Complexity of Satisfiability in Kochen-Specker Partial Boolean Algebras

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Abstract

The Kochen-Specker no-go theorem established that hidden-variable theories in quantum mechanics necessarily admit contextuality. This theorem is formally stated in terms of the partial Boolean algebra structure of projectors on a Hilbert space. Each partial Boolean algebra provides a semantics for interpreting propositional logic. In this paper, we examine the complexity of propositional satisfiability for various classes of partial Boolean algebras. We first show that the satisfiability problem for the class of non-trivial partial Boolean algebras is **NP**-complete. Next, we consider the satisfiability problem for the class of partial Boolean algebras arising from projectors on finite dimensional Hilbert spaces. For real Hilbert spaces of dimension greater 2 and any complex Hilbert spaces of dimension greater than 3, we demonstrate that the satisfiability problem is complete for the existential theory of the reals. Interestingly, the proofs of these results make use of Kochen-Specker sets as gadgets. As a corollary, we conclude that deciding quantum homomorphism in these fixed dimensions are also complete for the existential theory of the reals. Finally, we show that the satisfiability problems for the class of all Hilbert spaces and all finite-dimensional Hilbert spaces is undecidable.

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1 Introduction

The topic of this paper lies at the intersection of one notable aspect in each of twentieth century physics, mathematics, and computer science. In physics, this aspect is the inherent contextuality in dealing with quantum physical systems as exemplified by the Kochen-Specker no-go theorem. In mathematics, this aspect is the algebraic tradition of logic, where we study the partiality relaxation, arising from Kochen and Specker's work, of the standard Boolean algebra semantics for propositional logic. In computer science, this aspect is understanding how the statement of Cook-Levin's theorem, demonstrating that propositional satisfiability is **NP**-complete, changes under this partiality relaxation.

The Kochen-Specker theorem, appearing in [17] proved the impossibility of embedding quantum theory into any classical physical theory possibly involving hidden variables. This resolved an open question that had plagued quantum theory since its inception. A key step in formulating a mathematical resolution to the question was to establish a necessary condition on observables that would hold if quantum theory was embeddable into a classical theory. Observables, in standard quantum theory, are represented as operators on the Hilbert space

$\mathcal{H}$ of a quantum system's states. The necessary condition was that the partial algebra of operators $\mathbf{B}(\mathcal{H})$, where algebraic operations are only defined on pairs of operators which commute, could be embedded into a single commutative algebra $\mathcal{A}$. This formalisation addresses the question since only observables which are represented by commuting operators are physically commeasurable, i.e. can be measured together. An immediate observation made by Kochen and Specker was that any embedding from $\mathbf{B}(\mathcal{H})$ to a commutative algebra would induce a homomorphism of the partial Boolean algebra of idempotent elements $\mathbf{P}(\mathcal{H})$ into the two-element Boolean algebra. Thus, the Kochen-Specker theorem was proved by showing that for any Hilbert space $\mathcal{H}$ of dimension ≥ 3, there is no homomorphism from $\mathbf{P}(\mathcal{H})$ into the two-element Boolean algebra.

The idempotent elements of $\mathbf{B}(\mathcal{H})$ forming the partial Boolean algebra $\mathbf{P}(\mathcal{H})$ are the projectors of the Hilbert space $\mathcal{H}$. Projectors, or equivalently their associated subspace, can be identified with yes/no questions, i.e. propositions, about the quantum system corresponding to $\mathcal{H}$. The "and" and "or" operations $\mathbf{P}(\mathcal{H})$ are only defined on projectors which commute. This respects the postulate that only commuting observables can be jointly measured, i.e. are commeasurable.

More generally, a partial Boolean algebra is equipped with a binary, reflexive, symmetric commeasurability relation which determines the domain of definition for the operations. Since these operations are partial, the usual notion of substitution of variables in a term is adapted to *meaningful substitution*. Thus, each partial Boolean algebra provides an alternative semantics of propositional logic via this notion of meaningful substitution.

In a paper dedicated to Specker, Kochen [15] argues that since these semantics restrict to physically meaningful substitutions, the partial Boolean algebra approach can be used to reconstruct significant features of reasoning in quantum mechanics. Moreover, recent papers [25, 1] have been reviving research in this partial Boolean algebra semantics for propositional logic in order to provide a fresh perspective on "quantum logic". This would contrast with the historical Birkhoff-von Neumann approach of propositional quantum semantics in terms of orthomodular posets. In particular, the partial Boolean algebra viewpoint arises from topos approaches to contextuality [13, 10]. These papers highlight that a key consequence of the Kochen-Specker theorem is that there exist propositional tautologies which may fail to be true under this alternative semantics. A question the authors put forth in [1] is to obtain characterisation of all such Kochen-Specker paradoxes.

This characterisation question is the primary motivation for this paper. The traditional mathematical logic approach to this question had been partially investigated in [16]. There, Kochen and Specker exhibited a sound and complete proof system for determining which propositional formulas are valid in all non-trivial partial Boolean algebras. By contrast, we take a computer science approach to the question by investigating the computational complexity of deciding the validity problem in various interesting classes of partial Boolean algebras. More precisely, we work with the dual question by exhibiting algorithms for deciding if an input propositional formula is satisfiable in a given class of partial Boolean algebras. Proving complexity bounds for such an algorithm would in fact be a natural adaptation of a staple result of theoretical computer science: the Cook-Levin theorem. The Cook-Levin theorem established that the computational complexity of deciding whether a propositional formula is satisfiable in total Boolean algebras is complete for non-deterministic polynomial time. Thus, the topic of this paper is to establish the complexity bounds for different classes of partial Boolean algebras. Apart from considering classes in generality, we focus on particular partial Boolean algebras motivated by quantum computing. This includes the class of algebras of the form $\mathbf{P}(\mathcal{H})$ for an arbitrary Hilbert space $\mathcal{H}$ but also

specifically the algebras $\mathbf{P}(\mathbb{C}^d)$ for a fixed dimension d. To motivate the latter, consider the problem where the input-output behaviour of a quantum circuit operating on a fixed number of qubits is specified using a propositional formula.

Outline. In Section 2, we review notational preliminaries and background material on partial Boolean algebras which we use throughout the rest of the paper. Section 3 formulates the weak and strong satisfiability decision problems, and their dual validity decision problems, for arbitrary classes $\mathbf{C}$ of partial Boolean algebras. The section also makes some general observations about these problems and concludes with a sufficient condition on $\mathbf{C}$ to guarantee $\mathbf{C}$-weak and $\mathbf{C}$-strong satisfiability problems are at least as hard as the classical satisfiability problem. We also demonstrate the satisfiability problem in which a pBA is given as additional input is **NP**-complete. Section 4 prove that satisfiability problem for the class of all non-trivial partial Boolean algebras is **NP**-complete. Section 5 reviews Kochen-Specker proofs how they can be encoded as graphs and propositional formulas. We leverage these Kochen-Specker proofs in the next section by using them as gadgets in a hardness reduction. In Section 6, we show that for any fixed dimension $d \geq 3$, the satisfiability problem for $\mathbf{P}(\mathbb{R}^d)$ is complete for the existential theory of the reals. Similarly, we show that for any fixed dimension $d \geq 4$, the satisfiability problem for $\mathbf{P}(\mathbb{C}^d)$ is complete for the existential theory of the reals. This section also establishes the for dimension $d = 1$ or 2, the satisfiability problem is **NP**-complete. Section 7 show that the satisfiability problems for pBAs arising from the class of finite dimensional and the class of all Hilbert spaces are undecidable.

Related past work. Variations of our results have been studied in the previous literature. In particular, projectors over real and complex Hilbert space $\mathcal{H}$ can also be equipped with an orthomodular lattice structure $\mathbf{O}(\mathcal{H})$ rather than the partial Boolean algebra structure $\mathbf{P}(\mathcal{H})$. Orthomodular lattices satisfy a weakening of the distributivity axiom of Boolean algebras. The orthomodular lattice $\mathbf{O}(\mathcal{H})$ interprets $\vee$ and $\wedge$ as total operations which return the projector onto the union and intersection, respectively, of the input projectors' image subspaces. In contrast to the partial operations of $\mathbf{P}(\mathcal{H})$, the total operations of $\mathbf{O}(\mathcal{H})$ do not have a simple algebraic definition in terms of operator addition and composition, see e.g. [20] for details. Moreover, these operations arguably lack a physical interpretation [15]. Orthomodular lattices are the algebraic semantics used in the traditional Birkhoff-von Neumann approach to quantum logic. It is known that satisfiability of propositional formulas in $\mathbf{O}(\mathcal{H})$ is $\exists\mathbb{R}$-complete for any d-dimensional real or complex Hilbert space where $d \geq 3$ [12]. Theorem 6.9 and Theorem 6.16 establish the same $\exists\mathbb{R}$-completeness result for $\mathbf{P}(\mathcal{H})$. It has been shown that satisfiability in $\mathbf{O}(\mathcal{H})$ for any finite dimensional real or complex Hilbert space is undecidable [9]. Theorem 7.4 establishes the same undecidability result for $\mathbf{P}(\mathcal{H})$.

That there is no algorithm for deciding if a constraint satisfaction problem over a Boolean domain has a satisfying assignment of involutive operators in $\mathbf{B}(\mathcal{H})$ for some finite-dimensional Hilbert space $\mathcal{H}$ was shown in [3]. Constraint satisfaction problems over a Boolean domain can be identified with propositional formulas in conjunctive normal form and involutive operators in $\mathbf{B}(\mathcal{H})$ are in bijection with projectors $\mathbf{P}(\mathcal{H})$. Thus, Theorem 7.4 is an algebraic reformulation of one of the undecidability results in [3]. We discuss the precise connection with Theorem 7.4 in Remark 7.6.

Complexity bounds for deciding if a quantifier-free first-order formula is satisfied in a partial structure of a lattice-theoretic algebraic theory are established in [24]. In particular, it is shown that for the theory of Boolean algebras, this decision problem is **NP**-complete. However, the notion of partial Boolean algebra employed in this paper is different from the

Kochen-Specker notion of partial Boolean algebra we employ in this paper. In particular, partial Boolean algebras in the sense of [24] are not equipped with a commeasurability relation and are embeddable into a total Boolean algebra. In light of the Kochen-Specker theorem, these structures are uninteresting for the purpose of understanding contextuality.

2 Preliminaries

We use $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ to denote the set of natural, real and complex numbers, respectively. We use the notation $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$ as a field. For every $n \in \mathbb{N}$, $[n] = \{1, \ldots, n\}$. An n-tuple is a function $\vec{a}\colon [n] \to A$ whose range is denoted $[\vec{a}]$. We enumerate n-tuples using the notation $(a_1, \ldots, a_n)$ where $a_i = \vec{a}(i)$. Let A^I denote the set of functions of type $I \to A$. For $n \in \mathbb{N}$, we use the notation A^n when $I = [n]$ and for n-tuples $\vec{a}$, we use the notation $A^{\vec{a}}$ for $I = [\vec{a}]$. Given a function $f\colon A \to B$ and a subset $D \subseteq A^I$, we use $f|_D\colon D \to B^I$ to denote the function defined as $f|_D(z) = \lambda i \in I.f(z(i))$ for all $z \in D$.

We assume familiarity with the standard syntax and truth table semantics of propositional logic. We also assume familiarity with the axioms of Boolean algebras. Propositional formulas are formed from closing a countably infinite set of propositional variables Var and constants $\{\top\}$ under the unary symbol $\neg$ and binary symbols $\vee$,$\wedge$. We use the symbols $\rightarrow$, $\leftarrow$, $\leftrightarrow$, and $\oplus$ as standard abbreviations for implication, converse implication, bi-implication, and exclusive-or. Let $\mathsf{Var}(\phi)$ denote the set of propositional variables which appear in ϕ. The notation $\phi(p_1, \ldots, p_n)$ indicates that $\mathsf{Var}(\phi) \subseteq \{p_1, \ldots, p_n\}$. Let $\mathsf{Sub}(\phi)$ denote the set of subformulas of ϕ. A formula that is either a variable, a constant, or a negation $\neg$ applied to a variable or constant is a *literal*. We also assume familiarity with standard syntax and model-theoretic semantics of first-order logic. To distinguish from propositional formulas, first-order formulas and their variables are denoted using capitalised letters, i.e. $\Phi(P_1, \ldots, P_n)$.

▶ **Definition 2.1.** *A* partial Boolean algebra (pBA) A *is a set equipped with*

- *distinguished elements* $0_A, 1_A \in A$,
- *a reflexive, symmetric binary relation* $\odot_A \subseteq A \times A$,
- *a function* $\neg_A\colon A \to A$,
- *and functions* $\vee_A, \wedge_A\colon \odot_A \to A$

satisfying the following condition:

(Ext) *For every set* $S \subseteq A$ *of pairwise* $\odot$*-related elements, there exists a* $T \supseteq S \cup \{1_A, 0_A\}$ *of pairwise* $\odot$*-related elements such that* $(T, 0_A, 1_A, \neg|_T, \vee|_T, \wedge|_T)$ *is a Boolean algebra.*

The relation $\odot_A$ *is called the* commeasurability relation *and we say that* $a \in A$ and $b \in A$ are commeasurable *if* $a \odot_A b$.

A partial Boolean algebra A is *total* if $\odot_A = A^2$. We sometimes drop the adjective "total" when referring to total Boolean algebras. The trivial Boolean algebra, denoted **1**, is the unique Boolean algebra where $1_A = 0_A$. We use **Total** to denote the class of all total non-trivial Boolean algebras. We use **2** to denote the two-element total Boolean algebra.

A set function $f\colon A \to B$ between partial Boolean algebras is a *homomorphism or* **pBA** *morphism* if f preserves the interpretation of commeasurability relations, the distinguished elements, and the operations $\neg, \vee, \wedge$ whenever defined. We use **pBA** to denote both the category of partial Boolean algebras and partial Boolean algebra homomorphisms, and the class of all partial Boolean algebras. We use **BA** to denote the category of total Boolean algebras and Boolean algebra homomorphisms.

▶ **Definition 2.2.** *Given a partial Boolean algebra A and relation $\circledcirc \subseteq A \times A$, we can construct the* $\circledcirc$-extension of A, *denoted $A[\circledcirc]$, which satisfies the following two properties:*

(E1) *There exists a* **pBA** *morphism $\eta\colon A \to A[\circledcirc]$ satisfying $a \circledcirc b \Rightarrow \eta(a) \odot_{A[\circledcirc]} \eta(b)$.*

(E2) *For every* **pBA** *morphism $h\colon A \to B$ satisfying $a \circledcirc b \Rightarrow h(a) \odot_B h(b)$, there exists a unique morphism $\hat{h}\colon A[\circledcirc] \to B$ such that $\hat{h} \circ \eta = h$.*

Intuitively, $A[\circledcirc]$ forces elements in A which are $\circledcirc$-related to be commeasurable. Thus, $A[\circledcirc]$ freely adds to A new elements which witness the output of the operations $\vee_{A[\circledcirc]}$ and $\wedge_{A[\circledcirc]}$. The explicit construction of $A[\circledcirc]$ is given in Section 2.2 of [1]. We make use of the properties (E1)-(E2) which $A[\circledcirc]$ satisfies by Theorem 1 and Proposition 3 of [1].

The category **pBA** has general colimits. This fact was first proved in [25], but the proof appealed to the Adjoint Functor Theorem and thus did not yield an explicit construction. An explicit construction of colimits in terms of coproducts and a quotient construction satisfying a universal property similar to the $\circledcirc$-extension of A were given in [1]

▶ **Definition 2.3.** *Given a partial Boolean algebra A and binary relation $\circledcirc$, the* $\circledcirc$-quotient of A, *denoted $A\backslash\circledcirc$ is a partial Boolean algebra satisfying the following two properties:*

(Q1) *There exists a* **pBA** *morphism $\upsilon\colon A \to A\backslash\circledcirc$ satisfying $a \circledcirc b \Rightarrow \upsilon(a) = \upsilon(b)$.*

(Q2) *For every* **pBA** *morphism $h\colon A \to B$ satisfying $a \circledcirc b \Rightarrow h(a) = h(b)$, there exists a unique morphism $\bar{h}\colon A\backslash\circledcirc \to B$ such that $\bar{h} \circ \upsilon = h$.*

The explicit construction is a simple modification of the construction for $A[\circledcirc]$ and is also detailed in Section 2.2 of [1]. As with the extension construction $A[\circledcirc]$, we only make use of the properties (Q1)-(Q2) which $A\backslash\circledcirc$ satisfies by Theorem 5 of [1].

Given a formula $\phi(p_1, \ldots, p_n)$ and Boolean algebra B, there is a substitution function $\phi^B\colon B^{\vec{p}} \to B$, which gives the value of the formula, given any valuation of the variables. In this context of partial Boolean algebras, the notion of substitution function must be adapted to *meaningful substitution*, which we define next.

▶ **Definition 2.4.** *Given a propositional formula $\phi(\vec{p})$ with variables among $\vec{p} = (p_1, \ldots, p_n)$ and partial Boolean algebra A, we define a* meaningful domain $A^{\phi(\vec{p})} \subseteq A^{\vec{p}}$ *and* meaningful substitution function $\phi^A\colon A^{\phi(\vec{p})} \to A$ *mutually by structural recursion on propositional formulas $\phi(p_1, \ldots, p_n)$. For the base case,*

(1) *If $\phi(\vec{p}) = \top$, then $A^{\phi(\vec{p})} = A^{\vec{p}}$ and for all $\alpha \in A^{\phi(\vec{p})}$, $\phi^A(\alpha) = 1_A$.*

(2) *If $\phi(\vec{p}) = p_i$, then $A^{\phi(\vec{p})} = A^{\vec{p}}$ and for all $\alpha \in A^{\phi(\vec{p})}$, $\phi^A(\alpha) = \alpha(p_i)$.*

For the inductive steps,

(3) *If $\phi(\vec{p}) = \neg\psi(\vec{p})$, then $A^{\phi(\vec{p})} = A^{\psi(\vec{p})}$ and for all $\alpha \in A^{\phi(\vec{p})}$, $\phi^A(\alpha) = \neg_A \psi^A(\alpha)$.*

(4) *If $\phi(\vec{p}) = \psi_1(\vec{p}) \bowtie \psi_2(\vec{p})$ for connective $\bowtie \in \{\vee, \wedge\}$, then*

$$A^{\phi(\vec{p})} = \{\alpha \in A^{\psi_1(\vec{p})} \cap A^{\psi_2(\vec{p})} \mid \psi_1^A(\alpha) \odot_A \psi_2^A(\alpha)\}$$

and for all $\alpha \in A^{\phi(\vec{p})}$, $\phi^A(\alpha) = \psi_1^A(\alpha) \bowtie_A \psi_2^A(\alpha)$.

For cleaner notation, for a propositional formula $\phi(p_1, \ldots, p_n)$, we sometimes use $\phi^A(a_1, \ldots, a_n)$ to denote $\phi^A(\alpha)$ where $\alpha \in A^{\phi(\vec{p})}$ and for every $i \in [n]$, $\alpha(p_i) = a_i$.

In the next section, we use the notion of meaningful substitution to generalize propositional satisfiability and validity to any class of partial Boolean algebras.

In ordinary universal algebra, and in particular Boolean algebras, satisfiability of a formula is preserved and reflected by **BA** morphisms. The following proposition from [17] extends this result to meaningful substitutions and **pBA** morphisms.

▶ **Proposition 2.5.** *If $h\colon A \to B$ is a* **pBA** *morphism and $\phi(\vec{p})$ is a propositional formula, then for all $\alpha \in A^{\phi(\vec{p})}$,*

$$\phi^B(h|_{A^{\phi(\vec{p})}}(\alpha)) = h(\phi^A(\alpha)).$$

We also make use of Proposition 2.6 and Proposition 2.7 which generalize useful facts about Boolean algebras to partial Boolean algebras.

▶ **Proposition 2.6.** *Let A be a partial Boolean algebra, $\phi(\vec{p}), \psi(\vec{p})$ be propositional formulas and $\alpha \in A^{\phi(\vec{p})} \cap A^{\psi(\vec{p})}$.*

$$\phi^A(\alpha) = \psi^A(\alpha) \text{ if, and only if, } \nu^A(\alpha) = 1_A$$

where $\nu(\vec{p})$ is the bi-implication $\phi(\vec{p}) \leftrightarrow \psi(\vec{p})$.

For every partial Boolean algebra A, we define the relation $\leq_A$ as $a \leq_A b$ if $a \odot_A b$ and $a \wedge_A b = a$. For every Boolean subalgebra B of A, $\leq_A$ restricted to B coincides with the underlying partial order on the Boolean algebra B. The relation $\leq_A$ on a partial Boolean algebra is expressible in propositional logic, since we can define the formula

$$\phi_{\leq}(p,q) := p \wedge q \leftrightarrow p \tag{1}$$

which satisfies the following proposition.

▶ **Proposition 2.7.** *Let A be a partial Boolean algebra with $a, b \in A$.*

$$a \leq_A b \text{ if, and only if, } \phi^A_{\leq}(a,b) = 1_A$$

3 Formulation and observations

We start with the formulation of the primary motivation for our paper: satisfiability in classes of partial Boolean algebras. Using Definition 2.4, we obtain two sensible notions of satisfiability. Given a partial Boolean algebra A and propositional formula $\phi(\vec{p})$, we say

1. *ϕ is weakly satisfied in A* if there exists a $\alpha \in A^{\phi(\vec{p})}$ such that $\phi^A(\alpha) \neq 0_A$.
2. *ϕ is strongly satisfied in A* if there exists a $\alpha \in A^{\phi(\vec{p})}$ such that $\phi^A(\alpha) = 1_A$.

Similarly, if we are given a class of partial Boolean algebras $\mathbf{C}$,

1. *ϕ is $\mathbf{C}$-weakly-satisfiable* if ϕ is weakly satisfied in A for some $A \in \mathbf{C}$.
2. *ϕ is $\mathbf{C}$-strongly-satisfiable* if ϕ is strongly satisfied in A for some $A \in \mathbf{C}$.

We can also generalize the dual notion of validity to any class of partial Boolean algebras. We say that ϕ is $\mathbf{C}$-weakly-valid or $\mathbf{C}$-strongly-valid if for all $A \in \mathbf{C}$ and $\alpha \in A^{\phi(\vec{p})}$, $\phi(\alpha) \neq 0_A$ or $\phi(\alpha) = 1_A$, respectively. It is easily verified that, as long as $\mathbf{C}$ does not contain the trivial Boolean algebra, then any formula ϕ is $\mathbf{C}$-weakly-valid if, and only if, $\neg\phi$ is $\mathbf{C}$-strongly-satisfiable and similarly, ϕ is $\mathbf{C}$-strongly -valid if, and only if, $\neg\phi$ is $\mathbf{C}$-weakly-satisfiable.

Note that if A, B are isomorphic partial Boolean algebras, then A and B satisfy the same class of formulas. Thus, for the notions of satisfiability and validity we are studying, we assume without loss of generality throughout the rest of the paper that every class of partial Boolean algebras $\mathbf{C}$ is isomorphism-closed.

▶ **Proposition 3.1.** *Let $\mathbf{C}, \mathbf{C}'$ be classes of partial Boolean algebras such that $\mathbf{C} \subseteq \mathbf{C}'$ and ϕ is a propositional formula.*

1. *If ϕ is $\mathbf{C}$-weakly-satisfiable, then ϕ is $\mathbf{C}'$-weakly-satisfiable.*
2. *If ϕ is $\mathbf{C}'$-weakly-valid, then ϕ is $\mathbf{C}$-weakly-valid.*

Similar statements hold for the corresponding strong notions.

For some classes $\mathbf{C}$ of partial Boolean algebras, the collection of formulas ϕ which are $\mathbf{C}$-weakly-satisfiable and $\mathbf{C}$-strongly-satisfiable coincide. In these cases (which are all cases of interest in the subsequent sections), we can drop the qualifiers and say that a formula ϕ is $\mathbf{C}$-satisfiable or dually, is $\mathbf{C}$-valid. It is a well-known fact that **Total** is a class where **Total**-weakly-satisfiable and **Total**-strongly-satisfiable coincide. This is because **Total** is closed under taking a quotient $B\backslash\langle b\rangle$ of a Boolean algebra $B \in \mathbf{Total}$ by the filter generated from a non-zero element $b \in B$. The quotient $B\backslash\langle b\rangle$ "forces" $b \in B$ to be equal to one. This idea generalizes to an arbitrary class $\mathbf{C}$ which is closed under taking a $\circledcirc$-quotient $A\backslash\circledcirc$ of $A \in \mathbf{C}$ by $\circledcirc = \{(a, 1_A)\}$ for a non-zero element $a \in A$. The following definition isolates such classes $\mathbf{C}$.

▶ **Definition 3.2.** *A class* $\mathbf{C}$ *is* closed under collapse *if for all* $A \in \mathbf{C}$ *and* $a \in A$ *such that* $a \neq 0_A$, $A\backslash\circledcirc \in \mathbf{C}$ *where* $\circledcirc = \{(a, 1_A)\}$.

▶ **Proposition 3.3.** *If* $\mathbf{C}$ *is closed under collapse and* $\mathbf{1} \notin \mathbf{C}$, *then for any propositional formula* ϕ, ϕ *is* $\mathbf{C}$*-weakly-satisfiable if, and only if,* ϕ *is* $\mathbf{C}$*-strongly-satisfiable.*

From the notions of $\mathbf{C}$-weakly-satisfiable and $\mathbf{C}$-strongly-satisfiable, we can formulate the corresponding decision problems. $\mathbf{SAT}^{\neq 0}(\mathbf{C})$ and $\mathbf{SAT}^{=1}(\mathbf{C})$ denote the classes of $\mathbf{C}$-weakly-satisfiable or $\mathbf{C}$-strongly-satisfiable formulas, respectively. For classes $\mathbf{C}$ where these notions coincide, we use the notation $\mathbf{SAT}(\mathbf{C})$ for this decision problem. In the case where $\mathbf{C}$ is the singleton class $\{A\}$, we drop the superfluous braces when denoting these decision problems, i.e. $\mathbf{SAT}^{=1}(A)$.

Since the trivial Boolean algebra $\mathbf{1}$ satisfies every term, we observe that if $\mathbf{1} \in \mathbf{C}$, then $\mathbf{SAT}^{=1}(\mathbf{C})$ is trivial, i.e. it contains all formulas. By contrast, suppose $\mathbf{C}$ is such that $\mathbf{1} \notin \mathbf{C}$, then since any propositional formula which is **Total**-satisfied is satisfied in $\mathbf{2}$ and any non-trivial partial Boolean algebra contains $\mathbf{2}$ as a Boolean subalgebra, $\mathbf{SAT}^{\neq 0}(\mathbf{C})$ and $\mathbf{SAT}^{=1}(\mathbf{C})$ should be at least as computationally hard as $\mathbf{SAT}(\mathbf{Total})$. Corollary 3.5 of Proposition 3.4 below confirms this intuition.

▶ **Proposition 3.4.** *Let* $\mathbf{C}$ *be a class of partial Boolean algebras such that* $\mathbf{1} \notin \mathbf{C}$. *For every propositional formula* $\phi(\vec{p})$, *there exists a propositional formula* $\hat{\phi}(\vec{p})$ *of length* $O(|\phi(\vec{p})|^2)$ *such that the following are equivalent:*

(1) $\phi(\vec{p}) \in \mathbf{SAT}(\mathbf{Total})$.
(2) $\hat{\phi}(\vec{p}) \in \mathbf{SAT}^{=1}(\mathbf{C})$.
(3) $\hat{\phi}(\vec{p}) \in \mathbf{SAT}^{\neq 0}(\mathbf{C})$.

▶ **Corollary 3.5.** *If* $\mathbf{C}$ *is a class of partial Boolean algebras such that* $\mathbf{1} \notin \mathbf{C}$, *then* $\mathbf{SAT}^{\neq 0}(\mathbf{C})$ *and* $\mathbf{SAT}^{=1}(\mathbf{C})$ *is* **NP***-hard.*

Proof. Proposition 3.4 gives a polynomial time reduction from $\mathbf{SAT}(\mathbf{Total})$ to $\mathbf{SAT}^{\neq 0}(\mathbf{C})$ and $\mathbf{SAT}^{=1}(\mathbf{C})$. By the Cook-Levin theorem, $\mathbf{SAT}(\mathbf{Total})$ is **NP**-hard. ◀

Many of the classes $\mathbf{C}$ of partial Boolean algebras we consider satisfy Corollary 3.5, and so $\mathbf{SAT}(\mathbf{C})$ is **NP**-hard. However, it remains to establish upper bounds and, where relevant, tighter lower bounds for the complexity of $\mathbf{SAT}(\mathbf{C})$.

It is also worth observing at this point what the results tell us about the problem of *validity*, i.e. the class of propositional tautologies. Let $\mathbf{VAL}^{\neq 0}(\mathbf{C})$ and $\mathbf{VAL}^{=1}(\mathbf{C})$ denote the collection of formulas weakly valid and strongly valid in $\mathbf{C}$ respectively. By the duality noted above, these are computationally equivalent to the complements of the problems $\mathbf{SAT}^{=1}(\mathbf{C})$ and $\mathbf{SAT}^{\neq 0}(\mathbf{C})$ respectively. Thus, by Corollary 3.5 these problems are **coNP**-hard for any $\mathbf{C}$ such that $\mathbf{1} \notin \mathbf{C}$.

One strengthening of the Cook-Levin theorem, discovered by Tseitin in [23], showed that the **SAT**(**Total**) problem remains **NP**-complete when restricted to formulas $\phi(\vec{p})$ in 3-literal-per-clause conjunctive normal form (3CNF), i.e. $\phi(\vec{p})$ is a conjunction of disjunctive clauses each consisting of only at most 3 literals. Let $\mathbf{3CNFSAT}^{=1}(\mathbf{C})$ and $\mathbf{3CNFSAT}^{\neq 0}(\mathbf{C})$ denote the **C**-strong-satisfiability and **C**-weak-satisfiability decision problems where the inputs are restricted to CNF formulas.

The key insight of Tseitin was that every propositional formula $\phi(\vec{p})$ could be transformed, in polynomial time, into a **Total**-equisatisfiable, but not necessarily equivalent, CNF formula $\phi_{\mathsf{CNF}}(\vec{p},\vec{q})$. Proposition 3.6 shows that the Tseitin transformation preserves **C**-strong-satisfiability for any class **C** of partial Boolean algebras. The Tseitin transformation on a propositional formula $\phi(\vec{p})$ produces a CNF formula $\phi_{\mathsf{CNF}}(\vec{p},\vec{q})$ where $\vec{q}$ is such that

$$[\vec{q}] = \{q_\psi \mid \psi \text{ is non-variable subformula of } \phi\ \}.$$

We define ϕ_{CNF} and a set of clauses C_ϕ by induction on the structure of $\phi(\vec{p})$. For the base cases,

1. For $\phi(\vec{p}) = \top$, $\phi_{\mathsf{CNF}}(\vec{p},\vec{q}) = \top$ and $C_\phi = \varnothing$.
2. For $\phi(\vec{p}) = p_i$, $\phi_{\mathsf{CNF}}(\vec{p},\vec{q}) = p_i$ and $C_\phi = \varnothing$.

For the inductive steps, we define $\phi_{\mathsf{CNF}}(\vec{p},\vec{q})$ as $q_\phi \wedge \bigwedge C_\phi$ where C_ϕ depends on the inductive case. In the following, we assume $v_{p_i} = p_i$ and for non-variable subformulas ψ of ϕ, $v_\psi = q_\psi$.

1. For $\phi(\vec{p}) = \neg\psi(\vec{p})$, let $C_\phi = C_\psi \cup N$ where

   $$N = \{\neg v_\phi \vee \neg v_\psi, v_\phi \vee v_\psi\}.$$

   The conjunction of N is equivalent to $v_\phi \leftrightarrow \neg v_\psi$.
2. For $\phi = \psi_1 \vee \psi_2$, let $C_\phi = C_{\psi_1} \cup C_{\psi_2} \cup O$ where

   $$O = \{\neg v_\phi \vee v_{\psi_1} \vee v_{\psi_2}, v_\phi \vee \neg v_{\psi_1}, v_\phi \vee \neg v_{\psi_2}\}.$$

   The conjunction of O is equivalent to $v_\phi \leftrightarrow v_{\psi_1} \vee v_{\psi_2}$.
3. For $\phi = \psi_1 \wedge \psi_2$, let $C_\phi = C_{\psi_1} \cup C_{\psi_2} \cup W$ where

   $$W = \{v_\phi \vee \neg v_{\psi_1} \vee \neg v_{\psi_2}, \neg v_\phi \vee v_{\psi_1}, \neg v_\phi \vee v_{\psi_2}\}.$$

   The conjunction of W is equivalent to $v_\phi \leftrightarrow v_{\psi_1} \wedge v_{\psi_2}$.

▶ **Proposition 3.6.** *Let A be a partial Boolean algebra and $\phi(\vec{p})$ a propositional formula.*

$$\phi \text{ is strongly satisfied in } A \text{ iff } \phi_{\mathsf{CNF}} \text{ is strongly satisfied in } A$$

Thus, we can conclude that for all classes **C**, the problems $\mathbf{3CNFSAT}^{=1}(\mathbf{C})$ and $\mathbf{SAT}^{=1}(\mathbf{C})$ are in the same complexity class up to polynomial reductions.

Moreover, by modifying the formula used in the Proposition 3.4, we can prove a refinement of Corollary 3.5 demonstrating that $\mathbf{3CNFSAT}^{=1}(\mathbf{C})$ and $\mathbf{3CNFSAT}^{\neq 0}(\mathbf{C})$ are also **NP**-hard.

▶ **Theorem 3.7.** *If* **C** *is a class of partial Boolean algebras such that* $1 \notin \mathbf{C}$*, then* $\mathbf{3CNFSAT}^{=1}(\mathbf{C})$ *and* $\mathbf{3CNFSAT}^{\neq 0}(\mathbf{C})$ *is* **NP***-hard.*

Proof. In the proof of Proposition 3.4, for any **Total**-satisfiable formula $\phi(\vec{p})$, we defined a **C**-(weak/strong)-satisfiable formula as $\hat{\phi} = \phi(\vec{p}) \wedge \psi(\vec{p})$ where $\psi(\vec{p})$ was a classical tautology in which every pair of distinct variables $p_i, p_j \in [\vec{p}]$ appeared. Similarly, we define a classical CNF tautology $\psi_{\mathsf{CNF}}(\vec{p})$ with the same property on distinct variables:

$$\psi_{\mathsf{CNF}}(\vec{p}) = \bigwedge_{i \neq j} (\neg p_i \vee p_i \vee p_j) \wedge (\neg p_j \vee p_j \vee p_i).$$

Thus, starting with a **Total**-satisfiable CNF formula $\phi(\vec{p})$, we can produce a **C**-(weak/strong)-satisfiable CNF formula $\hat{\phi}(\vec{p}) = \phi(\vec{p}) \wedge \psi_{\mathsf{CNF}}(\vec{p})$. The problems $\mathbf{3CNFSAT}^{=1}(\mathbf{C})$ and $\mathbf{3CNFSAT}^{\neq 0}(\mathbf{C})$ are **NP**-hard as $\mathbf{3CNFSAT}(\mathbf{Total})$ is **NP**-complete. ◀

4 All satisfiability

Before we proceed to investigate the complexity of the problems $\mathbf{SAT}^{=1}(\mathbf{C})$ and $\mathbf{SAT}^{\neq 0}(\mathbf{C})$ for various classes **C** of pBAs, we first consider the decision problem **SOLUTION**. This consists of the set of tuples $(A, a, \phi(\vec{p}))$ where A is a finite pBA, $a \in A$, and $\phi(\vec{p})$ is a propositional formula such that there exists an $\alpha \in A^{\phi(\vec{p})}$ with $\phi^A(\alpha) = a$. The following proposition shows that **SOLUTION** is **NP**-complete.

▶ **Proposition 4.1.** **SOLUTION** *is* **NP***-complete.*

The first natural class to investigate is the class of all partial Boolean algebras **pBA**. However, **pBA** includes the trivial partial Boolean algebra **1** which satisfies every propositional formula. Therefore, we instead consider the class **All**, formally defined as

$$\mathbf{All} = \{A \in \mathbf{pBA} \mid 0_A \neq 1_A\},$$

of all non-trivial partial Boolean algebras.

Observe that by construction $\mathbf{1} \notin \mathbf{All}$ and so, **All** is closed under collapse (as in Definition 3.2). Therefore, by Proposition 3.3, the notions of **All**-strong-satisfiability and **All**-weak-satisfiability coincide, and we can consider the $\mathbf{SAT}(\mathbf{All})$ decision problem without any ambiguity. Our next observation is that if a $\phi(\vec{p})$ is **All**-satisfiable, then $\phi(\vec{p})$ is satisfiable in a finite **pBA**. Namely, we can freely construct a finite **pBA** F_ϕ for every formula $\phi(\vec{p})$ which always has a term corresponding to $\phi(\vec{p})$. This would then yield an algorithm for $\mathbf{SAT}(\mathbf{All})$ by deciding **SOLUTION** on the input pBA F_ϕ. By utilising extensions (Definition 2.2) and quotients (Definition 2.3), we construct F_ϕ by structural induction on ϕ.

(1) If $\phi = \top$, then $F_\phi = \mathbf{2}$.

(2) If $\phi = p_i$, then F_ϕ is the 4-element Boolean algebra on $\{0, p_i, \neg p_i, 1\}$

(3) If $\phi = \neg\psi$, then $F_\phi = F_\psi$. Note that $[\phi] = \neg[\psi] \in F_\phi$

(4) If $\phi = \psi_1 \bowtie \psi_2$ where $\bowtie \in \{\vee, \wedge\}$, then we break down the construction into three steps:

- $H_\phi = F_{\psi_1} \uplus F_{\psi_2}$ is the coproduct in **pBA** with coprojections $\iota_j \colon F_{\psi_j} \to F_{\psi_1} \uplus F_{\psi_2}$ for $j \in \{1, 2\}$.
- $G_\phi = H_\phi[\circledcirc]$ where $\circledcirc = \{(\iota_1([\psi_1]), \iota_2([\psi_2]))\}$. By the construction of $\circledcirc$-extensions, this means that there exists a term

  $$[\phi]_G = [\iota_1([\psi_1]) \bowtie \iota_2([\psi_2])] \in (F_{\psi_1} \uplus F_{\psi_2})[\circledcirc]$$

  By property (E1) of $\circledcirc$-extensions, there exists a morphism $\eta_\phi \colon H_\phi \to G_\phi$.
- $F_\phi = G_\phi \backslash \ominus$ where

  $$\ominus = \{(\iota_1([\gamma]), \iota_2([\gamma])) \mid \gamma \in \mathsf{Sub}(\psi_1) \cap \mathsf{Sub}(\psi_2)\}.$$

  There is a term $[\phi]_F \in F_\phi$ which is the equivalence class with the representative $[\phi]_G \in G_\phi$. By property (Q1) of $\ominus$-quotient, there exists a morphism $\nu_\psi \colon G_\phi \to F_\phi$.

Explicitly, $F_\phi = (F_{\psi_1} \uplus F_{\psi_2})[\circledcirc] \backslash \ominus$.

Intuitively, F_ϕ can be viewed as a **pBA** analogue to the free Boolean algebra construction in **BA**. This intuition is confirmed by the following proposition.

▶ **Proposition 4.2.** *If A is partial Boolean algebra, $\phi(p_1, \ldots, p_n)$ a propositional formula, and $\alpha \in A^{\phi(\vec{p})}$, then there exists a* **pBA** *morphism $\hat{\alpha}\colon F_\phi \to A$ such that $\hat{\alpha}([\phi]) = \phi^A(\alpha)$.*

The pBA F_ϕ is the minimal pBA which has an element $[\phi] \in F_\phi$ corresponding to the formula ϕ such that if ϕ has variables amongst $\vec{p} = (p_1, \ldots, p_n)$ there is a corresponding meaningful substitution γ where $\gamma(p_i) = [p_i] \in F_\phi$ and $\phi^{F_\phi}(\gamma) = [\phi]$. Thus, we can reduce **All**-satisfiability to weak satisfiability in F_ϕ via γ. Similarly, we can consider M_ϕ where M_ϕ is the quotient of F_ϕ by the relation $\{([\phi], 1_{F_\phi})\}$, and the corresponding meaningful substitution χ where $\chi(p_i) = [[p_i]] \in M_\phi$ and $\phi^{M_\phi}(\chi) = 1 \in M_\phi$. In this case, we can reduce **All**-satisfiability to non-triviality and strong satisfiability in M_ϕ via χ.

▶ **Proposition 4.3.** *Let $\phi(\vec{p})$ be a propositional formula, then the following are equivalent:*

(1) *ϕ is* **All***-satisfiable*

(2) *ϕ is weakly satisfied in F_ϕ via γ.*

(3) *ϕ is strongly satisfied in M_ϕ via χ and M_ϕ is non-trivial.*

In light of Proposition 4.3, and F_ϕ and M_ϕ being finite, there is an algorithm for **All**-satisfiability which constructs M_ϕ from $\phi(\vec{p})$, checks its non-triviality, then decides **SOLUTION** on input $(M_\phi, 1_{M_\phi}, \phi(\vec{p}))$. This algorithm demonstrates that **All**-satisfiability and its dual problem is decidable resolving an open question posed in [16]. However, in the worst case, F_ϕ and M_ϕ are doubly-exponential in the size of the input $\phi(\vec{p})$. Namely, if all variables in $\phi(\vec{p})$ appear pairwise in some subformula, then F_ϕ is just the free Boolean algebra on $[\vec{p}]$ which is of size $2^{2^{|\vec{p}|}}$. Thus, such an algorithm demonstrates that **SAT**(**All**) $\in$ **2EXPTIME**.

We can improve upon this result by observing that it is not necessary to construct the full **pBA** A to check that it is non-trivial and that it satisfies $\phi(\vec{p})$. Instead, we can augment the witness of **SOLUTION**, i.e. the meaningful substitution $\alpha \in A^{\vec{p}}$ with some additional data in order to demonstrate that A is non-trivial. To this end, suppose A is a pBA, $\phi(\vec{p})$ is a propositional formula, and $\alpha \in A^{\phi(\vec{p})}$ is a meaningful substitution, we consider the induced subgraph G_α of $(A, \odot_A)$ by the subformula witnesses arising from α. Explicitly, G_α is the induced subgraph of $(A, \odot_A)$ on the set:

$$V(G_\alpha) := \{\alpha_\psi \mid \psi \in \mathsf{Sub}(\phi(\vec{p}))\}.$$

where we define $\alpha_\psi := \psi^A(\alpha)$ for a cleaner notation. The graph G_α has a root α_ϕ corresponding to the full formula $\phi(\vec{p})$.

Let K be a family of cliques of G_α which cover every edge of G_α. For a clique $C \in K$, let $\mathsf{Eq}(C)$ be all the height-1 Boolean equations satisfied in $\langle C \rangle_A$ involving the elements in C. Note that the equation $\alpha_\phi = 1$ in $\mathsf{Eq}(C)$ if the root α_ϕ of G_α is in C. A function $\nu\colon C \to \{0, 1\}$ respects a set E of height-1 Boolean equations if for every equation $e \in E$, ν restricted to the variables of e is the interpretation of e in the two-element Boolean algebra $\mathbf{2}$, e.g. if e is $\alpha_\psi = \alpha_{\psi_1} \wedge \alpha_{\psi_2}$, then $\nu(\alpha_\psi) = \nu(\alpha_{\psi_1}) \wedge_{\mathbf{2}} \nu(\alpha_{\psi_2})$. By construction C is a pairwise commeasurable set of A, so by (Ext), C is contained within some total Boolean subalgebra of A, we denote the minimal such total subalgebra as $\langle C \rangle_A$. These notions are connected together in the following lemma and allow us to provide a polynomial size witness to the non-triviality of A.

▶ **Lemma 4.4.** *Let A be a* **pBA***, $\phi(\vec{p})$ a propositional formula, $\alpha \in A^{\phi(\vec{p})}$, and K be a non-empty clique-edge cover of G_α. The following are equivalent:*

(1) *A is non-trivial and $\phi^A(\alpha) = 1_A$.*

(2) *For every clique $C \in K$, $\langle C \rangle_A$ is non-trivial and satisfies $\mathsf{Eq}(C)$.*

(3) *For every clique $C \in K$, for every $a \not\leq_A b \in C$, there exists a function $\nu_{a,b}\colon C \to \{0, 1\}$ which respects $\mathsf{Eq}(C)$, $\nu_{a,b}(a) = 1$ and $\nu_{a,b}(b) = 0$.*

▶ **Theorem 4.5.** $\mathbf{SAT}(\mathbf{All})$ *is* $\mathbf{NP}$*-complete.*

Proof. Hardness follows from Corollary 3.5. To show membership in $\mathbf{NP}$, we need to produce a certificate of non-triviality of some pBA A and strong-satisfiability of the input ϕ via a meaningful substitution $\alpha \in A^{\phi(\vec{p})}$. The problem is the pair (A, α) is unlikely to be polynomial size. Instead, we use a certificate consisting of the following data which gives a partial view of the objects A and α:

- A compatibility graph G which view as isomorphic to G_α of size $\leq |\phi(\vec{p})|$
- A binary relation $\leq$ on $V(G)$ which view as a restriction of $\leq_A$ to $V(G_\alpha)$ and pulled back along the isomorphism $G \cong G_\alpha$.
- A non-empty clique edge cover K of G which encode as at most $|E(G)|$ bitstrings of length $|V(G)|$.
- For every $C \in K$, a set of equations $\mathsf{Eq}(C)$ of size $\leq |\mathsf{Sub}(\phi(\vec{p}))|$.
- For every $C \in K$, a set of at most $|C|^2$ functions $\nu_{a,b} \colon C \to \{0,1\}$ (one for every pair $a \not\leq b$) encoded as bitstrings of length $|C|$.

The size bounds demonstrate that our certificate is polynomial in the length of $\phi(\vec{p})$. By Lemma 4.4, these data are sufficient to prove non-triviality of A and satisfiability of ϕ via α. Checking the conditions in Lemma 4.4(3) can be done in polynomial time. $\mathbf{SAT}(\mathbf{All})$ is $\mathbf{NP}$-hard by $\mathbf{1} \notin \mathbf{All}$ and Proposition 3.4. ◀

▶ **Example 4.6.** In this example, we construct a polynomial size witness, as in Lemma 4.4(3), for the simplest example of a formula $\varphi(\vec{p})$ separating **Total**-satisfiability from **All**-satisfiability. This classical contradiction $\varphi(\vec{p})$ is

$$[(p_1 \leftrightarrow p_2) \leftrightarrow (p_3 \leftrightarrow p_4)] \oplus [(p_1 \leftrightarrow p_4) \leftrightarrow (p_2 \leftrightarrow p_3)] \tag{2}$$

The formula $\varphi(\vec{p})$ is the dual of the Kochen-Specker paradox that appeared in [16]. To demonstrate that $\varphi \in \mathbf{SAT}(\mathbf{All})$, we view the graph G_α as a gluing of the 7 cliques in its cover K. In the cover K, there are 3 cliques $\{p_1, p_2, q_{12}\}$, $\{p_3, p_4, q_{34}\}$, $\{q_{12}, q_{34}, z_{\mathsf{left}}\}$ corresponding to the 3 bi-implications in the left argument of $\oplus$, and similarly 3 cliques $\{p_1, p_4, q_{14}\}$, $\{p_2, p_3, q_{23}\}$,$\{q_{14}, q_{23}, z_{\mathsf{right}}\}$ representing the bi-implications on the right. The final seventh clique $T = \{z_\varphi, z_{\mathsf{left}}, z_{\mathsf{right}}\}$ corresponds to the $\oplus$ operation. For each of the cliques C not equal to T, we have the equation expressing the bi-implication, e.g. for $C = \{p_1, p_2, q_{12}\}$, we have $q_{12} = p_1 \leftrightarrow p_2$ in $\mathsf{Eq}(C)$. For the clique T, we have equations expressing $z_\varphi = 1$ and $z_\varphi = z_{\mathsf{left}} \oplus z_{\mathsf{right}}$ in $\mathsf{Eq}(T)$. Although technically if we unroll the definition $\leftrightarrow$ of $\oplus$, these equations are not height-1, we can treat them as such without a meaningful difference to the algorithm. Finally, for every clique $C \in K$, each pair of elements in C are incomparable, so we have 3 valuation functions $\nu_{a,b} \colon C \to \{0,1\}$ for every $C \in K$. For a clique C not equal to T, each valuation function ν has two choices, e.g. for $C = \{p_1, p_1, q_{12}\}$, ν_{p_1,p_2} either maps p_1 to 0, p_2 to 1 or vice-versa and the image of q_{12} is determined by the bi-implication in $\mathsf{Eq}(C)$. For the top clique T, the valuation functions are the unique 3 that respect $\mathsf{Eq}(T)$ and the separation condition.

By the duality previously noted, Theorem 4.5 also implies that $\mathbf{VAL}(\mathbf{All})$ is $\mathbf{coNP}$-complete.

5 Kochen-Specker Proofs

Partial Boolean algebras were defined by Kochen and Specker in order to formalize a necessary condition for non-contextual hidden-variable theories of quantum theory to exist. This condition involved the existence of a **pBA** morphism from pBAs arising in quantum theory to the two-element Boolean algebra **2**. Thus, to begin we define these motivating pBAs arising in quantum theory.

Recall that in the standard textbook formulation of quantum theory the pure states of a quantum system live in a vector space $\mathcal{H}$ over the field of complex numbers $\mathbb{C}$. Measurements of a quantum system with pure states in $\mathcal{H}$ are identified with the bounded self-adjoint, also called Hermitian, linear maps $M\colon \mathcal{H} \to \mathcal{H}$. The operation of measuring M on a state yields one of the eigenvalues λ of M and as M is self-adjoint, $\lambda \in \mathbb{R}$. After measurement, the state of the system collapses to one of the eigenvectors corresponding to the eigenvalue λ. We denote the set of bounded self-adjoint linear maps over $\mathcal{H}$ as $\mathbf{B}(\mathcal{H})$. By the spectral theorem, each outcome λ of the measurement $M \in \mathbf{B}(\mathcal{H})$ can be identified with the idempotent element $E_\lambda \in \mathbf{B}(\mathcal{H})$ which projects onto the eigenspace of the eigenvalue λ of M. We denote the set of idempotent self-adjoint linear maps over $\mathcal{H}$ as $\mathbf{P}(\mathcal{H})$. Projectors in $\mathbf{P}(\mathcal{H})$ have eigenvalues $0, 1$, i.e. answer yes/no questions, and are in bijection with subspaces of $\mathcal{H}$. Thus, "measuring" the yes/no question E_λ yields an answer to the question: Did measurement M have outcome λ? Given a projector $E \in \mathbf{P}(\mathcal{H})$, let $\mathbf{Im}(E)$ denote its image subspace and $\mathbf{rank}(E)$ denote the dimension of $\mathbf{Im}(E)$, i.e. the *rank* of E.

▶ **Definition 5.1.** *Given a real or complex Hilbert space $\mathcal{H}$, the* $\mathcal{H}$-projector partial Boolean algebra *is the set $Q = \mathbf{P}(\mathcal{H})$ where:*

- *1_Q is the projector $I_\mathcal{H}$ onto the whole space $\mathcal{H}$, 0_Q is the projector $0_\mathcal{H}$ onto the 0-dimensional subspace of $\mathcal{H}$,*
- *$E \odot_Q F$ if $[E, F] = EF - FE = 0_H$, i.e. two projectors are commeasurable if they commute,*
- *$\neg_Q E = \overline{E} = I_\mathcal{H} - E$,*
- *$E \vee_Q F = E + F - EF$,*
- *$E \wedge_Q F = EF$.*

Pure states of a quantum system are vectors (up to normalisation) and thus can be recovered from $\mathbf{P}(\mathcal{H})$ as the set of rank-1 projectors, denoted $\mathbf{P}_1(\mathcal{H})$.

For a non-contextual hidden variable theory of the quantum system described $\mathcal{H}$ to exist, there would necessarily be a **pBA**-morphism $f\colon \mathbf{P}(\mathcal{H}) \to \mathbf{2}$. Thus, the Kochen-Specker theorem demonstrates that there cannot exist such a **pBA**-morphism for real or complex Hilbert space $\mathcal{H}$ of dimension ≥ 3. Proofs of Kochen-Specker's theorem can be formulated as exhibiting an orthogonality graph G of vectors in $\mathcal{H}$ which lacks a certain two colouring of the vertices.

We make the notion of Kochen-Specker proof precise using a formalisation which differs slightly from the usual graph-theoretic formalisations, see e.g. [21] for details, by allowing vertices to be assigned to projectors of rank greater than 1 and not necessarily injectively. In the following definitions, a graph G is simple, undirected, without self-loops. A graph G has vertex set $V(G)$, edge set $E(G)$, and set of maximum cliques $\Omega(G)$.

▶ **Definition 5.2.** *Given a graph G and real or complex Hilbert space $\mathcal{H}$, an* orthogonal assignment *$f\colon G \to \mathbf{P}(\mathcal{H})$ is a function $f\colon V(G) \to \mathbf{P}(\mathcal{H})$ if the following two conditions hold:*

(O1) *For every $(v, w) \in E(G)$, $f(v)f(w) = 0_\mathcal{H}$*

(O2) *For every maximum clique $C \in \Omega(G)$ of G, $\sum_{v \in C} f(v) = I_\mathcal{H}$.*

We say f is a rank-1 orthogonal assignment *if $f\colon V(G) \to \mathbf{P}_1(\mathcal{H})$ and use the notation $f\colon G \to \mathbf{P}_1(\mathcal{H})$.*

An orthogonal assignment is a generalisation of the notion of non-contextual colouring.

▶ **Definition 5.3.** *Given a graph G, a function $f\colon V(G) \to \{0,1\}$ is a* non-contextual colouring of G *if the following two conditions hold:*

(C1) *For every edge $(v,w) \in E$, $f(v) + f(w) \leq 1$*

(C2) *For every maximum clique $C \in \Omega(G)$ of G, $\sum_{v \in C} f(v) = 1$.*

▶ **Definition 5.4.** *For a (possibly infinite) cardinal d, a graph G is a d-*dimensional Kochen-Specker proof *if G has an orthogonal assignment $f\colon G \to \mathbf{P}(\mathcal{H})$ to a real or complex d-dimensional Hilbert space $\mathcal{H}$ and G does **not** have a non-contextual colouring.*

If $S \subseteq \mathcal{H}$ is a set vectors, we can form its orthogonality graph G_S with vertices S and $\vec{v}, \vec{v'} \in S$ are adjacent if $\vec{v}^\dagger \vec{v'} = 0$, i.e. are orthogonal. Obviously, G_S has a rank-1 orthogonal assignment $f_S(\vec{v}) = E_{\vec{v}}$. We say S is basis-complete if it is equal to the union of a family of orthonormal bases $\mathcal{B}_S$. For any basis-complete set $S \subseteq \mathbb{K}^d$, the orthogonality graph $\omega(G_S) = d$ and every maximal clique is a maximum clique. Appropriating terminology from topology, we will say a graph G is a *d-facet graph* if $\omega(G) = d$ and every maximal clique is a maximum clique. If S is basis-complete, then G_S is a facet graph, so condition (C1) is redundant and we obtain the following equivalence:

▶ **Proposition 5.5.** *If S is basis-complete, then the following are equivalent:*

(1) *G_S does not have a non-contextual colouring.*

(2) *G_S has no colouring $f\colon S \to \{0,1\}$ satisfying condition (C2).*

(3) *For every function $f\colon S \to \{0,1\}$, there exists a maximum (equiv. maximal) clique C_f of G_S such that $\sum_{c \in C_f} f(u) \neq 1$.*

The original proof of the Kochen-Specker theorem [17] was a proof in the sense of Definition 5.4 with dimension 3. This proof was constructed as the orthogonality graph G_S of a set of 117 vectors $S \subseteq \mathbb{R}^3$ (or 120 vectors, if we take the basis completion of S). Subsequent work has resulted in Kochen-Specker proofs of lower cardinality. The next section involves a reduction which uses a basis-complete 3-dimensional Kochen-Specker proof G as a gadget. This reduction is entirely independent in the choice of basis-complete Kochen-Specker proof. Therefore, we opt to consider the smallest known example in dimension 3, the basis-completion of the Conway-Kochen set from [7] consisting of only 40 vectors which we denote as CK.

The contrapositive of [17, Theorem 4] states that the non-existence of **pBA** morphism $h\colon A \to \mathbf{2}$ is equivalent to exhibiting a classical propositional contradiction φ which is satisfied in the pBA A. It follows that from a finite d-dimensional Kochen-Specker proof G, we should be able to construct a propositional contradiction φ_G which is satisfied in $\mathbf{P}(\mathcal{H})$. For any finite graph G, we can associate a propositional formula φ_G:

$$\varphi_G(\vec{p}) := \bigwedge_{(v,v') \in E(G)} \neg(p_v \wedge p_{v'}) \wedge \bigwedge_{C \in \Omega(G)} \bigvee_{v \in V(C)} p_v.$$

where variables $[\vec{p}]$ are indexed by $V(G)$. The desired connection of φ_G to a Kochen-Specker proof G is verified in the following proposition.

▶ **Proposition 5.6.** *Let $Q = \mathbf{P}(\mathcal{H})$ for some real or complex Hilbert space. G has an orthogonal assignment $f\colon G \to Q$ iff $\varphi_G(\vec{p})$ is strongly satisfied in Q. In particular, G has non-contextual colouring iff $\varphi_G(\vec{p})$ is classically satisfiable.*

Proof. The conjunction $\bigwedge_{(v,v')} \neg(p_v \wedge p_{v'})$ asserts that $f(v)f(v') = 0$, i.e. condition (O1). Using $f(v)f(v') = 0$, the clause $\bigvee_{v \in V(C)} p_v$ requires that $\sum_{i \in [d]} f(v) = 1$, i.e condition (O2). In this case of $Q = \mathbf{P}(\mathbb{K}) \cong \mathbf{2}$, conditions (O1)-(O2) are simply conditions (C1)-(C2) of a non-contextual colouring. ◀

6 Fixed dimension quantum satisfiability

In this and the next section, we establish complexity bounds for the satisfiability problem for partial Boolean algebras which arise from the study of quantum contextuality. This section focuses on the case for quantum system whose states reside in a d-dimensional Hilbert space for some $d \in \mathbb{N}$.

Since we restrict ourselves to the finite-dimensional case in this section, we take $\mathcal{H} = \mathbb{K}^d$. In this case $\mathbf{B}(\mathbb{K}^d)$ is the set of $d \times d$ matrices $M \in M_{\mathbb{K}}(d,d)$ which satisfy the self-adjoint condition: M is equal to its conjugate transpose $M^\dagger$. The partial Boolean algebra $Q = \mathbf{P}(\mathbb{K}^d)$ consists of $d \times d$ projection matrices, $I_{\mathcal{H}}$ is the identity matrix, $0_{\mathcal{H}}$ is the all-zeros matrix, and the operations used in Definition 5.1 for defining $\neg_Q$, $\vee_Q$ and $\wedge_Q$ are matrix addition and multiplication.

Recall that each measurement has an associated set of orthogonal projectors called a PVM. In the maximal case, this PVM is associated to a basis of the Hilbert space $\mathcal{H}$. We define the formula $\mathsf{basis}_d(p_1, \ldots, p_d) := \varphi_{K_d}$ where K_d is the d-vertex complete graph to "capture" PVMs in $\mathcal{H}$.

▶ **Proposition 6.1.** *Let $Q = \mathbf{P}(\mathbb{K}^d)$, if $E_1, \ldots, E_d \in Q$ are non-zero projectors, then $\mathsf{basis}_d^Q(E_1, \ldots, E_d) = 1_Q$ if, and only if,*
(1) *for every $i \in [d]$, there exists unit vectors $\vec{v_i} \in \mathbb{K}^d$ where $E_i = \vec{v_i}\vec{v_i}^\dagger$, and*
(2) *$\{\vec{v_1}, \ldots, \vec{v_d}\}$ is an orthonormal basis of $\mathbb{K}^d$.*
In particular, $(p_1, \ldots, p_d) \mapsto (E_1, \ldots, E_d)$ is a rank-1 assignment.

Our first observation is that for $d = 1$ or $d = 2$, the weak and strong satisfiability problems are **NP**-complete.

▶ **Theorem 6.2.** *If $d \in \{1, 2\}$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, then problems $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^d))$ and $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{K}^d))$ are **NP**-complete.*

Proof. In all cases, the problems are equivalent to the classical satisfiability problem $\mathbf{SAT}(\mathbf{Total})$. For the case where $d = 1$, we note that $\mathbf{P}(\mathbb{R}) = \mathbf{P}(\mathbb{C}) = \mathbf{2}$ is the two-element Boolean algebra.

For the case where $d = 2$. By Theorem 4 of [17], a **pBA** M has a **pBA**-morphism $f\colon M \to \mathbf{2}$ if, and only if, every formula which is strongly satisfiable in M is classically satisfiable. In particular, since $\mathbf{P}(\mathbb{K}^2)$ has (uncountably many) morphisms to $\mathbf{2}$, i.e. by choosing which element in a pair of orthogonal rank-1 projectors is assigned to 1, every input to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^2))$ is classically satisfiable. For $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{K}^2))$, every yes-instance $\phi(\vec{p})$ of $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{K}^2))$ is either a yes-instance of $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^d))$ or the witness α is such that $\phi^{\mathbf{P}(\mathbb{K}^2)}(\alpha)$ is a rank-1 projector – and therefore a yes-instance to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K})) \cong \mathbf{2}$. Thus, all cases reduce to and from the classical satisfiability problem $\mathbf{SAT}(\mathbf{Total})$. ◀

6.1 Real case

Here, we prove that for every $d \geq 3$, the complexity of satisfiability in $\mathbf{P}(\mathbb{R}^d)$ is $\exists\mathbb{R}$-complete.

Thus, we began by defining the complexity class called the existential theory of the reals which is denoted $\exists\mathbb{R}$. Instead of defining $\exists\mathbb{R}$ directly, we give a more general definition that is helpful in subsequent sections.

We associate a complexity class $\exists(L, M)$ to any first-order signature L which contains the signature of rings $L_{\mathsf{rings}} = \{0, 1, +, *\}$ and L-structure M. For every such pair (L, M), we define

- $E(L)$ to be the set of first-order sentences Φ in L of the form

$$\exists X_1, \ldots, \exists X_n \Psi(X_1, \ldots, X_n)$$

  where $\Psi(X_1, \ldots, X_n)$ is a quantifier-free formula; and
- $E(L, M)$ to the set of sentences $\Phi \in E(L)$ which are true in M.

We can then define the complexity class $\exists(L, M)$ as the class of decision problems which are many-one polynomial-time reducible to $E(L, M)$. For example, **NP** can be defined as $\exists(L, M)$ where $L = L_{\mathsf{rings}}$ and M is the two-element field $(\mathbb{Z}_2, 0, 1, \oplus, \wedge)$. The class $\exists\mathbb{R}$ is $\exists(L, M)$ where $L = L_{\mathsf{rings}} \cup \{\leq\}$ is the signature of *ordered* rings and M is the ordered field of real numbers $(\mathbb{R}, 0, 1, +, *, \leq)$. As usual, a decision problem D is $\exists\mathbb{R}$-*hard* if every problem in $\exists\mathbb{R}$ polynomial-time many-one reduces to D and D is $\exists\mathbb{R}$-*complete* if D is in $\exists\mathbb{R}$ and $\exists\mathbb{R}$-hard. Many natural computational problems, usually related to discrete objects embedded in continuous spaces, are known to be $\exists\mathbb{R}$-complete, e.g. see the compendium [22] for details. The class $\exists\mathbb{R}$ can also be defined as the Boolean fragment of non-deterministic polynomial time on real Blum-Shub-Smale machines [5]. In relation to classic complexity classes, it is known that **NP** $\subseteq \exists\mathbb{R} \subseteq$ **PSPACE**. The inclusion **NP** $\subseteq \exists\mathbb{R}$ follows from the Cook-Levin theorem and the polynomial-time mapping, e.g. $\neg p \vee q \mapsto \exists P \exists Q(P = 0 \vee Q = 1)$, from a propositional formula to an existential sentence in the language of fields. The inclusion of $\exists\mathbb{R} \subseteq$ **PSPACE** was a seminal result proved by Canny [6].

Proposition 6.3 below demonstrates that the weak and strong satisfiability problems for $\mathbf{P}(\mathbb{R}^d)$ are in $\exists\mathbb{R}$. This is because, using Definition 5.1, a propositional formula $\phi(p_1, \ldots, p_n)$ being satisfied in $\mathbf{P}(\mathbb{R}^d)$ translates into the existence of self-adjoint solutions to a set S of (in)equations in the ring of matrices $M_{\mathbb{R}}(d, d)$. Hence, by unfolding the definitions of matrix addition and multiplication in $M_{\mathbb{R}}(d, d)$ in terms of their d^2 entries, these equations translate to $O(d^2|S|)$ equations over the field $\mathbb{R}$.

▶ **Proposition 6.3.** *For every $d \geq 1$, $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^d))$ and $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{R}^d))$ are in $\exists\mathbb{R}$.*

Now we proceed with the proof that $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^d))$ and $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{R}^d))$ are $\exists\mathbb{R}$-hard for any fixed $d \geq 3$. To accomplish this, we work by induction on dimension d. Hence, for the base case, we first prove that $\mathbf{SAT}(\mathbf{P}(\mathbb{R}^3))$ is $\exists\mathbb{R}$-hard. We construct reductions from the $\exists\mathbb{R}$-complete problem from [11] called cross-product term satisfiability over the real projective plane $\mathbb{RP}^2$. We denote this decision problem $\mathbf{XSAT}(\mathbb{RP}^2)$. In order to describe $\mathbf{XSAT}(\mathbb{RP}^2)$, recall that the right-handed cross product is a binary operation $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined coordinate-wise as:

$$\times(\vec{v}, \vec{w}) = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1). \tag{3}$$

for every $\vec{v} = (v_1, v_2, v_3), \vec{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$. The operation is defined so that if $w = u \times v$, then w is orthogonal to both u and v. This is a key fact which we exploit in our reduction. The operation $\times$ extends to a partial operation, which we also denote as $\times$, on the real projective plane $\mathbb{RP}^2$. Elements of $\mathbb{RP}^2$ are the lines $\mathbb{R}\vec{v} = \{\lambda\vec{v} \mid \lambda \neq 0\}$ for every $\vec{v} \in \mathbb{R}^3 \backslash \{\vec{0}\}$ in $\mathbb{R}^3$, and $\times$ is defined on representatives as $\mathbb{R}\vec{v} \times \mathbb{R}\vec{w} = \mathbb{R}(\vec{v} \times \vec{w})$ for distinct lines $\mathbb{R}v \neq \mathbb{R}w$. A *cross product term* $t(x_1, \ldots, x_n)$ with variables amongst $X = \{x_1, \ldots, x_n\}$ is recursively defined as either one of the variables x_i or $s \times s'$ for cross product terms s, s' with variables

also amongst X. Given a cross product term $t(x_1, \ldots, x_n)$ and $\mathbb{R}\vec{v}_1, \ldots, \mathbb{R}\vec{v}_n \in \mathbb{RP}^2$, the value $\langle t(\mathbb{R}\vec{v}_1, \ldots, \mathbb{R}\vec{v}_n)\rangle \in \mathbb{RP}^2$ is defined by inductively applying the definition of $\curlywedge$. A cross product term $t(x_1, \ldots, x_n)$ is *satisfied in* $\mathbb{RP}^2$ if there exists $\mathbb{R}\vec{v}_1, \ldots, \mathbb{R}\vec{v}_n \in \mathbb{RP}^2$ such that $\langle t(\mathbb{R}\vec{v}_1, \ldots, \mathbb{R}\vec{v}_n)\rangle$ is defined and $\langle t(\mathbb{R}\vec{v}_1, \ldots, \mathbb{R}\vec{v}_n)\rangle = \mathbb{R}\vec{v}_1 \neq 0$. The $\mathbf{XSAT}(\mathbb{RP}^2)$ problem takes as input a cross product term $t(x_1, \ldots, x_n)$ and decides if t is satisfied in $\mathbb{RP}^2$. In order to reduce $\mathbf{XSAT}(\mathbb{RP}^2)$ to $\mathbf{SAT}(\mathbf{P}(\mathbb{R}^3))$, we give a translation from a cross product term $t(x_1, \ldots, x_n)$, to a propositional formula $\theta_t^{\mathbb{R}}$.

Consider a non-variable subterm $s = r \curlywedge o$ of t. The lines $\mathbb{R}\langle s\rangle$, $\mathbb{R}\langle r\rangle$, and $\mathbb{R}\langle o\rangle$ are in bijection with the rank-1 projectors $E_{\langle s\rangle}, E_{\langle r\rangle}$, and $E_{\langle o\rangle}$ onto these lines. Since the line $\mathbb{R}\langle s\rangle$ is orthogonal to both $\mathbb{R}\langle r\rangle$ and $\mathbb{R}\langle o\rangle$, we need to express that the pairs $(E_{\langle s\rangle}, E_{\langle r\rangle})$ and $(E_{\langle s\rangle}, E_{\langle o\rangle})$ extend to triples of projectors, e.g. $(E_{\langle s\rangle}, E_{\langle r\rangle}) \mapsto (E_{\langle s\rangle}, E_{\langle r\rangle}, \overline{E_{\langle r\rangle} + E_{\langle s\rangle}})$ which correspond to orthonormal bases of $\mathbb{R}^3$.

From these observations, the naive construction would be to define for every non-variable subterm $s = r \curlywedge o$, a formula ϑ_s which is a conjunction of two basis_3 formulas. A witness that the term $t(x_1, \ldots, x_n)$ is satisfied in $\mathbb{RP}^2$ would then yield a meaningful substitution demonstrating that the conjunction of ϑ_s is in $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^3))$. However, such a meaningful substitution is not sufficient to produce a witness to t. We need to show that any meaningful substitution of the conjunction produces a substitution where every variable is assigned to a rank-1 projector. We can show that for each basis_3 conjunct of ϑ_s, there exists uncountably many rank-1 assignments (one for each orthonormal basis of $\mathbb{R}^3$). The problem arises when taking a conjunction of basis_3 formulas with shared variables. Namely, though for each conjunct $c = \mathsf{basis}_3(p_s, p_r, p_{s,r})$ we can break down an arbitrary assignment of variables in c into a rank-1 assignment, there are many inconsistent ways of collecting all these assignments to produce a rank-1-assignment for the entire conjunction. In order to overcome this issue, we must acknowledge the inherent contextuality in the $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^3))$ problem. Thus, as part of our reduction, we use the orthogonality graph G_{CK} of the basis-complete 40-vector Kochen-Specker set CK mentioned in section 5 as a gadget. The primary property of G_{CK} we use is that any orthogonal assignment is guaranteed to have a triangle which is assigned to only rank-1 projectors.

▶ **Proposition 6.4.** *If $f\colon G_{\mathsf{CK}} \to \mathbf{P}(\mathbb{R}^3)$ is an orthogonal assignment, then there exist orthonormal bases $\{\vec{x}, \vec{y}, \vec{z}\} \in \mathcal{B}_{\mathsf{CK}}$ and $\{\vec{u}_x, \vec{u}_y, \vec{u}_z\}$ such that $f(\vec{x}) = E_{\vec{u_x}}$, $f(\vec{y}) = E_{\vec{u_y}}$, and $f(\vec{z}) = E_{\vec{u_z}}$ are rank-1 projectors.*

We use two copies of K_3 to form the "consistency" gadget Cons depicted in Figure 1 to ensure that different vertices are assigned the same projector by any orthogonal assignment. The following proposition expresses the how Cons is used in the reduction.

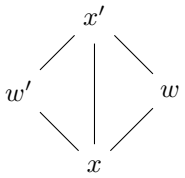

Figure 1 The gadget (Cons, w, w').

▶ **Proposition 6.5.** *If $f\colon \mathsf{Cons} \to \mathbf{P}(\mathbb{R}^3)$ is an orthogonal assignment such that $f(w) = E_{\vec{w}}$, $f(w') = E_{\vec{w'}}$, are rank-1 projectors corresponding to unit vectors $\vec{w}, \vec{w'} \in \mathbb{R}^3$, then $f(w') = f(w) = E_{\vec{w}}$.*

We now can convert every cross-product term $t(x_1, \ldots, x_n)$ into a graph G_t. The construction of G_t is through the following steps:

1. View $t(x_1, \ldots, x_n)$ as its rooted directed parse tree T_t.
2. Produce a directed-acyclic graph D_t by identifying all leaf nodes in T_t that are occurrences of the same variable x_i.
3. Produce a directed graph D'_t by identifying the root node t of D_t with the sink node representing variable x_1.
4. Produce an undirected graph G'_t by forgetting the orientations of edges in D'_t.
5. Produce G_t by taking the triangle completion of G'_t, i.e. for every edge $e = (u, v)$ add an additional vertex v_e incident to both u and v ensuring $\{u, v, v_e\}$ is a triangle.

For every graph G, we construct $\mathsf{CKE}(G)$ through the following steps:

1. Take the categorical product $G \times G_{\mathsf{CK}}$. Recall that the categorical product $G \times H$ of two graphs G and H has vertex set $V(G \times H) = V(G) \times V(H)$ and edge set defined as:

$$E(G \times H) = \{((v, v'), (w, w')) \mid (v, w) \in E(G) \text{ and } (v', w') \in E(H)\}$$

2. To construct $\mathsf{CKE}(G)$, for every $w \in V(G)$ and pair of distinct vectors $\vec{v}, \vec{u} \in \mathsf{CK}$ we add a consistency gadget $(\mathsf{Cons}, (w, \vec{v}), (w, \vec{u}))$ to $G \times G_{\mathsf{CK}}$.

The primary property of the $\mathsf{CKE}(\cdot)$ construction we use is stated in the following lemma.

▶ **Lemma 6.6.** *Let G be a 3-facet graph. For every orthogonal assignment $f \colon \mathsf{CKE}(G) \to \mathbf{P}(\mathbb{K}^3)$, there exists a rank-1 orthogonal assignment $\nu_f \colon G \to \mathbf{P}_1(\mathbb{K}^3)$.*

Finally, we use the propositional formula $\theta_t(\vec{p}) = \varphi_{\mathsf{CKE}(G_t)}(\vec{p})$, i.e. the orthogonal assignment formula of graph $\mathsf{CKE}(G_t)$ to map an instance of $\mathbf{XSAT}(\mathbb{RP}^2)$ to $\mathbf{SAT}(\mathbf{P}(\mathbb{R}^3))$.

▶ **Proposition 6.7.** *For every cross-product term t, the following are equivalent*

(1) $t(x_1, \ldots, x_n) \in \mathbf{XSAT}(\mathbb{RP}^2)$.
(2) *There exists an orthogonal assignment* $f \colon \mathsf{CKE}(G_t) \to \mathbf{P}(\mathbb{R}^3)$.
(3) $\theta_t(\vec{p}) \in \mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^3))$.
(4) $\theta_t(\vec{p}) \in \mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{R}^3))$.

The reduction exhibited in Proposition 6.7 establishes that $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^3))$ and $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{R}^3))$ are $\exists\mathbb{R}$-hard. In order to generalize these hardness results to dimension $d > 3$, we need a padding argument to obtain a reduction from $\mathbf{XSAT}(\mathbb{RP}^2)$ to the satisfiability problems over $\mathbf{P}(\mathbb{R}^d)$. This padding argument is encapsulated in Proposition 6.8 below.

For every propositional formula $\phi(p_1, \ldots, p_n)$ and dimension d, we define a *d-dimensional realisation* formula $\hat{\phi}_d(p_1, \ldots, p_n, q_1, \ldots, q_{d+1})$ defined as the conjunction of:

(D1) $\mathsf{basis}_{d+1}(q_1, \ldots, q_{d+1})$
(D2) $\bigwedge_{i \in [n]} \phi_{\leq}(p_i, \neg q_{d+1})$
(D3) $\phi(p_1, \ldots, p_n) \leftrightarrow \neg q_{d+1}$

The purpose of this formula is to show that we can embed the witnesses to satisfiability of $\phi(p_1, \ldots, p_n)$ in $\mathbb{K}^d$ into a d-dimensional subspace of $\mathbb{K}^{d+1}$. Suppose $\gamma \in Q^{\hat{\phi}}$ is a meaningful substitution. Item (D1), asserts that if the projectors $\gamma(q_i)$ assigned to the q variables are all non-zero, then they form a PVM of rank-1 projectors, i.e. pick out an orthonormal basis of $\mathbb{K}^{d+1}$. Item (D2) asserts that each of the projectors $\gamma(p_j)$ assigned to the p-variables is a projector onto some subspace of the d-dimensional subspace S orthogonal to the 1-dimensional subspace $\gamma(q_{d+1})$, i.e. the image of $\sum_{i \in [d]} \gamma(q_d)$. Item (D3) asserts that $\phi(p_1, \ldots, p_n)$ is assigned to the projector on S.

▶ **Proposition 6.8.** *Let* $\vec{p} = (p_1, \ldots, p_n)$ *and* $\vec{q} = (q_1, \ldots, q_{d+1})$. $\phi(\vec{p}) \in \mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^d))$ *if, and only if,* $\hat{\phi}_d(\vec{p}, \vec{q}) \in \mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^{d+1}))$.

▶ **Theorem 6.9.** *For every* $d \geq 3$, $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{R}^d))$ *and* $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^d))$ *are* $\exists\mathbb{R}$*-complete.*

Proof. Proposition 6.3 proves membership in $\exists\mathbb{R}$. For $\exists\mathbb{R}$-hardness of the base case $d = 3$, Proposition 6.7 yields reductions from the $\exists\mathbb{R}$-complete problem $\mathbf{XSAT}(\mathbb{RP}^2)$ to $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{R}^3))$ and $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^3))$. The size of $\mathsf{CKE}(G_t)$ is $O(|G_{\mathsf{CK}}||G_t|) = O(40|t|)$ and thus $\theta_t(\vec{p})$ is polynomial in the size of $t(x_1, \ldots, x_n)$. For $\exists\mathbb{R}$-hardness beyond the base case, we inductively apply Proposition 6.8 to obtain reduction from $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^d))$ to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^{d+1}))$. Finally, the $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{R}^d))$ problem is equivalent to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{R}^{d'}))$ for some $d' \leq d$. The d-dimensional padding formula $\hat{\phi}_d(\vec{p}, \vec{q})$ from Proposition 6.8 is linear in the size of $\phi(\vec{p})$. ◀

Dually, we have that $\mathbf{VAL}^{\neq 0}(\mathbf{P}(\mathbb{R}^d))$ and $\mathbf{VAL}^{=1}(\mathbf{P}(\mathbb{R}^d))$ are $\forall\mathbb{R}$-complete, where $\forall\mathbb{R}$ is the class of problems that are polynomial-time equivalent to the universal theory of the real field.

6.2 Complex case

We define the class $\exists\mathbb{C}$ as $\exists(L, M)$ where $L = L_{\mathsf{rings}}$ and M is the field of complex numbers $(\mathbb{C}, 0, 1, +, *)$. It is easy to see that $\exists\mathbb{C} \subseteq \exists\mathbb{R}$ since there is an isomorphism of rings mapping a complex number $z = x + iy$ to the 2×2-real matrix $\left(\begin{smallmatrix} x & y \\ -y & x \end{smallmatrix}\right)$. We also define $\exists\overline{\mathbb{C}}$ to be $\exists(L, M)$ where $L = L_{\mathsf{rings}} \cup \{u\}$ has an additional unary function symbol u and M is the complex field $(\mathbb{C}, 0, 1, +, *, (\cdot)^*)$ equipped with complex conjugation $(\cdot)^*$ as the interpretation of u. As the subfield $\mathbb{R}$ can be defined as the numbers fixed by complex conjugation, we obtain the equality between the complexity classes $\exists\overline{\mathbb{C}} = \exists\mathbb{R}$

▶ **Proposition 6.10.** $\exists\mathbb{C} \subseteq \exists\mathbb{R}$ *and* $\exists\overline{\mathbb{C}} = \exists\mathbb{R}$.

To show that satisfiability in $\mathbf{P}(\mathbb{C}^d)$ is in $\exists\mathbb{R}$, we use a similar proof to Proposition 6.3 to produce a sentence in $\exists\overline{\mathbb{C}}$ from an input propositional formula. The result then follows from Proposition 6.10.

▶ **Proposition 6.11.** *For every* $d \geq 1$, $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$ *and* $\mathbf{SAT}^{\neq 0}(\mathbf{P}(\mathbb{C}^d))$ *are in* $\exists\mathbb{R}$.

We now proceed to prove that $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$ for $d \geq 4$ is $\exists\mathbb{R}$-hard, and thus $\exists\mathbb{R}$-complete by Proposition 6.11. The reduction from $\mathbf{XSAT}(\mathbb{RP}^2)$ we employed in the real case (Proposition 6.7) does not straightforwardly translate to the complex case. This is because we need to produce orthogonal sets of real lines in $\mathbb{R}^3$ from a meaningful substitution which consists of complex projectors. That is we need to translate data in the form of operators over $\mathbb{C}^d$ to lines in $\mathbb{R}^3$. In order to accomplish this, we have to change the base case of our dimension from $d = 3$ to $d = 4$. However, the advantage of this reduction is that we also obtain a reduction from $\mathbf{XSAT}(\mathbb{RP}^2)$ to the exclusive-or satisfiability problem $\mathbf{XORSAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$ that we describe below.

The first ingredient in our reduction from $\mathbf{XSAT}(\mathbb{RP}^2)$ to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^4))$ is the Pauli vector isomorphism from $\mathbb{R}^3$ to the set $\mathbf{B}_0(\mathbb{C}^2)$ of traceless Hermitian matrices over $\mathbb{C}^2$. This isomorphism is the map $\vec{\sigma}\colon \mathbb{R}^3 \to \mathbf{B}_0(\mathbb{C}^2)$ given by $(a_1, a_2, a_3) \mapsto (a_1\sigma_x + a_2\sigma_y + a_3\sigma_z)$ where $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli matrices. The second ingredient in our reduction is to change perspective by using the pBA $\mathbf{I}(\mathbb{C}^d)$ consisting of self-adjoint involutions, i.e. $A^2 = I_d$ instead of $\mathbf{P}(\mathbb{C}^d)$. For every $d \in \mathbb{N}$, there is a bijection $b\colon \mathbf{P}(\mathbb{C}^d) \to \mathbf{I}(\mathbb{C}^d)$ given by $b(E) = I_d - 2E$ which preserves commutative pairs. This bijection allows us to reverse engineer the definitions of $\vee, \wedge$ for $\mathbf{I}(\mathbb{C}^d)$ from their definitions in the pBA $\mathbf{P}(\mathbb{C}^d)$. However,

our main reason for shifting to $\mathbf{I}(\mathbb{C}^d)$ is that the exclusive-or operation $\oplus$ on involutions $\mathbf{I}(\mathbb{C}^d)$ is matrix multiplication. Moreover, $\mathbf{I}(\mathbb{C}^2) \subseteq \mathbf{B}_0(\mathbb{C}^2)$ and contains the standard Pauli matrices. Note the in the classical case of $d = 1$, this amounts to moving the Boolean domain from $\{0, 1\}$ to $\{+1, -1\}$.

The final ingredient in our reduction is the so-called Peres-Mermin Magic Square [19]. The Peres-Mermin magic square is a witness to Kochen-Specker contextuality through quantum entanglement. The magic square M is a set of equations over the group $J_2 = (\{-1, +1\}, *) \cong (\mathbb{Z}_2, +)$ typically presented as the two table in equation (4) below.

$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & f \\ \hline g & h & i \\ \hline \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|} \hline \sigma_x \otimes I_2 & I_2 \otimes \sigma_x & \sigma_x \otimes \sigma_x \\ \hline I_2 \otimes \sigma_z & \sigma_z \otimes I_2 & \sigma_x \otimes \sigma_z \\ \hline \sigma_x \otimes \sigma_z & \sigma_z \otimes \sigma_x & \sigma_y \otimes \sigma_y \\ \hline \end{array} \tag{4}$$

In the table on the left, each cell represents a variable, every row and the first 2 columns represent a multiplicative equation equal to 1 (e.g. $abc = 1$), and the last column represents the equation $cfi = -1$ (indicated by the dotted column separator). These equations are satisfiable in $\mathbf{I}(\mathbb{C}^4)$ via the assignment expressed in the table on the right where $\sigma_x, \sigma_z, \sigma_y$ are the Pauli matrices acting on two qubits. These yield a satisfying assignment since every row and the first 2 columns multiply to I_4 and the entries of the last column multiply to $-I_4$. The magic square equations can be written as a propositional formula $\mu(\vec{a})$ involving the variables $[\vec{a}]$ listed in the table (4) and the exclusive or operation $\oplus$. Namely, with the exception of the last column, every row and column corresponds to a positive XOR-clause of the variables, e.g. the first row is $a \oplus b \oplus c$. The last column corresponds as negative XOR-clause $\neg(c \oplus f \oplus i)$. Since M does not have solution over $\mathbb{Z}_2$, we have that $\mu(\vec{a})$ is unsatisfiable in the two-element Boolean algebra $\mathbf{2}$. The assigment (4) demonstrates that $\mu(\vec{a})$ is strongly satisfiable in $\mathbf{I}(\mathbb{C}^4)$. We call such strongly satisfying assignments $\alpha \in Q^{\mu(\vec{a})}$ *solutions to the magic square* M.

In Lemma 6.13, we show that we can express the orthogonality of lines in $\mathbb{RP}^2$ using solutions to the magic square M. This is because an orthogonal set of lines in $\mathbb{R}^3$ corresponds to a qubit basis in $\mathbf{B}_0(\mathbb{C}^2)$ under the bijection $\vec{\sigma}$. A set of involutions $\{X, Y, Z\}$ is a *qubit basis* if the following equations are satisfied:

$$\begin{aligned} [X, Y] &= 2iZ \quad \{X, Y\} = 0 \\ [Z, X] &= 2iY \quad \{Z, X\} = 0 \\ [Y, Z] &= 2iX \quad \{Y, Z\} = 0 \end{aligned}$$

where $[X, Y] := XY - YX$ is the commutator operation and $\{X, Y\} := XY + YX$ is the anti-commutator operation for all $X, Y \in \mathbf{B}(\mathcal{H})$.

▶ **Proposition 6.12.** *Let $A, B, X, Y \in I(\mathcal{H})$ be involutions over $\mathbb{K}$ Hilbert space $\mathcal{H}$.*
(1) *$(AB)^2 = I_{\mathcal{H}}$ iff $AB = BA$, i.e. A and B commute.*
(2) *$(AB)^2 = -I_{\mathcal{H}}$ iff $AB = -BA$, i.e. A and B anti-commute.*
(3) *If $A \otimes B = X \otimes Y$, then $A = \pm X$ and $B = \pm Y$.*

▶ **Lemma 6.13.** *Suppose $\vec{x}, \vec{y}, \vec{z}$ are unit vectors in $\mathbb{R}^3$, $d \geq 1$, and $Q = \mathbf{I}(\mathbb{C}^{4d})$ then the following are equivalent:*
(1) *$\{\vec{x}, \vec{y}, \vec{z}\}$ is an orthonormal basis of $\mathbb{R}^3$.*
(2) *$\{\vec{\sigma}(\vec{x}), \vec{\sigma}(\vec{y}), \vec{\sigma}(\vec{z})\}$ is a qubit basis.*
(3) *There exists solution $\alpha \in Q^{\mu(\vec{a})}$ to the magic square expressed as:*

$\vec{\sigma}(\vec{x}) \otimes I \otimes I_d$	$I \otimes \vec{\sigma}(\vec{x}) \otimes I_d$	$\vec{\sigma}(\vec{x}) \otimes \vec{\sigma}(\vec{x}) \otimes I_d$
$I \otimes \vec{\sigma}(\vec{z}) \otimes I_d$	$\vec{\sigma}(\vec{z}) \otimes I \otimes I_d$	$\vec{\sigma}(\vec{z}) \otimes \vec{\sigma}(\vec{z}) \otimes I_d$
$\vec{\sigma}(\vec{x}) \otimes \vec{\sigma}(\vec{z}) \otimes I_d$	$\vec{\sigma}(\vec{z}) \otimes \vec{\sigma}(\vec{x}) \otimes I_d$	$\vec{\sigma}(\vec{y}) \otimes \vec{\sigma}(\vec{y}) \otimes I_d$

In addition to showing we can express cross product in terms of the magic square, we also need to show that any solution to the magic square can be forced into the form of Lemma 6.13 (3). The following lemma is proved by first by showing that any solution to the magic square is unique up to changing the basis of each qubit, i.e. up to unitary maps applied to each qubit. Therefore, we can diagonalize to the basis of one qubit. This lemma was inspired by similar uniqueness results in the self-testing literature [26, 27] though none were precisely suitable for our application.

▶ **Lemma 6.14.** *Let $Q = \mathbf{I}(\mathbb{C}^{4d})$ for $d \geq 1$. For every solution $\alpha \in Q^{\mu(\vec{a})}$ to the magic square, there exists a unitary $U \colon \mathbb{C}^2 \to \mathbb{C}^2$ and solution $\alpha_U \in Q^{\mu(\vec{a})}$ expressed as:*

$V(\sigma_x \otimes I_2)V^{-1} \otimes I_d$	$V(I \otimes \sigma_x)V^{-1} \otimes I_d$	$V(\sigma_x \otimes \sigma_x)V^{-1} \otimes I_d$
$V(I_2 \otimes \sigma_z)V^{-1} \otimes I_d$	$V(\sigma_z \otimes I)V^{-1} \otimes I_d$	$V(\sigma_z \otimes \sigma_z)V^{-1} \otimes I_d$
$V(\sigma_x \otimes \sigma_z)V^{-1} \otimes I_d$	$V(\sigma_z \otimes \sigma_x)V^{-1} \otimes I_d$	$V(\sigma_y \otimes \sigma_y)V^{-1} \otimes I_d$

where $V = U \otimes U$ and $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli matrices.

Let t be a cross-product term. For every non-variable subterm $s = r \measuredangle q$, a satisfying assignment of t yields two orthonormal bases. One basis $B_{s,r}$ containing the pair $\langle s\rangle$,$\langle r\rangle$, and the other containing the pair $\langle s\rangle$,$\langle q\rangle$. For each such basis B_ρ associated to the pair $\rho = (s, r)$, we create a copy M_ρ of the magic square:

$p_{r,1}$	$p_{1,r}$	$p_{r,r}$
$p_{1,s}$	$p_{s,1}$	$p_{s,s}$
$p_{r,s}$	$p_{s,r}$	$p_{r',r'}$

(5)

The indexing of the variables in the copy of M_ρ suggest the intended interpretation. That is, if t is satisfied such that the subterm s resolves to $\mathbb{R}\langle s\rangle = \mathbb{R}\langle r\rangle \measuredangle \mathbb{R}\langle q\rangle$, then e.g. $p_{r,s}$ assigned to $\vec{\sigma}(\langle r\rangle) \otimes \vec{\sigma}(\langle s\rangle)$, $p_{s,s}$ is assigned to $\vec{\sigma}(\langle s\rangle) \otimes \vec{\sigma}(\langle s\rangle)$, etc. Let $\mu_\rho(\vec{p_\rho})$ be the conjunction of the XOR formulas expressing the equations in M_ρ where $[\vec{p_\rho}]$ is the set of variables appearing in the table (5). Define $\vartheta_t(\vec{p})$ to the conjunction of the formulas $\mu_\rho(\vec{p_\rho})$ over the set of all pairs $\rho = (s, r)$ where s is a non-variable subterm of t and r is an immediate child of s. Since the term satisfiability problem decides if $\langle t(\mathbb{R}\vec{x_1}, \ldots, \mathbb{R}\vec{x_1})\rangle = \mathbb{R}\vec{x_1}$ we identify variables $p_{x_1,1}$, p_{x_1,x_1}, p_{1,x_1} with the variables $p_{t,1}$, $p_{t,t}$, $p_{1,t}$, respectively.

▶ **Proposition 6.15.** *Let $g \geq 1$. The following are equivalent:*

1. *$t(x_1, \ldots, x_n) \in \mathbf{XSAT}(\mathbb{RP}^2)$.*
2. *$\vartheta_t(\vec{p})$ is in $\mathbf{SAT}^{=1}(\mathbf{I}(\mathbb{C}^{4g}))$.*

▶ **Theorem 6.16.** *For every $d \geq 4$, $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$ is $\exists\mathbb{R}$-complete.*

Proof. Proposition 6.11 proves membership in $\exists\mathbb{R}$. For $\exists\mathbb{R}$-hardness of the base case $d = 3$, Proposition 6.15 yields reductions from the $\exists\mathbb{R}$-complete problem $\mathbf{XSAT}(\mathbb{RP}^2)$ to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^4))$. For $\exists\mathbb{R}$-hardness beyond the base case, we inductively apply Proposition 6.8 to obtain a reduction from $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$ to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^{d+1}))$. ◀

Another consequence of Proposition 6.15, is that we obtain a reduction from $\mathbf{XSAT}(\mathbb{RP}^2)$ to a restricted version of the satisfiability problem to conjunctions of $\oplus$-clauses, i.e. XOR-formulas. The XOR fragment of propositional logic is sometimes called the linear fragment as classically it encodes systems of linear equations over $\mathbb{Z}_2$. For a class of pBAs $\mathbf{C}$, let $\mathbf{XORSAT}^{=1}(\mathbf{C})$ denote the decision problem which takes an input an XOR-formula $\varphi(\vec{p})$ and decides if $\varphi(\vec{p})$ is strongly satisfied in some $A \in \mathbf{C}$. Observe that since the formula $\vartheta_t(\vec{p})$ constructed in Proposition 6.15 is an XOR-formula, we obtain the following $\exists\mathbb{R}$-completeness result.

▶ **Theorem 6.17.** *For every $g \geq 1$, $\mathbf{XORSAT}^{=1}(\mathbf{P}(\mathbb{C}^{4g}))$ is $\exists\mathbb{R}$-complete.*

Theorem 6.17 exhibits a contrast to the classical case. Classically, $\mathbf{XORSAT}(\mathbf{Total})$ is in $\mathbf{PTIME}$ and $\mathbf{SAT}(\mathbf{Total})$ is $\mathbf{NP}$-complete. By contrast, for the case of dimensions d divisible by 4, both $\mathbf{XORSAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$ and $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$ are $\exists\mathbb{R}$-complete.

▶ Remark 6.18. An inspection of the proofs in [4] reveals a dimension preserving reduction from quantum isomorphism of graphs to $\mathbf{XORSAT}^{=1}(\mathbf{P}(\mathbb{C}^d))$. Thus, a corollary of Theorem 6.17 is that for every d divisible by 4, deciding if two input graphs have a quantum isomorphism witnessed by d-dimensional projectors is $\exists\mathbb{R}$-complete

7 Quantum Homomorphisms

We begin this section by introducing a notion of quantum homomorphism between relational structures from [2]. Quantum homomorphisms correspond to operator solutions of constraint satisfaction problems from [3] analogous to the correspondence between ordinary solutions and ordinary homomorphisms between relational structures. With this notion, we consider the corresponding decision problem $\mathbf{QHOM}(\mathcal{H})$ and exhibit reductions to/from the satisfiability problem $\mathbf{SAT}(\mathbf{P}(\mathcal{H}))$. Since reductions preserve the Hilbert space and thus the dimension, we are able to obtain the following corollaries:

- $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{<\omega}))$ is undecidable, where $\mathbf{P}(\mathbb{K}^{<\omega}) = \{\mathbf{P}(\mathbb{K}^d) \mid d \in \mathbb{N}\}$ is the class of pBAs arising from finite-dimensional $\mathbb{K}$-Hilbert spaces.
- $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{\infty}))$ is undecidable, where $\mathbf{P}(\mathbb{K}^{\infty})$ is the class of pBAs arising from all $\mathbb{K}$-Hilbert spaces.
- For every fixed $d \geq b(\mathbb{K})$, $\mathbf{QHOM}(\mathbb{K}^d)$ is $\exists\mathbb{R}$-complete where $b(\mathbb{R}) = 3$ and $b(\mathbb{C}) = 4$.

In the following, we say that a *signature* σ is a finite set of relational symbols $R \in \sigma$ which each have an associated positive arity $r > 0$. Given a signature σ, a *σ-structure* $\mathcal{M}$ is a set M paired with interpretations $R^{\mathcal{M}} \subseteq A^r$ for every r-ary relational symbol $R \in \sigma$. Given two σ-structures, a *σ-homomorphism* $f\colon \mathcal{M} \to \mathcal{N}$ is a set function $M \to N$ which preserves the interpretations of relations. The following definition from [2] generalizes the notion of σ-homomorphism to the quantum setting.

▶ **Definition 7.1.** *Let $\mathcal{H}$ be a real or complex Hilbert space. A* $\mathcal{H}$-quantum σ-homomorphism *between two σ-structures $\mathcal{M}$ and $\mathcal{N}$, denoted $\mathbf{F}\colon \mathcal{M} \xrightarrow{\mathcal{H}} \mathcal{N}$, is family of projectors $\mathbf{F} = \{F_{m,n}\}_{m\in M, n\in N} \subseteq \mathbf{P}(\mathcal{H})$ satisfying the following conditions:*

(QH1) *For all $m \in M$, $\sum_{n\in N} F_{m,n} = I_{\mathcal{H}}$.*

(QH2) *If m, m' appear in some relational tuple of $\mathcal{M}$ and $n, n' \in \mathcal{N}$, then $F_{m,n} \odot_{\mathbf{P}(\mathcal{H})} F_{m',n'}$.*

(QH3) *For every r-ary symbol $R \in \sigma$, $(m_1, \ldots, m_r) \in R^{\mathcal{M}}$ and $(n_1, \ldots, m_r) \notin R^{\mathcal{N}}$, then $F_{m_1,n_1} \ldots F_{m_r,n_r} = 0_{\mathcal{H}}$.*

▶ Remark 7.2. A similar notion of quantum homomorphism for undirected simple graphs was originally introduced in [18]. Though every quantum homomorphism between graphs in our sense results in a quantum homomorphism in the sense of [18], the converse does not hold, see e.g. [14] for details.

The associated decision problem $\mathbf{QHOM}(\mathcal{H})$ is the set of pairs $(\mathcal{M},\mathcal{N})$ of finite σ-structures such that there exists a $\mathcal{H}$-quantum homomorphism $\mathbf{F}\colon \mathcal{M} \xrightarrow{\mathcal{H}} \mathcal{N}$.

7.1 Homomorphism to Satisfiability

In the section, we demonstrate that the standard reduction from σ-homomorphism to $\mathbf{SAT}(\mathbf{Total})$ generalizes to the quantum setting. For brevity of presentation, we assume our signature σ has a single r-ary relation symbol R. For every pair of σ-structures $\mathcal{M},\mathcal{N}$, we can rewrite the conditions in Definition 7.1 as a propositional formula $\phi_{\mathcal{M},\mathcal{N}}(\vec{p})$ with variables amongst $[\vec{p}] = \{p_{m,n} \mid m \in M, n \in N\}$. The formula $\phi_{\mathcal{M},\mathcal{N}}(\vec{p})$ is defined as the conjunction $\varphi_{\mathsf{func}}(\vec{p}) \wedge \varphi_{\mathsf{rel}}(\vec{p})$ where

$$\varphi_{\mathsf{func}}(\vec{p}) := \bigwedge_{m\in M} \bigvee_{n\in N} \left(p_{m,n} \wedge \bigwedge_{n\neq n'\in N} \neg p_{m,n'} \right),$$

$$\varphi_{\mathsf{rel}}(\vec{p}) := \bigwedge_{(m_1,\ldots,m_r)\in R^{\mathcal{M}}} \bigwedge_{(n_1,\ldots,n_r)\notin R^{\mathcal{N}}} \neg(p_{m_1,n_1} \wedge \cdots \wedge p_{m_r,n_r}).$$

The propositional formula $\phi_{\mathcal{M},\mathcal{N}}(\vec{p})$ yields the desired reduction from $\mathbf{QHOM}(\mathcal{H})$ to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$.

▶ **Proposition 7.3.** *Let $\mathcal{M},\mathcal{N}$ be two σ-structures, then $\phi_{\mathcal{M},\mathcal{N}}(\vec{p})$ is strongly satisfied in $\mathbf{P}(\mathcal{H})$ iff there exists a $\mathcal{H}$-quantum homomorphism $\mathbf{F}\colon \mathcal{M} \xrightarrow{\mathcal{H}} \mathcal{N}$.*

Let $\mathbf{QHOM}(\mathbb{K}^{\omega})$ (resp. $\mathbf{QHOM}(\mathbb{K}^{\infty})$) denote the decision problem consisting of pairs $(\mathcal{M},\mathcal{N})$ for which there exists a $\mathcal{H}$-quantum homomorphism for any finite-dimensional (resp. any) $\mathbb{K}$-Hilbert space $\mathcal{H}$. From various results in the literature, we can conclude that $\mathbf{QHOM}(\mathbb{K}^{<\omega})$ and $\mathbf{QHOM}(\mathbb{K}^{\infty})$ are undecidable [3, 4], and Proposition 7.3 exhibits a computable reduction from these problems to $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{<\omega}))$ and $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{\infty}))$ respectively. Thus, we obtain the following theorem.

▶ **Theorem 7.4.** *For $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{<\omega}))$ and $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{\infty}))$ are undecidable.*

As a corollary from our results, we can conclude that $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{<\omega}))$ is complete for the class of recursively enumerable (**RE**) languages.

▶ **Corollary 7.5.** *For $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{<\omega}))$ is **RE**-complete.*

Proof. To establish **RE**-hardness, observe that the proof of Theorem 7.4 uses a chain of reductions that starts with the halting problem for Turing machines (without an oracle) and ends with $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{<\omega}))$. To establish membership in **RE**, Proposition 6.3 and Proposition 6.11 yield that $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{d}))$ is computable for every fixed dimension $d \geq 1$. Thus, by textbook computation theory, there exists a first-order formula $\Phi(d, x_\varphi)$ in the language of arithmetic describing the computation for an input propositional formula φ encoded as a natural number $x_\varphi \in \mathbb{N}$. By existentially quantifying d in $\Phi(d, x_\varphi)$, we obtain that $\mathbf{SAT}(\mathbf{P}(\mathbb{K}^{<\omega}))$ is in **RE**. ◀

▶ Remark 7.6. A result related to Theorem 7.4 was proved in [3]. Here, they did not use the formulation of partial Boolean algebras, but instead operator assignments for constraint satisfaction problems with Boolean domain $\{-1,1\}$. In this formulation, a propositional formula $\phi(p_1,\ldots,p_n)$ in CNF form is turned into a constraint satisfaction problem $\mathcal{X}_\phi = (X_\phi, C_\phi)$ with variables $X = \{x_1,\ldots,x_n\}$, and constraints

$$C_\phi = \{(Z_c, R_c) \mid c \text{ clause in } \phi\},$$

where each clause c of ϕ with variables $p_{i_1}, \ldots, p_{i_r}$ gives rise to a constraint with scope $Z_c = \{x_{i_1}, \ldots, x_{i_r}\}$ and constraint $R_c \in \{-1, +1\}^r$ consisting of the set of satisfying tuples of c. An assignment $f\colon X_\phi \to \{-1, +1\}$ is satisfying of the CSP $\mathcal{X}_\phi$ if $(f(x_{i_1}), \ldots, f(x_{i_n}) \in R_c$ for all clauses c. This satisfying assignment is in fact a substitution $\phi^{\mathbf{2}}\colon \mathbf{2}^{\vec{p}} \to \mathbf{2}$. To generalize this to operator assignments, they represent a constraint (Z, R) with scope $Z = \{x_{i_1}, \ldots, x_{i_r}\}$ as a characteristic multilinear polynomial $P_R(x_{i_1}, \ldots, x_{i_r})$ with the property that $P_R(a_{i_1}, \ldots, a_{i_m}) = -1$ if $(a_{i_1}, \ldots, a_{i_r}) \in R_c$ for $a_{i_z} \in \{-1, 1\}$ and $+1$ otherwise. An operator assignment for $\mathcal{H}$ is just an assignment $f\colon X_\phi \to \mathbf{I}(\mathcal{H})$ of involutions $\mathbf{I}(\mathcal{H})$ on $\mathcal{H}$ to each element of X_ϕ. From such operator assignments, we construct meaningful substitutions that witness the strong satisfiability of ϕ in $\mathbf{I}(\mathcal{H})$.

7.2 Satisfiability to Homomorphism

In this section, we show that the standard reduction from **SAT**(**Total**) to deciding if there exists a σ-homomorphism generalizes to the quantum setting. By Proposition 3.6, we can restrict our reduction to the case of propositional formulas in 3CNF form. Given that the formula $\phi(\vec{p})$ is in CNF form, recall the classical reduction constructs two σ-structures $\mathcal{V}_\phi$ and $\mathcal{T}_\phi$ in a signature σ_ϕ. The signature σ_ϕ has a relational symbol R_c for every clause c in $\phi(\vec{p})$ with arity equal to the number of literals, i.e. ≤ 3, in c. The universe of $\mathcal{V}_\phi$ is the set of variables $[\vec{p}] = \{p_1, \ldots, p_n\}$ and for the clause $c = l_{i_1} \vee l_{i_2} \vee l_{i_3}$ where l_{i_j} is a literal in p_{i_j}, the interpretation of $R_c^{\mathcal{V}_\phi}$ is the singleton $\{(p_{i_1}, p_{i_2}, p_{i_3})\}$. The universe of $\mathcal{T}_\phi$ is $\{0, 1\}$ and for the clause c, $R_c^{\mathcal{T}_\phi} = \{0, 1\}^3 \backslash \{(\partial(l_{i_1}), \partial(l_{i_2}), \partial(l_{i_3}))\}$ where $\delta(l_{i_j}) = 0$ if l_{i_j} is a positive literal and $\delta(l_{i_j}) = 1$ if l_{i_j} is a negative literal.

In order to generalize this construction the quantum case, we simply have to replace $\mathcal{T}$ with the structure $\mathbf{Q}_d(\mathcal{T})$ where $\mathbf{Q}_d$ is a functorial (in fact monadic) construction from [2]. We recall the $\mathbf{Q}_d$ construction here. For any σ-structure $\mathcal{M}$, the universe of $\mathbf{Q}_d(\mathcal{M})$ is the set of M-labelled PVMs:

$$\mathbf{Q}_d(M) = \{h\colon M \to \mathbf{P}(\mathbb{K}^d) \mid \sum_{m \in M} h(m) = I_d, \mathsf{supp}(h) \text{ is finite } \}$$

where $\mathsf{supp}(h)$ is the set of $m \in M$ such that $h(m) \neq 0_d$. Elements $h \in \mathbf{Q}_d(\mathcal{M})$ can be expressed as formal sums, i.e. $\sum_{m \in M} h(m).m$. For every relational symbol $R \in \sigma$ of arity r, $(h_1, \ldots, h_r) \in R^{\mathbf{Q}_d(\mathcal{M})}$ if

(QR1) for all $i, j \in [r]$ and $m, m' \in M$, $h_i(a) \odot_{\mathbf{P}(\mathbb{K}^d)} h_j(m')$.

(QR2) if $(m_1, \ldots, m_r) \notin R^{\mathcal{M}}$, then $\prod_{i \in [r]} h_i(m_i) = 0_d$.

The relevant property of $\mathbf{Q}_d$ we use is [2, Proposition 8]: there exists an $\mathbb{K}^d$-quantum σ-homomorphism $\mathcal{M} \xrightarrow{\mathbb{K}^d} \mathcal{N}$ if, and only if, there exists an ordinary σ-homomorphism $\mathcal{M} \to \mathbf{Q}_d(\mathcal{N})$.

From a 3CNF propositional formula $\phi(\vec{p})$, we construct a signature σ_ϕ and σ_ϕ-structures $\mathcal{V}_\phi, \mathcal{T}_\phi$ satisfying the following proposition.

▶ **Proposition 7.7.** *Let $\phi(\vec{p})$ be a 3CNF-formula, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $d \geq 1$, and $\mathcal{H} = \mathbb{K}^d$. $\mathbf{P}(\mathcal{H})$ satisfies $\phi(\vec{p})$ if, and only if, there is a quantum homomorphism $\mathbf{F}\colon \mathcal{V}_\phi \xrightarrow{\mathcal{H}} \mathcal{T}_\phi$.*

▶ **Theorem 7.8.** *Let $b(\mathbb{R}) = 3$ and $b(\mathbb{C}) = 4$. For every $d \geq b(\mathbb{K})$, $\mathbf{QHOM}(\mathbb{K}^d)$ is $\exists\mathbb{R}$-complete.*

Proof. Theorem 6.9 and Theorem 6.16 demonstrate that $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^d))$ is an $\exists\mathbb{R}$-complete problem. Proposition 7.3 is a polynomial-time reduction from $\mathbf{QHOM}(\mathbb{K}^d)$ to $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^d))$. Proposition 7.7 is a polynomial-time reduction from $\mathbf{3CNFSAT}^{=1}(\mathbf{P}(\mathbb{K}^d))$ and $\mathbf{SAT}^{=1}(\mathbf{P}(\mathbb{K}^d))$ (by Proposition 3.6) to $\mathbf{QHOM}(\mathbb{K}^d)$. ◀

8 Conclusion

The Cook-Levin theorem is a classical result of computational complexity that tells us that the problem of deciding whether a propositional formula is satisfiable is **NP**-complete. Or, dually, the class of propositional tautologies is **coNP**-complete. What the Kochen-Specker theorem tells us is that if we interpret the propositions not as classical truth values but as measurement outcomes in a quantum system, then not every classical tautology is always true. Equivalently, there are propositional formulas that are satisfiable in such systems that are not classically satisfiable. What is then the complexity of the class of satisfiable formulas? This is the question that we set out to address.

When we interpret satisfiability to mean satisfiable in some non-trivial partial Boolean algebra, the problem is again **NP**-complete. The hardness is a direct consequence of the **NP**-hardness of classical satisfiability but the upper bound is non-trivial to establish as the natural witness to satisfiability can be of doubly exponential size. Our main contribution here, in Theorem 4.5, is to construct a suitable polynomial-size witness that can be efficiently verified.

If we restrict ourselves to the partial Boolean algebras that motivated the question, namely those of projectors on a Hilbert space, the picture is more complicated. We show that for any finite dimension $d \geq 4$, the problem of deciding satisfiability in such a projector space of dimension d is complete for the existential theory of the reals. However, the problem of determining whether a formula is satisfiable in the algebra of projectors of some finite-dimensional Hilbert space is undecidable. The $\exists\mathbb{R}$-completeness result situates this problem in an interesting place in the complexity landscape and relates it to a number of natural problems in computational geometry. This opens up avenues of further research relating quantum complexity classes to such problems.

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