The Uniformisation of Monadic Second-Order Logic over Countable Ordinals

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Abstract

We study the uniformisation problem for monadic second-order logic (MSO) over countable ordinal chains. Given a formula defining a relation between subsets of the input structure, the question is whether there exists a formula that defines a function selecting, for every set in the domain of the relation, a unique set such that the pair belongs to the relation. It is known, due to Lifsches and Shelah [9], that MSO cannot, in general, be uniformised over the class of countable ordinals.

We show that the maximal uniformisation degree is reached by extending the logic with a predicate that, given a set, selects (when possible) a cofinal subset of order type ω. Equivalently, every MSO formula can be uniformised over the class of countable ordinal chains using a formula in this extended logic.

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1 Introduction

A uniformiser for a binary relation $R \subseteq A \times B$ is a maximal subrelation f which is the graph of a partial function and has the same domain as R. Equivalently, let us call the *cross-section at a* for $a \in A$, the set $R_a = \{b \mid (a, b) \in R\}$, then a uniformiser for R is a relation $f \subseteq R$ such that for all $a \in A$, if R_a is non-empty, then there exists exactly one b such that $(a, b) \in f$. See an illustration in Figure 1.

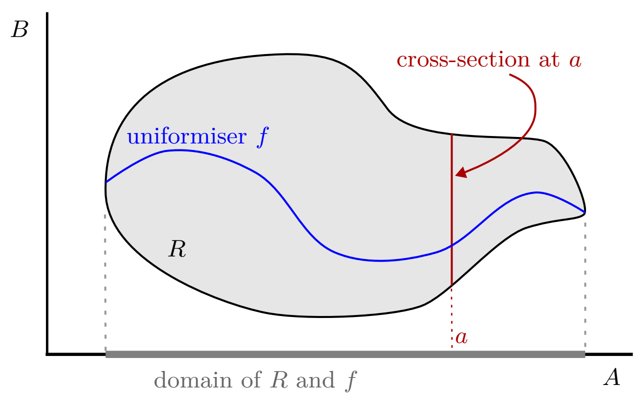

Figure 1 f uniformises R.

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The axiom of choice implies that a uniformiser always exists; however, it is often important that a uniformiser has some "nice" properties. This is where the uniformisation theorems come into action. They guarantee that if R satisfies certain properties, then it has a uniformiser that has other desirable properties.

Lifsches and Shelah [9] considered uniformisation for relations definable in monadic second-order logic[1] (MSO).

Let $\mathcal{M}$ be a structure with domain U. An MSO formula $\varphi(X,Y)$ with free set variables X and Y defines a binary relation between subsets of the universe, that is, a subset of $\mathcal{P}(U) \times \mathcal{P}(U)$ (the definition extends in a straightforward way to formulas with arbitrary tuples $\overline{X}, \overline{Y}$ of set variables, see Section 4). A formula $\psi(X,Y)$ is a uniformiser for $\varphi(X,Y)$ in $\mathcal{M}$ if the relation defined by ψ is a uniformiser of the relation defined by φ.

A formula $\psi(X,Y)$ is a uniformiser for $\varphi(X,Y)$ over a class $\mathcal{C}$ of structures if ψ is a uniformiser for φ in every structure $\mathcal{M} \in \mathcal{C}$.

We say that a structure $\mathcal{M}$ (respectively, a class $\mathcal{C}$) has the uniformisation property (for MSO) if every MSO formula admits an MSO uniformiser in $\mathcal{M}$ (respectively, over $\mathcal{C}$).

Lifsches and Shelah [9] proved that an ordinal α has the uniformisation property if and only if $\alpha < \omega^\omega$.

In particular, the formula $\psi_\omega(X,Y)$ expressing that

"Y has order type ω and is a cofinal subset of X"

does not admit any MSO uniformiser over the class of countable ordinals (and in particular over any ordinal $\alpha \geq \omega^\omega$) [9].

Now, consider the formula ψ_{ω^2} stating that

"Y is cofinal in X and has order type ω^2."

This formula also does not admit an MSO-uniformiser over ordinals $\alpha \geq \omega^\omega$.

Are there further interesting relationships between ψ_ω and ψ_{ω^2}? For instance, can we say that ψ_{ω^2} is, in some sense, "harder" to uniformise than ψ_ω – or perhaps the other way around? Does the example ψ_{ω^2} introduce a genuinely "new idea" in our understanding of uniformisation?

Note the following observation. For any subset Y of order type ω^2, the set of its limit points is cofinal in Y and has order type ω. Therefore, from any cofinal subset of X of order type ω^2, one can define (in MSO) a cofinal subset of X of order type ω. This shows that ψ_ω is in fact easier to uniformise than ψ_{ω^2}.

To make such comparisons precise, we introduce a notion of reducibility between formulas. Intuitively, a formula ϕ is easier to uniformise than a formula ψ if any method for uniformising ψ can be transformed into a method for uniformising ϕ.

Let $\mathcal{M}$ be a structure in a signature Σ, and let ψ and ϕ be formulas defining binary relations R and S on $\mathcal{M}$. We say that ϕ is *easier to uniformise* than ψ in $\mathcal{M}$ if there exists a formula χ in the expansion of Σ by a new binary relation symbol r, such that χ defines a uniformiser for S in every expansion of $\mathcal{M}$ where r interprets a uniformiser of R. We call such a χ a *reduction* from ϕ to ψ in $\mathcal{M}$.

[1] Let us recall that monadic second-order logic is an extension of first-order logic by set variables (which range over the subset of the domain of a structure), atomic formulas $x \in Y$, where x is a first-order variable and Y a set variable, and the quantifiers over the set variables.

For example, for every ordinal α, one can define a uniformiser for ψ_ω from any uniformiser for ψ_{ω^2}. Hence, there is a reduction from ψ_ω to ψ_{ω^2} in α.

We then say that ϕ is *easier to uniformise* than ψ over a class $\mathcal{C}$ of structures, denoted $\phi \preceq \psi$, if there exists a uniform reduction χ from ϕ to ψ that works in every $\mathcal{M} \in \mathcal{C}$. This defines a preorder on formulas (modulo logical equivalence), and the resulting equivalence classes are called the *uniformisation degrees*.

Contributions

In this work, we prove that over the class of countable ordinals every formula is easier to uniformise than "Y has the order type ω and is a cofinal subset of X." Hence, ψ_ω has the maximal uniformisation degree. More formally, let $\mathrm{U}_{\omega\mathrm{cof}}$ be a partial function that for every subset X of an ordinal returns a cofinal in X subset Y of the order type ω. Let $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ be the expansion of MSO by $\mathrm{U}_{\omega\mathrm{cof}}(X)$. The *main theorem* is expressed as follows:

▶ **Theorem 1.1** (Maximal uniformisation degree)**.** *For every $\varphi \in \mathrm{MSO}$ there is $\psi \in \mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ such that ψ is a uniformiser for φ over the class of countable ordinals. Moreover, there is an algorithm that computes ψ from φ.*

Technical tools

Algebraic techniques related to ordinal monoids, introduced under the name ω_1-monoid in [2], will play a key role in our proofs. In particular, we use it in order to reformulate the uniformisation problem for MSO in algebraic terms.

Our argument also builds on a variation of the notion of *factorization trees*, introduced by Simon [15] and extended to infinite words in [6], which plays a central role in our construction. We reformulate these results in terms of MSO-definable condensations, which are more convenient to manipulate within logical frameworks.

Significance

The study of uniformization over infinite structures has received considerable attention in recent years; see, e.g., [4, 10, 8, 7, 11, 12]. Uniformization asks whether a relational specification can be replaced by a definable functional one. In computational terms, this corresponds to transforming a nondeterministic specification into an unambiguous one, which is central in areas such as definability, synthesis, and the algorithmic manipulation of logical specifications.

In our setting, the question is not only whether a given MSO-definable relation over countable ordinals can be uniformized, but also whether one can identify the precise source of difficulty of uniformization over this class. Our main result shows that this difficulty is completely captured by a single canonical obstruction: choosing an ω-cofinal subset. Equivalently, the formula expressing that "Y is an ω-cofinal subset of X" has maximal uniformization degree.

We view this as the main conceptual contribution of the paper: it isolates a single principle that explains all failures of MSO uniformization over countable ordinals.

Importantly, the result is also effective: we provide an algorithm that constructs uniformizers in the extension $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$. Together with the algebraic reformulation via ordinal monoids, this places the result at the interface between automata theory, word combinatorics, and logic.

Related Work

The notion of selection over a class $\mathcal{C}$ is the restriction of the uniformization problem to formulas $\phi(Y)$ without domain variables.

Recall that the uniformisation problem depends not only on the formula but also on a partition of its free variables into *domain variables* $\overline{X}$ and *image variables* $\overline{Y}$. Selection corresponds to the special case in which there are *no* domain variables.

▶ **Definition 1.2** (Selection). *Let $\phi(Y)$ and $\psi(Y)$ be formulas, and let $\mathcal{C}$ be a class of structures. We say that ψ selects (or is a selector for) ϕ* over *$\mathcal{C}$ if for every $\mathcal{M} \in \mathcal{C}$:*

1. *either both ϕ and ψ are not satisfied in $\mathcal{M}$, or*
2. *there exists a unique P such that $\mathcal{M} \models \psi(P)$, and this P also satisfies $\mathcal{M} \models \phi(P)$.*

We say that $\mathcal{C}$ has the selection property if every formula $\phi(Y)$ has a selector over $\mathcal{C}$. A structure $\mathcal{M}$ has the selection property if the singleton class $\{\mathcal{M}\}$ does.

In [13], the selection problem was investigated over countable ordinals. It was shown that the selection property fails in every ordinal $\alpha \geq \omega^\omega$. In particular, the formula "Y is a cofinal subset of order type ω" has no selector in ω^ω.

Furthermore, it was proved that for every countable ordinal α, there exists an algorithm that decides whether a given MSO-formula $\phi(Y)$ has a selector in α, and if so, computes one.

In the same work, the notion of selection degrees was introduced. A formula $\phi(Y)$ is said to be *easier to select* than a formula $\psi(Y)$ in a structure $\mathcal{M}$ (denoted $\phi \preceq \psi$) if there exists an MSO-formula $\chi(Y', Y)$ such that whenever Y' satisfies ψ, there is a unique Y such that $\chi(Y', Y)$ holds in $\mathcal{M}$, and Y satisfies ϕ.

For every ordinal α, the relation $\preceq$ is a preorder on formulas, and its equivalence classes are called *selection degrees* in α. It was shown that every countable ordinal $\alpha \geq \omega^\omega$ has exactly two selection degrees. In particular, over ω^ω, every formula is either selectable or has the same degree as the formula "Y is a cofinal subset of order type ω".

In [14], the above results were extended to classes of countable ordinals. A uniformization problem with restricted domain variables was also considered there.

Fix ordinals δ and α. One can consider a restricted form of uniformization in which the input sets are bounded in complexity. A formula $\psi(\overline{X}, \overline{Y})$ is said to uniformize a formula $\varphi(\overline{X}, \overline{Y})$ over $(\alpha, <)$ under a δ-restriction if the usual uniformization conditions hold when the variables $\overline{X}$ are required to range only over subsets of α of order type strictly less than δ.

In [14], it was shown that this bounded uniformization problem is decidable: given an ordinal $\alpha \in [\omega^\omega, \omega_1]$, an ordinal $\delta < \omega^\omega$, and a formula $\varphi(\overline{X}, \overline{Y})$, there is an algorithm that determines whether such a restricted uniformizer exists over $(\alpha, <)$, and constructs one whenever it does.

Structure of the Paper

Section 2 introduces the necessary preliminaries and definitions.

In Section 3, we introduce the notion of *E-simple* (idempotent-simple) words – words whose factorizations exhibit a regular structure with respect to a fixed idempotent element E of a finite monoid. This section also states the height-decreasing condensation theorem (Theorem 3.4) inspired by Simon's factorization forests [15]. The height-decreasing condensation theorem guarantees that every word admits an MSO-definable condensation into an E-simple word of strictly smaller height. This result is a crucial ingredient in the proof of our main theorem.

Section 4 defines the notion of *uniformisation* and presents its reformulation in algebraic terms. These algebraic tools are then used in Section 5 to show that an MSO-definable relation has an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-definable uniformiser (Theorem 5.1).

Section 6 concludes the paper.

The appendix (Appendix A) contains the missing proofs.

2 Preliminaries

In this section, we recall standard definitions, notions, and results concerning ordinals and ordinal words (Section 2.1), ordinal monoids (Section 2.2), Green relations (Section 2.3) and monadic second-order logic (Section 2.4).

We use standard notations. In particular, we denote by $[h]$ the set of integers $\{0, \dots, h-1\}$, and by $\mathcal{P}(D)$ the powerset of the set D.

2.1 Ordinals chains and words

We use the standard terminology for ordinals. An *ordinal* is, up to isomorphism, a set equipped with a well-founded total order $<$. We follow standard ordinal notation; for instance, ω denotes the first infinite ordinal.

A *limit ordinal* is a non-empty ordinal with no maximal element. An ordinal is called a *successor ordinal* if it is nonzero and has a maximal element.

Given a subset X of an ordinal α, we denote by $\alpha|_X$ the restriction of α to X. The *order type* of X is then the ordinal $\alpha|_X$.

We use standard notations for intervals: in particular, $[x, y)$ denotes the set $\{z \in \alpha \mid x \leqslant z < y\}$, and $[x, \infty)$ denotes the set $\{z \in \alpha \mid x \leqslant z\}$.

A set X is *cofinal* in Y if $X \subseteq Y$, and for every $y \in Y$, there exists $x \in X$ with $x \geqslant y$. The *cofinality* of α is the least order type of a cofinal subset of α. It is well known that every countable limit ordinal has cofinality ω. A cofinal subset of order type ω is called *ω-cofinal*.

In this work, the term *chain* refers to a linear order, and the two terms are used interchangeably.

A *convex subset* S of a chain is a set such that for all $x < y < z$, if x and z belong to S, then y does as well. In other words, a set is convex if and only if it is an interval.

A *condensation* of a chain is an equivalence relation on it such that all equivalence classes (called *condensation classes*) are convex. The condensation classes themselves form a chain.

Given a subset X of a chain, it naturally induces an equivalence relation $\sim_X$ defined as follows: for $x < y$, we declare $x \sim_X y$ if $[x, y] \subseteq X$ or $[x, y] \cap X = \emptyset$. The equivalence classes of $\sim_X$ are convex, so $\sim_X$ defines a condensation. It is well known that every condensation $\mathcal{F}$ arises in this way for some set X, i.e., there exists X such that $\sim_X = \mathcal{F}$.

Given an alphabet Σ, an *ordinal countable Σ-word* u – or simply a *Σ-word* in this paper – is a map from a countable ordinal (referred to as the *domain*) to Σ.

The domain of the word (a countable chain) is denoted $\mathrm{dom}(u)$, and its elements are called *positions*. Positions that are not minimal and have no predecessor are called *limit positions*. The letter at position x is written $u(x)$. Words are considered up to isomorphism of their domains.

Given two Σ-words u and v, their *concatenation* uv is defined in the usual way. A word v is an *infix* (or *factor*) of a word u if $u = wvw'$ for some words w, w'. If $(u_i)_{i\in\alpha}$ is a family of words indexed by an ordinal α, we denote by $\prod_{i\in\alpha} u_i$ their concatenation.

For a given alphabet Σ, we write Σ^{ord} for the set of Σ-words, and $\Sigma^{\mathrm{ord}+}$ for the set of non-empty ones. The empty word (whose domain is empty) is denoted ε.

Given a word u and a subset S of its domain, the restriction of u to S is denoted $u|_S$.

2.2 Ordinal monoids

Ordinal monoids provide an algebraic framework for recognisable properties of countable words, which are effectively equivalent to MSO-definability.

In categorical terms, an ordinal monoid is an Eilenberg-Moore algebra for the monad of countable words. It means that it is a set M equipped with an infinitary operation which, given a M-word produces an element of M, that satisfies suitable identities. We provide here a direct definition for it. The object was introduced by Bedon and Carton as ω_1-monoids [2].

An *ordinal monoid* is a set M together with a map

$$\pi\colon M^{\mathrm{ord}} \to M \ ,$$

such that $\pi(a) = a$ for all $a \in M$, and for all sequence $(u_i)_{i\in\alpha}$ of M-words indexed by a countable ordinal α,

$$\pi\left(\prod_{i\in\alpha} u_i\right) = \pi\left(\prod_{i\in\alpha} \pi(u_i)\right) \ .$$

This identity is called *generalised associativity*. It can be equivalently expressed as the property that for all M-words u and condensations $\sim$ of its domain, $\pi(u) = \pi(u/^{\pi}_{\sim})$, in which $u/^{\pi}_{\sim}$ (called the *condensation of u by $\sim$*) is the word of domain the condensation classes for $\sim$, ordered as in u, and such that each condensation class C is labelled $\pi(u|_C)$.

From the map π, we can define three *derived operations*: the constant $1 = \pi(\varepsilon)$, the binary operation $a \cdot b = \pi(ab)$, and the unary operation $a^{\omega} = \pi(\overbrace{aaa\cdots}^{\omega})$. It is also known that two finite ordinal monoids that coincide on their carrier set and on the operations 1, $\cdot$ and ω, have the same function π (consequence of Theorem 12 in [2]).

An element $e \in M$ is an *idempotent* if $e \cdot e = e$. It is an *ω-idempotent* if furthermore $e^{\omega} = e$. An idempotent e is *ω-stable* if $e^{\omega} \cdot a = e$ for some $a \in M$.

Ordinal monoids can be used to describe sets of Σ-words as follows. Given a map $h\colon \Sigma \to M$ and a set $F \subseteq M$ (called the *accepting set*), the *language recognised by* M, h, F is the set $\{u \in \Sigma^{\mathrm{ord}} \mid \pi(\tilde{h}(u)) \in F\}$, in which $\tilde{h}$ is the unique extension of h into a ordinal monoid morphism. It is known that being recognisable by a finite ordinal monoid is equivalent to being monadic definable (see Theorems 2.9 and 2.10). In particular, the set of types of quantifier depth at most k form an ordinal monoid [2].

▶ **Example 2.1.** Consider $L \subseteq \{a, b\}^{\mathrm{ord}}$ that contains the words such that:

- the order type of the positions carrying the letter a is ω, and
- the positions carrying the letter a is cofinal in u.

An ordinal monoid that recognises this language (in fact the smallest one) has carrier set $M = \{1, a, b, a^{\omega}, 0\}$ with the intended meaning that a word:

- has value 1 if it is empty,
- has vaule b if it contains only b's, and at least one b,
- has value a if it contains only b's and a's, with at least one occurrence of a, and finitely many of them,
- has value a^{ω} if the order type of a's is ω, and a letters are cofinal in u,
- has value 0 in all other cases.

For showing that this is possible, we define the map h which sends a to a and b to b, and the product $\pi(u)$ for $u \in M^{\mathrm{ord}}$ as:

$$\pi(u) = \begin{cases} 1 & \text{if } u \in \{1\}^{\mathrm{ord}}, \\ b & \text{if } u \in \{1,b\}^{\mathrm{ord}}, \text{ with at least one } b, \\ a & \text{if } u \in \{1,a,b\}^{\mathrm{ord}}, \text{ with at least one } a, \\ & \text{and finitely many of them,} \\ a^\omega & \text{if } u = va^\omega w \text{ with } \pi(v) \in \{1,b,a\} \text{ and } \pi(w) = 1, \text{ or} \\ & u = v_1 v_2 \dots w \text{ with } \pi(v_i) = a \text{ for } i \in \omega \text{ and } \pi(w) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is a tedious verification to check that π satisfies generalised associativity, which is a necessary step for showing that $M = (M, \pi)$ is an ordinal monoid. Finally, set $F = \{a^\omega\}$. One easily sees that M, h, F recognises the language L. The derived operation are presented as a table:

$\cdot$	1	b	a	a^ω	0	$-^\omega$
1	1	b	a	a^ω	0	1
b	b	b	a	a^ω	0	b
a	a	a	a	a^ω	0	a^ω
a^ω	a^ω	0	0	0	0	0
0	0	0	0	0	0	0

One can for instance check that the concatenation of two words that have finitely many a's and at least one, has also this property. Hence $a \cdot a = a$. However, if one concatenates an a to the right of a word in the language, we fall definitely out of the language, hence $a^\omega \cdot a = 0$. Etc. The accepted words are the one of value in the accepting set $F = \{a^\omega\}$.

Given an ordinal monoid M, its *powerset ordinal monoid* $\mathcal{P}(M)$ is the ordinal monoid of carrier set $\mathcal{P}(M)$, and product:

$$\begin{aligned} \pi \colon \mathcal{P}(M)^{\mathrm{ord}} &\to \mathcal{P}(M) \\ U &\mapsto \{\pi(u) \mid u \in^{\mathrm{ord}} U\} \ , \end{aligned}$$

in which $u \in^{\mathrm{ord}} U$ holds if u and U have same domain, and for all positions x, $u(x) \in U(x)$.

Note here that if M, h, F recognise $L \subseteq \Sigma^{\mathrm{ord}}$ and $g \colon \Sigma \to \Gamma$ is extended into the morphism $\tilde{g} \colon \Sigma^{\mathrm{ord}} \to \Gamma^{\mathrm{ord}}$, then $\mathcal{P}(M), \pi \circ \tilde{h} \circ \widetilde{g^{-1}}, \{A \in \mathcal{P}(M) \mid F \cap A \neq \emptyset\}$ recognise the language $\tilde{g}(L) \subseteq \Gamma^{\mathrm{ord}}$. We call it the *g-projection* of L.

We conclude with a classical lemma concerning $\mathcal{P}(M)$:

▶ **Lemma 2.2.** *Let $E \in \mathcal{P}(M)$ be idempotent, and $a \in E^\omega$, then there exist $b, e \in E$ such that e is idempotent and $a = b \cdot e^\omega$.*

Proof. Since $a \in E^\omega$, this means that there exists $u \in^{\mathrm{ord}} \overbrace{EE\cdots}^{\omega}$ such that $\pi(u) = a$. By Ramsey, u can be factorised as $u = u_0 u_1 u_2 \dots$ in which all the $\pi(u_i u_{i+1} \dots u_j)$ for $1 \leq i \leq j$ have same value, say e. Set $b := \pi(u_0)$. Since $e \cdot e = \pi(u_1) \cdot \pi(u_2) = \pi(u_1 u_2) = e$, e is indeed an idempotent. Note that since $E = E \cdot E$ and each u_i is finite, we have $\pi(u_i) \in E$ for all $i \geq 0$. Hence $b \in E$ and $e \in E$. Finally, $b \cdot e^\omega = \pi(bee\cdots) = \pi(\pi(u_0)\pi(u_1)\cdots) = \pi(u_0 u_1 \cdots) = \pi(u) = a$. ◀

2.3 Green's relations and some related results

We recall here the Green relations, that are defined generally on all monoids. We also provide some useful results about Green relations in the context of ordinal monoids. Note that only the $\leqslant_{\mathcal{J}}$-relation will be used outside of this section.

Given a monoid M, the *Green relations* [1] are defined as follows:

$$\begin{array}{rrl} & a \leqslant_{\mathcal{L}} b & \text{if } a = x \cdot b \text{ for some } x \in M, \\ & a \mathrel{\mathcal{L}} b & \text{if } a \leqslant_{\mathcal{L}} b \text{ and } b \leqslant_{\mathcal{L}} a, \\ & a \leqslant_{\mathcal{R}} b & \text{if } a = b \cdot y \text{ for some } y \in M, \\ & a \mathrel{\mathcal{R}} b & \text{if } a \leqslant_{\mathcal{R}} b \text{ and } b \leqslant_{\mathcal{R}} a, \\ & a \leqslant_{\mathcal{J}} b & \text{if } a = x \cdot b \cdot y \text{ for some } x, y \in M, \\ & a \mathrel{\mathcal{J}} b & \text{if } a \leqslant_{\mathcal{J}} b \text{ and } b \leqslant_{\mathcal{J}} a \\ \text{and} & a \mathrel{\mathcal{H}} b & \text{if } a \mathrel{\mathcal{L}} b \text{ and } a \mathrel{\mathcal{R}} b. \end{array}$$

These relations describe the structure of monoids, and in particular of finite monoids, for which we have $\mathcal{J}=\mathcal{L} \circ \mathcal{R}=\mathcal{R} \circ \mathcal{L}$. We do not develop the full theory here, but recall only the key results needed for finite ordinal monoids.

▶ **Lemma 2.3.** *If $e \mathrel{\mathcal{J}} f$ for e, f idempotents, then $e^\omega \mathrel{\mathcal{L}} f^\omega$.*

Proof. Indeed, since $e \mathrel{\mathcal{J}} f$, it is classical that there exists a, b such that $e = a \cdot b$ and $f = b \cdot a$. We have $e^\omega = (a \cdot b)^\omega = a \cdot (b \cdot a)^\omega = a \cdot f^\omega \leqslant_{\mathcal{L}} f^\omega$. The other inequality is by symmetry. ◀

▶ **Lemma 2.4.** *If $a \mathrel{\mathcal{J}} a^\omega$ then $a \mathrel{\mathcal{R}} a^\omega$, a is idempotent and $a \mathrel{\mathcal{H}} b$ implies $a = b$.*

Proof. We have $a^\omega = a \cdot a^\omega \leqslant_{\mathcal{R}} a$, which with $a \mathrel{\mathcal{J}} a^\omega$ implies $a \mathrel{\mathcal{R}} a^\omega$ [1]. Hence $a = a^\omega \cdot y$ for some y. We now have $a \cdot a = a \cdot a^\omega \cdot y = a^\omega \cdot y = a$, and thus a is an idempotent. Assume now $a \mathrel{\mathcal{H}} b$, then, since a is an idempotent, the $\mathcal{H}$-class of a is a group of unit a [1]. Hence $a = b^k$ for some k, and hence $b^\omega = (b^k)^\omega = a^\omega \mathrel{\mathcal{J}} a \mathrel{\mathcal{H}} b$, and hence b is also an idempotent, which means $a = b$ (a group has exactly one idempotent: its unit). ◀

▶ **Lemma 2.5.** *Let $e \mathrel{\mathcal{J}} f$ be idempotents, and a, b elements such that $a \cdot f^\omega \cdot b = e$, then $e^\omega \cdot b = e$.*

Proof. Since $a \cdot f^\omega \cdot b = e$, we have $e \leqslant_{\mathcal{L}} f^\omega \cdot b$. By Lemma 2.3, we also have $f^\omega \mathrel{\mathcal{L}} e^\omega$. Hence $e \leqslant_{\mathcal{L}} e^\omega \cdot b$. Since we also have $e^\omega \cdot b \leqslant_{\mathcal{R}} e$, we get $e^\omega \cdot b \mathrel{\mathcal{H}} e$. Together with Lemma 2.4, we get $e^\omega \cdot b = e$. ◀

▶ **Lemma 2.6.** *If e and f are idempotent and $a \cdot f^\omega \cdot b = e$ and $e^\omega \cdot b \neq e$, then $e <_{\mathcal{J}} f$.*

Proof. Since $a \cdot f^\omega \cdot b = e$, we have $e \leqslant_{\mathcal{J}} f$. Let us assume for the sake of contradiction that $e \mathrel{\mathcal{J}} f$. By Lemma 2.5, we would get $e^\omega \cdot b = e$. This is a contradiction. ◀

2.4 Monadic Second-Order Logic of Order

We use standard notations and terminology for monadic second-order logic of order.

The *monadic second-order logic of order* (also called *monadic logic of order* or simply *monadic logic*, abbreviated as MSO) is an extension of first-order logic over the signature $\{<\} \cup \{P_1, \ldots, P_k\}$, where $<$ is a binary relation symbol interpreted as a total order, and each P_i is a *monadic predicate symbol*.

The logic uses two types of variables: first-order variables $x, y, \ldots$ ranging over elements of the domain, and second-order (monadic) variables $X, Y, \ldots$ ranging over subsets of the domain.

The atomic formulas are of the form $x < y$, $x \in X$, and $P_i(x)$. Formulas are built from atomic formulas using Boolean connectives, first-order quantifiers $\forall x.\, \psi$ and $\exists x.\, \psi$, and second-order quantifiers $\forall X.\, \psi$ and $\exists X.\, \psi$.

In this paper, we interpret this logic exclusively over ordinal words.

A Σ-word u over an ordinal α is identified with the Σ-labeled linear order $\mathcal{M} = (\alpha, <, \{P_\sigma \mid \sigma \in \Sigma\})$, where each predicate symbol P_σ is interpreted as the set $\{\beta \in \alpha \mid u(\beta) = \sigma\}$.

We write $\mathcal{M} \models \varphi(U_1, \ldots, U_k, v_1, \ldots, v_\ell)$ if the *monadic formula* $\varphi(X_1, \ldots, X_k, y_1, \ldots, y_\ell)$ holds in the structure $\mathcal{M}$ under the valuation that assigns each X_i the set U_i and each y_j the element v_j.

A relation $R \subseteq \big(\mathcal{P}(\alpha)\big)^k \times \alpha^\ell$ is said to be *definable by* a formula φ in a model $\mathcal{M}$ if

$$R = \{(U_1, \ldots, U_k, v_1, \ldots, v_\ell) \mid \mathcal{M} \models \varphi(U_1, \ldots, U_k, v_1, \ldots, v_\ell)\}.$$

We say that a relation is MSO-definable in $\mathcal{M}$ if it is definable by an MSO formula.

We shall use the classical technique of *relativisation* of formulas.

▶ **Lemma 2.7** (relativisation). *Let $\varphi(Y_1, \ldots, Y_\ell)$ be an* MSO *formula and let U be a fresh variable not appearing in φ. Then we can compute a formula $\varphi^U(Y_1, \ldots, Y_\ell, U)$ such that for every model $\mathcal{M}$, every non-empty subset $D \subseteq M$, and every ℓ-tuple $\bar{P}$ of subsets of D,*

$$\mathcal{M} \models \varphi^U(\bar{P}, D) \quad \text{if and only if} \quad \mathcal{M}|_D \models \varphi(\bar{P}),$$

where $\mathcal{M}|_D$ is the restriction of $\mathcal{M}$ to the domain D.

When this holds, we say that φ holds in $(\mathcal{M}, \bar{P})$ *relativised to D*.

For convenience, we use overlined variables like $\bar{X}$, $\bar{Y}$ to denote tuples of monadic variables. We also allow shorthand such as $X \subseteq Y$ to denote set inclusion, and more generally use any informal abbreviation when it is clear that it corresponds to a valid MSO formula. For example, we write expressions such as “X is cofinal” or “X has order type ω”.

We sometimes use variables F that represent functions mapping the domain of a structure to a finite alphabet Δ. Such a function is encoded by a Δ-tuple of monadic predicate variables $(X_\delta)_{\delta \in \Delta}$, where each X_δ is interpreted as the preimage $F^{-1}(\delta)$. Formally, $z \in X_\delta$ if and only if $F(z) = \delta$.

The following theorem, due to Büchi and Siefkes [3], summarizes the decidability results of MSO over countable ordinals:

▶ **Theorem 2.8.** *The* MSO*-theory of the class of countable ordinals is decidable. Moreover, the* MSO*-theory of every countable ordinal is decidable.*

The following two theorems state the effective equivalence between the expressive power of MSO over countable ordinal words and recognisability by ordinal monoids.

▶ **Theorem 2.9** ([5]). *Let M be an ordinal monoid. For every $a \in M$, there is an MSO-sentence φ_a such that for every M-word v:*

$$v \models \varphi_a \quad \text{if and only if} \quad \pi(v) = a.$$

▶ **Theorem 2.10** ([2]). *Let Φ be an MSO-sentence, there exists effectively an ordinal monoid M, a map $h\colon \Sigma \to M$, and a set $F \subseteq M$ such that the language recognised by M, h, F is the set of Σ-words that model Φ.*

3 Simple Words and Height-Decreasing Condensation

In this section, we introduce a tool that reduces certain problems concerning “M-words” to more manageable cases: words of length at most ω, and the so-called E-simple (“idempotent-simple”) words. Our approach is inspired by Simon’s factorisation forest theorem [15], which we adapt here in terms of successive condensations. The resulting height-decreasing condensation theorem, Theorem 3.4, will play a central role in the proof of our main result, Theorem 5.1.

We begin with the introduction of e-simple words.

▶ **Definition 3.1** (e-simple words). *Let $e \in M$ be an idempotent and let $S \subseteq M$ be such that $e^\omega \cdot a = e$ for every $a \in S$. A word over $\{e\} \cup S$ is called e-S-simple if the following conditions hold:*

1. *the first letter is e, and every successor position is labelled by e,*
2. *every limit position is labelled by a letter from S.*

A word is e-simple if it is e-S-simple for some S. A word is simple if it is e-simple for some idempotent e.

A first lemma states that e-simple are indeed “simple.”

▶ **Lemma 3.2.** *Let u be an e-simple word. Then:*

$$\pi(u) = \begin{cases} e, & \text{if the length of } u \text{ is a successor ordinal,} \\ e^\omega, & \text{if the length of } u \text{ is a limit ordinal.} \end{cases}$$

Given some word $u \in M^{\mathrm{ord}+}$ and a condensation $\sim$ of its domain, we denote by $u/_\sim$ a word that has the $\sim$-condensation classes as positions, ordered in the natural way, and each $\sim$-condensation class C is mapped to $\pi(u|_C)$.

▶ **Definition 3.3** (Definable condensation). *Let $\theta(X)$ be a formula with a free variable X that ranges over intervals. Let $\mathcal{M}$ be a structure. The family of intervals defined by θ on $\mathcal{M}$ is:*

$$Int(\mathcal{M}, \theta) := \{S \text{ is an interval on the domain of } \mathcal{M} \mid \mathcal{M} \models \theta(S)\}.$$

θ defines[2] a condensation on $\mathcal{M}$ if $Int(\mathcal{M}, \theta)$ is a partition of the domain of $\mathcal{M}$. θ defines a condensation on a class of structures $\mathcal{C}$ if θ defines a condensation on every $\mathcal{M} \in \mathcal{C}$.

Theorem 3.4, that we call the *height-decreasing condensation theorem*, is a variant of Theorem 3 in [6], rephrased in terms of MSO-definable condensations. The motivation for replacing Ramsey splits in the original statement with MSO-definable condensations is that condensations are significantly more convenient to manipulate within a logical framework.

Let $(I, <)$ be a linear order. A family $\mathcal{F}$ of non-empty intervals on I is *tree like* if $I \in \mathcal{F}$ and for every $J_1, J_2 \in \mathcal{F}$ either $J_1 \cap J_2 = \emptyset$ or one is included in the other. A tree like family $\mathcal{F}$ *defines* an ordered tree $(\mathcal{F}, \sqsubseteq_\mathcal{F}, <)$, where the ancestor order $\sqsubseteq_\mathcal{F}$ is defined as: $J_1 \sqsubseteq_\mathcal{F} J_2$ if $J_1 \supseteq J_2$ (so that I is the minimal element, and hence the root of the tree). We say that J' is a *child* of J if $J \sqsubset_\mathcal{F} J'$ and there exist no $\mathcal{F}$-interval J'' such that $J \sqsubset_\mathcal{F} J'' \sqsubset_\mathcal{F} J'$. The children of a node are ordered by $\leq$ defined as: $J < J'$ if $a \in J$ and $a' \in J'$ implies

[2] Let $\Theta(y, z)$ be a formula that defines a convex equivalence relation on every $\mathcal{M} \in \mathcal{C}$. Let $\theta(X)$ express that “X is an equivalence class defined by $\Theta(y, z)$ in $\mathcal{M}$.” The family of intervals defined by $\theta(X)$ in $\mathcal{M}$ is the same as the family of equivalence classes defined by θ.

$a < a'$. It is clear that $<$ linearly orders the children of a node. A tree like family $\mathcal{F}$ is a *tree decomposition* of $(I, <)$ if for every $J \in \mathcal{F}$ that contains at least two elements, the union of the children of J is J, and $\mathcal{F}$ contains all one point intervals (that we call the *leaves* of the tree). A tree has *height* $n \in \mathbb{N}$ if the branch of maximal size contains $n + 1$ nodes.

Let u be an M-word over the domain $(I, <)$ and J be an interval of I. The value of J is $\mathit{Val}(J) := \pi(u|_J)$.

Theorem 3.4 implies that for every finite ordinal monoid M there is n_M such that for all M-words u over an ordinal $(\alpha, <)$ there is a tree decomposition $\mathcal{F}$ of $(\alpha, <)$ of height at most n_M and such that for all $J \in \mathcal{F}$ with children

$$J_0 < J_1 < \cdots < J_\gamma < \cdots \qquad (\gamma \in \beta)$$

either $\beta \leq \omega$ or the M-word

$$\mathit{Val}(J_0)\,\mathit{Val}(J_1) \cdots \mathit{Val}(J_\gamma) \cdots$$

is E-simple for some E.

In addition Theorem 3.4 explains how to provide an MSO description of $\mathcal{F}$ in u.

▶ **Theorem 3.4** (Height-decreasing condensation). *For every finite ordinal monoid M, there exist $n \in \mathbb{N}$, a monadically definable map*

$$\mathrm{height}\colon M^{\mathrm{ord}+} \to [n],$$

and a monadically definable condensation $\sim$, such that for all $u \in M^{\mathrm{ord}+}$, the following hold:

- *$u/_\sim$ is either simple or has order type at most ω; and*
- *either $|u| = 1$, or $\mathrm{height}(u|_C) < \mathrm{height}(u)$ for all $\sim$-equivalence classes C.*

Moreover, n and MSO formulas defining height and $\sim$ are effectively computable.

4 Uniformisation and its Algebraic Reformulation

In this section, we introduce the core definitions related to uniformisation, along with their reformulations in algebraic terms (see Lemma 4.4). These notions will be used in Section 5, where we show that the uniformisability of MSO can be expressed within the extended logic $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$ (see Theorem 5.1).

Recall that a *uniformiser* for a binary relation $R \subseteq A \times B$ is a subrelation $f \subseteq R$ that is the graph of a partial function from A to B with the same domain as R.

Let $\mathcal{M}$ be a structure over a domain D, and let

$$\varphi(X_1, \ldots, X_\ell, Y_1, \ldots, Y_k)$$

be a formula. Then φ *defines* a relation

$$R_\varphi \subseteq \mathcal{P}(D)^\ell \times \mathcal{P}(D)^k.$$

A formula $\psi(X_1, \ldots, X_\ell, Y_1, \ldots, Y_k)$ *uniformises* φ in $\mathcal{M}$ if the relation it defines is a uniformiser of the relation defined by φ in $\mathcal{M}$. We say that ψ is a *uniformiser for φ over a class $\mathcal{C}$* of structures if ψ uniformises φ in every $\mathcal{M} \in \mathcal{C}$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two logics. We say that a structure $\mathcal{M}$ (or a class of structures $\mathcal{C}$) has the *$\mathcal{L}_1$-$\mathcal{L}_2$ uniformisation property* if for every $\mathcal{L}_1$-formula φ, there exists an $\mathcal{L}_2$-formula ψ that uniformises φ in $\mathcal{M}$ (respectively, over every $\mathcal{M} \in \mathcal{C}$). The problem of deciding whether an $\mathcal{L}_1$-formula φ admits an $\mathcal{L}_2$-formula ψ that uniformises it is called the $\mathcal{L}_1$-$\mathcal{L}_2$ *uniformisation problem*.

An important instance in the development of this paper is the notion of uniformisers for the formula $\omega\mathrm{cof}(X,Y)$, which expresses that Y has order type ω and is cofinal subset of X:

$$\omega\mathrm{cof}(X,Y) := Y \subseteq X \wedge \mathtt{OrderType}{-}\omega(Y) \wedge \mathtt{CoFinal}(Y,X)\,.$$

in which $\mathtt{OrderType}{-}\omega(X)$ is an MSO-formula that expresses that the order type of X is ω, and $\mathtt{CoFinal}(Y,X)$ is an MSO-formula expressing that Y is cofinal in X.

▶ **Theorem 4.1** ([9]). *There is no monadic uniformiser for $\omega\mathrm{cof}(X,Y)$ over countable ordinal chains.*

Since $\omega\mathrm{cof}(X,Y)$ is definable by an MSO formula, Theorem 4.1 implies that the class of countable ordinal chains does not have the MSO–MSO uniformisation property.

On the other hand, our main result, Theorem 1.1, can be rephrased as stating that the class of countable ordinals has the MSO–MSO($\mathrm{U}_{\omega\mathrm{cof}}$) uniformisation property.

Algebraic reformulation

We now reformulate the uniformisability question for MSO using the framework of ordinal monoids.

Given two alphabets Σ and Γ and a class $\mathcal{C}$ of Σ-words, a *relabelling* R from $\mathcal{C}$ to Γ-words is a function that maps each Σ-word in $\mathcal{C}$ to a Γ-word with the same domain.

Let $\mathcal{L}$ be a logic (typically MSO or its extension MSO($\mathrm{U}_{\omega\mathrm{cof}}$)). We say that R is a *$\mathcal{L}$-relabelling* (for $\mathcal{C}$ and Γ) if there exists a family of $\mathcal{L}$-formulas $(\gamma_a(x))_{a\in\Gamma}$ such that, for every Σ-word $u \in \mathcal{C}$ and every position x in the domain of u, there exists a unique $a \in \Gamma$ such that $u \models \gamma_a(x)$. In this case, the *relabelled word $R(u)$* is defined as the Γ-word with the same domain as u, where $R(u)(x) = a$ if $u \models \gamma_a(x)$ for all x in the domain.

Alternatively, an *$\mathcal{L}$-relabelling* can be defined as follows. Let $\Phi(G)$ be an $\mathcal{L}$-formula where G is a function variable ranging over maps from the domain of u to Γ. We say that R is definable by $\Phi(G)$ if, for every $u \in \mathcal{C}$, there exists a unique function G such that $u \models \Phi(G)$, and this G coincides with $R(u)$.

Let $\mathcal{L}$ be MSO, MSO($\mathrm{U}_{\omega\mathrm{cof}}$), or a reasonable extension of MSO. The following statement shows that the two descriptions above coincide.

▶ **Lemma 4.2.** *A relabelling R is $\mathcal{L}$-definable by a formula $\Phi(G)$ if and only if it is $\mathcal{L}$-definable by a family of formulas $(\gamma_a(x))_{a\in\Gamma}$.*

We will describe relabellings either by a single $\mathcal{L}$-formula $\Phi(G)$ or by a family of formulas $(\gamma_a(x))_{a\in\Gamma}$, depending on context.

A *monadic relabelling* refers to an MSO-relabelling.

▶ **Definition 4.3** (Algebraic uniformiser). *A relabelling R is an algebraic uniformiser for $F \subseteq M$, for a $\mathcal{P}(M)$-word U, if*

$$\pi(U) \cap F \neq \emptyset \quad \textit{implies} \quad R(U) \in^{\mathrm{ord}} U \textit{ and } \pi(R(U)) \in (\pi(U) \cap F).$$

R is an algebraic uniformiser for $F \subseteq M$ over a class $\mathcal{C}$ of $\mathcal{P}(M)$-words if it is an algebraic uniformiser for every $U \in \mathcal{C}$.

An algebraic uniformiser R over a class $\mathcal{C}$ of $\mathcal{P}(M)$-words is called an algebraic $\mathcal{L}$-uniformiser if R is a $\mathcal{L}$-relabelling.

For $\Sigma \subseteq \mathcal{P}(M)$ and $F \subseteq M$, we say that R is an algebraic $\mathcal{L}$-uniformiser for M, Σ, F if R is an algebraic $\mathcal{L}$-uniformiser for F over the class of Σ-words.

R is an algebraic uniformiser (respectively, an algebraic $\mathcal{L}$-uniformiser) if it is an algebraic uniformiser (respectively, an algebraic $\mathcal{L}$-uniformiser) over the class of all $\mathcal{P}(M)$-ordinal words.

The following lemma reformulates the notion of $\mathcal{L}$-uniformisation in terms of algebraic $\mathcal{L}$-uniformisers, using classical constructions.

▶ **Lemma 4.4** (Logical to Algebraic Uniformisation)**.** *For every* MSO *formula*

$$\varphi(X_1, \dots, X_\ell, Y_1, \dots, Y_k) \ ,$$

one can effectively construct an ordinal monoid M, a subset $\Sigma \subseteq \mathcal{P}(M)$, and a subset $F \subseteq M$, such that the following statements are effectively equivalent:

- *there exists an $\mathcal{L}$-uniformiser for φ;*
- *there exists an algebraic $\mathcal{L}$-uniformiser for M, Σ, F.*

Proof. Consider an MSO formula $\varphi(\overline{X}, \overline{Y})$, where $\overline{X} = \langle X_1, \dots, X_\ell \rangle$ and $\overline{Y} = \langle Y_1, \dots, Y_k \rangle$.

We define the language L_φ over the alphabet $\Gamma = \{0,1\}^{\ell+k}$ as:

$$L_\varphi := \{\langle a_1, \dots, a_\ell, b_1, \dots, b_k \rangle \in \Gamma^\alpha \mid \begin{array}{l} \alpha \text{ is an ordinal and} \\ \alpha, \overline{A}, \overline{B} \models \varphi(\overline{X}, \overline{Y})\}, \end{array}$$

where $A_i \subseteq \alpha$ contains those positions β such that the β-th letter of the word has $a_i = 1$, and B_j is defined similarly.

It is well known that L_φ is recognisable by some finite ordinal monoid M, a morphism $h : \Gamma \to M$, and an accepting set $F \subseteq M$; moreover, M, h, and F are effectively computable from φ.

Define the map:

$$H : \{0,1\}^\ell \to \mathcal{P}(M), \quad H(\overline{a}) := \left\{ h(\overline{a}, \overline{b}) \mid \overline{b} \in \{0,1\}^k \right\},$$

and set

$$\Sigma := \left\{ H(\overline{a}) \mid \overline{a} \in \{0,1\}^\ell \right\} \subseteq \mathcal{P}(M).$$

We now show that M, Σ, F satisfy the conditions of the lemma.

We say that $\overline{a} \in \{0,1\}^\ell$ is *consistent* with $\sigma \in \Sigma$ if $H(\overline{a}) = \sigma$. Since H is not necessarily injective, we fix a linear order $<$ on $\{0,1\}^\ell$ and say that $\overline{a}$ is *min-consistent* with σ if $H(\overline{a}) = \sigma$ and $\overline{a}$ is minimal among all such tuples. A sequence of subsets $X_1, \dots, X_\ell \subseteq \alpha$ is said to be *min-consistent* with an α-word U over Σ if for every $\beta < \alpha$, the tuple $\langle X_1(\beta), \dots, X_\ell(\beta) \rangle$ is min-consistent with $U(\beta)$. It is easy to verify that both consistency and min-consistency are MSO-definable.

Direct implication. Assume that $\psi(\overline{X}, \overline{Y})$ is a $\mathcal{L}$-uniformiser for φ.

We define a family of relabelling formulas $\gamma = \{\gamma_a(z) \mid a \in M\}$ over Σ-words by:

$$U \models \gamma_a(z) \iff \begin{array}{l} \text{there exists } \overline{X} \text{ min-consistent with } U \text{ and } \overline{Y} \\ \text{such that } \psi(\overline{X}, \overline{Y}) \text{ holds,} \\ \text{and } h(X_1(z), \dots, X_\ell(z), Y_1(z), \dots, Y_k(z)) = a. \end{array}$$

If $\mathcal{L}$ extends MSO and ψ is $\mathcal{L}$-definable, then so is each γ_a, hence γ defines a $\mathcal{L}$-relabelling.

Moreover, if $\pi(U) \cap F \neq \emptyset$, then $\pi(u) \in \pi(U) \cap F$ for u defined by relabelling γ.

Reverse implication. Suppose now that γ is an algebraic $\mathcal{L}$-uniformiser for M, Σ, F.

Given α and an Σ-word U of domain α induced by $\overline{X}$ (i.e., $U(\beta) = H(\overline{X}(\beta))$), let u be the relabelling of U by γ.

We define $\overline{Y}(\beta) := \langle Y_1(\beta), \dots, Y_k(\beta)\rangle$ to be the *lexicographically minimal* tuple $\overline{c} \in \{0,1\}^k$ such that:

$$h(\overline{X}(\beta), \overline{c}) = u(\beta).$$

Since this choice is deterministic and based on an effective monoid morphism, it can be defined by a $\mathcal{L}$-formula $\psi'(\overline{X}, \overline{Y})$.

If $\pi(U) \cap F \neq \emptyset$, then $\pi(u) \in F$, and hence $\alpha, \overline{X}, \overline{Y} \models \varphi$.

Thus, the formula $\psi'(\overline{X}, \overline{Y}) \wedge \varphi(\overline{X}, \overline{Y})$ is a $\mathcal{L}$-uniformiser for φ. ◀

▶ **Corollary 4.5.** *If, for every ordinal monoid M and every element $a \in M$, there exists an algebraic $\mathcal{L}$-uniformiser for $(M, \mathcal{P}(M), \{a\})$ over countable words, then every* MSO*-formula admits an $\mathcal{L}$-uniformiser.*

Proof. Assume the assumptions of Corollary 4.5 hold. For each $a \in M$, let R_a be an algebraic $\mathcal{L}$-uniformiser for $(M, \mathcal{P}(M), \{a\})$ over countable words. By Lemma 4.4, it suffices to show that for every $F \subseteq M$, there exists an algebraic $\mathcal{L}$-uniformiser for $(M, \mathcal{P}(M), F)$ over countable words. Fix an arbitrary linear order $\prec$ on the elements of M. Define the uniformiser R_F as follows: for each input U, let $R_F(U) := R_a(U)$, where $a \in F$ is the $\prec$-minimal element such that $R_a(U)$ is defined; if no such a exists, then $R_F(U)$ is undefined. Clearly, R_F is algebraic and $\mathcal{L}$-definable, and it uniformises $(M, \mathcal{P}(M), F)$ as required. ◀

5 Uniformising with a built-in uniformiser for ω-cofinality

In this section, we establish our main theorem, Theorem 5.1, which states that every MSO-formula admits an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-uniformiser.

5.1 Statement and structure of the proof

We denote by $\mathrm{U}_{\omega\mathrm{cof}}(X)$ a uniformiser for $\omega\mathrm{cof}(X, Y)$, i.e., a partial function that, given a set X, returns (when it exists) a subset $Y \subseteq X$ such that Y has order type ω and is cofinal in X. Let $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$ denote monadic second-order logic extended with the ability to use the map $\mathrm{U}_{\omega\mathrm{cof}}$; that is, $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$ extends MSO by allowing atomic formulas of the form $\mathrm{U}_{\omega\mathrm{cof}}(X, Y)$. Note of course that the semantic of this logic depends on the choice of $\mathrm{U}_{\omega\mathrm{cof}}$, and that in this respect the logic is not completely specified. This is not a problem in our case since in this paper, we only build formulae in this logic that exhibit the expected behaviour whatever choice of $\mathrm{U}_{\omega\mathrm{cof}}$ is made.

This section is devoted to the proof of the main result of our paper:

▶ **Theorem 5.1.** *For all monadic formulae $\varphi(\overline{X}, \overline{Y})$, there exists an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-uniformiser for φ over the class of countable ordinals, and it can be constructed effectively.*

▶ **Definition 5.2** ($\mathcal{L}$-relabelling property)**.** *Let M be a finite ordinal monoid, and $\mathcal{C}$ be a class of $\mathcal{P}(M)$-words. We say $\mathcal{C}$ has the $\mathcal{L}$-relabelling property if for every $a \in M$, there is an $\mathcal{L}$-relabelling R_a such that for every $U \in \mathcal{C}$ if $a \in \pi(U)$, then $R_a(U)(x) \in U(x)$ for every position x, and $\pi(R_a(U)) = a$. We say that M has the $\mathcal{L}$-relabelling property if the class of all countable $\mathcal{P}(M)$-words has the $\mathcal{L}$-relabelling property.*

In view of Corollary 4.5, the proof of our main result reduces to establishing the following theorem:

▶ **Theorem 5.3** ($\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property)**.** *Every finite ordinal monoid has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property.*

▶ Remark 5.4 (Computability). Our proof of Theorem 5.3 is constructive. One can extract from the proof an algorithm that for every M and $a \in M$ constructs an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling R such that for all $\mathcal{P}(M)$-words U if $a \in \pi(U)$, then $R(U) \in^{\mathrm{ord}} U$ and $\pi(R(U)) = a$.

The proof of Theorem 5.3 uses two key ingredients: (1) the treatment of the special case of simple words, which is dealt with in Section 5.2, and (2) the use Theorem 3.4 and the known case of words of length ω (Lemma 5.7) for extending it to the general case, which is done in Section 5.3

From now, the ordinal monoid M is fixed.

In the remainder of the proof, all lemmas follow a similar pattern, essentially corresponding to instances of Theorem 5.3 for restricted classes of words U and elements a. We also adopt an informal description of the relabellings. We illustrate this informal description in the simplest case of two-letter words:

▶ **Lemma 5.5.** *For all $a \in M$, there exists a monadic relabelling R such that for all $\mathcal{P}(M)$-words U of length 2 such that $a \in \pi(U)$, $R(U) \in^{\mathrm{ord}} U$, and $\pi(R(u)) = a$.*

Proof. Since U has length 2, $U = BC$ for $B, C \in \mathcal{P}(M)$. Since $a \in \pi(U) = B \cdot C$, there exists $b \in B$, $c \in C$, that we can assume fixed given B and C, such that $b \cdot c = a$. We set $R(U) = bc$. This clearly satisfies the conclusion of the lemmas (see following remark). ◀

▶ Remark 5.6. In the above proof, we have used an informal description of the relabelling. Writing in detail the formulae would be straightforward, but more tedious. Let us do it in the particularly simple case of Lemma 5.5. First, we assume fixed $b_{a,B,C}$ and $c_{a,B,C}$ for all a, B, C, which are defined if $a \in B \cdot C$, and such that $b_{a,B,C} \in B$, $c_{a,B,C} \in C$, and $b_{a,B,C} \cdot c_{a,B,C} = a$. The formula that describes when and where the relabelling R decide to output the letter $d \in M$ is:

$$\begin{aligned}\gamma_d(x) := \exists y. &\left(\mathrm{Two}(x,y) \wedge \bigvee_{b_{a,B,C}=d} B(x) \wedge C(y)\right)\\ \vee &\left(\mathrm{Two}(y,x) \wedge \bigvee_{c_{a,B,C}=d} B(y) \wedge C(x)\right)\end{aligned}$$

in which $\mathrm{Two}(x, y)$ holds over U if its domain has exactly size 2, the first position is x, and the second is y.

Alternatively, we can define the relabelling using a formula $\Phi(u)$, where u ranges over M-words of length two.

Let $\prec$ be any linear order on M, and let $\prec_2$ be its lexicographic extension to M^2, defined by $(a, b) \prec_2 (a', b') \iff a \prec a' \quad \text{or} \quad a = a'$ and $b \prec b'$. Let $\Phi(u)$ express that u is $\prec_2$-minimal among those satisfying condition, say $u(1)u(2) = a$, where 1 (respectively, 2) is the minimal (respectively, maximal) element of the domain of u. Then the formula $\Phi(u)$ defines the desired relabelling.

In the rest of the paper, we systematically adopt this informal way to describe the relabelling, leaving to the reader the task to check the definability in the expected logic.

In fact, we know from the results of Lifsches and Shelah [9] how to solve a more general case.

▶ **Lemma 5.7** (uniformisation for finite and ω-words)**.** *For all $a \in M$, there exists a monadic relabelling R such that for all $\mathcal{P}(M)$-words U of length at most ω such that $a \in \pi(U)$, $R(U) \in^{\mathrm{ord}} U$, and $\pi(R(u)) = a$. Moreover, the same statement holds when ω is replaced by any ordinal $\alpha < \omega^\omega$.*

5.2 The case of simple words

The goal of this section is to establish Theorem 5.3 for E-simple $\mathcal{P}(M)$-words (see Definition 3.1), namely:

▶ **Lemma 5.8** ($\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property for E-simple words)**.** *Let M be a finite ordinal monoid, and $E \in \mathcal{P}(M)$ be an idempotent, the class of E-simple $\mathcal{P}(M)$-words has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property.*

We have to prove that for every $a \in E$ (respectively, $a \in E^\omega$), there exists an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling R_a such that $R_a(U) = a$ for all E-simple words U of successor length (respectively, of limit length).

We first consider a very simple case where a is ω-insensitive to all letters in the word (see Definition 5.9 and Lemma 5.10). We then extend the construction to the case where a is idempotent (Lemma 5.12). Finally, we construct R_a for an arbitrary $a \in E$ (Lemma 5.13).

▶ **Definition 5.9** (ω-insensitive)**.** *An idempotent e is ω-insensitive to $A \in \mathcal{P}(M)$ if there exists $a \in A$ such that $e^\omega \cdot a = e$. Otherwise e is ω-sensitive to A.*

A *renaming* is a function from an alphabet Σ to an alphabet Δ. A renaming ρ is pointwise extended to a function from words over Σ to words over Δ, which also will be denoted by ρ. It is clear that such a function is an instance of monadic relabelling.

▶ **Lemma 5.10** (ω-insensitive case)**.** *Assume that an idempotent $e \in E$ is ω-insensitive to every $A \in S$. Then there is a renaming R_e that maps every E-S-simple word of limit length to e^ω and every E-S-simple word of successor length to e.*

Proof. For every $A \in S$ there is $e_A \in A$ such that $e^\omega \cdot e_A = e$. Define R_e as follows: $R_e(E) = e$ and for $A \in S$, $R_e(A) = e_A$. It is easy to verify, by transfinite induction, that R_e has the expected properties. ◀

We shall now extend Lemma 5.10 by removing the assumption of ω-insensitivity. Note that the renaming is now replaced by an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling. The next lemma makes explicit a key argument which state that in the case of an element e which is ω-sensitive to some A, if A occurs, then the relabelling problem reduces to some idempotent f which is $\mathcal{J}$-above.

▶ **Lemma 5.11.** *Let A be such that $E^\omega \cdot A = E$, and $e \in E$ be an idempotent wich is ω-sensitive to A. Then there exist $b, f \in E$ and $a \in A$ such that f is idempotent, $e <_{\mathcal{J}} f$ and $b \cdot f^\omega \cdot a = e$.*

Proof. Since $E^\omega \cdot A = E$, $e \in E$ and E is idempotent, this means that there exist $b, f \in E$ and $a \in A$ such that f is idempotent and $c \cdot f^\omega \cdot a = e$. By Lemma 2.6 we get $e <_{\mathcal{J}} f$. ◀

▶ **Lemma 5.12.** *Let $e \in E$ be an idempotent, there exists an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling R_e that maps every E-simple word of limit length to a word of value e^ω, and every E-simple word that has a last letter to a word of value e.*

Proof. We prove the lemma by induction on the $\leqslant_{\mathcal{J}}$-order of e. Let $e \in E$ be an idempotent. We assume that an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling R_f that satisfy the conclusions of the lemma is already known for all idempotents $f \in E$ with $f >_{\mathcal{J}} e$, and we have to construct an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling R_e as expected.

Let U be an E-simple word; we proceed by case distinction. Note that determining which case applies is expressible in MSO. In other words, for each case, there is an MSO-sentence that holds exactly when that case applies.

Case 1 If e is ω-insensitive to all A that appear at limit positions, then Lemma 5.10 provides a suitable relabelling R_e.

Case 2 If $U = VA$ for V that has a limit domain and some A such that e is ω-sensitive to A. We aim at defining $R_e(U)$.
The relabelling $R_e(U)$ first $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-defines a factorisation of V into E followed by ω factors $(U_i)_{i\in\omega}$ that are simple and have a successor domain:

$$V = E\,U_0\,U_1\ldots$$

This is possible since $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$ has the builtin possibility to define an ω-cofinal sequence. Note that by Lemma 3.2 $\pi(U_i) = E$ for all $i \in \omega$.
We know by Lemma 5.11 that there exist $b, f \in E$ and $a \in A$ such that f is idempotent, $e <_{\mathcal{J}} f$ and $b \cdot f^\omega \cdot a = e$.
The relabelling $R_e(U)$ chooses arbitrarily such a triple b, f, a (for instance the minimal that fulfills the conclusion for some arbitrary total order). In particular, since $e <_{\mathcal{J}} f$, by induction hypothesis, we have an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling R_f such that $\pi(R_f(U_i)) = f$ for all $i \in \omega$.
The relabelling R_e then simply outputs the word:

$$R_e(U) := b\,R_f(U_0)\,R_f(U_1)\,\cdots\,a\ .$$

Clearly, the above description is definable in $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$. We also immediately have $R_e(U) \in^{\mathrm{ord}} EU_0U_1\cdots A = U$. Finally, we have $\pi(R_e(U)) = b \cdot f^\omega \cdot a = e$. The induction step is established.

Case 3 If U has a limit domain and e is ω-sensitive to some A that occurs at positions that are cofinal in the domain of U. It means that U can be decomposed as an ω-sequence

$$U = U_0\,A\,U_1\,A\,\cdots$$

where the U_i's all have a limit ordinal domain.
Note that such a decomposition is $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-definable, again thanks to the builtin possibility to define an ω-cofinal sequence.
Using the second case, we have an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling R'_e such that $R'_e(U_iA) \in^{\mathrm{ord}} U_iA$ and $\pi(R'_e(U_iA)) = e$ for all $i \in \omega$.
Our relabelling R_e applied to U is defined in order to produce

$$R_e(U) := R'_e(U_0\,A)\,R'_e(U_1\,A)\,\cdots$$

Clearly, the above description is definable in $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$. We also immediately have $R_e(U) \in^{\mathrm{ord}} U_0AU_1A\cdots = U$. Finally, we have $\pi(R_e(U)) = e^\omega$. The induction step is established.

Case 4 If U can be decomposed as $U = VBW$, in which V is subject to the third case, and W to the first one, and e is insensitive to $B \in \mathcal{P}(M)$. Such a decomposition is MSO-definable (take the shortest V), and we define the relabelling as $R_e(U) := R'_e(V)e_BR''_e(W)$ where R'_e is the relabelling of the third case and R''_e the one of the first case, and $e_B \in B$ satisfies $e^\omega \cdot e_B = e$.

Case 5 The last case is when $U = VAW$, where VA is subject to the second case, A is such that e is ω-sensitive to A, and W is subject to the first case. Such a decomposition is MSO-definable (take the shortest V). We set $R_e(U) := R'_e(VA)R''_e(W)$ in which R'_e is the relabelling obtained from the second case, and R''_e is the one from the first case. ◀

What is left now is to remove the idempotency assumption for e. This is the subject of the next lemma.

▶ **Lemma 5.13.** *There exists an* $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$*-relabelling* R_a *such that for all* E*-simple words* U *if* $a \in \pi(U)$*, then* $R_a(U) \in^{\mathrm{ord}} U$ *and* $\pi(R_a(U)) = a$*.*

Proof. We proceed by case distinction.

Case 1 If U has a limit domain, then by Lemma 3.2, $\pi(U) = E^\omega$. By Lemma 2.2, there exist $b, e \in E$ such that e is an idempotent and $a = b \cdot e^\omega$. We then proceed as for the second case of Lemma 5.12. We fix such b and e, and use the predicate $\mathrm{U}_{\omega\mathrm{cof}}$ of $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$ to define a decomposition of U as

$$E\, U_0\, U_1\, \ldots,$$

where each U_i has a domain of successor length. We then define

$$R_a(U) := b\, R_e(U_0)\, R_e(U_1)\, \ldots,$$

where R_e is the relabelling map from Lemma 5.12. It is straightforward to verify that this relabelling satisfies the conclusion of the lemma.

Case 2 If U has finite length, one can use any relabelling that works for all finite words (Eg, choose the lexicographically least choice of letters such that it evaluates to a).

Case 3 The last case is when $U = VAW$ for V of limit length and AW of finite length (this decomposition is unique and is MSO-definable). Since $a \in E = \pi(V) \cdot \pi(AW)$ there are $b \in \pi(V) = E^\omega$ and $c \in \pi(AW)$ such that $a = b \cdot d$. Define $R_a(U) := R_b(V)\ R_c(AW)$ where R_b is the relabelling of the first case, and R_c the one of the second case. This relabelling clearly fulfills the conclusions of the lemma. ◀

It is easy to check that all the lemmas stated in this section are effective and their algorithmic versions hold. Hence, we proved Lemma 5.8.

5.3 General case

In this section, we prove Theorem 5.3 by reducing the general case to the special case of simple words. The central technical tool in our proof is the Height-Decreasing Condensation Theorem (Theorem 3.4).

Throughout this section, we use the term $\mathcal{L}$-relabelling to refer either to MSO-relabelling or to $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling, although the results apply to any reasonable extension $\mathcal{L}$ of MSO.

Let $\sim$ be a condensation on $u \in M^{\mathrm{ord}+}$. Recall that $u/^{\pi}_{\sim}$ or just $u/_{\sim}$ is a word over the $\sim$-equivalence classes, where each $\sim$-equivalence class C is mapped to $\pi(C)$. We denote by $\mathrm{Fct}(u, \sim) := \{u|_C \mid C \text{ is a } \sim\text{-equivalence class on } u\}$.

▶ **Notation 5.14.** *Let* $\mathcal{C}$ *be a class of* M*-words and* θ *be a formula that defines a condensation* $\sim_\theta$ *on* $\mathcal{C}$*. We write* Ind *for the induced index structures and* Fct *for the corresponding factors.*

$\mathrm{Ind}(\mathcal{C}, \theta) := \{u/_{\sim_\theta} \mid u \in \mathcal{C}\}$,

$\mathrm{Fct}(\mathcal{C}, \theta) := \{u|_S \mid u \in \mathcal{C} \textit{ and } S \textit{ is a } \sim_\theta \textit{-equivalence class of } u\}$.

▶ **Lemma 5.15** (Inheritance Lemma). *Let $\mathcal{C}$ be a class of $\mathcal{P}(M)$-words and θ be an* MSO*-formula that defines a condensation $\sim_\theta$ on $\mathcal{C}$. If* $\mathrm{Fct}(\mathcal{C},\theta)$ *and* $\mathrm{Ind}(\mathcal{C},\theta)$ *have the $\mathcal{L}$-relabelling property, then so does $\mathcal{C}$.*

Proof. We outline the definition of the relabelling.

Let R_a^{Fct} and R_a^{Ind} for $a \in M$ be relabellings for the classes $\mathrm{Fct}(\mathcal{C},\theta)$ and $\mathrm{Ind}(\mathcal{C},\theta)$, respectively.

We define a relabelling R_a for $a \in M$ over the class $\mathcal{C}$ as follows. Let $U \in \mathcal{C}$. Consider the quotient $U/{\sim_\theta} \in \mathrm{Ind}(\mathcal{C},\theta)$. As the domain of $U/{\sim_\theta}$, we take a set I containing exactly one representative from each $\sim_\theta$-equivalence class. Since the domain of U is an ordinal, we may choose I to be the set of minimal elements of the $\sim_\theta$-classes. This I is MSO-definable.

We now assign to each $S \in I$ the value of the subword of U corresponding to the equivalence class containing S.

This assignment can be obtained using Theorem 2.9. Hence, the word $U/{\sim_\theta}$ is MSO-definable in U. We can now apply the relabelling R_a^{Ind} to $U/{\sim_\theta}$. This assigns to each element $S \in I$ a value $b \in M$.

Next, for each $b \in M$ and each $S \in I$ labeled by b, we apply R_b^{Fct} to the equivalence class corresponding to S in U. This assigns a label to every element of each equivalence class, and therefore to every element of U.

It is clear that $\pi(R_a(U)) = a$. Moreover, since both R^{Fct} and R^{Ind} are $\mathcal{L}$-relabellings, where $\mathcal{L}$ is an extension of MSO, the relabelling R_a described above can be formalized within $\mathcal{L}$ using the standard relativization technique (see Lemma 2.7). ◀

▶ **Notation 5.16.** *Let $\mathcal{C}$ and $\mathcal{I}$ be classes of M-words, and let θ be an* MSO*-formula that defines a condensation $\sim_\theta$ on countable M-words. Define:*

- $\mathcal{C}^0 := \mathcal{C}$, *and*
- $\mathcal{C}^{m+1} := \{u \mid u/{\sim_\theta} \in \mathcal{I} \text{ and } \mathrm{Fct}(u,\sim_\theta) \subseteq \mathcal{C}^m\}$.

Strictly speaking, we should use the notation $\mathcal{C}^m(\mathcal{I},\sim_\theta)$, since $\mathcal{C}^m$ depends on $\mathcal{C}$, $\mathcal{I}$, and $\sim_\theta$. However, $\mathcal{I}$ and $\sim_\theta$ will be clear from the context.

▶ **Lemma 5.17.** *If $\mathcal{C}$ and $\mathcal{I}$ are* MSO*-definable, then $\mathcal{C}^m$ is* MSO*-definable for every m.*

Proof. This follows directly from the definition of $\mathcal{C}^m$ by straightforward formalization. ◀

▶ **Lemma 5.18.** *Let $\mathcal{C}$ and $\mathcal{I}$ be* MSO*-definable classes of $\mathcal{P}(M)$-words, and let θ be an* MSO*-formula that defines a condensation $\sim_\theta$ on countable $\mathcal{P}(M)$-words. Assume that both $\mathcal{C}$ and $\mathcal{I}$ have the $\mathcal{L}$-relabelling property. Then $\mathcal{C}^m$ also has the $\mathcal{L}$-relabelling property.*

Proof. The claim follows by induction on m, using the Inheritance Lemma (Lemma 5.15), the definition of $\mathcal{C}^m$ and Lemma 5.17 to pass from m to $m+1$. ◀

The following lemma is immediate.

▶ **Lemma 5.19** (Closure under union). *Assume D_1 and D_2 are* MSO*-definable classes and each has the* $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$*-relabelling property. Then $D_1 \cup D_2$ also has the* $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$*-relabelling property.*

Proof. Let α_i be an MSO sentence defining D_i for $i \in \{1,2\}$, and let Φ_i uniformise φ over D_i. Consider

$$\Phi \;:=\; (\alpha_1 \wedge \Phi_1) \;\vee\; (\neg\alpha_1 \wedge \alpha_2 \wedge \Phi_2).$$

By construction, Φ coincides with Φ_1 on D_1 and with Φ_2 on $D_2 \setminus D_1$, hence it uniformises φ over $D_1 \cup D_2$. Since $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$ is closed under Boolean connectives, Φ is an $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling witnessing the claim. ◀

We are now ready to prove an algebraic reformulation of our main result.

▶ **Theorem 5.3** ($\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property)**.** *Every finite ordinal monoid has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property.*

Proof. Let M be a finite ordinal monoid. Let $\mathcal{C}$ be the class of one-element $\mathcal{P}(M)$-words, and let $\mathcal{I}$ be the union of the class of simple $\mathcal{P}(M)$-words and the class of $\mathcal{P}(M)$-words of length at most ω.

It is clear that $\mathcal{C}$ has the MSO-relabelling property.

The class of simple $\mathcal{P}(M)$-words has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property by Lemma 5.8, and the class of $\mathcal{P}(M)$-words of length at most ω has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property by Lemma 5.7. Since both these classes are MSO-definable, we conclude that $\mathcal{I}$ has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property.

Apply Height-decreasing condensation theorem (Theorem 3.4) to $\mathcal{P}(M)$, and let n and a condensation $\sim$ be as in Theorem 3.4. Then, the union $\bigcup_{m=0}^{n} \mathcal{C}^m$ is the set of all $\mathcal{P}(M)$-words over countable ordinals.

By Lemma 5.18, $\mathcal{C}^m$ has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property for $m \in \mathbb{N}$.

Since each $\mathcal{C}^m$ is MSO-definable by Lemma 5.17, and each $\mathcal{C}^m$ has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property, we conclude, by Lemma 5.19, that $\bigcup_{m=0}^{n} \mathcal{C}^m$ has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property. Hence, the class of all $\mathcal{P}(M)$-words over countable ordinals has the $\mathrm{MSO}(\mathrm{U}_{\omega\mathrm{cof}})$-relabelling property. ◀

6 Conclusion

We have investigated the uniformisation for monadic second-order logic (MSO) over countable ordinal chains.

Our main result demonstrates that the maximal uniformisation degree is attained by expanding MSO with an operator that, given a set, selects a cofinal subset of order type ω. More precisely, let $\mathrm{U}_{\omega\mathrm{cof}}$ be a partial operation that maps every subset X of an ordinal α to a subset $Y = \mathrm{U}_{\omega\mathrm{cof}}(X)$ such that $Y \subseteq X$, Y is cofinal in X, and Y has order type ω. Let $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ denote the expansion of MSO by the predicate $\mathrm{U}_{\omega\mathrm{cof}}(X) = Y$.

We have proven that for every formula $\varphi \in \mathrm{MSO}$, there exists a formula $\psi \in \mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ that uniformises φ over the class of countable ordinals. Furthermore, there is an algorithm that effectively computes ψ from φ.

Note that the semantics of $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ depend on the specific choice of the operation $\mathrm{U}_{\omega\mathrm{cof}}$; thus, the logic is not fully specified until $\mathrm{U}_{\omega\mathrm{cof}}$ is fixed. This dependency, however, does not affect the robustness of our results: the uniformisers constructed in this paper remain correct for *every* admissible choice of $\mathrm{U}_{\omega\mathrm{cof}}$.

▶ **Open Question 1.** *Does there exist an operation $\mathrm{U}_{\omega\mathrm{cof}}$ such that $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ is decidable over the class of countable ordinals?*

▶ **Open Question 2.** *Fix an arbitrary operation $\mathrm{U}_{\omega\mathrm{cof}}$ as defined above. We have shown that every MSO formula admits a uniformiser definable in $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$. However, it is conceivable that $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ itself is not closed under uniformisation; that is, there may exist a formula $\theta(\overline{X}, \overline{Y}) \in \mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ that lacks an $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$-definable uniformiser.*

Does there exist an operation $\mathrm{U}_{\omega\mathrm{cof}}$ such that $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ has the uniformisation property *over countable ordinals (i.e., every $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ formula can be uniformised by an $\mathrm{MSO}[\mathrm{U}_{\omega\mathrm{cof}}]$ formula)?*

▶ **Open Question 3.** *Is it decidable, given an* MSO *formula* $\varphi(\overline{X}, \overline{Y})$*, whether* φ *admits an* MSO*-definable uniformiser over the class of countable ordinals?*

▶ **Open Question 4.** *What constitutes a maximal uniformisation degree for countable linear orders that are not ordinals?*

Beyond ordinals, new phenomena emerge – most notably the presence of non-trivial automorphisms. It is evident that novel constructions are required to uniformise arbitrary MSO-definable relations in this broader setting. For instance, one can show that the property "Y is dense and so is its complement" cannot be uniformised using only the cofinal ω-set predicate. At present, it remains unknown which additional predicates, or which general class thereof, would be sufficient to uniformise all MSO-definable relations over countable linear orders.

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A Appendix

In this appendix, we present the missing proof of the height-decreasing condensation theorem (Theorem 3.4).

A *split of height n* of a linear ordering β is a function $g\colon \beta \to \{1,\dots,n\}$. Two elements $x, y \in \beta$ are called *k-neighbours* if $g(x) = g(y) = k$ and $g(z) \leq k$ for all $z \in [x, y] \cup [y, x]$. Two elements are *neightbours* if they are k-neighbours for some k. Note that the neighbourhood relation is an equivalence relation, but not a condensation in general.

The split g is *Ramsey* for u if, for each equivalence class X of the neighbourhood relation, there exists an idempotent $e \in M$ such that $\pi(u|_{(x,y]}) = e$ for all $x < y$ in X.

Theorem A.1 was proved by Colcombet [6]. However, it was stated without definability clause.

▶ **Theorem A.1** (Existence of MSO-definable Ramsey splits [6]). *For every finite ordinal monoid M, there exists $n \in \mathbb{N}$ such that for all M-words u over an countable ordinal domain, there exists a Ramsey split g for u of height at most n.*

Moreover, the split g is MSO-definable over the class of M-words.

Proof. *Existence* follows from Colcombet's factorization–forest theorem for all linear orders (Theorem 4 in [6]). In the following, we establish that a Ramsey split is MSO-definable on ordinals, relying solely on the assumption that such a split exists.

First note that the statement "$X_1, \dots, X_n$ encodes a Ramsey split for u" is MSO-expressible: for each k one can define the k-neighbour equivalence on X_k, and for each equivalence class $X \subseteq X_k$ require (by a finite disjunction over idempotents $e \in M$) that $\pi\big(u|_{(x,y]}\big) = e$ for all $x < y$ in X.

We build a *canonical* Ramsey split of height $\leq n$ by induction on the height.

Base case ($n = 1$). Trivial.

Inductive step ($n \Rightarrow n{+}1$). Assume u admits some Ramsey split of height $\leq n{+}1$. For such a split g, write $X^g_{n+1} := g^{-1}(n{+}1)$ for the top layer. We define, in a uniform MSO way, a *canonical* top layer X_{n+1} and then apply the height-n induction on the convex pieces cut out by X_{n+1}.

For an idempotent $e \in M$ put

$$\mathcal{X}^e_{n+1} \;:=\; \Big\{ X^g_{n+1} \;\Big|\; \begin{array}{l} g \text{ is a Ramsey split, and} \\ \pi\big(u \restriction (x, x']\big) = e \;\text{ for all } x < x' \in X^g_{n+1}. \end{array} \Big\}$$

At least one $\mathcal{X}^e_{n+1}$ is nonempty. Fix once and for all a linear order on the elements of M, and let $\widehat{e}$ be the least idempotent with $\mathcal{X}^{\widehat{e}}_{n+1} \neq \varnothing$.

We define a *canonical* set $X^{\widehat{e}}_{n+1} = \{x_\beta\}_{\beta \in I}$ (by transfinite recursion on its index set I) as follows.

- *First element.*

$$x_0 \;:=\; \min\big\{\, x \mid \exists X \in \mathcal{X}^{\widehat{e}}_{n+1}\ (x \in X) \,\big\}.$$

- *Successor step.* Given x_β, let

$$S_{\beta+1} \;:=\; \{\, y > x_\beta \mid \exists X \in \mathcal{X}^{\widehat{e}}_{n+1} \;\text{ with } x_\beta \in X \text{ and } y \in X\}$$

If $S_{\beta+1} = \varnothing$ *or* x_β is the maximal element of some $X \in \mathcal{X}^{\widehat{e}}_{n+1}$, then the tail $u \restriction (x_\beta, \infty)$ admits a height-n Ramsey split. In this case set $X^{\widehat{e}}_{n+1} = \{x_\gamma \mid \gamma \le \beta\}$ and terminate the construction of the top layer. Otherwise, put

$$x_{\beta+1} \;:=\; \min S_{\beta+1}$$

- *Limit step.* Let λ be a limit index and suppose $\{x_\beta\}_{\beta<\lambda}$ has been defined strictly increasing; set $x := \sup_{\beta<\lambda} x_\beta$. For each $\beta < \lambda$, let

$$L_\beta := \{y \ge x \mid \text{ there exists } X \in \mathcal{X}^{\widehat{e}}_{n+1} \text{ with } x_\beta \in X \text{ and } y \in X\}$$

If some $L_\beta = \varnothing$, then the tail $u \restriction [x, \infty)$ has a height-n Ramsey split; set $X^{\widehat{e}}_{n+1} = \{x_\gamma \mid \gamma < \lambda\}$ and finish. Otherwise, let $y_\beta := \min L_\beta$ and define $a_\beta := \pi\big(u \restriction [x, y_\beta]\big)$. Since M is finite, some $a \in M$ occurs cofinally often in $(a_\beta)_{\beta<\lambda}$; let a_λ be the least in $\{a \mid a$ occurs cofinally often in $(a_\beta)_{\beta<\lambda}\}$; put

$$\Lambda := \{\beta < \lambda \mid a_\beta = a_\lambda\}$$

$$x_\lambda := \inf\{y_\beta \mid \beta \in \Lambda\}$$

Note that $(x_\beta)_{\beta\in\Lambda}$ is cofinal in $[0, x)$.

The following claim is immediate from the definition of successor step:

▷ Claim. For every consecutive pair $x < x'$ in $X^{\widehat{e}}_{n+1}$ one has $\pi\big(u \restriction (x, x']\big) = \widehat{e}$.

▷ Claim. For every pair of indices $\beta < \alpha$ in I one has

$$\pi\big(u \restriction (x_\beta, x_\alpha]\big) = \widehat{e}.$$

Proof of the claim. Fix β and argue by transfinite induction on $\lambda > \beta$.

Successor step. If $\lambda = \gamma + 1$, then by the induction hypothesis $\pi\big(u \restriction (x_\beta, x_\gamma]\big) = \widehat{e}$, and by the successor-clause of the construction, $\pi\big(u \restriction (x_\gamma, x_{\gamma+1}]\big) = \widehat{e}$ (since $x_{\gamma+1}$ is chosen as the next top-layer point after x_γ within some $X \in \mathcal{X}^{\widehat{e}}_{n+1}$). Hence

$$\pi\big(u \restriction (x_\beta, x_{\gamma+1}]\big) \;=\; \pi\big(u \restriction (x_\beta, x_\gamma]\big) \cdot \pi\big(u \restriction (x_\gamma, x_{\gamma+1}]\big) \;=\; \widehat{e} \cdot \widehat{e} \;=\; \widehat{e},$$

using idempotency of $\widehat{e}$.

Limit step. Let λ be a limit index. By construction of x_λ for every $\gamma \in \Lambda$ we have

$$\begin{aligned}
\pi\big(u \restriction (x_\gamma, x_\lambda]\big) &= \pi\big(u \restriction (x_\gamma, x)\big) \cdot \pi\big(u \restriction [x, x_\lambda]\big) \\
&= \pi\big(u \restriction (x_\gamma, x)\big) \cdot a_\lambda \\
&= \pi\big(u \restriction (x_\gamma, x)\big) \cdot a_\gamma \\
&= \pi\big(u \restriction (x_\gamma, x)\big) \cdot \pi\big(u \restriction [x, y_\gamma]\big) \\
&= \pi\big(u \restriction (x_\gamma, y_\gamma]\big) \\
&= \widehat{e}.
\end{aligned}$$

Since $(x_\beta)_{\beta\in\Lambda}$ is cofinal in $[0,x)$, there is $\gamma \in \Lambda$ such that $x_\gamma \in (x_\beta, x)$

By the induction hypothesis, $\pi\big(u \restriction (x_\beta, x_\gamma]\big) = \widehat{e}$, hence again by idempotency,

$$\pi\big(u \restriction (x_\beta, x_\lambda]\big) \;=\; \pi\big(u \restriction (x_\beta, x_\gamma]\big) \cdot \pi\big(u \restriction (x_\gamma, x_\lambda]\big) \;=\; \widehat{e} \cdot \widehat{e} \;=\; \widehat{e}.$$

This completes the induction and proves the claim. ◁

Each step in the construction is given by an MSO-definable minimization over an MSO-definable set; consequently, the resulting $X^{\widehat{e}}_{n+1}$ is uniquely defined by an MSO formula. Finally, on each interval between consecutive points of $X^{\widehat{e}}_{n+1}$ (and on the initial or final segment, if present) apply the height-n construction and assemble with the level $n{+}1$ layer to obtain the global split. ◀

Now we are going to prove Theorem 3.4.

Given a Ramseian split g for an M-word u, we define the function height and the condensation $\sim$ according to the following cases.

Let $k := \max g$. Let I be the minimal interval containing all elements z such that $g(z) = k$. Let J (respectively, A) be the set of all elements in the domain of g that precede (respectively, follow) all elements of I. Recall that the domain of the split g is obtained by extending the domain D of the M-word u with a new element that is minimal.

- **Case 1:** I has only one element.
  In this case, set $\mathrm{height}(u) := 3k$.
  If $I \cap D = \emptyset$ (i.e., g assigns the value k only to the minimal element of the domain g), define $\sim$ to split D into two intervals: the first contains only the minimal element of D, and the second contains all other elements.
  Otherwise, define $\sim$ to split D into three intervals: $J \cap D$, $I \cap D$, and $A \cap D$ (note that $J \cap D$ or $A \cap D$ may be empty).
- **Case 2:** I has at least two elements and $J = A = \emptyset$.
  In this case, set $\mathrm{height}(u) := 3k + 1$, and define $\sim$ to have one equivalence class for each point z with $g(z) = k$.
  Define $x \sim z$ if $x \leq z$ and $g(v) < k$ for every $v \in [x, z)$. In particular, each equivalence class intersects the g-preimage of k in exactly one point.
- **Case 3:** I is a proper interval of the domain of g and has at least two elements.
  In this case, set $\mathrm{height}(u) := 3k+2$ and define $\sim$ to split u into the intervals $J \cap D$, $I \cap D$, and $A \cap D$.