Existential Positive Transductions of Sparse Graphs

Nikolas Mählmann
University of Warsaw, Poland

Sebastian Siebertz
University of Bremen, Germany

Abstract

Monadic stability generalizes many tameness notions from structural graph theory such as planarity, bounded degree, bounded tree-width, and nowhere density. The *sparsification conjecture* predicts that the (possibly dense) monadically stable graph classes are exactly those that can be logically encoded by first-order (FO) transductions in the (always sparse) nowhere dense classes. So far this conjecture has been verified for several special cases, such as for classes of bounded shrub-depth, and for the monadically stable fragments of bounded (linear) clique-width, twin-width, and merge-width.

In this work we propose the *existential positive sparsification conjecture*, predicting that the more restricted co-matching-free, monadically stable classes are exactly those that can be transduced from nowhere dense classes using only existential positive FO formulas. While the general conjecture remains open, we verify its truth for all known special cases of the original conjecture. Even stronger, we find the sparse preimages as subgraphs of the dense input graphs.

As a key ingredient, we introduce a new combinatorial operation, called *subflip*, that arises as the natural co-matching-free analog of the flip operation, which is a central tool in the characterization of monadic stability. Using subflips, we characterize the co-matching-free fragment of monadic stability by appropriate strengthenings of the known flip-flatness and flipper game characterizations for monadic stability.

In an attempt to generalize our results to the more expressive MSO logic, we discover (rediscover?) that on relational structures (existential) positive MSO has the same expressive power as (existential) positive FO.

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1 Introduction

First-order transductions have emerged as a fundamental tool in structural graph theory and finite model theory, providing a convenient way to encode one class of structures inside another using logic. A *transduction* T_φ is an operation specified by a binary first-order formula $\varphi(x, y)$ over the signature of colored graphs. It maps an input graph G to the set $\mathsf{T}_\varphi(G)$ containing every graph H that can be obtained through the following three-step process. (See Figure 1 for an example.)

1. $G \to G'$: color G with vertex colors appearing in φ.
2. $G' \to H'$: let φ define a new edge relation: $uv \in E(H') \Leftrightarrow G' \models \varphi(u, v)$.
3. $H' \to H$: choose H to be any induced subgraph of H' and remove the vertex colors.

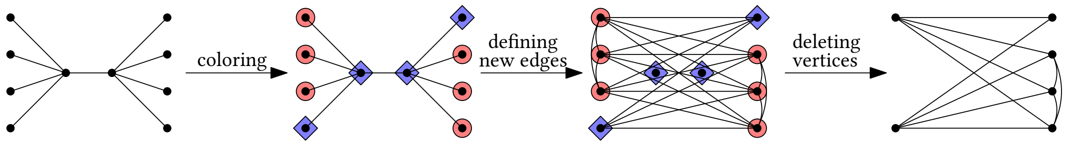

Figure 1 Example of a transduction. On the very left: the graph G. On the very right: a graph $H \in \mathsf{T}_\varphi(G)$ for the formula $\varphi(x,y) = (\mathrm{dist}(x,y) = 3) \vee (\mathrm{Red}(x) \wedge \mathrm{Red}(y))$.

Further graph transformations that can be expressed by transductions include the complement and the square operation, specified by the formulas $\neg E(x,y)$, and $E(x,y) \vee \exists z \left(E(x,z) \wedge E(z,y)\right)$, respectively. This motivates the study of properties of graph classes that are preserved under transductions. A popular example of such a *transduction-closed* class property is bounded clique-width: for every graph class $\mathcal{C}$ whose clique-width is bounded by a constant and every transduction T, the clique-width of the class $\mathsf{T}(\mathcal{C}) := \bigcup_{G \in \mathcal{C}} \mathsf{T}(G)$ is again bounded by a (possibly larger) constant. For every class $\mathcal{D} \subseteq \mathsf{T}(\mathcal{C})$, we say that $\mathcal{D}$ is a *transduction* of $\mathcal{C}$, and that $\mathcal{C}$ *transduces* $\mathcal{D}$. Further examples of transduction-closed class properties include bounded shrub-depth, bounded linear clique-width, bounded twin-width, and bounded merge-width. Subsuming all previous examples, the most general non-trivial, transduction-closed class property is *monadic dependence*: a class is *monadically dependent* if it does not transduce the class of all graphs. In this paper we study the relationship between three fragments of monadic dependence:

1. the *biclique-free* fragment (a.k.a. *weakly sparse* fragment),
2. the *semi-ladder-free* fragment, and
3. the *half-graph-free* fragment (a.k.a. *stable* fragment).

▶ **Definition 1.** *We call a bipartite graph G on vertices $a_1, \dots, a_n$ and $b_1, \dots, b_n$*

- *the* biclique $K_{n,n}$ *of order n, if for all $i, j \in [n]$,*

$$a_i b_j \in E(G),$$

- *the* half-graph *of order n, if for all $i, j \in [n]$,*

$$a_i b_j \in E(G) \Leftrightarrow i \leqslant j,$$

- *the* co-matching *of order n, if for all $i, j \in [n]$*

$$a_i b_j \in E(G) \Leftrightarrow i \neq j.$$

A bipartite graph H is a semi-induced subgraph *of a graph G, if there are $A, B \subseteq V(G)$ such that the bipartite subgraph of G induced between the sides A and B is isomorphic to H. A graph class is* biclique-free / half-graph-free / co-matching-free *if it does not contain arbitrarily large bicliques / half-graphs / co-matchings as semi-induced subgraphs. The class is* semi-ladder-free *if it is both half-graph-free and co-matching-free. See Figure 2 for an illustration.*

The *biclique-free* fragment of monadic dependence (also called the *weakly sparse* fragment) consists exactly of the *nowhere dense* classes. The notion of nowhere density is the main object of study in sparsity theory initiated by Nešetřil and Ossona de Mendez [24]. It generalizes many tameness notions for sparse graphs such as planarity, bounded tree-width, excluded minors, bounded degree, and bounded expansion. The equivalence of the combinatorial

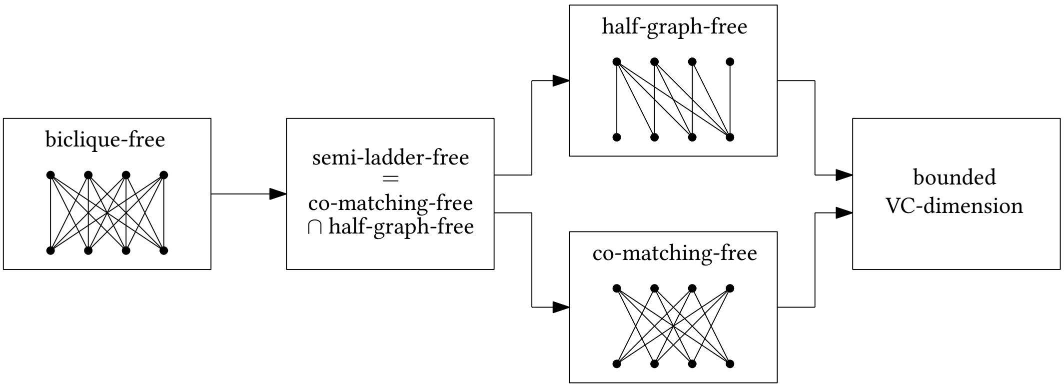

Figure 2 A hierarchy of properties of graph classes. An arrow $P_1 \to P_2$ between two properties means that every graph class that has property P_1 also has property P_2.

notion of nowhere density and the logical notion of monadic dependence in the biclique-free setting is one of the key results that initially connected sparsity theory and model theory. It follows from the results of Podewski and Ziegler [30], Adler and Adler [1], and Dvořák [14].

Generalizing nowhere density, the *half-graph-free* fragment of monadic dependence consists exactly of the *monadically stable* classes [25], which are those that do not transduce the class of all half-graphs [2]. While nowhere density is limited to sparse classes, monadic stability also includes dense classes like map graphs and every class of bounded shrub-depth. Monadic stability is transduction-closed and hence, every transduction of a nowhere dense class is monadically stable. The *sparsification conjecture* predicts an even stronger link between the biclique-free and the half-graph-free fragments of monadic dependence, asserting that the reverse should hold as well.

▶ **Conjecture 2** (see [28])**.** *For every graph class $\mathcal{C}$ the following are equivalent.*
1. *$\mathcal{C}$ is monadically stable.*
2. *$\mathcal{C}$ is a transduction of a nowhere dense class.*

So far the sparsification conjecture has been confirmed for all half-graph-free classes $\mathcal{C}$ that have bounded shrub-depth [19, 20], linear clique-width [26], clique-width [25], twin-width [18], or merge-width [8]. In each of these cases, actually something stronger holds. For example, every half-graph-free class of bounded clique-width can be transduced from a class that is not only nowhere dense, but even has bounded tree-width [25] (which is the same as having both bounded clique-width and being biclique-free [5, 21]). This suggests the following refinement of Conjecture 2.

▶ **Conjecture 3.** *Let $\mathcal{P}$ be a non-trivial and transduction-closed class property. For every graph class $\mathcal{C}$, the following are equivalent.*
1. *$\mathcal{C}$ is a half-graph-free class with $\mathcal{P}$.*
2. *$\mathcal{C}$ is a transduction of a biclique-free class with $\mathcal{P}$.*

Note that this conjecture implies the sparsification conjecture by choosing $\mathcal{P}$ to be monadic dependence. The solved cases of the sparsification conjecture can now be stated as follows.

▶ **Theorem 4.** *Conjecture 3 is true for the class property* bounded $\mathcal{P}$ *for all $\mathcal{P} \in \{$shrub-depth, linear clique-width, clique-width, twin-width, merge-width$\}$. In particular, every half-graph-free class $\mathcal{C}$ is transducible from a class $\mathcal{D}$ that satisfies all of the following.*
1. *$\mathcal{C}$ has bd. shrub-depth $\Rightarrow$ $\mathcal{D}$ has bd. tree-depth [19, 20].*
2. *$\mathcal{C}$ has bd. linear clique-width $\Rightarrow$ $\mathcal{D}$ has bd. path-width [26].*

3. $\mathcal{C}$ *has bd. clique-width* $\Rightarrow$ $\mathcal{D}$ *has bd. tree-width [25].*
4. $\mathcal{C}$ *has bd. twin-width* $\Rightarrow$ $\mathcal{D}$ *has bd. sparse twin-width [18].*
5. $\mathcal{C}$ *has bd. merge-width* $\Rightarrow$ $\mathcal{D}$ *has bd. expansion [8].*

The "in particular" part of the theorem uses the following fact.

▶ **Theorem 5.** *Every biclique-free class* $\mathcal{C}$ *with*
1. *bounded shrub-depth has bounded tree-depth [19, 20],*
2. *bounded linear clique-width has bounded path-width [21],*
3. *bounded clique-width has bounded tree-width [5, 21],*
4. *bounded twin-width has bounded sparse twin-width [4],*
5. *bounded merge-width has bounded expansion [13].*

Contribution 1: Existential positive sparsification

The main object of our study is the *semi-ladder-free* fragment of monadic dependence, which strictly subsumes the nowhere dense classes and is strictly contained in the monadically stable classes. A class is semi-ladder-free if and only if it is half-graph-free and co-matching-free [15]. Prominent examples of semi-ladder-free monadically dependent classes are the class of map graphs and fixed powers of nowhere dense classes [15]. Analogously to the sparsification conjecture, we conjecture that the semi-ladder-free monadically dependent classes are exactly the *existential positive* transductions of nowhere dense classes. More formally, a formula is *positive* if it contains no negation symbol ($\neg$), and in particular no non-equality ($\neq$). A positive formula is moreover *existential* if it contains no universal quantifier ($\forall$). In this case we call it an *EP-formula.* We call a transduction an *EP-transduction* if it is specified by an EP-formula.

It turns out that when studying EP-transductions it is crucial that all vertices have self-loops (i.e., every vertex is adjacent to itself). Intuitively, the reason for this is that otherwise the information that two vertices are connected by an edge implicitly carries the non-positive information that these vertices are non-equal. We will discuss this phenomenon in more detail in the next subsection. In the following, a *reflexive graph* is a graph in which every vertex has a self-loop. For additional clarity, we mark statements requiring this assumption with a $\circlearrowleft$ symbol. On the other hand, if we do not mention reflexivity explicitly, this means we allow self-loops in our graphs, but do not require them. Sometimes we use the term *partially reflexive graph*, to stress that we are working in this more relaxed setting.

We propose the following *existential positive sparsification conjecture.*

▶ **Conjecture 6** **($\circlearrowleft$).** *For every class* $\mathcal{C}$ *of reflexive graphs, the following are equivalent.*
1. $\mathcal{C}$ *is semi-ladder-free and monadically dependent.*
2. $\mathcal{C}$ *is an EP-transduction of a nowhere dense class of reflexive graphs.*

Note that according to [25] and [15] the first condition that $\mathcal{C}$ is semi-ladder-free and monadically dependent is equivalent to the condition that $\mathcal{C}$ is co-matching-free and monadically stable.

Again, the statement can be refined to subproperties of monadic dependence.

▶ **Conjecture 7** **($\circlearrowleft$).** *Let* $\mathcal{P}$ *be a non-trivial and transduction-closed class property. For every class* $\mathcal{C}$ *of reflexive graphs, the following are equivalent.*
1. $\mathcal{C}$ *is a semi-ladder-free class with* $\mathcal{P}$*.*
2. $\mathcal{C}$ *is an EP-transduction of a biclique-free class of reflexive graphs with* $\mathcal{P}$*.*

As our first main result, we establish the implication (2. $\Rightarrow$ 1.) of Conjecture 6. For this direction no assumptions on self-loops are required.

▶ **Theorem 8.** *Let $\mathcal{C}$ be a nowhere dense class of partially reflexive graphs. Every EP-transduction of $\mathcal{C}$ is semi-ladder-free and monadically dependent.*

As a corollary we obtain that also the implication (2. $\Rightarrow$ 1.) of Conjecture 7 is true.

▶ **Corollary 9.** *The implication (2. $\Rightarrow$ 1.) of Conjecture 7 is true.*

Proof. Let $\mathcal{C}$ be an EP-transduction of a biclique-free class $\mathcal{D}$ with some non-trivial, transduction-closed class property $\mathcal{P}$. $\mathcal{D}$ must be monadically dependent, because monadic dependence is the most general non-trivial, transduction-closed class property. Then by biclique-freeness, $\mathcal{D}$ is nowhere dense. By Theorem 8, $\mathcal{C}$ is semi-ladder-free. As $\mathcal{P}$ is transduction-closed, $\mathcal{C}$ still has $\mathcal{P}$. ◀

Consider the following algorithmic application of our structural result. It was proved in [9] that the Dominating Set and Independent Set problems admit a polynomial kernel (with respect to solution size k) on every co-matching-free monadically stable class of graphs. We conclude that polynomial kernels for these problems exist on all EP-transductions of nowhere dense classes.

With the implication (2. $\Rightarrow$ 1.) of the existential positive sparsification conjecture settled, we continue to prove the full equivalence for all special cases for which the original sparsification conjecture is known to be true.

▶ **Theorem 10** (↻). *Conjecture 7 is true for the class property* bounded $\mathcal{P}$ *for all* $\mathcal{P} \in$ *{shrub-depth, linear clique-width, clique-width, twin-width, merge-width}. In particular, for every semi-ladder-free class $\mathcal{C}$ of reflexive graphs there are EP-transductions* Sparsify *and* Recover *such that every graph $G \in \mathcal{C}$ contains a reflexive subgraph G^* such that*

$$G^* \in \mathsf{Sparsify}(G) \text{ and } G \in \mathsf{Recover}(G^*),$$

and the class $\mathcal{D} := \{G^* : G \in \mathcal{C}\}$ satisfies all of the following.

1. *$\mathcal{C}$ has bd. shrub-depth $\Rightarrow$ $\mathcal{D}$ has bd. tree-depth.*
2. *$\mathcal{C}$ has bd. linear clique-width $\Rightarrow$ $\mathcal{D}$ has bd. path-width.*
3. *$\mathcal{C}$ has bd. clique-width $\Rightarrow$ $\mathcal{D}$ has bd. tree-width.*
4. *$\mathcal{C}$ has bd. twin-width $\Rightarrow$ $\mathcal{D}$ has bd. sparse twin-width.*
5. *$\mathcal{C}$ has bd. merge-width $\Rightarrow$ $\mathcal{D}$ has bd. expansion.*

Note that in the above theorem the sparsified graphs are obtained as *subgraphs of the original input graphs*. This is in contrast to the known cases of the general sparsification conjecture (Theorem 4), where one merely obtains the existence of some graph class with the desired structural properties that is transduction-equivalent to $\mathcal{C}$, without any guarantee that the witnessing graphs are subgraphs of the original ones. Thus, for all known cases of the conjecture, the theorem establishes a substantially stronger, canonical form of sparsification, in which the structure of G is preserved and refined rather than abstractly re-encoded.

An overview of the solved cases of the (existential positive) sparsification conjecture and of the relationships between the appearing class properties is given in Figure 3.

Figure 3 A hierarchy of properties of graph classes. An arrow $P_1 \to P_2$ between two properties means that every graph class that has property P_1 also has property P_2. The highlighted semi-ladder-free column represents our results on the existential positive sparsification conjecture.

Contribution 2: The non-equality property

We take some time to distill the model theoretic core, that helps to explain both the connection between semi-ladder-freeness and EP-transductions, and the need to work with reflexive graphs.

Let $\varphi(\bar{x}, \bar{y})$ be an FO formula over the signature of colored graphs, where $\bar{x}$ and $\bar{y}$ are two tuple variables of the same length t. We say that φ has the *monadic order-property* on a graph class $\mathcal{C}$ if for every $\ell \in \mathbb{N}$ there is a graph coloring G_ℓ^+ of a graph $G_\ell \in \mathcal{C}$ and tuples $\bar{a}_1, \ldots, \bar{a}_\ell \in V(G_\ell)^t$ such that for all $i, j \in [\ell]$

$$G_\ell^+ \models \varphi(\bar{a}_i, \bar{a}_j) \iff i \leqslant j. \tag{$*$}$$

While we have previously defined monadic stability via transductions, the original (and equivalent) definition by Baldwin and Shelah [2] states that a graph class $\mathcal{C}$ is monadically stable if no formula has the monadic order-property on $\mathcal{C}$. It is now natural to define the *monadic non-equality-property* by replacing $(*)$ in the definition of the monadic order property with

$$G_\ell^+ \models \varphi(\bar{a}_i, \bar{a}_j) \iff i \neq j.$$

The simple formula $\varphi(x, y) = \neg(x = y)$ has the monadic non-equality-property on every graph class that contains arbitrarily large graphs. Hence, the monadic non-equality-property is only interesting for positive formulas. We show that for reflexive graphs, the monadic non-equality-property for EP-formulas characterizes the semi-ladder-free fragment of monadic dependence.

▶ **Theorem 11** (↻)**.** *For every class $\mathcal{C}$ of reflexive graphs the following are equivalent.*
1. *$\mathcal{C}$ is semi-ladder-free and monadically dependent.*
2. *No EP-formula has the monadic non-equality-property on $\mathcal{C}$.*

Above, assuming reflexiveness is crucial, and it is illustrative to see the difference to the general partially reflexive case, where we prove the following.

▶ **Theorem 12.** *For every class $\mathcal{C}$ of partially reflexive graphs the following are equivalent.*
1. *$\mathcal{C}$ is semi-ladder-free, irreflexive-clique-free, and monadically dependent.*
2. *No EP-formula has the monadic non-equality-property on $\mathcal{C}$.*

Here, an *irreflexive* graph is a graph, in which no vertex has a self-loop. We call a graph class *irreflexive-clique-free* if it does not contain all irreflexive cliques as induced subgraphs. Clearly, on the class of irreflexive cliques, the edge relation $E(x, y)$ has the monadic non-equality-property. This hints at the reason why in the setting of positive logic, reflexive graphs behave nicer than irreflexive ones: in irreflexive graphs, the positive information of being connected by an edge carries the negative information of being non-equal.

Contribution 3: Subflips

As a foundation for the logical results presented in Contributions 1 and 2, we initiate the systematic combinatorial study of the semi-ladder-free fragment of monadic dependence. We obtain multiple characterizations through an operation, which we call *subflip*. We believe to be of independent interest. Our starting point are the more general monadically stable classes (i.e., the half-graph-free fragment of monadic dependence). Only recently, fixed-parameter tractability of the FO model checking problem was established for all monadically stable classes [7, 10]. Key to these algorithmic results was the emergence of a multitude of characterizations of monadic stability that are combinatorial rather than model theoretic in nature. At the heart of these combinatorial characterizations lies the *flip* operation. Given a graph G, a k*-flip* of G is obtained by choosing a partition $\mathcal{P}$ of $V(G)$ of size at most k and complementing the edge relation between arbitrary pairs of parts from $\mathcal{P}$ (possibly complementing parts with themselves, but preserving self-loops). See Figure 4 for an example.

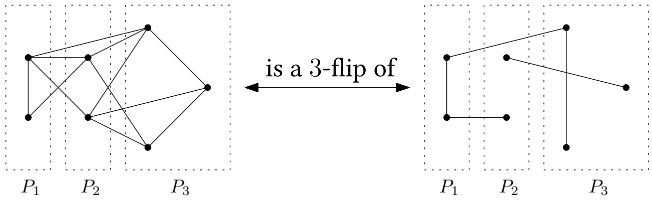

Figure 4 Two graphs that are 3-flips of each other. This is witnessed by the partition $\{P_1, P_2, P_3\}$ where the pairs (P_1, P_2), (P_2, P_3), and (P_3, P_3) were flipped.

The goal is usually to separate sets of vertices by putting them at high distance, using a k-flip for some small k. Exemplary for this is the following characterization of monadic stability by a property called *flip-flatness*, which is one of the main tools for the algorithmic treatment of monadically stable classes.

▶ **Definition 13.** *A graph class $\mathcal{C}$ is* flip-flat *if for every $r \in \mathbb{N}$ there is a function $M_r : \mathbb{N} \to \mathbb{N}$ and a constant $k_r \in \mathbb{N}$, such that for all $m \in \mathbb{N}$, $G \in \mathcal{C}$, and $W \subseteq V(G)$ with $|W| \geqslant M_r(m)$, there is a subset $A \subseteq W$ with $|A| \geqslant m$ and a k_r-flip H of G, such that for every two distinct vertices $u, v \in A$*

$$\mathrm{dist}_H(u, v) \geqslant r.$$

▶ **Theorem 14** ([11])**.** *A graph class is monadically stable if and only if it is flip-flat.*

Intuitively, flip-flatness states that in every huge set W, we find a still large subset A whose vertices are pairwise at high distance (“well-separated”) after performing a small flip.

In this work we consider a restriction of the flip operation that we call *subflip*. A graph H is a k*-subflip* of a graph G if it is both a k-flip and a subgraph of G. This means subflips are only allowed to remove edges, so we can only complement the edge relation between two parts P and Q of the flip partition $\mathcal{P}$ of $V(G)$ if

- P and Q are the same part and the graph induced on $P = Q$ is a clique, or
- P and Q are distinct parts and the graph semi-induced between P and Q is a biclique.

Being more restricted, subflips have some advantages over flips. To give an example, the next lemma from [3] states that multiple flips of the same graph can be aggregated into a single flip that roughly achieves the same separation. This aggregation of flips is lossy and highly non-trivial to prove.

▶ **Lemma 15** ([3, Lem. 8])**.** *Let G be a graph and $H_1, \ldots, H_m$ be k-flips of G. There is an $\mathcal{O}(k^m \cdot 2^{k^m})$-flip H of G such that for all $u, v \in V(G)$ and $i \in [m]$*

$$6 \cdot \mathrm{dist}_H(u, v) \geqslant \mathrm{dist}_{H_i}(u, v).$$

In contrast, aggregation of subflips is simple and lossless by working with the common refinement of the two flip partitions. (We later prove a strengthening of this observation in Lemma 27.)

▶ **Observation 16.** *Let G be a graph and $H_1, \ldots, H_m$ be k-subflips of G. There is a k^m-subflip H of G that is a subgraph of all the H_i.*

This tameness comes at the cost of subflips being less powerful than unrestricted flips. A 2-flip can turn any co-matching into a matching, by flipping between the two parts of the bipartition. This flip achieves very strong separation: after the flip, every vertex has a single neighbor and distance ∞ to all other vertices. This is not possible to achieve with a small subflip. However, it turns out that co-matchings are in some sense the only graphs where subflips perform worse than flips. As one of our main results, we show that in co-matching-free classes, subflips can always approximate flips.

The *co-matching-index* of a graph G is the largest number t such that G contains a semi-induced co-matching of order t.

▶ **Theorem 17.** *Let G be a graph with co-matching-index less than t and let H be a k-flip of G. There exists a kt^k-subflip H' of G such that for all $u, v \in V(G)$*

$$3t \cdot \mathrm{dist}_{H'}(u, v) \geqslant \mathrm{dist}_H(u, v).$$

Leveraging the above theorem, we obtain multiple combinatorial characterizations of semi-ladder-free monadically dependent classes, by replacing flips with subflips in known characterizations of monadic stability. We need to make no assumptions about self-loops here.

▶ **Theorem 18.** *For every class $\mathcal{C}$ the following are equivalent.*
1. *$\mathcal{C}$ is semi-ladder-free and monadically dependent.*
2. *$\mathcal{C}$ is subflip-flat.*
3. *Subflipper wins the subflipper-game on $\mathcal{C}$.*
4. *$\mathcal{C}$ is semi-ladder-free and for every $r, k \in \mathbb{N}$ both of the following hold:*
   - *There is a star r-crossing that is not a k-subflip of any induced subgraph from $\mathcal{C}$.*
   - *There is a clique r-crossing that is not a k-subflip of any induced subgraph from $\mathcal{C}$.*

Naturally, the definition of *subflip-flatness* is obtained by replacing flips with subflips in the definition of flip-flatness (Definition 13). The *subflipper-game* is similarly derived from the *flipper-game*, which is a game theoretic characterization of monadic stability [17]. The last item restricts a characterization of monadic stability by forbidden induced subgraphs [7, 12] to the semi-ladder-free setting. We provide the formal definitions in the respective sections.

Contribution 4: Collapse of positive MSO

Transductions, monadic stability, and monadic dependence can also be defined for monadic second-order logic (MSO) instead of first-order logic, and the same holds for the model theoretic properties occurring in this paper. In this context, again beautiful connections with structural graph theory arise:

- Graph classes with bounded shrub-depth are exactly those without the *monadic MSO-order-property.* [23]
- Graph classes with bounded clique-width are exactly those without the *monadic CMSO-independence-property* (which is conjectured to collapse with the *monadic MSO-independence-property*). [6]

In this context it is natural to ask whether also the *monadic MSO-non-equality property* corresponds to some tameness notion from structural graph theory. Studying this question, we uncover that on relational structures (existential) positive MSO collapses to (existential) positive FO.

▶ **Theorem 19.** *Every (existential) positive MSO formula $\varphi(\bar{x}, \bar{X})$ is equivalent to an (existential) positive FO formula.*

This theorem is not hard to prove. However, to the best of our knowledge, it was not known before in the literature, and we find it worth stating.

Organization of the paper

We give preliminaries in Section 2. In Sections 3–6 we introduce subflips and prove the combinatorial characterizations of the semi-ladder-free fragment of monadic stability, that amount to Theorem 18. In Sections 7 and 8, we characterize the monadic non-equality-property. In Section 9 we show how to find sparse preimages for EP-transductions as subgraphs (Theorem 10). In Section 10, we show how (existential) positive MSO collapses to (existential) positive FO. This is an extended abstract. Statements whose proofs were omitted are marked with a ♠ symbol. Complete proofs of all statements can be found in the full paper.

2 Short preliminaries

We use standard terminology from graph and (finite) model theory. As discussed in the introduction, we deviate from the standard by allowing (but in general not requiring) self-loops on vertices. As a consequence we do not demand the specifying formula $\varphi(x, y)$ of a transduction to be irreflexive (and also not reflexive). This means the transduction $\mathsf{T}_\varphi(G)$ creates self-loops exactly on those vertices v of G that satisfy $G^+ \models \varphi(v, v)$ in the chosen coloring G^+ of G. We observe that the following transitivity property from general transductions (see for example [16]) is preserved in the existential positive setting.

▶ **Lemma 20.** *For every two EP-transductions $\mathsf{T}_1, \mathsf{T}_2$ there is an EP-transduction T, such that for every graph G*

$$\mathsf{T}(G) \supseteq (\mathsf{T}_2 \circ \mathsf{T}_1)(G).$$

We use $N_r^G[v] := \{u \in V(G) : \mathrm{dist}_G(u, v) \leqslant r\}$ to denote the (closed) radius-r neighborhood of a vertex v in a graph G.

3 Subflips

In this section, we introduce the new notion of a *subflip*. For this purpose, let us first review the notion of a *flip* from the literature.

3.1 Flips and partitions

Let $\mathcal{P}$ be a partition of a set V. For every element $v \in V$, we denote by $\mathcal{P}(v) \in \mathcal{P}$ the part of $\mathcal{P}$ that contains v. For a set $S \subseteq V$ we denote by

$$\mathcal{P}|_S := \{P \cap S : P \in \mathcal{P}\}$$

the *restriction* of $\mathcal{P}$ to S, which is a partition of S. For another partition $\mathcal{Q}$ on V we denote the *common refinement* of $\mathcal{P}$ and $\mathcal{Q}$ by

$$\mathcal{P} \wedge \mathcal{Q} := \{P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q}\}.$$

Fix a graph G and a partition $\mathcal{P}$ of its vertices. Let $F \subseteq \mathcal{P}^2$ be a symmetric relation. We define $H := G \oplus (\mathcal{P}, F)$ to be the graph with vertex set $V(G)$, where we define the edges between distinct vertices $u, v \in V(G)$ using the following condition:

$$uv \in E(H) \Leftrightarrow \begin{cases} uv \notin E(G) & \text{if } (\mathcal{P}(u), \mathcal{P}(v)) \in F, \\ uv \in E(G) & \text{otherwise.} \end{cases}$$

Self-loops are preserved, i.e., for every $v \in V(G)$ we have $vv \in E(H) \Leftrightarrow vv \in E(G)$. We call H a *$\mathcal{P}$-flip* of G. If $\mathcal{P}$ has at most k parts, we say that H is a *k-flip* of G. See Figure 4 in the introduction for an example.

3.2 Subflips

▶ **Definition 21** (Fully (non-)adjacent sets)**.** *For a graph G and two sets $A, B \subseteq V(G)$, with possibly $A = B$, we say A and B are* fully adjacent *(*fully non-adjacent*) if every two distinct vertices $a \in A$ and $b \in B$ are adjacent in G (non-adjacent in G). If A and B are disjoint, this means they semi-induce a biclique in G (an edgeless graph in G). If $A = B$, this means A induces a clique (an independent set).*

Because we only speak about distinct vertices, the above definition makes no assumptions about self-loops.

▶ **Definition 22** (Subflips)**.** *For a graph G and partition $\mathcal{P}$ of $V(G)$, we define the $\mathcal{P}$-*subflip of G *to be the graph*

$$G \ominus \mathcal{P} := G \oplus (\mathcal{P}, F),$$

*where $F := \{(A, B) : A$ and B are fully adjacent in $G\}$. If $\mathcal{P}$ has at most k parts, we say that H a k-*subflip *of G.*

Note that the structure of a subflip differs from that of a flip. We fix a partition $\mathcal{P}$ but do not supply a flip relation F. Instead, we always use the relation F that removes the most edges. The following crucial observation gives the subflips their name.

▶ **Observation 23.** *If H is a subflip of G, then H is a subgraph of G.*

In the full paper, we establish a toolbox for working with subflips. In the next subsections, we present the most important properties of subflips.

3.3 EP-transducibility

In reflexive graphs we can transduce back and forth between a graph and its subflips using quantifier-free, positive formulas.

▶ **Lemma 24** (↻, ♠)**.** *For every $k \in \mathbb{N}$ there is a quantifier-free, positive formula $\varphi(x, y)$ such that for every reflexive graph G and k-subflip H of G*

$$H \ \varphi\text{-transduces } G \quad \text{and} \quad G \ \varphi\text{-transduces } H\ .$$

In the setting of partially reflexive graphs, we can still recover a graph from its subflips if we forbid large irreflexive cliques.

▶ **Lemma 25** (♠)**.** *For all $k, t \in \mathbb{N}$ there is a quantifier-free, positive formula $\varphi(x, y)$ with the following property. Let G be a partially reflexive graph that excludes the irreflexive K_t as an induced subgraph, and let H be a k-subflip of G. Then H φ-transduces G.*

3.4 Transitivity

General flips satisfy the following transitivity property:

▶ **Observation 26.** *If H_1 is a $\mathcal{P}_1$-flip of G and H_2 is a $\mathcal{P}_2$-flip of H_1, then H_2 is a $(\mathcal{P}_1 \wedge \mathcal{P}_2)$-flip of G.*

This property fails for subflips. The finer parts of $\mathcal{P}_1 \wedge \mathcal{P}_2$ can be fully adjacent, while their containing coarser parts in $\mathcal{P}_1$ and $\mathcal{P}_2$ are not. In this case the subflip specified by $\mathcal{P}_1 \wedge \mathcal{P}_2$ removes even more edges. We can still work with refinements, using the following property.

▶ **Lemma 27 (♠).** *For every graph G, partition $\mathcal{P}$ of $V(G)$, and refinement $\mathcal{R}$ of $\mathcal{P}$ we have*

$$(G \ominus \mathcal{P}) \ominus \mathcal{R} = G \ominus \mathcal{R}.$$

3.5 Approximating flips through subflips

A *semi-induced matching* of size t in a graph G is a set of vertices $a_1, \dots, a_t$ and $b_1, \dots, b_t$ such that for all $i, j \in [t]$: $a_i b_j \in E(G) \Leftrightarrow i = j$. In the more restricted notion of an *induced matching*, we additionally demand that all the a_is form an independent set, and all the b_is form an independent set. Distinguishing between the two will be important when making the following observation.

▶ **Observation 28.** *Let G be a graph, B be a bipartite graph semi-induced in G, and $\overline{B}$ be the bipartite complement of B. If G has co-matching-index less than t, then $\overline{B}$ contains no induced matching of size t.*

The observation fails if we replace "induced matching" with "semi-induced matching". For this reason, we state the next lemma for forbidden induced matchings, even though for forbidden semi-induced matchings a better bound of $2t$ would be possible.

▶ **Lemma 29.** *Let G be a graph that contains no induced matching of size t. Then G contains less than t connected components of size at least 2 and each of these components has diameter less than $3t$.*

Proof. Let $C_1, \dots, C_m$ be the connected components of G of size at least 2. Each component C_i contains at least one edge $u_i v_i$. The u_is and v_is induce a matching of order m. Hence, $m < t$.

Assume towards a contradiction that one of these components has diameter at least $3t$. This is witnessed by an induced path with $3t$ edges of which t induce a matching of order t, a contradiction. ◀

▶ **Lemma 30.** *For every graph G with co-matching-index less than t and size-k partition $\mathcal{P}$ of $V(G)$, there exists a refinement $\mathcal{Q}$ of $\mathcal{P}$ of size at most $k \cdot t^k$, such that for every $\mathcal{P}$-flip H of G and all vertices $u, v \in V(G)$*

$$G \ominus \mathcal{Q} \models E(u, v) \text{ implies } \operatorname{dist}_H(u, v) < 3t.$$

▶ Remark 31. Lemma 30 implies for every $r \in \mathbb{N}$ and $v \in V(G)$:

$$N_r^{G \ominus \mathcal{Q}}[v] \subseteq N_{3tr}^{H}[v].$$

Proof of Lemma 30. We construct $\mathcal{Q}$ by individually refining each part $P \in \mathcal{P}$. For all pairs of parts $P, Q \in \mathcal{P}$ (possibly $P = Q$) we will define a partition $\mathcal{X}(P, Q)$ of P of size at most t, which naturally lifts to a partition $\mathcal{X}'(P, Q) := \mathcal{X}(P, Q) \cup \{V(G) \setminus P\}$ of the whole vertex set $V(G)$. We define $\mathcal{Q}$ to be the common refinement of all the $\mathcal{X}'(P, Q)$. As each $\mathcal{X}'(P, Q)$ only refines a single part P, this bounds the size of $\mathcal{Q}$ by $k \cdot t^k$.

Construction of $\mathcal{X}(P,Q)$ for $P \neq Q$. Consider the bipartite graph B semi-induced by the parts P and Q in G and let $\overline{B}$ be its bipartite complement. First, we add to $\mathcal{X}(P,Q)$ the *isolation part* $X_0(P,Q)$ containing all vertices of P that are isolated in $\overline{B}$. Second, let $C_1,\ldots,C_m$ be the connected components of $\overline{B}$ of size at least two. For each C_i, we add to $\mathcal{X}(P,Q)$ the *component part* $X_i(P,Q) := C_i \cap P$. Since G has co-matching-index less than t, $\overline{B}$ cannot contain an induced matching of size t (Observation 28). By Lemma 29, we conclude that $m < t$ and $|\mathcal{X}(P,Q)| = m + 1 \leqslant t$.

Construction of $\mathcal{X}(P,P) =: \mathcal{X}(P)$. Consider the complement $\overline{G[P]}$ of the graph $G[P]$ induced by P in G. We proceed as in the case above: we add to $\mathcal{X}(P)$ the *isolation part* $X_0(P)$ containing all vertices isolated in $\overline{G[P]}$ and each connected component C_i of $\overline{G[P]}$ of size at least two gets its own *component part* $X_i(P) := C_i$. Again by Lemma 29, $|\mathcal{X}(P)| \leqslant t$.

Analysis for edges between distinct parts. Consider an edge uv in $G \ominus \mathcal{Q}$ with endpoints in distinct parts $u \in P \in \mathcal{P}$ and $v \in Q \in \mathcal{P}$ and an arbitrary $\mathcal{P}$-flip H of G. Our goal is to show that $\text{dist}_H(u,v) < 3t$. If u and v are also adjacent in H we are done, so assume they are non-adjacent. As they are adjacent in the subgraph $G \ominus \mathcal{Q}$ of G, they must be also adjacent in G. This means the adjacency between P and Q was flipped in H and the bipartite graph $\overline{B}$ semi-induced between P and Q in H is the same as the bipartite complement of the graph B semi-induced between P and Q in G. Note that the isolation part $X_0(P,Q)$ and Q are fully non-adjacent in $\overline{B}$, which means they are fully adjacent in G, which means they are fully non-adjacent in $G \ominus \mathcal{Q}$. As u is adjacent to $v \in Q$ in $G \ominus \mathcal{Q}$, we must have $u \notin X_0(P,Q)$ and symmetrically $v \notin X_0(Q,P)$. Then u and v must be contained in component parts $u \in X_i(P,Q)$ and $v \in X_j(Q,P)$. If u and v are contained in different connected components of $\overline{B}$, then $X_i(P,Q)$ and $X_j(Q,P)$ are fully non-adjacent in $\overline{B}$ and also in $G \ominus \mathcal{Q}$; a contradiction. This means u and v are contained in the same connected component of $\overline{B}$. By Lemma 29, this component has diameter less than $3t$ in $\overline{B}$, which yields the desired bound $\text{dist}_H(u,v) < 3t$.

Analysis for edges inside the same part. Similar as above. If $\{u,v\} \subseteq P \in \mathcal{P}$ are adjacent in $G \ominus \mathcal{Q}$ but non-adjacent in H, then the part P was flipped with itself. Every two distinct parts of $\mathcal{X}(P)$ are fully non-adjacent in $\overline{G[P]}$, so fully adjacent in G, so fully non-adjacent in $G \ominus \mathcal{Q}$. This means u and v must be contained in the same part of $\mathcal{X}(P)$. The isolation part $X_0(P)$ forms a clique in G and therefore an independent set in $G \ominus \mathcal{Q}$. Hence, u and v must be contained in the same component part, and we conclude by Lemma 29. ◀

▶ **Lemma 32.** *For every graph G and partition $\mathcal{P}$ of $V(G)$ the following holds:*

1. *For all $t > 1$, if* co-matching-index$(G \ominus \mathcal{P}) \geqslant t \cdot |\mathcal{P}|^2$*, then* co-matching-index$(G) \geqslant t$.
2. *For all $t \geqslant 0$, if* co-matching-index$(G) \geqslant t \cdot |\mathcal{P}|^2$*, then* co-matching-index$(G \ominus \mathcal{P}) \geqslant t$.

Proof. For the first implication let $A, B \subseteq V(G)$ be sets of size $t \cdot |\mathcal{P}|^2$ that semi-induce a co-matching in $G \ominus \mathcal{P}$. Applying the pigeonhole principle twice, we get two parts $P, Q \in \mathcal{P}$ and subsets $A' \subseteq A \cap P$ and $B' \subseteq B \cap Q$ with $|A'| = |B'| = t$ that still semi-induce a co-matching in $G \ominus \mathcal{P}$. Hence, P and Q are neither fully-adjacent in $G \ominus \mathcal{P}$ nor in G (here we used $t > 1$, as a co-matching of size 1 would just be a non-edge, so P and Q could be fully adjacent). This means the adjacency between A' and B' is the same in G and $G \ominus \mathcal{P}$, where they semi-induce a co-matching of size t.

For the second implication with $t > 1$, swap the roles of $G \ominus \mathcal{P}$ and G in the proof above. The case $t = 0$ is vacuous and for $t = 1$ notice that every co-matching of size 1 in G (i.e., every non-edge in G) is also a co-matching of size 1 in $G \ominus \mathcal{P}$. ◀

4 Subflip-flatness

▶ **Definition 33** ((Sub)flip-flatness)**.** *For $r \in \mathbb{N} \cup \{\infty\}$, a graph class $\mathcal{C}$ is* r-flip-flat *(*r-subflip-flat*), if there exists a* margin *function $M : \mathbb{N} \to \mathbb{N}$ and a* budget *$k \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ and set $W \subseteq V(G)$ of size at least $M(m)$, there is a k-flip (k-subflip) H of G and a size m set $A \subseteq W$ such that for all distinct $u, v \in A$ we have*

$$\mathrm{dist}_H(u, v) > r.$$

(For $r = \infty$, we demand that u and v are in different connected components of H.)

Dreier, Mählmann, Siebertz, and Toruńczyk in [11] proved the following combinatorial characterization of monadic stability.

▶ **Theorem 34** ([11])**.** *For every class $\mathcal{C}$, the following are equivalent:*
1. *$\mathcal{C}$ is monadically stable.*
2. *$\mathcal{C}$ is r-flip-flat for every $r \in \mathbb{N}$.*

The goal of this section is to prove the following analog for co-matching-free classes.

▶ **Theorem 35.** *For every class $\mathcal{C}$, the following are equivalent:*
1. *$\mathcal{C}$ is monadically stable and co-matching-free.*
2. *$\mathcal{C}$ is r-subflip-flat for every $r \in \mathbb{N}$.*

The forward direction is a simple application of Lemma 30.

▶ **Lemma 36.** *For every graph class $\mathcal{C}$ with co-matching-index less than $t \in \mathbb{N}$ and for every $r \in \mathbb{N} \cup \{\infty\}$: If $\mathcal{C}$ is $3tr$-flip-flat, then it is also r-subflip-flat.*

Proof. Let $M : \mathbb{N} \to \mathbb{N}$ and $k \in \mathbb{N}$ be the margin and budget for which $\mathcal{C}$ is $3tr$-flip-flat. We claim that $\mathcal{C}$ is r-subflip-flat for margin M and budget $k' := k \cdot t^k$. Given a size $M(m)$ vertex subset W in a graph $G \in \mathcal{C}$, we can apply $3tr$-flip-flatness, which yields a size m subset A of W and a k-flip H in which all elements from A are at pairwise distance greater than $3tr$ from each other. Using Lemma 30, we obtain from H a kt^k-subflip H' of G in which all elements from A are at pairwise distance greater than r from each other. (In particular if $r = \infty$, then elements in different connected components of H are still in different connected components of H'.) ◀

For the backward direction, notice that if a graph class is not monadically stable, then, since it is not even flip-flat, it is for sure not subflip-flat. It remains to prove that classes with unbounded co-matching-index are not subflip-flat.

▶ **Lemma 37.** *No class with unbounded co-matching-index is 2-subflip-flat.*

Proof. Assume towards a contradiction the existence of a graph class $\mathcal{C}$ with unbounded co-matching-index that is 2-subflip-flat with margin M and budget k. Let $G \in \mathcal{C}$ be a graph with co-matching-index at least $\ell := M(3k^2)$ and let $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ be the vertices that semi-induce a co-matching of order ℓ. Applying subflip-flatness to G and the set $W = \{a_1, \ldots, a_\ell\}$ yields a partition $\mathcal{P}$ of size at most k and a size $\ell' := 3k^2$ subset A of W (we can assume $A = \{a_1, \ldots, a_{\ell'}\}$) such that the vertices in A are at pairwise distance greater than 2 in $G \ominus \mathcal{P}$. Applying the pigeonhole principle twice, we can assume a_1, a_2, a_3 are in the same part P of $\mathcal{P}$ and b_1, b_2, b_3 are in the same part Q of $\mathcal{P}$. This means P and Q are not fully adjacent in G, so the edges between P and Q were not deleted in the subflip. This means the six vertices still semi-induce a co-matching in $G \ominus \mathcal{P}$. Then a_1 and a_2 have distance at most two from each other; a contradiction. ◀

This finishes the proof of Theorem 35.

5 Subflipper-rank

In this section we will characterize the semi-ladder-free fragment of monadic dependence as those classes that have bounded *subflipper-rank*. Given a graph G, an integer r, and a vertex $v \in V(G)$ we write $\mathrm{Ball}_r^G(v)$ as a shorthand for the graph $G[N_r^G[v]]$ induced by the (closed) r-neighborhood of v in G.

▶ **Definition 38** ((Sub)flipper-rank). *Fix $r \in \mathbb{N}$ and $k \in \mathbb{N}$. For the single vertex graph K_1 we define the* flipper-rank $\mathrm{frk}_{r,k}(K_1) := 0$. *For every other graph G we define*

$$\mathrm{frk}_{r,k}(G) := 1 + \min_{\substack{H \text{ is a} \\ k\text{-flip of } G}} \max_{v \in V(H)} \mathrm{frk}_{r,k}(\mathrm{Ball}_r^H(v)).$$

A graph class $\mathcal{C}$ has bounded r-flipper-rank *if there are $k, \ell \in \mathbb{N}$ such that $\mathrm{frk}_{r,k}(G) \leqslant \ell$ for all $G \in \mathcal{C}$. The definition of the* subflipper-rank $\mathrm{sfrk}_{r,k}(G)$ *is derived by demanding that H is also a k-subflip of G. We define* bounded r-subflipper-rank *as expected.*

Intuitively, we can understand the flipper-rank (and also the subflipper-rank) as a game called the *flipper-game* [17]. Played on a graph G between two players *flipper* and *localizer*, in every round of the game, flipper applies a k-flip to the current graph (with the goal of decomposing it as fast as possible), and localizer shrinks the arena to an r-neighborhood (with the goal of surviving as long as possible). The flipper-rank then denotes the number of rounds needed for flipper to finish the game. The following theorem is a key ingredient for the first-order model checking algorithm for monadically stable classes.

▶ **Theorem 39** ([17]). *For every class $\mathcal{C}$, the following are equivalent:*
1. *$\mathcal{C}$ is monadically stable.*
2. *$\mathcal{C}$ has bounded r-flipper-rank for every $r \in \mathbb{N}$.*

We prove the following analog for co-matching-free classes.

▶ **Theorem 40** (♠). *For every class $\mathcal{C}$, the following are equivalent:*
1. *$\mathcal{C}$ is monadically stable and co-matching-free.*
2. *$\mathcal{C}$ has bounded r-subflipper-rank for every $r \in \mathbb{N}$.*

The forward direction is implied by the following lemma.

▶ **Lemma 41** (♠). *Fix $r \in \mathbb{N} \cup \{\infty\}$. If a graph class $\mathcal{C}$ has bounded $3tr$-flipper-rank and co-matching-index less than t, then $\mathcal{C}$ also has bounded r-subflipper-rank.*

The lemma is proven by induction on the flipper-rank, where we transfer flipper's moves in the flipper-game to the subflipper-game using Lemma 30.

6 Forbidden induced subflips

The following characterization of half-graph-free monadically dependent classes (i.e., monadically stable classes) by forbidden induced subgraphs was originally proved in [7]. We refer to [22, 29] for the following formulation of the result. The definition of a *star/clique r-crossing* is given below.

▶ **Theorem 42** ([7]). *For every half-graph-free graph class $\mathcal{C}$ the following are equivalent.*
1. *$\mathcal{C}$ is monadically dependent.*
2. *For every $r, k \geqslant 1$ there exists $t \in \mathbb{N}$ such that*
   - *the star r-crossing of order t is not a k-flip of any induced subgraph in $\mathcal{C}$, and*
   - *the clique r-crossing of order t is not a k-flip of any induced subgraph in $\mathcal{C}$.*

Similar to the definition of subflip-flatness, we obtain our characterization by replacing flips with subflips.

▶ **Theorem 43** (♠). *For every semi-ladder-free graph class $\mathcal{C}$ the following are equivalent.*

1. *$\mathcal{C}$ is monadically dependent.*
2. *For every $r, k \geqslant 1$ there exists $t \in \mathbb{N}$ such that*
   - *the star r-crossing of order t is not a k-subflip of any induced subgraph in $\mathcal{C}$, and*
   - *the clique r-crossing of order t is not a k-subflip of any induced subgraph in $\mathcal{C}$.*

For $r \geqslant 1$, the *star r-crossing* of order t is the r-subdivision of $K_{t,t}$ (the biclique of order t). More precisely, it consists of *roots* $a_1, \ldots, a_t$ and $b_1, \ldots, b_t$ together with t^2 many pairwise vertex-disjoint r-vertex paths $\{\pi_{i,j} : i, j \in [t]\}$, whose endpoints we denote as $\mathrm{start}(\pi_{i,j})$ and $\mathrm{end}(\pi_{i,j})$. Each root a_i is adjacent to $\{\mathrm{start}(\pi_{i,j}) : j \in [t]\}$, and each root b_j is adjacent to $\{\mathrm{end}(\pi_{i,j}) : i \in [t]\}$. See Figure 5. The *clique r-crossing* of order t is the graph obtained from the star r-crossing of order t by turning the neighborhood of each root into a clique.

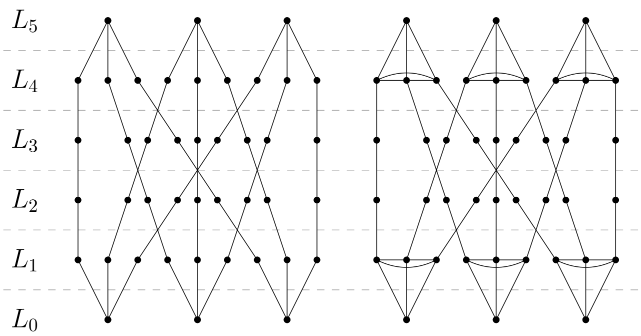

Figure 5 star 4-crossing and clique 4-crossing of order 3.

For the proof of Theorem 43 the flip partitions of the r-crossings are regularized into layers (as depicted in Figure 5). This is already done in [7] and [12]. Then it is argued that in the absence of co-matchings, all flipped pairs must be fully adjacent. Hence, we actually have a subflip. As a corollary, we get the following.

▶ **Corollary 44** (♠). *For every semi-ladder-free class $\mathcal{C}$ of partially reflexive graphs, the following are equivalent.*

1. *$\mathcal{C}$ is monadically dependent.*
2. *$\mathcal{C}$ does not EP-transduce the class of all reflexive graphs.*

7 A normal form for EP-formulas

In this section we establish a normal form for EP-formulas that we will use for our further arguments. We remark that this normal form can also be established for general relational Σ-structures by considering distances in the Gaifman graph of G; we refrain from doing so for consistency of presentation.

▶ **Definition 45** (Clique formulas). *A formula $\varphi(\bar{x})$ is an r-*clique formula *if for every graph G and every tuple $\bar{v} \in V(G)^{|\bar{x}|}$*

$$G \models \varphi(\bar{v}) \quad \Rightarrow \quad G \models \bigwedge_{1 \leqslant i < j \leqslant |\bar{x}|} \mathrm{dist}(v_i, v_j) \leqslant r.$$

This means that the (instantiations of the) free variables of φ form a distance-r clique *in G.*

In the full paper we prove the following normal form.

▶ **Theorem 46** (♠)**.** *Every EP-formula $\varphi(\bar{x})$ with quantifier rank q is equivalent to an EP-formula that is a Boolean combination of 2^q-clique formulas $\chi_i(\bar{z}_i)$ of quantifier rank at most q, where the $\bar{z}_i$ are subtuples of $\bar{x}$.*

Proof sketch. We argue by structural induction on φ. Atomic formulas and positive Boolean combinations are immediate. For $\varphi(\bar{x}) = \exists y\, \psi(\bar{x}, y)$, use the induction hypothesis to write ψ as a positive DNF of r-clique formulas, push $\exists y$ inside disjunctions, and discard conjuncts not containing y. Any satisfying assignment then forces all variables in $\bar{x}$ to be pairwise at distance $\leqslant 2r$, hence φ is a $2r$-clique formula; iterating yields a Boolean combination of 2^q-clique formulas. ◀

8 The monadic non-equality-property

▶ **Definition 47.** *Let $\varphi(\bar{x}, \bar{y})$ be an FO formula over the signature of colored graphs, where $\bar{x}$ and $\bar{y}$ are two tuple variables of the same length t. We say that φ has the* monadic non-equality-property *on a graph class $\mathcal{C}$ if for every $\ell \in \mathbb{N}$ there is a graph coloring G_ℓ^+ of a graph $G_\ell \in \mathcal{C}$ and tuples $\bar{a}_1, \dots, \bar{a}_\ell \in V(G_\ell)^t$ such that for all $i, j \in [\ell]$*

$$G_\ell^+ \models \varphi(\bar{a}_i, \bar{a}_j) \iff i \neq j.$$

The goal of this section is to prove Theorem 12 from the introduction, which we restate here for convenience.

▶ **Theorem 12.** *For every class $\mathcal{C}$ of partially reflexive graphs the following are equivalent.*

1. *$\mathcal{C}$ is semi-ladder-free, irreflexive-clique-free, and monadically dependent.*
2. *No EP-formula has the monadic non-equality-property on $\mathcal{C}$.*

As reflexive graphs contain no irreflexive cliques, the simpler version for reflexive graphs (Theorem 11) follows immediately.

We prove both implications of Theorem 12 separately. Towards showing the non-structure implication (2. ⇒ 1.), observe that, by contraposition and a simple formula rewriting argument, *not* having the monadic non-equality-property for all EP-formulas is preserved by EP-transductions.

▶ **Observation 48.** *For every EP-formula $\varphi(x, y)$ and EP-transduction T there exists an EP-formula $\psi(x, y)$ with the following property. For every graph class $\mathcal{C}$: If φ has the monadic non-equality-property on $\mathsf{T}(\mathcal{C})$, then ψ has the monadic non-equality-property on $\mathcal{C}$.*

Proof of (2. ⇒ 1.) of Theorem 12. By contraposition, assume $\mathcal{C}$ is either not irreflexive-clique-free, not semi-ladder-free, or not monadically dependent. In the first case, $E(x, y)$ has the monadic non-equality-property on $\mathcal{C}$.

In the second case, $\mathcal{C}$ EP-transduces either the class of all co-matchings or the class of all half-graphs. By Observation 48, it suffices to show that there are EP-formulas φ and ψ with the monadic non-equality-property on the class of all co-matchings and on the class of all half-graphs, respectively. We can pick

$$\varphi(x_1x_2, y_1y_2) := E(x_1, y_2) \text{ and } \psi(x_1x_2, y_1y_2) := E(x_1, y_2) \vee E(y_1, x_2).$$

See Figure 6 on how to choose witnessing tuples.

In the third case, if $\mathcal{C}$ is semi-ladder-free, but not monadically dependent then it EP-transduces the class of all reflexive graphs by Corollary 44, so we conclude as in the second case. ◀

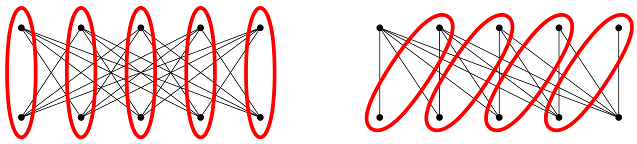

Figure 6 Witnesses for the monadic non-equality-property.

Proof sketch of (1. $\Rightarrow$ 2.) of Theorem 12. We sketch the argument for formulas $\varphi(x, y)$ on singletons. Assume there is an EP-formula $\varphi(x, y)$ and arbitrarily large sequences of vertices $A = a_1, \dots, a_\ell$ with $G^+ \models \varphi(a_i, a_j) \Leftrightarrow i \neq j$. Using subflip-flatness of $\mathcal{C}$, we pass to a subflip H in which (many of) these vertices are pairwise at distance $> 2^q$, where q is the quantifier-rank of φ. Using Lemma 25, we obtain an EP-formula φ^* witnessing the same pattern on (a coloring of) H. By the EP normal form (Theorem 46), φ^* is a DNF of 2^q-clique formulas, so no satisfied disjunct can have a conjunct involving two of the pairwise far vertices in A. Thus, there is a satisfied disjunct $\alpha(x, y)$, consisting only of conjuncts that each involve only one of x and y. Thus, the truth of α depends only on the q-types of x and y individually. Since there are only finitely many q-types, there must be two vertices $a_i, a_j \in A$ of the same q-type satisfying $H \models \alpha(a_i, a_j)$. Then also $H \models \alpha(a_i, a_i)$ and $H \models \varphi^*(a_i, a_i)$, contradicting the monadic non-equality-property. ◀

As a corollary, we obtain Theorem 8 from the introduction.

▶ **Theorem 8.** *Let $\mathcal{C}$ be a nowhere dense class of partially reflexive graphs. Every EP-transduction of $\mathcal{C}$ is semi-ladder-free and monadically dependent.*

Proof. As $\mathcal{C}$ is nowhere dense, it satisfies all conditions of Theorem 12. Hence, no EP-formula has the monadic non-equality-property on $\mathcal{C}$. By Observation 48, the latter also holds for every class $\mathcal{D}$ that is EP-transducible from $\mathcal{C}$. Then again by Theorem 12, $\mathcal{D}$ must be semi-ladder-free. ◀

9 EP-sparsification

As a stepping stone to Theorem 10, we define the graph parameter *subflip-depth* and show that every class of bounded subflip-depth can be EP-transduced from a class of bounded tree-depth. (The reverse is true as well by Theorem 8.)

9.1 Definition and basic properties

We define *flip-depth* and *subflip-depth* through the (sub)flipper-rank for radius $r = \infty$. For every graph G and $k \in \mathbb{N}$, we define $\text{flip-depth}_k(G) := \text{frk}_{\infty,k}(G)$ and $\text{subflip-depth}_k(G) := \text{sfrk}_{\infty,k}(G)$. We recall the definition of the flipper-rank for the special case of radius $r = \infty$ here for convenience.

▶ **Definition 49** ((Sub)flip-depth). *For every $k \in \mathbb{N}$ we define the k-*flip-depth *of the single vertex graph K_1 as* $\text{flip-depth}_k(K_1) := 0$*. For every other graph G we define*

$$\text{flip-depth}_k(G) := 1 + \min_{\substack{H \text{ is a} \\ k\text{-flip of } G}} \max_{C \in \text{Conn}(H)} \text{flip-depth}_k(C),$$

where $\text{Conn}(H)$ *are the connected components of H. We naturally obtain the definition of* k-subflip-depth *by replacing k-flips with k-subflips in the above. A graph class $\mathcal{C}$ has* bounded (sub)flip-depth*, if there are $k, d \in \mathbb{N}$ such that every graph $G \in \mathcal{C}$ has k-(sub)flip-depth at most d.*

Better known than flip-depth is the parameter *SC-depth* [19]. We refrain from defining this parameter here, but mention the folklore result that flip-depth and SC-depth are functionally equivalent.

▶ **Lemma 50** (♠). *For every graph G and $k \geqslant 2$*

$$\text{flip-depth}_k(G) \leqslant \text{SC-depth}(G) \leqslant 3k^2 \cdot \text{flip-depth}_k(G).$$

Through the above equivalence, the following facts about SC-depth carry over to flip-depth.

▶ **Theorem 51** ([19, 20]). *For every k, d there is $t \in \mathbb{N}$ such that no graph G with k-flip-depth d contains an induced path of length t.*

▶ **Theorem 52** ([12]). *A graph class $\mathcal{C}$ has bounded flip-depth if and only if it is ∞-flip-flat.*

Utilizing Lemma 41 and Lemma 36 we prove the following characterizations of subflip-depth in the full paper.

▶ **Theorem 53** (♠). *For every class $\mathcal{C}$ the following are equivalent.*

1. *$\mathcal{C}$ is co-matching-free and has bounded flip-depth.*
2. *$\mathcal{C}$ has bounded subflip-depth.*
3. *$\mathcal{C}$ is ∞-subflip-flat.*

9.2 Sparsification for subflip-depth

The goal of this subsection is to prove the following theorem.

▶ **Theorem 54** (↻). *For every $k, d \in \mathbb{N}$ there are EP-transductions $\mathsf{Sparsify}_{k,d}$ and $\mathsf{Recover}_{k,d}$ such that for every reflexive graph G with k-subflip-depth at most d there is a reflexive subgraph G^* of G with tree-depth at most $k \cdot d$ such that*

$$G^* \in \mathsf{Sparsify}_{k,d}(G) \text{ and } G \in \mathsf{Recover}_{k,d}(G^*).$$

We recall the definition of tree-depth.

▶ **Definition 55** (Tree-depth). *We define the* tree-depth *of the single vertex graph K_1 as* $\text{tree-depth}(K_1) := 0$. *For every other graph G we define*

$$\text{tree-depth}(G) := 1 + \min_{v \in V(G)} \max_{C \in \text{Conn}(G-v)} \text{tree-depth}(C),$$

where $G - v$ is the graph G with v removed and $\text{Conn}(G - v)$ are the connected components of $G - v$.

Before we can start sparsifying graphs of bounded subflip-depth, we need some additional tools. First, we can use EP-transductions to add and remove edge sets that have a small vertex cover. A set of vertices $S \subseteq V$ is a *vertex cover* for a set of edges $X \subseteq V \times V$, if for each $uv \in X$ we have $\{u, v\} \cap S \neq \emptyset$.

▶ **Lemma 56** (↻, ♠). *For every $k \in \mathbb{N}$, there is a quantifier-free, positive transduction* $\mathsf{VertexCover}$ *such that for every reflexive graph $G = (V, E)$ and edge set $X \subseteq E$ with a vertex cover of size at most k*

$$H \in \mathsf{VertexCover}(G) \text{ and } G \in \mathsf{VertexCover}(H),$$

where $H := (V, E - X)$.

Second, we can apply a transduction to all connected components of a graph in parallel, if all components have bounded diameter.

▶ **Lemma 57 (♠).** *For every $d \in \mathbb{N}$ and EP-transduction T, there is a EP-transduction P with the following property. For every two partially reflexive graphs G and G^*, if they satisfy*
- *$H_1, \dots, H_m$ are the connected components of G,*
- *every H_i has diameter at most d,*
- *$H_1^*, \dots, H_m^*$ are the connected components of G^*, and*
- *every H_i^* is contained in $\mathsf{T}(H_i)$,*

then $G^* \in \mathsf{P}(G)$.

We are now ready to prove Theorem 54.

Proof of Theorem 54. We first prove, by induction on d, a slight variation of the lemma, where we additionally
- assume that the input G is connected, and
- demand that the result G^* is connected.

For the base case $d = 0$, we can choose $G^* := G = K_1$ and $\mathsf{Sparsify}_{k,0}$ and $\mathsf{Recover}_{k,0}$ to be the identity transduction. For the inductive step, assume G has k-subflip-depth at most $d+1$. Let $\mathcal{P}$ be the partition of size at most k such that the connected components $H_1, \dots, H_m$ of the graph

$$H := H_1 \uplus \dots \uplus H_m = G \ominus \mathcal{P}$$

all have k-subflip-depth at most d. Applying induction to these components, we construct the graph

$$H^* := H_1^* \uplus \dots \uplus H_m^*$$

We build G^* by adding to H^* the edges X defined as follows. Pick an arbitrary *leader* vertex $\ell(P) \in P$ from each part $P \in \mathcal{P}$. Build X by including:
- for each part P that forms a clique in G: an edge from $\ell(P)$ to each other vertex of P, and
- for each pair of distinct parts P, Q that are fully adjacent in G: an edge from $\ell(P)$ to each vertex of Q and an edge $\ell(Q)$ to each vertex from P.

▶ **Observation 58.** *The set of leaders forms a vertex cover of size at most k for X.*

We now check that G^* satisfies the statement of the lemma.

▷ Claim 59. G^* is a connected subgraph of G.

Proof. All the H_i^* are connected subgraphs of G by induction. Clearly G^* is a subgraph of G: G^* only adds to the H_i^* the edges X edges that were already present in G.

It remains to show that G^* is connected. As G is connected, it suffices to show that for every edge uv in G, there is a path between u and v in G^*. If u and v are in the same H_i^*, this follows by induction. Otherwise, u and v are in different components H_i and H_j of $G \ominus \mathcal{P}$, so the edge uv was removed by applying the subflip. Hence, either u and v are in the same part $P \in \mathcal{P}$ that is a clique in G, or they are in two distinct parts P and Q that are fully adjacent in G. In both cases we can verify that there is a path of length at most 3 using only edges from X in G^*. ◁

▷ Claim 60. G^* has tree-depth at most $k \cdot (d+1)$.

Proof. H^* is a disjoint union of graphs of tree-depth at most $k \cdot d$ by induction. It follows from the definition of tree-depth that adding the edges X which have a vertex cover of size at most k can increase the tree-depth by at most k. ◁

▷ Claim 61. There is an EP-transduction $\mathsf{Sparsify}_{k,d+1}$ that produces G^* from G.

Proof. As subflips are transducible (Lemma 24), we can transduce H from G. By induction, $\mathsf{Sparsify}_{k,d}$ produces H_i^* from H_i for all $i \in [m]$. As each of the H_i has bounded diameter (Theorem 51), we can apply $\mathsf{Sparsify}_{k,d}$ in parallel to transduce H^* from H (Lemma 57). We can transduce G^* from H^* by adding the additional edges X through Lemma 56. By transitivity (Lemma 20), we can chain these transductions to transduce G^* from G. ◁

▷ Claim 62. There is an EP-transduction $\mathsf{Recover}_{k,d+1}$ that produces G from G^*.

Proof. We first transduce H^* from G^* by removing the edges $E(G^*) - E(H^*) = X$ (Lemma 56). Recall that $H^* = H_1^* \uplus \ldots \uplus H_m^*$. By induction, $\mathsf{Recover}_{k,d}$ produces H_i from H_i^* for all $i \in [m]$. Crucially, each H_i^* is connected and has bounded tree-depth and thus bounded diameter (Theorem 51). We can therefore apply $\mathsf{Recover}_{k,d}$ in parallel to all H_i^* (Lemma 57). This yields H. As the last step, we undo the subflip and obtain G (Lemma 24). ◁

This finishes the proof of the lemma, where we assumed that G is connected. By Lemma 57, the original statement follows. ◀

9.3 Sparsification beyond subflip-depth

Theorem 54, which we proved in the previous section, settles the bounded shrub-depth case of Theorem 10. (This is because co-matching-free bounded shrub-depth and bounded subflip-depth are equivalent by Theorem 53.) To lift this result to more general classes, we will work with *covers*.

▶ **Definition 63** (p-covers)**.** *Fix a universe V. A collection of sets $U_1, \ldots, U_s \subseteq V$ forms a* p-cover *of V if for every subset $X \subseteq V$ of size at most p, there is $i \in [s]$ with $X \subseteq U_i$.*

▶ **Definition 64** ($\mathcal{P}$-covers)**.** *For a class property $\mathcal{P}$ and a graph class $\mathcal{C}$, we say $\mathcal{C}$ has* $\mathcal{P}$-covers *if for every $p \in \mathbb{N}$ there exists $s \in \mathbb{N}$ such that for every $G \in \mathcal{C}$ there is a size-s p-cover $U_1, \ldots, U_s$ of $V(G)$ such that the class $\mathcal{D} = \{G[U_i] \mid G \in \mathcal{C}, i \in [s]\}$ has property $\mathcal{P}$.*

A graph class has *structurally bounded expansion* if it is transducible (by a general transduction, not necessarily EP-transducible) from a class of *bounded expansion.* We will use the following characterizations through covers as definitions.

▶ **Theorem 65.** *For every graph class $\mathcal{C}$, all of the following hold.*
1. *$\mathcal{C}$ has bd. expansion $\Leftrightarrow$ $\mathcal{C}$ has bd.-tree-depth-covers. [27]*
2. *$\mathcal{C}$ has struc. bd. expansion $\Leftrightarrow$ $\mathcal{C}$ has bd.-SC-depth-covers. [16]*

We contribute to this the following equivalence.

▶ **Theorem 66** (♠)**.** *For every class $\mathcal{C}$, the following are equivalent.*
1. *$\mathcal{C}$ is co-matching-free and has structurally bd. expansion.*
2. *$\mathcal{C}$ has bounded-subflip-depth-covers.*

The following lemma proves the essence of Theorem 10, because half-graph-free classes of bounded shrub-depth / (linear) clique-width / twin-width / merge-width all have structurally bounded expansion.

▶ **Lemma 67** (⟳, ♠). *For every co-matching-free class $\mathcal{C}$ of reflexive graphs of structurally bounded expansion, there are EP-transductions* $\mathsf{Sparsify}$ *and* $\mathsf{Recover}$ *such that every graph $G \in \mathcal{C}$ contains a reflexive subgraph G^* such that*

$$G^* \in \mathsf{Sparsify}(G) \quad \text{and} \quad G \in \mathsf{Recover}(G^*),$$

and the class $\mathcal{D} := \{G^* : G \in \mathcal{C}\}$ has bounded expansion.

Proof sketch. By Theorem 66 we obtain bounded-subflip-depth covers of $G \in \mathcal{C}$. Each cover set is sparsified to bounded tree-depth using Theorem 54. The resulting graphs are glued, which yields the desired sparsification. G is recovered by going to an appropriate subgraph (untangling the unions over the cover) and reversing the construction using the recovering transduction of Theorem 54. ◀

10 Positive MSO

We close the paper with this easy to prove observation.

▶ **Theorem 68** (♠). *Every (existential) positive MSO formula $\varphi(\bar{x}, \bar{X})$ is equivalent to an (existential) positive FO formula.*

Proof sketch. We eliminate the set quantifiers one by one. First assume $\varphi = \exists Y\, \psi(\bar{x}, Y, \bar{X})$ where ψ is a positive formula. Define $\psi^\top(\bar{x}, \bar{X})$ as the formula obtained from $\psi(\bar{x}, Y, \bar{X})$ by replacing every atom $Y(t)$ by $\top$ (equivalently, interpreting Y as the full vertex set). We claim that for every G, every tuple $\bar{a}$ and tuple $\bar{B}$ we have

$$G \models \exists Y\, \psi(\bar{a}, Y, \bar{B}) \quad \Longleftrightarrow \quad G \models \psi^\top(\bar{a}, \bar{B}).$$

To see this, observe that if $G \models \exists Y\, \psi(\bar{a}, Y, \bar{B})$, then by positivity of ψ also $G \models \psi^\top(\bar{a}, \bar{B})$. Conversely, if $G \models \psi^\top(\bar{a}, \bar{B})$, then choosing $Y = V(G)$ witnesses the existential quantifier, hence $G \models \exists Y\, \psi(\bar{a}, Y, \bar{B})$.

Now assume $\varphi = \forall Y\, \psi(\bar{x}, Y, \bar{X})$ with ψ positive. Let $\psi^\perp(\bar{x}, \bar{X})$ be the formula obtained from $\psi(\bar{x}, Y, \bar{X})$ by replacing every atom $Y(t)$ by $\perp$ (equivalently, interpreting Y as the empty set). Then for every $G, \bar{a}$, and $\bar{B}$,

$$G \models \forall Y\, \psi(\bar{a}, Y, \bar{B}) \quad \Longleftrightarrow \quad G \models \psi^\perp(\bar{a}, \bar{B}).$$

For the forward direction, take $Y = \emptyset$. For the reverse direction, if $G \models \psi^\perp(\bar{a}, \bar{B})$ then by positivity of ψ we have $G \models \psi(\bar{a}, Y, \bar{B})$ for every $Y \supseteq \emptyset$, i.e., for every set Y, hence $G \models \forall Y\, \psi(\bar{a}, Y, \bar{B})$. ◀

11 Outlook

In this paper we have proposed the existential positive sparsification conjecture predicting that co-matching-free, monadically stable classes are exactly those that can be transduced from nowhere dense classes by existential positive formulas. We were able to confirm this conjecture for multiple special cases. Along the way we have gathered insight on

1. the expressive power of existential positive first-order logic in nowhere dense classes, and
2. the combinatorial structure of co-matching-free, monadically stable classes.

These two directions each open up a natural direction for future work.

Positive FO. During our studies we were neither able to prove nor disprove, whether the restriction to *existential* positive logic is necessary. We believe that also positive transductions of nowhere dense classes (using both existential and universal quantifiers) are well-behaved.

▶ **Conjecture 69** (↻). *For every class $\mathcal{C}$ of reflexive graphs, the following are equivalent.*
1. *$\mathcal{C}$ is semi-ladder-free and monadically dependent.*
2. *$\mathcal{C}$ is a positive transduction of a nowhere dense class of reflexive graphs.*

Proving the (2) ⇒ (1) direction, i.e., the non-existential analog of Theorem 8, would already suffice to lift our partial results to the positive case.

Co-matching-free monadic dependence. We have seen that co-matching-free monadic stability admits multiple nice characterizations through subflips, that mirror characterizations of monadic stability through flips. This begs the question, whether also co-matching-free monadic dependence admits meaningful characterizations. Monadic dependence admits flip-based characterizations in the form of *flip-breakability* [12] and *flip-separability* [3]. One can naturally define *subflip-breakability* and *subflip-separability*, but it is easy to prove that the class of co-matchings is both subflip-breakable and -separable. Thus, the two conditions are not sufficient to characterize co-matching-free monadic dependence.

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