010110	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{134} + x_{124} + x_{123} \mod 2$
326	1111111000010111	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{13} + x_{134} + x_{12} + x_{124} \mod 3$
327	1111111001000000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{134} + x_{124} \mod 2$
328	1111111001000001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + x_{134} + x_{124} \mod 3$
329	1111111001000010	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + x_{134} + x_{124} + 2x_{123} \mod 3$
330	1111111001000011	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{134} + x_{124} + x_{123} \mod 2$
331	1111111001010000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + x_{124} \mod 3$
332	1111111001010001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{124} \mod 2$
333	1111111001010010	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{124} + x_{123} \mod 2$
334	1111111001010011	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + x_{12} + x_{123} \mod 3$
335	1111111001010100	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} \mod 2$
336	1111111001010101	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + x_{12} + 2x_{124} + 2x_{123} \mod 3$
337	1111111001010110	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + x_{12} + 2x_{124} + x_{123} \mod 3$
338	1111111001010111	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{123} \mod 2$
339	1111111001100000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + x_{124} + x_{123} \mod 3$
340	1111111001100001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{124} + x_{123} \mod 2$
341	1111111001100100	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{123} \mod 2$
342	1111111001100101	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + x_{12} + 2x_{124} \mod 3$
343	1111111001100110	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + x_{12} + 2x_{124} + 2x_{123} \mod 3$
344	1111111001100111	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} \mod 2$
345	1111111001101000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{12} \mod 2$
346	1111111001101001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + 2x_{12} + 2x_{124} + 2x_{123} \mod 3$
347	1111111001110000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{134} + x_{124} + x_{123} \mod 2$
348	1111111001110100	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + x_{134} + x_{123} \mod 3$
349	1111111001110110	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{134} \mod 2$
350	1111111001111000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + x_{134} + 2x_{12} + 2x_{123} \mod 3$
351	1111111001111001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{134} + x_{12} \mod 2$
352	1111111001111100	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{134} + x_{12} + x_{124} \mod 2$
353	1111111001111101	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + x_{134} + 2x_{12} + x_{124} + 2x_{123} \mod 3$
354	1111111001111110	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + 2x_{14} + 2x_{13} + x_{134} + 2x_{12} + x_{124} + x_{123} \mod 3$
355	1111111001111111	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_1 + x_{14} + x_{13} + x_{134} + x_{12} + x_{124} + x_{123} \mod 2$
356	1111111010000000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_{14} + x_{13} + x_{134} + x_{12} + x_{124} + x_{123} \mod 2$
357	1111111010000001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_{14} + x_{13} + x_{12} + 2x_{123} \mod 3$
358	1111111010010000	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_{14} + x_{13} + x_{134} + x_{12} \mod 3$
359	1111111010010001	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_{14} + x_{13} + x_{12} + x_{124} + x_{123} \mod 2$
360	1111111010010100	3	$(0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$	3	$x_{234} + x_{14} + x_{13} + x_{12} + x_{123} \mod 2$

Num.	Predicate	OR	OR Cert.	Deg	Polynomial
361	1111111010010101	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{14}+x_{13}+x_{134}+x_{12}+x_{124} \mod 3$
362	1111111010010110	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{14}+x_{13}+x_{134}+x_{12}+x_{124}+x_{123} \mod 3$
363	1111111010010111	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{14}+x_{13}+x_{12} \mod 2$
364	1111111011000000	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{13}+x_{12}+2x_{123} \mod 3$
365	1111111011000001	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{13}+x_{12}+x_{123} \mod 2$
366	1111111011000010	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{13}+x_{12} \mod 2$
367	1111111011000011	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{13}+2x_{12}+2x_{124} \mod 3$
368	1111111011010000	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{13}+x_{134}+x_{12}+x_{123} \mod 2$
369	1111111011010010	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{13}+2x_{134}+x_{12}+x_{124}+x_{123} \mod 3$
370	1111111011010011	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{13}+x_{134}+x_{12} \mod 2$
371	1111111011100000	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+x_{12} \mod 2$
372	1111111011100001	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+x_{12} \mod 3$
373	1111111011100100	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+x_{12}+2x_{124} \mod 3$
374	1111111011100101	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+x_{12}+x_{124} \mod 2$
375	1111111011100110	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+x_{12}+x_{124}+x_{123} \mod 2$
376	1111111011100111	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+2x_{12}+x_{124}+x_{123} \mod 3$
377	1111111011101000	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+x_{124}+x_{123} \mod 3$
378	1111111011101001	3	$(0,\bar{x}_1,\bar{x}_2,\bar{x}_3)$	3	$x_{234}+x_{134}+x_{124}+x_{123} \mod 2$
379	1111111100000000	1	$(\bar{x}_1,0,0,0)$	1	$x_1 \mod 2$
380	1111111110000000	2	$(\bar{x}_1,0,0,\bar{x}_2)$	2	$x_{14}+x_{13}+x_{12} \mod 5$
381	1111111110000001	2	$(\bar{x}_1,0,0,\bar{x}_2)$	2	$x_{14}+x_{13}+x_{12} \mod 3$
382	1111111110010000	2	$(\bar{x}_1,0,0,\bar{x}_2)$	2	$x_{14}+4x_{13}+2x_{12} \mod 5$
383	1111111110010100	2	$(\bar{x}_1,0,0,\bar{x}_2)$	2	$x_{14}+2x_{13}+2x_{12} \mod 3$
384	1111111110010110	2	$(\bar{x}_1,0,0,\bar{x}_2)$	2	$3x_{14}+3x_{13}+3x_{12} \mod 6$
385	1111111111000000	2	$(\bar{x}_1,0,\bar{x}_2,0)$	2	$x_{13}+x_{12} \mod 3$
386	1111111111000010	3	$(\bar{x}_1,\bar{x}_2,\bar{x}_2,\bar{x}_3)$	3	$x_{13}+x_{12}+x_{124}+x_{123} \mod 3$
387	1111111111000011	2	$(\bar{x}_1,0,\bar{x}_2,0)$	2	$x_{13}+x_{12} \mod 2$
388	1111111111100000	3	$(\bar{x}_1,0,\bar{x}_2,\bar{x}_3)$	3	$x_{134}+x_{12} \mod 3$
389	1111111111100001	3	$(\bar{x}_1,0,\bar{x}_2,\bar{x}_3)$	3	$x_{134}+x_{12} \mod 2$
390	1111111111100100	3	$(\bar{x}_1,0,\bar{x}_2,\bar{x}_3)$	3	$x_{134}+x_{12}+x_{124} \mod 2$
391	1111111111100110	3	$(\bar{x}_1,0,\bar{x}_2,\bar{x}_3)$	3	$x_{134}+2x_{12}+x_{124}+x_{123} \mod 3$
392	1111111111100111	3	$(\bar{x}_1,0,\bar{x}_2,\bar{x}_3)$	3	$x_{134}+x_{12}+x_{124}+x_{123} \mod 2$
393	1111111111101000	3	$(\bar{x}_1,0,\bar{x}_2,\bar{x}_3)$	3	$x_{134}+x_{124}+x_{123} \mod 2$
394	1111111111101001	3	$(\bar{x}_1,0,\bar{x}_2,\bar{x}_3)$	3	$x_{134}+x_{124}+x_{123} \mod 3$
395	1111111111110000	2	$(\bar{x}_1,\bar{x}_2,0,0)$	2	$x_{12} \mod 2$
396	1111111111111000	3	$(\bar{x}_1,\bar{x}_2,0,\bar{x}_3)$	3	$x_{124}+x_{123} \mod 3$
397	1111111111111001	3	$(\bar{x}_1,\bar{x}_2,0,\bar{x}_3)$	3	$x_{124}+x_{123} \mod 2$
398	1111111111111100	3	$(\bar{x}_1,\bar{x}_2,\bar{x}_3,0)$	3	$x_{123} \mod 2$
399	1111111111111110	4	$(\bar{x}_1,\bar{x}_2,\bar{x}_3,\bar{x}_4)$	4	$x_{1234} \mod 2$