Distinguishing Graphs by Counting Homomorphisms from Sparse Graphs

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Abstract

Lovász (1967) showed that two graphs G and H are isomorphic if, and only if, they are *homomorphism indistinguishable* over all graphs, i.e., G and H admit the same number of number of homomorphisms from every graph F. Subsequently, a substantial line of work studied homomorphism indistinguishability over restricted graph classes. For example, homomorphism indistinguishability over minor-closed graph classes $\mathcal{F}$ such as the class of planar graphs, the class of graphs of treewidth $\leq k$, pathwidth $\leq k$, or treedepth $\leq k$, was shown to be equivalent to quantum isomorphism and equivalences with respect to counting logic fragments, respectively.

Via such characterisations, the distinguishing power of e.g. logical or quantum graph isomorphism relaxations can be studied with graph-theoretic means. In this vein, Roberson (2022) conjectured that homomorphism indistinguishability over every graph class excluding some minor is not the same as isomorphism. We prove this conjecture for all vortex-free graph classes. In particular, homomorphism indistinguishability over graphs of bounded Euler genus is not the same as isomorphism. As a negative result, we show that Roberson's conjecture fails when generalised to graph classes excluding a topological minor.

Furthermore, we show homomorphism distinguishing closedness for several graph classes including all topological-minor-closed and union-closed classes of forests, and show that homomorphism indistinguishability over graphs of genus $\leq g$ (and other parameters) forms a strict hierarchy.

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1 Introduction

Lovász [44] showed that two graphs G and H are isomorphic if, and only if, they admit the same number of homomorphisms from every graph F. Decades later, Dvořák [24] proved that two graphs G and H satisfy the same sentences in the k-variable fragment C^k of first-order logic with counting quantifiers if, and only if, they admit the same number of homomorphisms from every graph F of treewidth $< k$. Finally, a celebrated theorem of Mančinska and Roberson [46] asserts that two graphs G and H are quantum isomorphic if, and only if, they admit the same number of homomorphisms from every planar graph F.

These results motivate the following notion: Two graphs G and H are *homomorphism indistinguishable* over a graph class $\mathcal{F}$, in symbols $G \equiv_{\mathcal{F}} H$, if they admit the same number of homomorphisms $\hom(F, G) = \hom(F, H)$ from every graph $F \in \mathcal{F}$. In addition to the initially mentioned results, homomorphism indistinguishability over graph classes such as the class of graphs of pathwidth $\leq k$, treedepth $\leq k$, the class of outerplanar graphs, the class of graphs of maximum degree $\leq d$ etc. has been characterised in terms of graph isomorphism relaxations from areas ranging from finite model theory [24, 22, 32, 29, 61, 47, 17, 6, 61], optimisation [22, 35, 7, 34, 45, 56, 38], category theory [47, 17, 1, 61], algebraic graph theory [22, 35, 54], and invariant theory [14, 75, 15] to machine learning [74, 48, 76, 30, 70] and quantum information theory [46, 38, 65].[1]

The plenitude of such characterisations motivates studying how properties of the homomorphism indistinguishability relation $\equiv_{\mathcal{F}}$ are related to properties of the graph class $\mathcal{F}$ [64]. Concretely, the computational complexity [10, 62, 71] and the distinguishing power [55] of $\equiv_{\mathcal{F}}$ have been studied from this perspective. This paper is concerned with connections between notions of sparsity of $\mathcal{F}$ and the distinguishing power of $\equiv_{\mathcal{F}}$.

1.1 Weak Roberson Conjecture: Separation from Isomorphism

A systematic study of the distinguishing power of $\equiv_{\mathcal{F}}$ was initiated by Roberson in [55]. Arguably, the most fundamental question in this context is whether $\equiv_{\mathcal{F}}$ is the same as the isomorphism relation when $\mathcal{F}$ is some proper graph class. For example, it is not hard to see [44] that two graphs are homomorphism indistinguishable over all connected graphs if, and only if, they are isomorphic. A more involved example is given by Dvořák [24], who showed that two graphs G and H are homomorphism indistinguishable over all 2-degenerate graphs if, and only if, they are isomorphic.

In contrast, there exist, for every $k \geq 0$, non-isomorphic graphs G and H that are homomorphism indistinguishability over the class of graphs of treewidth $\leq k$. This is a consequence of the seminal result of Cai, Fürer, and Immerman [13] that the logic C^{k+1}, and equivalently the k-dimensional Weisfeiler–Leman algorithm, fails to distinguish all pairs of non-isomorphic graphs. Similarly, the known characterisation in terms of quantum isomorphisms [46, 8] implies that homomorphism indistinguishability over the class of planar graphs is weaker than the isomorphism relation [3]. Inspired by these two examples, Roberson proposed the following conjecture:

▶ **Conjecture 1.1** (Roberson [55, Conjecture 5])**.** *For every $k \geq 0$, there exist non-isomorphic graphs G and H that are homomorphism indistinguishable over all graphs of Hadwiger number $\leq k$.*

Here, the *Hadwiger number* of a graph F is the largest integer $k \in \mathbb{N}$ such that F contains the k-vertex complete graph K_k as a minor. In other words, Conjecture 1.1 asserts that, for every graph class $\mathcal{F}$ excluding some minor, there exist non-isomorphic graphs G and H such that $G \equiv_{\mathcal{F}} H$. Thus, the conjecture vastly generalises the aforementioned examples of planar graphs and graphs of bounded treewidth. We stress that, prior to this work, no further examples were known to support Conjecture 1.1.[2]

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[1] For further references, visit the *Homomorphism Indistinguishability Zoo* at tseppelt.github.io/homind-database.

[2] In [55], a proof of Conjecture 1.1 for $k = 4$ due to Richter, Roberson, and Thomassen was announced. However, this result is to our knowledge still unpublished. Our Theorems 1.2 and 4.22 covers this case.

As our first main result, we prove Conjecture 1.1 for all minor-closed graph classes that do not exhibit a certain ingredient of the Robertson–Seymour structure theorem [60] for minor-free graphs. By this seminal theorem, for every graph H, every H-minor-free graph can be obtained by clique-sums of graphs that can be almost embedded on a surface Σ of bounded Euler genus. Intuitively, a graph is *almost embeddable* on Σ if it can be obtained from a graph embeddable on Σ by attaching bounded-pathwidth graphs (so-called *vortices*) at a bounded number of faces and adding a bounded number of vertices with arbitrary neighborhood (so-called *apices*). Our main theorem applies to every minor-closed graph class omitting vortices.

▶ **Theorem 1.2.** *For every $k \geq 0$, there exist non-isomorphic graphs G and H that are homomorphism indistinguishable over all graphs of vertex-free Hadwiger number $\leq k$.*

The parameter *vortex-free Hadwiger number* was introduced by Thilikos and Wiederrecht [69], who also gave a characterisation of this parameter in terms of a parametric family of forbidden minors analogous to the grid minor theorem for treewidth [58]. Just like the Hadwiger number, the vortex-free Hadwiger number is a minor-monotone graph parameter. The most notable special case of Theorem 1.2 are graphs of bounded genus. It implies that, for every $g \geq 0$, there exist non-isomorphic graphs G and H that are homomorphism indistinguishable over all graphs that can be embedded on a (orientable or non-orientable) surface of genus $\leq g$. Another special case of Theorem 1.2 are graph classes that exclude a single-crossing minor [59].

It is natural to ask what happens beyond classes excluding a minor such as classes excluding a topological minor and other well-established notions in the theory of sparse graphs [53, 66]. For example, it is known that classes of bounded degree yield homomorphism indistinguishability relations that are weaker than isomorphism [55], and characterisations have recently been obtained in [15]. Note that graphs of maximum degree 3 contain every graph as a minor. On the other hand, Dvořák [24] obtained a graph class $\mathcal{F}$ of bounded expansion for which $\equiv_{\mathcal{F}}$ is the same as isomorphism. Additionally, the graph class $\mathcal{F}$ is easily seen to have bounded local treewidth (see [28, 31]). As our second main result, we strengthen those negative results and show that variants of Conjecture 1.1 fail for many natural notions that go beyond excluding a minor or bounding the degree, such as excluding a topological minor.

▶ **Theorem 1.3.** *Two graphs are isomorphic if, and only if, they are homomorphism indistinguishable over all graphs excluding K_5 as a topological minor.*

Theorem 1.3 gives further evidence for the special role that minor-closed graph classes play in the realm of homomorphism indistinguishability, e.g., as a demarcation line in Figure 1. In fact, this is further highlighted by our approach to prove Theorem 1.3: We say a graph class $\mathcal{F}$ is *closed under* 2*-sums with triangles* if, for every $F \in \mathcal{F}$ and every edge $e = vw \in E(F)$, we have $F \oplus_e K_3 \in \mathcal{F}$. Here, $F \oplus_e K_3$ denotes the graph obtained from F by adding a fresh vertex x_e connected to exactly v, w (i.e., we glue a triangle to the edge e). Note that many natural graph classes are closed under 2-sums with triangles, e.g., planar graphs, graphs of treewidth $\leq k$ for all $k \geq 2$, and the class of graphs excluding K_5 as a topological minor. As our main technical insight, we prove that, for a class $\mathcal{F}$ closed under subgraphs and 2-sums with triangles, we can recover homomorphism counts $\hom(F', \star)$ for every graph F' that is a minor of some graph $F \in \mathcal{F}$, given only the homomorphism counts $\hom(F, \star)$ from graphs $F \in \mathcal{F}$; see Lemma 6.1.

At this point, we directly obtain Theorem 1.3 by observing that the graphs of maximum degree 3 exclude K_5 as a topological minor, and every graph is a minor of a graph of maximum degree 3. More concretely, our arguments yield a graph class $\mathcal{D}_3^*$ over which homomorphism

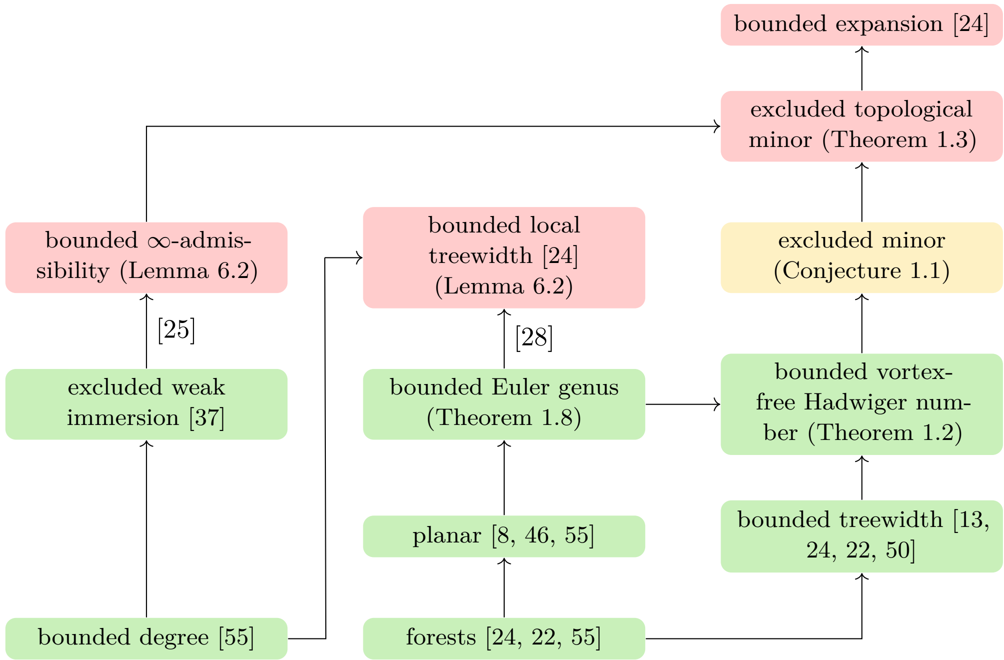


Figure 1 Properties of a graph class $\mathcal{F}$ that do (green) or do not (red) guarantee that there exist non-isomorphic graphs G and H that are homomorphism indistinguishable over $\mathcal{F}$.

indistinguishability is the same as isomorphism. In addition to the fact that $\mathcal{D}_3^*$ excludes K_5 as a topological minor, it also has bounded ∞-admissibility (see [26, 66]) and bounded local treewidth.

Finally, we note that in a concurrent and independent work, Jiménez, Moore, Quiroz, and Yoo [37] show that for graph classes $\mathcal{F}$ that exclude a fixed graph as a weak immersion, the relation $\equiv_{\mathcal{F}}$ is not isomorphism. Graph classes with an excluded weak immersion sit between classes of bounded degree and classes of bounded ∞-admissibility [25]; see Figure 1.

1.2 Strong Roberson Conjecture: Separating Homomorphism Indistinguishability Relations

Once a homomorphism indistinguishability relation $\equiv_{\mathcal{F}}$ is separated from isomorphism, the question arises of how $\equiv_{\mathcal{F}}$ relates to other homomorphism indistinguishability relations $\equiv_{\mathcal{F}'}$. Clearly, if $\mathcal{F} \supseteq \mathcal{F}'$, then the relation $\equiv_{\mathcal{F}}$ refines $\equiv_{\mathcal{F}'}$. When does the converse hold?

Answers to this question have been used to pinpoint the distinguishing power of graph isomorphism relaxations from finite model theory [29, 2, 50, 61], optimisation [56], and machine learning [76, 30, 70].

The central definition for answering this question is the following: A graph class $\mathcal{F}$ is *homomorphism distinguishing closed* [55] if, for all $F' \notin \mathcal{F}$, there exist graphs G and H that are homomorphism indistinguishable over $\mathcal{F}$ and satisfy $\hom(F', G) \neq \hom(F', H)$, i.e., G and H admit a different number of homomorphisms from F'. In other words, homomorphism distinguishing closed graph classes $\mathcal{F}$ are maximal in the sense that, for every $F' \notin \mathcal{F}$, the relation $\equiv_{\mathcal{F} \cup \{F'\}}$ is strictly finer than $\equiv_{\mathcal{F}}$.

This definition is significant since, if $\mathcal{F}$ is homomorphism distinguishing closed, then it holds that $\equiv_{\mathcal{F}}$ refines $\equiv_{\mathcal{F}'}$ if, and only if, $\mathcal{F} \supseteq \mathcal{F}'$. Hence, in this case, the relations $\equiv_{\mathcal{F}}$ and $\equiv_{\mathcal{F}'}$ can be compared simply by comparing the graph classes $\mathcal{F}$ and $\mathcal{F}'$. The aforementioned articles [29, 2, 61, 56, 76, 30, 70] made use of precisely this property. Hence, it is desirable to exhibit homomorphism distinguishing closed graph classes. To that end, Roberson [55] proposed the following conjecture.

▶ **Conjecture 1.4** (Roberson [55, Conjecture 4]). *Every minor-closed and union-closed*[3] *graph class $\mathcal{F}$ is homomorphism distinguishing closed.*

Clearly, Conjecture 1.4 implies Conjecture 1.1. Note, however, that Conjecture 1.1 admits a certain monotonicity that Conjecture 1.4 lacks: A proof of Conjecture 1.1 for some $k \in \mathbb{N}$ immediately yields a proof for all $k' \leq k$. In the case of Conjecture 1.4, if a graph class $\mathcal{F}$ is homomorphism distinguishing closed, then its sub- and super-classes may or may not be homomorphism distinguishing closed, see [63, Section 6].

Conjecture 1.4 is known to be true for the class of planar graphs [55], forests [55], disjoint unions of paths (and cycles) [55], graphs of treewidth $\leq k$ [50], treedepth $\leq k$ [29], pathwidth $\leq k$ [63], graphs with bounded-depth pebble forest covers [2, 61], and essentially finite graph classes [64], see [63, Section 6]. Most of these instances are motivated by the study of the expressive power of counting logic fragments [50, 29, 2, 63, 61], and in fact the proofs for homomorphism distinguishing closedness often heavily rely on known characterisations. This is a clear drawback towards verifying Conjecture 1.4, since not all minor-classed classes $\mathcal{F}$ are likely to admit natural characterisations for $\equiv_{\mathcal{F}}$. In contrast, we obtain several new results in the realm of Conjecture 1.4 that do not rely on such characterisations and reprove some of the existing results using more direct and simpler arguments.

First, we resolve Conjecture 1.4 for all classes of forests. In fact, our result for classes $\mathcal{F}$ of forests even hold under the weaker assumption that $\mathcal{F}$ is only closed under topological minors.

▶ **Theorem 1.5.** *Every union-closed class of forests closed under taking topological minors is homomorphism distinguishing closed.*

For example, this generalises the known result that the class of forests of maximum degree d is homomorphism distinguishing closed [55], which strengthened a previous result from [35]. Moreover, as a direct corollary, we precisely determine the distinguishing power of the graph neural network architecture proposed in [70, B.3] since they can be characterised by $\equiv_{\mathcal{F}}$ for certain classes $\mathcal{F}$ of forests.

Furthermore, we develop a combinatorial toolkit for proving that graph classes constructed via standard operations from sparsity theory such as taking clique-sums or adding apices are homomorphism distinguishing closed. As a corollary, we extend the list of known homomorphism distinguishing closed graph classes by the following examples:

▶ **Theorem 1.6.** *The following graph classes are homomorphism distinguishing closed.*

1. *the class of cactus graphs,*
2. *the class of outerplanar graphs,*
3. *the class of $K_{3,3}$-minor-free graphs,*
4. *the class of K_5-minor-free graphs,*
5. *for $k \geq 0$, the class of disjoint unions of k-apex planar graphs,*
6. *for $k \geq 0$, the class of disjoint unions of graphs of vertex cover number $\leq k$, and,*
7. *for $k \geq 0$, the class of disjoint unions of graphs of feedback vertex set number $\leq k$.*

[3] A graph class $\mathcal{F}$ is *union-closed* if $F_1, F_2 \in \mathcal{F}$ implies that their disjoint union $F_1 + F_2$ is in $\mathcal{F}$. It is not hard to see that every homomorphism distinguishing closed graph class is necessarily union-closed [55].

Item 2 has an immediate corollary via results of [56, 24, 22] and thus answers a question of Roberson and Seppelt [56].

▶ **Corollary 1.7.** *There exist graphs G and H such that the first-level Lasserre relaxation of the graph isomorphism integer program for G and H is feasible but G and H are distinguished by the 2-dimensional Weisfeiler–Leman algorithm.*

Indeed, it was shown in [56] that the first-level Lasserre SDP relaxation of the graph isomorphism integer program for graphs G and H is feasible if, and only if, G and H are homomorphism indistinguishable over all outerplanar graphs. Since the 2-dimensional Weisfeiler–Leman algorithms distinguishes G and H if, and only if, there is a graph F of treewidth ≤ 2 such that $\hom(F, G) \neq \hom(F, H)$, Corollary 1.7 follows from Item 2 of Theorem 1.6 as the complete bipartite graph $K_{2,3}$ has treewidth 2 and is not outerplanar.

Finally, we consider the class $\mathcal{E}_g$ of graphs of Euler genus $\leq g$ and the class $\mathcal{H}_k$ of graphs of vortex-free Hadwiger number $\leq k$ for $g, k \geq 0$. Although we are unable to show that $\mathcal{E}_g$ and $\mathcal{H}_k$ are homomorphism distinguishing closed, we prove that their homomorphism indistinguishability relations are distinct.

▶ **Theorem 1.8.** *For all $g \geq 0$, there exist graphs G, H such that $G \equiv_{\mathcal{E}_g} H$ and $G \not\equiv_{\mathcal{E}_{g+1}} H$.*

▶ **Theorem 1.9.** *For all $k \geq 0$, there exist graphs G, H such that $G \equiv_{\mathcal{H}_k} H$ and $G \not\equiv_{\mathcal{H}_{k+1}} H$.*

This yields two new provably infinite hierarchies of graph isomorphism relaxations. Observe that Theorem 1.2 is a direct corollary of Theorem 1.9, since G, H are in particular non-isomorphic.

1.3 Analysing CFI Graphs via Oddomorphisms

Having stated our main results, we proceed to describing the techniques underlying their proofs. The central challenge that has to be overcome is that, for most of the graph classes $\mathcal{F}$ considered in our results, no characterisations of the homomorphism indistinguishability relations $\equiv_{\mathcal{F}}$ are known. Since it is unreasonable to expect that the homomorphism indistinguishability relations of all minor-closed graph classes admit practical characterisations, we do not attempt to establish such characterisations towards Conjectures 1.1 and 1.4 but work directly with homomorphism counts.

Spelled out, proving that $\mathcal{F}$ is homomorphism distinguishing closed amounts to constructing, for every graph $G \notin \mathcal{F}$, two graphs G_0 and G_1 such that $G_0 \equiv_{\mathcal{F}} G_1$ and $\hom(G, G_1) \neq \hom(G, G_1)$. The main source for such highly similar graphs G_0 and G_1 is the CFI construction of Cai, Fürer, and Immerman [13], who devised it to show that the k-dimensional Weisfeiler–Leman algorithm does not distinguish all non-isomorphic graphs. Given a connected graph G, the CFI construction yields two non-isomorphic graphs, the *even CFI graph* G_0 and the *odd CFI graph* G_1. Originally, Cai, Fürer, and Immerman [13] showed that, if G is sufficiently connected, then G_0 and G_1 are not distinguished by the k-dimensional Weisfeiler–Leman algorithm. Subsequently, this argument was refined [19, 4, 5, 50] to the statement that, if the treewidth of G is greater than k, then G_0 and G_1 are not distinguished by the k-dimensional Weisfeiler–Leman algorithm, and equivalently [24, 22], they are homomorphism indistinguishable over all graphs of treewidth at most k. Furthermore, it was shown in [3, 8, 46, 55] that, if G is non-planar, then G_0 and G_1 are quantum isomorphic, i.e., homomorphism indistinguishable over all planar graphs.

These results already indicate that CFI graphs are a rather universal construction in light of Conjectures 1.1 and 1.4. Roberson [55] formalised this observation by giving a combinatorial criterion for the CFI graphs G_0 and G_1 of some base graph G to be homomorphism indistinguishable over a graph class $\mathcal{F}$.

More precisely, Roberson [55, Theorem 3.13] showed that a graph F and a connected graph G satisfy that $\hom(F, G_0) \neq \hom(F, G_1)$ if, and only if, there exists a *weak oddomorphism* $\varphi\colon F \to G$. A weak oddomorphism is a homomorphism satisfying certain parity constraints, see Definition 2.3 and Figure 2. Thus, in order to show that $\mathcal{F}$ is homomorphism distinguishing closed, it suffices to show that $\mathcal{F}$ is *closed under weak oddomorphisms*, i.e., if $\varphi\colon F \to G$ is a weak oddomorphism and $F \in \mathcal{F}$, then $G \in \mathcal{F}$; see Theorem 2.6. Graph classes known to be closed under weak oddomorphisms include the class of planar graphs [55], the class of graphs of maximum degree $\leq d$ [55], treewidth $\leq k$ [50], treedepth $\leq k$ [29], and pathwidth $\leq k$ [63].

These results were established mostly via a model-theoretic or algebraic analysis of CFI graphs and using characterisations of what it means for two graphs to be homomorphism indistinguishable over the respective graph classes. The technical contribution of this paper is a set of purely combinatorial tools for establishing that a graph class is closed under weak oddomorphisms.

▶ **Lemma 1.10.** *Let $\mathcal{F}$ be closed under weak oddomorphisms and taking subgraphs. Let $d \geq 0$. The following graph classes are closed under weak oddomorphisms:*

1. *the class of graphs of deletion distance $\leq d$ to $\mathcal{F}$,*
2. *the class of graphs of elimination distance $\leq d$ to $\mathcal{F}$,*
3. *the closure $\mathcal{F}^{\oplus 1}$ of $\mathcal{F}$ under 1-sums, and,*
4. *if $\mathcal{F}$ is minor-closed, the closure $\mathcal{F}^{\oplus 2}$ of $\mathcal{F}$ under 2-sums.*

Several of our main results follow as corollaries from this lemma. For example, Item 1 yields that the class of disjoint unions of k-apex planar graphs is homomorphism distinguishing closed, and Item 2 gives a self-contained combinatorial proof of the fact that the class of graphs of treedepth $\leq k$ is homomorphism distinguishing closed [29].

A crucial tool in graph minor theory are clique-sums, i.e., the operation of gluing together two graphs at cliques. A key technical result of this paper is Lemma 4.1, which allows us to deal with weak oddomorphisms in light of clique-sums. Intuitively, it asserts that, if $F_1 \oplus F_2 \to G$ is a weak oddomorphism from a clique-sum $F_1 \oplus F_2$ into a sufficiently connected graph G, then there exists a minor F' of F_1 or F_2 admitting a weak oddomorphism $F' \to G$. Items 3 and 4 of Lemma 1.10 follow from this observation. For higher-arity clique-sums, we give an approximate result in Corollary 4.2 strong enough to yield Theorems 1.2, 1.8, and 1.9.

By studying the combinatorial properties of oddomorphisms, we also contribute to the study of CFI graphs. This line of work has recently received much attention [41, 33] and enjoys connections to various other areas such as proof complexity [21], counting complexity [16], and algebraic complexity [20, 18, 27].

2 Preliminaries

2.1 Graphs

We use standard graph notation (see, e.g., [23]). All graphs in this article are finite, undirected, loopless, and without multiple edges. We write $V(G)$ and $E(G)$ for the vertex and edge set of a graph G, respectively. For $v \in V(G)$, we write $N_G(v)$ for the set of neighbours of v. Also, for $X \subseteq V(G)$ we write $G[X]$ for the subgraph of G induced by X, and $G - X := G[V(G) \setminus X]$. For $k \geq 1$, a graph G is *k-connected* if $|V(G)| \geq k+1$ and the graph $G - S$ is connected for every subset $S \subseteq V(G)$ such that $|S| < k$.

A *tree decomposition* of a graph G is a pair (T, β) of a tree T and a map $\beta \colon V(T) \to 2^{V(G)}$ such that

1. for every edge $uv \in E(G)$, there exists a $t \in V(T)$ such that $u, v \in \beta(t)$, and
2. for every $v \in V(G)$, the subgraph of T induced by the vertices $t \in V(T)$ such that $v \in \beta(t)$ is non-empty and connected.

The *width* of (T, β) is $\max_{t \in V(T)} |\beta(t)| - 1$. The *treewidth* $\mathrm{tw}(G)$ of G is the minimum width of a tree decomposition of G. For $tt' \in E(T)$, we define $\sigma(t, t') \coloneqq |\beta(t) \cap \beta(t')|$ to be the *adhesion* of tt'. The *adhesion* of (T, β) is $\max_{tt' \in E(T)} \sigma(t, t')$.

Let G_1 and G_2 be two graphs whose vertex sets intersect in the set S. Suppose that $G_1[S]$ and $G_2[S]$ are cliques. We write $G_1 \oplus_S G_2$ for any graph obtained from the union of G_1 and G_2 by possibly deleting edges between vertices from S. Such a graph is called a *clique-sum* or $|S|$-*sum* of G_1 and G_2. Note that a 0-sum is merely a disjoint union.

For a graph class $\mathcal{C}$, we write $\mathcal{C}^{\oplus s}$ for the closure of $\mathcal{C}$ under ($\leq s$)-sums. This can also be formalised via tree decompositions. For $X \subseteq V(G)$, we define the *torso of G on X* to be the graph $G[\![X]\!]$ with vertex set $V(G[\![X]\!]) \coloneqq X$ and edge set $E(G[\![X]\!])$ given by

$$\{vw \in E(G) \mid v, w \in X\} \cup \{xy \mid x, y \in N_G(C),\ C \text{ connected component of } G - X\}.$$

▶ **Fact 2.1.** *Let $\mathcal{C}$ be a class of graphs and $s \geq 0$. Then $\mathcal{C}^{\oplus s}$ is the class of graphs G such that there is a tree decomposition (T, β) of G such that*

1. *$\sigma(t, t') \leq s$ for every $tt' \in E(T)$, and*
2. *$G[\![\beta(t)]\!] \in \mathcal{C}$ for every $t \in V(T)$.*

▶ **Fact 2.2.** *Let $\mathcal{F}$ be a graph class and $k \geq 0$. If $\mathcal{F}$ is minor-closed, then so is $\mathcal{F}^{\oplus k}$.*

Proof. The proof is by structural induction on the graphs in $\mathcal{F}^{\oplus k}$. The base case holds by assumption. Now assume that M is a minor of $F_1 \oplus_S F_2$ for $F_1, F_2 \in \mathcal{F}^{\oplus k}$ of lesser complexity. It follows from [56, Lemma 4.14.2] that M can be written as the k-sum of two graphs M_1 and M_2 such that M_1 is a minor of F_1 and M_2 is a minor of F_2. By the induction hypothesis, $M_1, M_2 \in \mathcal{F}^{\oplus k}$ and hence $M \in \mathcal{F}^{\oplus k}$. ◀

2.2 Homomorphism Indistinguishability

A *homomorphism* from a graph F to a graph G is a map $\varphi \colon V(F) \to V(G)$ such that $\varphi(u)\varphi(v) \in E(G)$ for all $uv \in E(F)$. Write $\hom(F, G)$ for the number of homomorphisms $F \to G$. Two graphs G and H are *homomorphism indistinguishable* over a graph class $\mathcal{F}$, in symbols $G \equiv_{\mathcal{F}} H$, if $\hom(F, G) = \hom(F, H)$ for every $F \in \mathcal{F}$. The *homomorphism distinguishing closure* [55] of a graph class $\mathcal{F}$ is the graph class $\mathrm{cl}(\mathcal{F})$ containing all graphs F such that, for all graphs G and H, if $G \equiv_{\mathcal{F}} H$, then $\hom(F, G) = \hom(F, H)$. In other words, $\mathrm{cl}(\mathcal{F})$ is the maximal graph class such that $\equiv_{\mathcal{F}}$ and $\equiv_{\mathrm{cl}(\mathcal{F})}$ coincide. A graph class $\mathcal{F}$ is *homomorphism distinguishing closed* if $\mathcal{F} = \mathrm{cl}(\mathcal{F})$. We note that $\mathrm{cl}(\cdot)$ is a closure operator in the sense that

$$\mathrm{cl}(\mathcal{F}) = \mathrm{cl}(\mathrm{cl}(\mathcal{F})), \tag{1}$$

$$\mathrm{cl}(\mathcal{F}) \subseteq \mathrm{cl}(\mathcal{F}') \tag{2}$$

for all graph classes $\mathcal{F} \subseteq \mathcal{F}'$. We refer to [63] for further background.

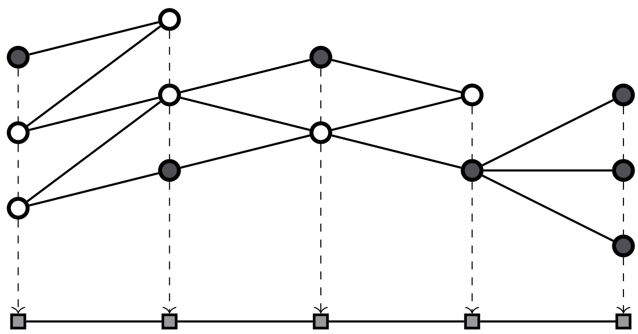


Figure 2 An oddomorphism (- - - - ->) to the 5-vertex path. Even vertices are depicted as ○ and odd vertices as ●.

2.3 Oddomorphisms

As already indicated above, a key notion underlying many of our results is that of an *oddomorphism* introduced by Roberson [55][4] to give a combinatorial criterion for the homomorphism indistinguishability of CFI graphs. We recall the essential definitions and properties; see Figure 2 for an example.

▶ **Definition 2.3** ([55, Definition 3.9]). *Let F and G be graphs and $\varphi\colon F \to G$ a homomorphism. A vertex $a \in V(F)$ is* odd/even with respect to φ *if $|N_F(a) \cap \varphi^{-1}(v)|$ is odd/even for every $v \in N_G(\varphi(a))$. The homomorphism φ is an* oddomorphism *if*
1. *every vertex of F is even or odd with respect to φ,*
2. *for every $v \in V(G)$, the set $\varphi^{-1}(v)$ contains an odd number of odd vertices.*

The homomorphism $\varphi\colon F \to G$ is a weak oddomorphism *if there is a subgraph F' of F such that $\varphi|_{F'}$ is an oddomorphism from F' to G.*

If a vertex $a \in V(F)$ is odd or even with respect to φ, it is referred to as *φ-odd* or *φ-even*, respectively. The sets $\varphi^{-1}(v) \subseteq V(F)$ for $v \in V(G)$ are called the *fibres* of φ.

▶ **Example 2.4.** For every graph F, the identity map $\mathrm{id}\colon F \to F$ is an oddomorphism.

▶ **Observation 2.5.** *Every weak oddomorphism $\varphi\colon F \to G$ is surjective on edges and vertices.*

Our motivation for studying (weak) oddomorphisms is the following Theorem 2.6. We say a graph class $\mathcal{F}$ is *closed under (weak) oddomorphisms* if, for every $F \in \mathcal{F}$ and (weak) oddomorphism $\varphi\colon F \to G$, it holds that $G \in \mathcal{F}$. Also, a graph class $\mathcal{F}$ is *componental* if, for all graphs F_1, F_2, it holds that $F_1 + F_2 \in \mathcal{F}$ if, and only if, $F_1, F_2 \in \mathcal{F}$.

▶ **Theorem 2.6** ([55, Theorem 6.2]). *Let $\mathcal{F}$ be a componental graph class. If $\mathcal{F}$ is closed under weak oddomorphisms, then $\mathcal{F}$ is homomorphism distinguishing closed.*

We recall the following lemma, which implies via Theorem 2.6 that the class of planar graphs, the class of graphs of treewidth $\leq k$, and the class of graphs F of maximum degree $\Delta(F) \leq d$ are homomorphism distinguishing closed. While the first two statements are involved and rely on characterisations [46, 24, 22] of homomorphism indistinguishability

[4] The paper [55] is currently only published as a preprint. However, its results are accepted in the community and, according to the author, it is in the process of being published in the Journal of Combinatorial Theory, Series B.

relations, the third statement is rather immediate from Definition 2.3. For our purposes, the second assertion is required only for $\mathrm{tw}(F) \leq 2$, for which we give a self-contained proof in Theorems 4.15 and 4.19.

▶ **Lemma 2.7** ([55, Lemmas 4.1 and 8.2], [50, Corollary 13]). *The following holds for every weak oddomorphism $\varphi\colon F \to G$.*
(a) *If F is planar, then G is planar.*
(b) $\mathrm{tw}(F) \geq \mathrm{tw}(G)$.
(c) $\Delta(F) \geq \Delta(G)$.

We recall the following properties of oddomorphisms.

▶ **Lemma 2.8** ([55, Lemmas 5.2, 5.6, 5.8, and Theorem 3.13]). *Let $\varphi\colon F \to G$ be a weak oddomorphism.*
(a) *For every subgraph G' of G, the induced map $\varphi|_{\varphi^{-1}(G')}\colon F[\varphi^{-1}(G')] \to G'$ is a weak oddomorphism. If φ is an oddomorphism, then so is $\varphi|_{\varphi^{-1}(G')}$.*
(b) *For every connected subgraph G' of G, there exists a connected subgraph F' of $F[\varphi^{-1}(G')]$ admitting an oddomorphism $F' \to G'$.*
(c) *For every minor G' of G, there exists a minor F' of F admitting an oddomorphism $F' \to G'$. If G' has less vertices than G, then F' has less vertices than F.*

In order to simplify some subsequent arguments, we make the following observation.

▶ **Observation 2.9.** *For every graph class $\mathcal{F}$ closed under taking subgraphs, the following are equivalent:*
1. *$\mathcal{F}$ is closed under weak oddomorphisms,*
2. *$\mathcal{F}$ is closed under oddomorphisms.*

Proof. For the converse direction, let $F \to G$ be a weak oddomorphism and $F \in \mathcal{F}$. Then there exists an oddomorphism $F' \to G$ from some subgraph F' of F by Definition 2.3. In particular, $F' \in \mathcal{F}$ and thus $G \in \mathcal{F}$. ◀

Finally, we shall use the following Lemma 2.10 to separate homomorphism indistinguishability relations.

▶ **Lemma 2.10** ([55, Theorem 3.13 and Lemma 3.14]). *Let $\mathcal{F}$ and $\mathcal{F}'$ be graph classes. If there is a connected graph $F' \in \mathcal{F}'$ such that no graph $F \in \mathcal{F}$ admits a weak oddomorphism to F', then there exist graphs G, H such that $G \equiv_{\mathcal{F}} H$, but $G \not\equiv_{\mathcal{F}'} H$.*

3 Deletion and Elimination Distance

In this section, we show that, if a graph class $\mathcal{F}$ is closed under weak oddomorphisms, then so are the classes of graphs of bounded deletion and elimination distance to $\mathcal{F}$. As a corollary, we show that the classes of k-apex planar graphs and of graphs of vertex cover or feedback vertex set number $\leq k$ are homomorphism distinguishing closed (Items 5–7 of Theorem 1.6). We also give a self-contained combinatorial argument for the fact that the classes of graphs of treedepth $\leq k$ are homomorphism distinguishing closed [29].

▶ **Definition 3.1** ([11]). *Let $\mathcal{F}$ be a graph class. The* deletion distance $\mathrm{dd}_{\mathcal{F}}(F)$ *to $\mathcal{F}$ of a graph F is defined as the least number k such that there exist vertices $v_1, \dots, v_k \in V(F)$ such that $F - \{v_1, \dots, v_k\} \in \mathcal{F}$.*

For example, the *vertex cover number* of a graph is equal to its deletion distance to the class of edge-less graphs. The *feedback vertex set number* of a graph is equal to its deletion distance to forests.

▶ **Lemma 3.2.** *Let $\mathcal{F}$ be closed under weak oddomorphisms and taking subgraphs. If $F \to G$ is a weak oddomorphism, then $\mathrm{dd}_{\mathcal{F}}(F) \geq \mathrm{dd}_{\mathcal{F}}(G)$.*

Proof. Let $S \subseteq V(F)$ be such that $F - S \in \mathcal{F}$. Let $\varphi \colon F \to G$ be a weak oddomorphism. Let $T := \varphi(S)$. Clearly, $|T| \leq |S|$. We show that $G - T \in \mathcal{F}$. By Lemma 2.8, there exists a weak oddomorphism $F' \to G - T$ for some subgraph F' of $F - S$. Since $\mathcal{F}$ is closed under taking subgraphs, it holds that $F' \in \mathcal{F}$ and hence $G - T \in \mathcal{F}$. ◀

Lemma 3.2 has the following corollary, which implies Items 5–7 of Theorem 1.6 via Lemma 2.7.

▶ **Corollary 3.3.** *Let $\mathcal{F}$ be closed under weak oddomorphisms and taking subgraphs. For every $k \geq 0$, the class of graphs whose connected components have deletion distance at most k to $\mathcal{F}$ is closed under weak oddomorphisms. In particular, it is homomorphism distinguishing closed.*

Proof. Let $F \to G$ be an oddomorphism and suppose that F is the disjoint union of graphs whose deletion distance to $\mathcal{F}$ is at most k. Towards concluding that G is the disjoint union of graphs whose deletion distance to $\mathcal{F}$ is at most k, we may suppose, by Lemma 2.8, that F and G are connected. By Lemma 3.2, $\mathrm{dd}_{\mathcal{F}}(G) \leq \mathrm{dd}_{\mathcal{F}}(F) \leq k$, as desired. The final assertion follows from Theorem 2.6. ◀

Next, we consider the more general notion of elimination distance, which is inspired by treedepth [52]. See [36] for computational aspects.

▶ **Definition 3.4** ([11]). *The* elimination distance *to a graph class $\mathcal{F}$ of a graph F is defined as*

$$\mathrm{ed}_{\mathcal{F}}(F) := \begin{cases} 0, & \text{if } F \in \mathcal{F}, \\ 1 + \min_{v \in V(F)} \mathrm{ed}_{\mathcal{F}}(F - v), & \text{if } F \text{ is connected}, \\ \max_i \mathrm{ed}_{\mathcal{F}}(F_i), & \text{if } F = \sum F_i \text{ is disconnected.} \end{cases}$$

The *treedepth* [52] of a graph is equal to its elimination distance to the graph class containing only the empty graph. We recall the following lemma.

▶ **Lemma 3.5** ([12, p. 154]). *Let $\mathcal{F}$ be closed under taking subgraphs. If F' is a subgraph of F, then $\mathrm{ed}_{\mathcal{F}}(F') \leq \mathrm{ed}_{\mathcal{F}}(F)$.*

The following Theorem 3.6 generalises a result from [29] asserting that the class of graphs of treedepth $\leq q$ is closed under weak oddomorphisms. Furthermore, in contrast to [29], our proof of Theorem 3.6 is purely combinatorial and does not rely on Ehrenfeucht–Fraïssé or graph searching games.

▶ **Theorem 3.6.** *Let $d \geq 0$ and $\mathcal{F}$ be closed under taking subgraphs. If $\mathcal{F}$ is closed under weak oddomorphisms, then so is the class of all graphs F with $\mathrm{ed}_{\mathcal{F}}(F) \leq d$.*

Proof. The proof is by induction on d. First consider the case $d = 0$. Let F be such that $\mathrm{ed}_{\mathcal{F}}(F) = 0$ and let $F \to G$ be a weak oddomorphism. Let G' be a connected component of G. By Lemma 2.8, there exists a connected subgraph F' of F admitting an oddomorphism $F' \to G'$. By assumption, $F' \in \mathcal{F}$ and hence $G' \in \mathcal{F}$. Since the elimination distance to $\mathcal{F}$ does not increase under taking disjoint unions, it follows that $G \in \mathcal{F}$.

For the inductive step, first suppose that F is connected and that $|V(F)| \geq 2$. Since surjective homomorphisms map connected graphs to connected graphs, it follows that G is connected. Without loss of generality, it may be supposed that $|V(G)| \geq 2$.

It holds that $\operatorname{ed}_{\mathcal{F}}(F) = 1 + \min_{a \in V(F)} \operatorname{ed}_{\mathcal{F}}(F - a)$. Let $a \in V(F)$ be such that $\operatorname{ed}_{\mathcal{F}}(F) = 1 + \operatorname{ed}_{\mathcal{F}}(F - a)$. Define $G' \coloneqq G - \varphi(a)$ and $F' \coloneqq F[\varphi^{-1}(G)] = F - \varphi^{-1}(\varphi(a))$. By Lemma 2.8, the map $\varphi' \colon F' \to G'$ induced by φ is a weak oddomorphism. Note that F' is a subgraph of $F - a$. By Lemma 3.5, $\operatorname{ed}_{\mathcal{F}}(F') \leq \operatorname{ed}_{\mathcal{F}}(F - a) = \operatorname{ed}_{\mathcal{F}}(F) - 1$. By the inductive hypothesis, $\operatorname{ed}_{\mathcal{F}}(G') \leq \operatorname{ed}_{\mathcal{F}}(F')$. Combining the above,

$$\operatorname{ed}_{\mathcal{F}}(G) \leq 1 + \operatorname{ed}_{\mathcal{F}}(G') \leq 1 + \operatorname{ed}_{\mathcal{F}}(F') \leq \operatorname{ed}_{\mathcal{F}}(F).$$

It remains to consider the case when F is disconnected. Write $G = G_1 + \dots + G_n$ as disjoint union of connected components. Note that it may be the case that G is connected and consists of a single connected component. By Lemma 2.8, for every $i \in [n]$, there exists a connected subgraph F_i of F admitting an oddomorphism $\varphi_i \colon F_i \to G_i$. By Lemma 3.5, $\operatorname{ed}_{\mathcal{F}}(F_i) \leq \operatorname{ed}_{\mathcal{F}}(F)$. Since F_i is connected, the previous case applies. It follows that

$$\operatorname{ed}_{\mathcal{F}}(G) = \max_{i \in [n]} \operatorname{ed}_{\mathcal{F}}(G_i) \leq \max_{i \in [n]} \operatorname{ed}_{\mathcal{F}}(F_i) \leq \operatorname{ed}_{\mathcal{F}}(F). \qquad ◀$$

4 Separators

This section is concerned with the interaction of oddomorphisms $F \to G$ with separators in the graph F. The main technical contribution is Lemma 4.1. As corollaries, we show Theorem 1.5 and Items 1, 3, and 4 of Theorem 1.6.

▶ **Lemma 4.1** (Separator lemma). *Let $\varphi \colon F \to G$ be an oddomorphism. For a set of vertices $S \subseteq V(F)$, write $C_1, \dots, C_n$ for the connected components of $F - S$ and $D_1, \dots, D_m$ for the connected components of $G - \varphi(S)$.*

If $n > m$, then there exists a minor F' of the graph obtained from $F[C \cup S]$ by adding a clique on S such that F' admits an oddomorphism to G. Here, $C \coloneqq \bigcup_{i \in I} C_i$ for some proper subset $I \subsetneq [n]$.

Lemma 4.1 will be applied in Sections 4.2 and 5.2 for showing that a minor-closed graph class $\mathcal{F}$ formed by taking clique-sums is closed under weak oddomorphisms. Its prototypical application is the following Corollary 4.2.

▶ **Corollary 4.2.** *Let $s \geq 0$ and let $\mathcal{F}$ be a minor-closed graph class. Let G be an $(s+1)$-connected graph. If there exists a graph $F \in \mathcal{F}^{\oplus s}$ admitting an oddomorphism to G, then there exists a graph $F' \in \mathcal{F}$ admitting an oddomorphism to G.*

Proof (assuming Lemma 4.1). Let $F \in \mathcal{F}^{\oplus s}$ have the minimal number of vertices among graphs in $\mathcal{F}^{\oplus s}$ admitting an oddomorphism to G. Write $\varphi \colon F \to G$ for the oddomorphism.

If $F \notin \mathcal{F}$, then there exists a set $S \subseteq V(F)$ such that $|S| \leq s$, the graph $F - S$ is disconnected, and $F = F_1 \oplus_S F_2$ for some $F_1, F_2 \in \mathcal{F}^{\oplus s}$. Since G is $(s+1)$-connected and $|\varphi(S)| \leq |S| \leq s$, the graph $G - \varphi(S)$ is connected. By Fact 2.2 and Lemma 4.1, there exists a graph $F' \in \mathcal{F}^{\oplus s}$ on less vertices than F admitting an oddomorphism to G, contradicting the minimality of F. Hence, $F \in \mathcal{F}$. ◀

As a warm-up for Lemma 4.1, we prove the following lemma for oddomorphism and cut vertices. It will be used in the proof of Lemma 4.12.

▶ **Lemma 4.3** (Cut vertex lemma). *Let $\varphi\colon F \to G$ be an oddomorphism. Let $a \in V(F)$ and write $C_1, \dots, C_n$ for the connected components of $F - a$. If the graph $G - \varphi(a)$ is connected, then there exists an oddomorphism $\varphi_i\colon F[C_i \cup \{a\}] \to G$ for some $i \in [n]$ such that*
1. *$\varphi_i(b) = \varphi(b)$ for all $b \in C_i \cup \{a\}$,*
2. *for all $b \in C_i$, the vertex b is φ-odd if, and only if, it is φ_i-odd,*
3. *if a is φ_i-odd, then there exists a φ-odd vertex $a^* \in V(F) \setminus C_i$ such that $\varphi(a) = \varphi(a^*)$.*

Proof. As the fist step, we select the index $i \in [n]$.

▷ Claim 4.4. There exists an index $i \in [n]$ such that, for every $x \in V(G) \setminus \{\varphi(a)\}$, the set $\varphi^{-1}(x) \cap C_i$ contains an odd number of φ-odd vertices.

Proof. This essentially follows from arguments in the proofs of [55, Lemmas 3.12 and 5.2]. Concretely, let $x \in V(G) \setminus \{\varphi(a)\}$ be arbitrary. Since $\varphi^{-1}(x) = \biguplus_{i=1}^{n} C_i \cap \varphi^{-1}(x)$ contains an odd number of φ-odd vertices, there exists an index $i \in [n]$ such that $C_i \cap \varphi^{-1}(x)$ contains an odd number of φ-odd vertices. Let $y \in V(G) \setminus \{\varphi(a)\}$ be a neighbour of x. Since $C_i \cap \varphi^{-1}(x)$ contains an odd number of φ-odd vertices, the number of edges between $C_i \cap \varphi^{-1}(x)$ and $C_i \cap \varphi^{-1}(y)$ is odd. Hence, $C_i \cap \varphi^{-1}(y)$ also contains an odd number of φ-odd vertices. Since $G - \varphi(a)$ is connected, the claim follows by iterating this argument. ◁

Define φ_i as the restriction of φ to $F[C_i \cup \{a\}]$. This map is a homomorphism and satisfies the first assertion. For the second assertion, observe that, for every $b \in C_i$, it holds that $N_{F[C_i \cup S]}(b) = N_F(b)$. Hence, for all $b \in C_i$, the vertex b is φ-odd if, and only if, it is φ_i-odd.

In order to show that φ_i is an oddomorphism, it remains to consider the parity of the vertex a. Let $x \in N_G(\varphi(a))$. Since $\varphi^{-1}(x) \cap C_i$ contains an odd number of φ-odd vertices, the number of edges in F between $\varphi^{-1}(x) \cap C_i$ and $\varphi^{-1}(\varphi(a)) \cap (C_i \cup \{a\})$ is odd. The parity of a is equal to the number of edges between $\varphi^{-1}(x) \cap C_i$ and $\varphi^{-1}(\varphi(a)) \cap \{a\}$, whose parity depends only on the number of φ-odd vertices in $\varphi^{-1}(\varphi(a)) \cap C_i$ and not on x. Hence, a is either φ_i-even or φ_i-odd, and φ_i is an oddomorphism.

For the last assertion, suppose that a is φ_i-odd and φ-even. For $x \in N_G(\varphi(a))$, observe the following:
1. The number α of edges in F between $\varphi^{-1}(x)$ and $\varphi^{-1}(\varphi(a))$ is odd, since φ is an oddomorphism.
2. The number β of edges in F between $\varphi^{-1}(x) \cap C_i$ and $\varphi^{-1}(\varphi(a)) \cap (C_i \cup \{a\})$ is odd, since $\varphi^{-1}(x) \cap C_i$ contains an odd number of φ-odd vertices and $N_F(C_i) \subseteq C_i \cup \{a\}$.
3. The vertex a has an even number γ of neighbours in $\varphi^{-1}(x)$ and an odd number δ of neighbours in $\varphi^{-1}(x) \cap C_i$. Hence, it has an odd number $\gamma - \delta$ of neighbours in $\varphi^{-1}(x) \setminus C_i$.

The number of edges from $\varphi^{-1}(x) \setminus C_i$ to $\varphi^{-1}(\varphi(a)) \setminus (C_i \cup \{a\})$ is $\alpha - \beta - (\gamma - \delta) \equiv 1 \mod 2$. Hence, there is an odd number of φ-odd vertices in $\varphi^{-1}(\varphi(a)) \setminus (C_i \cup \{a\})$, as desired. ◀

Analogous to Corollary 4.2, we conclude the following Corollary 4.5 from Lemma 4.3.

▶ **Corollary 4.5.** *Let $\mathcal{F}$ be closed under taking subgraphs. Let G be a 2-connected graph. If there exists a graph $F \in \mathcal{F}^{\oplus 1}$ admitting an oddomorphism to G, then there exists a graph $F' \in \mathcal{F}$ admitting an oddomorphism to G.*

Proof. Let $F \in \mathcal{F}^{\oplus 1}$ have the minimal number of vertices of any graph in $\mathcal{F}^{\oplus 1}$ admitting an oddomorphism to G. Write $\varphi\colon F \to G$ for the oddomorphism.

If $F \notin \mathcal{F}$, then there exists a vertex $a \in V(F)$ such that $F - a$ is disconnected. Since G is 2-connected, the graph $G - \varphi(a)$ is connected. By Lemma 4.3, there exists a graph $F' \in \mathcal{F}^{\oplus 1}$ on less vertices than F admitting an oddomorphism to G, contradicting the minimality of F. Hence, $F \in \mathcal{F}$. ◀

This concludes the preparations for the proof of Lemma 4.1.

Proof of Lemma 4.1. In order to construct the minor F', we select a proper subset of the connected components $C_1, \dots, C_n$ of $F - S$ (analogous to Claim 4.4). Crucially, this selection must be such that every fibre $\varphi^{-1}(x)$ for $x \in V(G) \setminus \varphi(S)$ contains an odd number of φ-odd vertices from this set. To that end, write $\mathbb{F}_2$ for the 2-element field. Define a matrix $P \in \mathbb{F}_2^{m \times n}$ whose entry P_{ji} is equal to the parity of the number of φ-odd vertex in $\varphi^{-1}(x) \cap C_i$ for any $x \in D_j$.

▷ Claim 4.6. The matrix P is well-defined.

Proof. Fix $i \in [n]$ and $j \in [m]$. Let $x, y \in D_j$ be adjacent in G. The parity of the number of edges with endpoints in $\varphi^{-1}(x) \cap C_i$ and $\varphi^{-1}(y) \cap C_i$ is equal to the number of φ-odd vertices in $\varphi^{-1}(x) \cap C_i$ and equal to the number of φ-odd vertices in $\varphi^{-1}(y) \cap C_i$. Since $G[D_j]$ is connected, this quantity does not depend on the choice $x \in D_j$ but merely on j. ◁

▷ Claim 4.7. There exists a proper subset $I \subsetneq [n]$ satisfying that $\sum_{i \in I} P_{ji} \equiv 1 \mod 2$ for all $j \in [m]$.

Proof. Throughout this proof, arithmetic is over $\mathbb{F}_2$. The matrix P has the property that $P\mathbf{1} = \mathbf{1}$, i.e. $\sum_{i \in [n]} P_{ji} = 1$ for all $j \in [m]$. This is because every φ-fibre $\varphi^{-1}(x)$ for $x \in V(G) \setminus \varphi(S)$ contains an odd number of φ-odd vertices and the $C_1 \uplus \dots \uplus C_n$ partition the fibre.

Hence, $Pe_1 + \dots + Pe_n = \mathbf{1}$ where $e_i \in \mathbb{F}_2^n$ denotes the i-th standard basis vector. Since $n > m$, the vectors $Pe_1, \dots, Pe_n \in \mathbb{F}_2^m$ are linearly dependent. Thus, there exists a non-empty set $I' \subseteq [n]$ such that $\sum_{i \in I'} Pe_i = 0$. Hence,

$$\mathbf{1} = Pe_1 + \dots + Pe_n = \sum_{i \in I'} Pe_i + \sum_{i \in [n] \setminus I'} Pe_i = \sum_{i \in [n] \setminus I'} Pe_i.$$

Thus, the set $I \coloneqq [n] \setminus I'$ is as desired. ◁

Let $I \subsetneq [n]$ denote the set from Claim 4.7. Define $C \coloneqq \bigcup_{i \in I} C_i$. By Claim 4.7, $C \cap \varphi^{-1}(x)$ contains an odd number of φ-odd vertices for every $x \in V(G) \setminus \varphi(S)$.

Having picked C, we now define the minor F' of the graph obtained from $F[C \cup S]$ by adding a clique on S. To that end, let $\mathfrak{S} \coloneqq \left\{ S \cap \varphi^{-1}(y) \mid y \in \varphi(S) \right\}$ denote the partition of S induced by φ. The vertex set of F' is $C \uplus \mathfrak{S}$. Its edges are specified as follows:

- all edges of $F[C]$ are edges of F',
- $vT \in E(F')$ for every $v \in C$ and $T \in \mathfrak{S}$ such that the number $|E_F(v, T)|$ of edges in F from v to T is odd,
- two distinct vertices $T, T' \in \mathfrak{S}$ are adjacent if $|E_F(T, T')|$ is odd.

Let $\varphi' \colon F' \to G$ denote the homomorphism induced by φ. By the definition of $\mathfrak{S}$, the map $\varphi' \colon V(F') \to V(G)$ is well-defined. Inspecting the three cases of the definition of the edges of F', shows that φ' is a homomorphism. Clearly, F' is a minor of the graph obtained from $F[C \cup S]$ by adding a clique on S. It remains to verify that φ' is an oddomorphism.

▷ Claim 4.8. If $v \in C$ is φ-even, respectively φ-odd, then it is φ'-even, respectively φ'-odd.

Proof. Since C is a union of connected components of $F - S$, all neighbours of $v \in C$ in F lie in $C \uplus S$. Let x be a neighbour of $\varphi(v) = \varphi'(v)$. Hence,

$$\begin{aligned} |N_F(v) \cap \varphi^{-1}(x)| &= |N_F(v) \cap \varphi^{-1}(x) \cap C| + |N_F(v) \cap \varphi^{-1}(x) \cap S| \\ &= |N_{F'}(v) \cap \varphi^{-1}(x) \cap C| + |E_F(v, \varphi^{-1}(x) \cap S)| \\ &\equiv |N_{F'}(v) \cap \varphi^{-1}(x)| \mod 2. \end{aligned}$$

For the last equality, observe that, if $x \notin \varphi(S)$, then $\varphi^{-1}(x) \cap S = \emptyset$. If $x \in \varphi(S)$, let $T \coloneqq \varphi^{-1}(x) \cap S$. By definition, $vT \in E(F')$ if, and only if, $|E_F(v, T)|$ is odd. ◁

▷ Claim 4.9. Let $T \in \mathfrak{S}$. The φ'-parity of T is $1+|\{a \in C \cap \varphi'^{-1}(\varphi'(T)) \mid a \text{ is } \varphi\text{-odd}\}| \bmod 2$.

Proof. Let x be adjacent to $\varphi'(T)$. First suppose that $x \notin \varphi(S)$. The following observations are all via double counting. The number of edges between $\varphi'^{-1}(x)$ and $\varphi'^{-1}(\varphi'(T))$ is odd by Claim 4.10 and the fact that φ is an oddomorphism. Furthermore, these edges can be partitioned into those between $\varphi'^{-1}(x)$ and T and those between $\varphi'^{-1}(x)$ and $\varphi'^{-1}(\varphi'(T)) \cap C$. The latter set of edges has the same parity as the number of φ-odd vertices in $\varphi'^{-1}(\varphi'(T)) \cap C$. In symbols,

$$\begin{aligned}
&|N_{F'}(T) \cap \varphi'^{-1}(x)| \\
&\equiv \sum_{a \in \varphi'^{-1}(x)} |E_F(a, T)| \\
&= |E_F(\varphi'^{-1}(x) \cap C, T)| \\
&= |E_F(\varphi'^{-1}(x) \cap C, \varphi'^{-1}(\varphi'(T)))| - |E_F(\varphi'^{-1}(x) \cap C, \varphi'^{-1}(\varphi'(T)) \cap C)| \\
&\equiv 1 + |\{a \in C \cap \varphi'^{-1}(\varphi'(T)) \mid a \text{ is } \varphi\text{-odd}\}| \mod 2.
\end{aligned}$$

Now suppose that $x \in \varphi(S)$. Write $T' \coloneqq \varphi^{-1}(x) \cap S$. Once again, the edges in F' between $\varphi'^{-1}(x)$ and $\varphi'^{-1}(\varphi(T))$ are double counted.

$$\begin{aligned}
|N_{F'}(T) \cap \varphi'^{-1}(x)| &\equiv |E_F(T, T')| + \sum_{a \in \varphi^{-1}(x) \cap C} |E_F(a, T)| \\
&\equiv 1 + |\{a \in C \cap \varphi'^{-1}(\varphi'(T)) \mid a \text{ is } \varphi\text{-odd}\}| \mod 2.
\end{aligned}$$ ◁

▷ Claim 4.10. For every $v \in V(G) \setminus \varphi(S)$, the fibre $\varphi'^{-1}(v)$ contains an odd number of φ'-odd vertices.

Proof. By Claim 4.8, the φ- and φ'-parities of the vertices in question are the same. The claim follows from Claim 4.7. ◁

By Claims 4.8 and 4.9, every vertex of F' is odd or even with respect to φ'. By Claim 4.10, it remains to verify the number of odd vertices in the fibres of the vertices in $\varphi(S)$. Let $x \in \varphi(S)$ and $T \coloneqq \varphi^{-1}(x) \cap S$. By Claims 4.8 and 4.9, the number of φ'-odd vertices in $\varphi^{-1}(x)$ is equal to the sum of $|\{a \in C \cap \varphi'^{-1}(\varphi'(T)) \mid a \text{ is } \varphi\text{-odd}\}|$ and the parity of T, i.e.,

$$1 + 2|\{a \in C \cap \varphi'^{-1}(\varphi'(T)) \mid a \text{ is } \varphi\text{-odd}\}| \equiv 1 \mod 2.$$

Hence, φ' is an oddomorphism. ◀

4.1 Classes of Forests Closed under Topological Minors

Applying Lemma 4.3, we prove Theorem 1.5 and thus resolve Conjecture 1.4 for all classes of forests. Let F and G be graphs. We say that $\rho\colon V(G) \to V(F)$ is a *topological model* of G in F if there is a collection of internally vertex-disjoint paths $(P_{vw})_{vw \in E(G)}$ in F from $\rho(v)$ to $\rho(w)$. If there exists a topological model of G in F, then G is said to be a *topological minor* of F.

▶ **Theorem 4.11** (Theorem 1.5). *Every class $\mathcal{F}$ of forests closed under topological minors is closed under weak oddomorphisms. If $\mathcal{F}$ is closed under disjoint unions, then $\mathcal{F}$ is homomorphism distinguishing closed.*

The theorem follows from Lemma 2.8, Theorem 2.6, and the following Lemma 4.12.

▶ **Lemma 4.12.** *Let $\varphi\colon F \to G$ be an oddomorphism. If F is a tree, then G is a topological minor of F.*

Proof. By Lemma 2.7, the graph G is a tree. We prove by induction on $|V(F)|$ that there is a topological model $\rho\colon V(G) \to V(F)$ of G in F such that

1. $\varphi(\rho(v)) = v$ for all $v \in V(G)$, and
2. $\rho(v)$ is φ-odd for all $v \in V(G)$.

In the base case $|V(F)| = 1$, it holds that $F \cong K_1 \cong G$ since φ is surjective. So suppose that $|V(G)| \geq 2$ and let $v \in V(G)$ be a leaf of G. Let $X \coloneqq \varphi^{-1}(v)$ be the fibre of v.

We distinguish two cases. First suppose that every $a \in X$ is a leaf of F, i.e., $\deg_F(a) = 1$ for every $a \in X$. Then $F - X$ is a tree. By Lemma 2.8, the map $\varphi|_{F-X}$ is an oddomorphism from $F-X$ to $G-v$. By the inductive hypothesis, there is a topological model $\rho\colon V(G)\setminus\{v\} \to V(F) \setminus X$ of $G - v$ in $F - X$ satisfying Items 1 and 2. Let w be the unique neighbour of v in G, and let $b \coloneqq \rho(w)$. Note that b is φ-odd, so it has a neighbour $a \in X$. Since X only contains vertices of degree 1, we conclude that a is φ-odd. We set $\rho(v) \coloneqq a$ and obtain the desired topological model of G in F.

Next, suppose that there is some $a \in X$ such that $\deg_F(a) \geq 2$. In particular, $F - a$ is disconnected, since F is a tree. Let $C_1, \dots, C_n$ denote the vertex sets of the connected components of $F - a$. By Lemma 4.3, there some $i \in [n]$ and an oddomorphism $\varphi_i\colon F[C_i \cup \{a\}] \to G$ such that

1. $\varphi_i(b) = \varphi(b)$ for all $b \in C_i \cup \{a\}$,
2. b is φ_i-odd if and only if b is φ-odd for all $b \in C_i$, and
3. if a is φ_i-odd, but a is φ-even, then there is some $i \neq j \in [s]$ such that $C_j \cap X$ contains a φ-odd vertex a^*.

We apply the inductive hypothesis to φ_i and obtain a topological model $\rho_i\colon V(G) \to C_i \cup \{a\}$. We define the map ρ to be identical to ρ_i except when a is φ_i-odd and a is φ-even. In this case, let $\rho(v) \coloneqq a^*$. Since $a^* \in C_j \neq C_i$, there is a path from the a^* to a whose internal vertices lie in C_j. This path is internally vertex-disjoint from the other paths in ρ_i. ◀

4.2 Clique-Sums

This section is concerned with proving, under certain assumptions on $\mathcal{F}$, that if $\mathcal{F}$ is closed under weak oddomorphisms, then so is the class $\mathcal{F}^{\oplus s}$ of graphs that are s-sums of graphs in $\mathcal{F}$. For $s \leq 2$, we show that the minimal forbidden subgraphs or minors of $\mathcal{F}^{\oplus s}$ must be $(s+1)$-connected. Then, Corollary 4.2 implies that any oddomorphism from $\mathcal{F}^{\oplus s}$ to a graph $G \notin \mathcal{F}^{\oplus s}$ yields an oddomorphism from $\mathcal{F}$ to G.

4.2.1 1-Sums

In the case of 1-sums, it suffices to assume that $\mathcal{F}$ is a closed under taking subgraphs. A *minimal excluded subgraph* of a graph class $\mathcal{F}$ is a graph $G \notin \mathcal{F}$ such that every proper subgraph G' of G is in $\mathcal{F}$.

▶ **Lemma 4.13.** *If a graph class $\mathcal{F}$ is closed under taking subgraphs, then every minimal excluded subgraph of $\mathcal{F}^{\oplus 1}$ is 2-connected.*

Proof. Let G be a minimal forbidden subgraph of $\mathcal{F}^{\oplus 1}$. If G is disconnected, then its connected components are in $\mathcal{F}^{\oplus 1}$. Since $\mathcal{F}^{\oplus 1}$ is closed under disjoint unions, it follows that $G \in \mathcal{F}^{\oplus 1}$, a contradiction.

If G is connected and not 2-connected, then there exists a cut vertex $x \in V(G)$. In particular, G is a 1-sum as $G = G_1 \oplus_{\{x\}} \cdots \oplus_{\{x\}} G_m$ for some subgraphs $G_1, \dots, G_m$ of G. By assumption, each of the $G_1, \dots, G_m$ are in $\mathcal{F}^{\oplus 1}$. Since $\mathcal{F}^{\oplus 1}$ is closed under 1-sums, it follows that $G \in \mathcal{F}^{\oplus 1}$, a contradiction. ◀

▶ **Lemma 4.14.** *Let $\mathcal{F}$ be closed under taking subgraphs. If $\mathcal{F}$ is closed under weak oddomorphisms, then $\mathcal{F}^{\oplus 1}$ is closed under weak oddomorphisms.*

Proof. Let $F \to G$ be a weak oddomorphism for $F \in \mathcal{F}^{\oplus 1}$. If $G \notin \mathcal{F}^{\oplus 1}$, then G contains a 2-connected subgraph $G' \notin \mathcal{F}^{\oplus 1}$ by Lemma 4.13. By Lemma 2.8, there exists a subgraph $F' \in \mathcal{F}^{\oplus 1}$ of F and an oddomorphism $F' \to G'$. By Corollary 4.5, there exists a graph $F'' \in \mathcal{F}$ admitting an oddomorphism to G'. By assumption, $G' \in \mathcal{F}$, a contradiction. ◀

Lemma 4.14 readily gives an alternative proof of [55, Corollary 6.8] asserting that the class of all forests is homomorphism distinguishing closed, see Lemma 2.7. Indeed, forests are precisely the 0- and 1-sums of copies of K_2. By Observation 2.5, the subgraph-closure $\{K_2, K_1+K_1, K_1\}$ of $\{K_2\}$ is closed under weak oddomorphisms. Hence, Lemma 4.14 applies. This idea is generalised by the following theorem which subsumes the case $k = 1$ of forests.

▶ **Theorem 4.15.** *For $k \geq 1$, the class of all graphs admitting a tree decomposition of width $\leq k$ and adhesion ≤ 1 is homomorphism distinguishing closed.*

Proof. Let $\mathcal{F}$ be the componental graph class generated by all graphs on at most $k+1$ vertices. This graph class is closed under weak oddomorphisms. Indeed, if $\varphi\colon F \to G$ is a weak oddomorphism and $F \in \mathcal{F}$. By Lemma 2.8, G and F can be assumed to be connected. Hence, $k \geq |V(F)| \geq |V(G)|$ since φ is surjective. The graph class in question is $\mathcal{F}^{\oplus 1}$, which is closed under weak oddomorphisms by Lemma 4.14. Finally, apply Theorem 2.6 to deduce that $\mathcal{F}^{\oplus 1}$ is homomorphism distinguishing closed. ◀

As a second example, we obtain the following Theorem 4.16 by applying Lemmas 2.7 and 4.14 and Theorem 2.6. Here, a *cactus graph* is a graph in which any two simple cycles have at most one vertex in common. Cactus graphs are outerplanar and characterised by the forbidden minor $K_4 - e$.

▶ **Theorem 4.16.** *For $d \geq 1$, the class $\{F \mid \Delta(F) \leq d\}^{\oplus 1}$ of 1-sums of graphs of maximum degree $\leq d$ is closed under weak oddomorphisms and homomorphism distinguishing closed.*

In particular, the class of cactus graphs ($d = 2$) is closed under weak oddomorphisms and homomorphism distinguishing closed.

4.2.2 2-Sums

The arguments from the previous section can be lifted to 2-sums, here, however under the assumption that $\mathcal{F}$ is minor-closed. As we shall see in Lemma 6.1, this restriction is necessary.

First, we strengthen Lemma 4.13. A *minimal excluded minor* of a graph class $\mathcal{F}$ is a graph $G \notin \mathcal{F}$ such that every proper minor G' of G is in $\mathcal{F}$. Note that it does not hold in general that, if $\mathcal{F}$ is minor-closed, then the minimal excluded minors of $\mathcal{F}^{\oplus s}$ are $(s+1)$-connected. For example, the class of forests is closed under clique-sums of any arity. However, its minimal excluded minor K_3 is not 3-connected.

▶ **Lemma 4.17.** *If a graph class $\mathcal{F}$ is minor-closed, then every minimal excluded minor of $\mathcal{F}^{\oplus 2}$ is 3-connected.*

Proof. Let G be a minimal excluded minor of $\mathcal{F}^{\oplus 2}$. By Lemma 4.13, G is 2-connected. If it is not 3-connected, then there exist two vertices $x_1 \neq x_2$ of G such that $G - \{x_1, x_2\}$ is disconnected. Write $D_1, \dots, D_m$ for the connected components of $G - \{x_1, x_2\}$ where $m \geq 2$. Since $G - x_1$ and $G - x_2$ are connected, it holds that $x_1, x_2 \in N_G(D_j)$ for all $j \in [m]$. For all $j \in [m]$, write G_j for the minor of G obtained by contracting all D_ℓ for $\ell \neq j$ to the edge x_1x_2. Then G is the 2-sum of $G_1, \dots, G_m$ at the clique $\{x_1, x_2\}$. It follows that $G_1, \dots, G_m \in \mathcal{F}^{\oplus 2}$ and hence $G \in \mathcal{F}^{\oplus 2}$, a contradiction. ◀

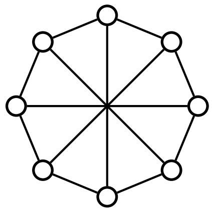


Figure 3 The Wagner graph V_8.

▶ **Lemma 4.18.** *Let $\mathcal{F}$ be minor-closed. If $\mathcal{F}$ is closed under weak oddomorphisms, then $\mathcal{F}^{\oplus 2}$ is closed under weak oddomorphisms.*

Proof. Let $F \to G$ be a weak oddomorphism for $F \in \mathcal{F}^{\oplus 2}$. If $G \notin \mathcal{F}^{\oplus 2}$, then there exists a 3-connected minor $G' \notin \mathcal{F}^{\oplus 2}$ of G. By Lemma 2.8, there exists an oddomorphism $F' \to G'$ from some minor $F' \in \mathcal{F}^{\oplus 2}$ of F. By Corollary 4.2, there exists a graph $F'' \in \mathcal{F}$ admitting an oddomorphism to G'. Hence, $G' \in \mathcal{F}$ contradicting that $G' \notin \mathcal{F}^{\oplus 2}$. ◀

As in the case of 1-sums, it follows readily that the class of graphs of treewidth ≤ 2 is closed under weak oddomorphisms as first shown in [55, Corollary 6.9], see Lemma 2.7. More generally, Theorem 4.19 follows via Lemma 4.18 as in Theorem 4.15.

▶ **Theorem 4.19.** *For $k \geq 1$, the class of all graphs admitting a tree decomposition of width $\leq k$ and adhesion ≤ 2 is homomorphism distinguishing closed.*

Employing a result of Wagner [73], we furthermore prove the following, that is, Item 3 of Theorem 1.6.

▶ **Theorem 4.20.** *The class of $K_{3,3}$-minor-free graphs is closed under weak oddomorphisms and homomorphism distinguishing closed.*

Proof. Wagner [73] proved that the $K_{3,3}$-minor-free graphs are precisely those that are 2-sums of planar graphs and K_5, see [40, Table 1]. Let $\mathcal{F}$ denote the union of the class of planar graphs with $\{K_5\}$. In order to apply Lemma 4.18, it must be argued that $\mathcal{F}$ is closed under weak oddomorphisms and taking minors. Let $F \in \mathcal{F}$ and $\varphi\colon F \to G$ be a weak oddomorphism. If F is planar, then so is G by Lemma 2.7. If F is non-planar, then $F \cong K_5$. Since φ is a surjective homomorphism, $G \cong K_5 \in \mathcal{F}$. Hence, the claim follows from Lemma 4.18 and Theorem 2.6. ◀

4.2.3 3-Sums

We finally give an application of Lemma 4.1 for 3-sums by showing that the class of K_5-minor-free graphs is homomorphism distinguishing closed. Wagner [72] showed that these graphs are precisely the 3-sums of planar graphs $\mathcal{P}$ and the Wagner graph V_8, see Figure 3, [40]. All graphs subject to Lemma 4.21 have at most 8 vertices, so the next lemma can be verified by exhaustive search.

▶ **Lemma 4.21.** *No minor of V_8 admits an oddomorphism to K_5.*

Proof. Let F be a minor of V_8 and $\varphi\colon F \to K_5$ be an oddomorphism. Each φ-odd vertex has degree at least 4. Hence, F must contain at least 5 vertices of degree at least 4. This contradicts the fact that V_8 is a 3-regular 8-vertex graph. ◀

▶ **Theorem 4.22.** *The class of K_5-minor-free graphs is closed under weak oddomorphisms and homomorphism distinguishing closed.*

Proof. By [72], the class of K_5-minor-free graphs is $(\mathcal{P} \cup \mathcal{V})^{\oplus 3}$ where $\mathcal{V}$ denotes the minor closure of V_8. Let $F \in (\mathcal{P} \cup \mathcal{V})^{\oplus 3}$ and $F \to G$ be a weak oddomorphism. If K_5 is a minor of G, then there exists, by Lemma 2.8, a graph $F' \in (\mathcal{P} \cup \mathcal{V})^{\oplus 3}$ admitting an oddomorphism to K_5. By definition, the graph class $\mathcal{P} \cup \mathcal{V}$ is minor-closed. Since K_5 is 4-connected, there exists, by Corollary 4.2, a graph $F'' \in \mathcal{P} \cup \mathcal{V}$ admitting an oddomorphism to K_5. By Lemma 2.7, the graph F'' is non-planar and thus a minor of V_8, contradicting Lemma 4.21. ◀

4.3 Outerplanar Graphs

As another application of Lemma 4.1, we show, for every $h \geq 3$, that the class of graphs of treewidth at most two that do not have the complete bipartite graph $K_{2,h}$ as a minor is homomorphism distinguishing closed. The best known example of such a graph class is the class of outerplanar graphs, which is characterised [68] by the forbidden minors K_4 and $K_{2,3}$.

▶ **Lemma 4.23.** *Let $h \geq 3$. If F is a graph of treewidth at most two admitting an oddomorphism $\varphi\colon F \to K_{2,h}$, then $K_{2,h}$ is a minor of F.*

Lemma 4.23 yields the following corollary via Theorems 2.6 and 4.19 and Lemma 2.8, i.e. Item 2 of Theorem 1.6. The proofs are deferred to the full version [51] or the second author's dissertation [63, Section 6.3.4].

▶ **Theorem 4.24.** *Let $h \geq 3$. The class of $\{K_4, K_{2,h}\}$-minor-free graphs is closed under weak oddomorphisms and homomorphism distinguishing closed. In particular, the class of outerplanar graphs is closed under weak oddomorphisms and homomorphism distinguishing closed.*

5 Excluding Minors

Next, we use the tools developed in previous sections to prove Theorems 1.2, 1.8, and 1.9. Towards this end, we first consider graph classes of bounded genus, and subsequently graph classes of bounded vortex-free Hadwiger number.

5.1 Bounded Genus

We recall the following definitions from [57]. A *surface* Σ is a non-null compact 2-manifold with possibly null boundary which is possibly disconnected. For a surface Σ, write $\widehat{\Sigma}$ for the surface obtained from Σ by pasting a closed disc onto each component of the boundary. If Σ is a connected surface and Σ is orientable, then its *genus* is defined as the number of handles one must add to 2-sphere to obtain $\widehat{\Sigma}$. If Σ is non-orientable, then its *genus* is the number of crosscaps one must add to a 2-sphere to obtain $\widehat{\Sigma}$. The *genus* of a disconnected surface is the sum of the genera of its components. Finally, the *genus* of a graph G is the minimal g such that G can be drawn on a surface Σ of genus g without edge crossings. Write $\mathcal{E}_g$ for the class of graphs of genus $\leq g$.

Since the forbidden minors characterising $\mathcal{E}_g$ for $g \geq 1$ are unknown [49], we do not work with $\mathcal{E}_g$ directly but rather with superclasses that are described by a concise list of forbidden minors. Recall that we write $\mathcal{P}$ for the class of planar graphs. Also recall that, by Wagner's theorem [72, 39], a graph is planar if, and only if, it does not contain K_5 and $K_{3,3}$ as a minor. For $k \geq 1$, a k-*Kuratowski graph* [57] is a graph with exactly k connected components, each of which isomorphic to K_5 or $K_{3,3}$. We write $\mathcal{P}^{(k)}$ for the class of graphs that exclude all k-Kuratowski graphs as a minor. It is known that graphs of genus g exclude all $(g+1)$-Kuratowski graphs as a minor, see also [57, (1.3)].

▶ **Lemma 5.1** ([9, Theorem 1]). *For every $g \geq 0$, it holds that $\mathcal{E}_g \subseteq \mathcal{P}^{(g+1)}$.*

We show that $\mathcal{P}^{(g)}$ is closed under weak oddomorphisms and homomorphism distinguishing closed by applying the following lemma.

▶ **Lemma 5.2.** *Let $M_1, \dots, M_\ell$ be connected graphs, and let $\mathcal{F}$ be the class of $\{M_1, \dots, M_\ell\}$-minor-free graphs. For $k \geq 1$, let $\mathcal{F}^{(k)}$ denote the class of graphs that exclude all minors that are the disjoint union of exactly k not necessarily distinct graphs in $\{M_1, \dots, M_\ell\}$. If $\mathcal{F}$ is closed under weak oddomorphisms, then so is $\mathcal{F}^{(k)}$.*

Proof. Let $F \in \mathcal{F}^{(k)}$ and $\varphi \colon F \to G$ be an oddomorphism, see Observation 2.9. If $G \notin \mathcal{F}^{(k)}$, then there exists a minor $M_{i_1} + \dots + M_{i_k}$ of G. By Lemma 2.8, there exists a minor F' of F admitting an oddomorphism $F' \to M_{i_1} + \dots + M_{i_k}$. By Lemma 2.8, there exist pairwise disjoint subgraphs $F'_1, \dots, F'_k$ of F' admitting oddomorphisms $F'_j \to M_{i_j}$ for $j \in [k]$. Since $\mathcal{F}$ is closed under oddomorphisms, it follows that $F'_j \notin \mathcal{F}$ for all $j \in [k]$. Hence, each F'_j contains at least one of the forbidden minors defining $\mathcal{F}$. In particular, $F' \notin \mathcal{F}^{(k)}$ which implies that $F \notin \mathcal{F}^{(k)}$, as desired. ◀

By Lemma 2.7, the class of planar graphs is closed under oddomorphisms. Hence, the following Corollary 5.3 follows from Lemmas 2.8 and 5.2 and Theorem 2.6.

▶ **Corollary 5.3.** *For every $k \geq 1$, the class $\mathcal{P}^{(k)}$ is closed under weak oddomorphisms. In particular, the disjoint union closure of $\mathcal{P}^{(k)}$ is homomorphism distinguishing closed.*

Lemma 5.1 and Corollary 5.3 imply that, for every $g \geq 0$, there exist non-isomorphic graphs G and H that are homomorphism indistinguishable over $\mathcal{E}_g$. Refining this, Theorem 1.8 shows that the relations $\equiv_{\mathcal{E}_g}$ for $g \geq 0$ form a strictly refining chain of graph isomorphism relaxations.

▶ **Theorem 1.8.** *For all $g \geq 0$, there exist graphs G, H such that $G \equiv_{\mathcal{E}_g} H$ and $G \not\equiv_{\mathcal{E}_{g+1}} H$.*

Proof. Let F be the graph obtained by taking $g+1$ copies of K_5 and connecting one of the vertices in each copy to a fresh vertex x. Since F contains a $(g+1)$-Kuratowski graph as a minor, $F \notin \mathcal{P}^{(g+1)}$. In particular, by Corollary 5.3 and Lemma 5.1, we conclude that no graph in $\mathcal{E}_g$ admits a weak oddomorphism to F.

By [9, Lemma 1], the genus of the graph F' obtained from F by contracting all edges incident to x has genus $\leq g+1$. Hence, F has genus $\leq g+1$. Now, the theorem follows from Lemma 2.10. ◀

5.2 Bounded Vortex-Free Hadwiger Number

Next, we generalise Theorem 1.8 to graphs of bounded vortex-free Hadwiger number. For $d, g \geq 0$, we write $\mathcal{C}_{g,d}$ for the class of all graphs G such that $\mathrm{dd}_{\mathcal{E}_g}(G) \leq d$, i.e., there is a set $S \subseteq V(G)$ of size at most d such that $G - S \in \mathcal{E}_g$. Also, we write $\mathcal{C}_{g,d}^{\oplus s}$ for the closure of $\mathcal{C}_{g,d}$ under clique-sums of size at most s. The *vortex-free Hadwiger number* [69] of a graph G is the minimum $k \geq 0$ such that $G \in \mathcal{C}_{k,k}^{\oplus k}$. Recall that we write $\mathcal{H}_k := \mathcal{C}_{k,k}^{\oplus k}$ for the class of graphs of vortex-free Hadwiger number at most k.

Lemma 5.1 generalises to also allow for apices. Alternatively, it is also possible to use the tools from Section 3 to deal with apices.

▶ **Lemma 5.4.** *For every $g, d \geq 0$, it holds that $\mathcal{C}_{g,d} \subseteq \mathcal{P}^{(g+1+d)}$.*

Proof. This directly follows from Lemma 5.1. Indeed, if a graph $G \in \mathcal{C}_{g,d}$ contains a $(g+1+d)$-Kuratowski graph as a minor, and $S \subseteq V(G)$ is a set of size at most d, then $G - S$ contains a $(g+1)$-Kuratowski graph as a minor. ◀

Now, we can put everything together to obtain the following.

▶ **Theorem 1.9.** *For all $k \geq 0$, there exist graphs G, H such that $G \equiv_{\mathcal{H}_k} H$ and $G \not\equiv_{\mathcal{H}_{k+1}} H$.*

Proof. Let $G' := (2k+1)K_5$ denote the disjoint union of $2k+1$ many K_5, and let G be the graph obtained from by adding universal vertices $u_1, \dots, u_{k+1}$, i.e., each u_i is connected to all other vertices of G. Observe that G is $(k+1)$-connected. Indeed, let $S \subseteq V(G)$ be a set of size at most k. Then there is some $i \in [k+1]$ such that $u_i \notin S$, and we obtain that $G - S$ is connected since u_i is connected to all vertices to G.

We first claim that there is no graph $F \in \mathcal{H}_k$ that admits an oddomorphism to G. Suppose towards a contradiction that such a graph $F \in \mathcal{H}_k$ exists. Since G is $(k+1)$-connected, by Corollary 4.2, we may assume that $F \in \mathcal{C}_{k,k}$. So $F \in \mathcal{P}^{(2k+1)}$ by Lemma 5.4. By Corollary 5.3, we conclude that $G \in \mathcal{P}^{(2k+1)}$. But this is a contradiction, since $(2k+1)K_5$ is a subgraph of G.

Next, we argue that $G \in \mathcal{H}_{k+1}$. Indeed, G is a $(k+1)$-sum of $2k+1$ many copies of K_{k+6}. Since $\mathcal{H}_{k+1}$ is closed under $(k+1)$-sums, it suffices to show that $K_{k+6} \in \mathcal{C}_{k+1,k+1}$. But this follows directly from the fact that $K_5 \in \mathcal{E}_1 \subseteq \mathcal{E}_{k+1}$, i.e. the graph K_5 can be drawn on the torus. Overall, the statement now follows from Lemma 2.10. ◀

Theorem 1.2 is a direct consequence of Theorem 1.9 since G, H are in particular non-isomorphic.

6 Beyond Excluding Minors

In this section, we give further evidence of the special role of minor-closed graph classes in the context of homomorphism indistinguishability, see [64]. In particular, we show that Conjecture 1.1 fails when generalised to topological minors. The key technical insight is the following Lemma 6.1, which also shows that the minor-closedness assumption in Lemma 4.18 is necessary.

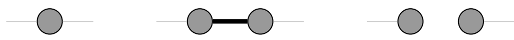
▶ **Lemma 6.1.** *If a graph class $\mathcal{F}$ is closed under taking subgraphs and 2-sums with K_3, then its homomorphism distinguishing closure $\mathrm{cl}(\mathcal{F})$ is minor-closed.*

Moreover, if Conjecture 1.4 holds and $\mathcal{F}$ is union-closed, then $\mathrm{cl}(\mathcal{F})$ is in fact equal to the minor closure of $\mathcal{F}$. Before proving Lemma 6.1, we first discuss some of its consequences including a proof of Theorem 1.3.

Let $\mathcal{D}_3$ denote the class of graphs of maximum degree ≤ 3. We write $\mathcal{D}_3^*$ for the closure of $\mathcal{D}_3$ under repeated 2-sums with triangles. More formally, a graph G is contained in $\mathcal{D}_3^*$ if, and only if, there is a tree decomposition (T, β) of G with a designated root node $r \in V(T)$ such that

1. the torso $G[\![\beta(r)]\!]$ has maximum degree ≤ 3, and
2. $|\beta(t)| \leq 3$ for every $r \neq t \in V(T)$.

▶ **Lemma 6.2.** *Two graphs are isomorphic if, and only if, they are homomorphism indistinguishable over $\mathcal{D}_3^*$.*



(a) $\boldsymbol{I}$. (b) $\boldsymbol{A}$. (c) $\boldsymbol{J}$.

Figure 4 Bilabelled graphs used in the proof of Lemma 6.1 in wire notation of [46]. The gray wires to the left/right indicate the position of the first/second label. Actual edges are depicted in black.

Proof (assuming Lemma 6.1). The class $\mathcal{D}_3^*$ is closed under taking subgraphs and 2-sums with K_3. Hence, $\text{cl}(\mathcal{D}_3^*)$ is minor-closed by Lemma 6.1. Since $\mathcal{D}_3 \subseteq \mathcal{D}_3^* \subseteq \text{cl}(\mathcal{D}_3^*)$ and every graph is a minor of a graph of maximum degree 3, we conclude that $\text{cl}(\mathcal{D}_3^*)$ contains all graphs. By [44], $\equiv_{\mathcal{D}_3^*}$ is the isomorphism relation $\cong$. ◀

Now, we directly obtain Theorem 1.3 by observing that every graph in $\mathcal{D}_3^*$ excludes K_5 as a topological minor.

▶ **Theorem 1.3.** *Two graphs are isomorphic if, and only if, they are homomorphism indistinguishable over all graphs excluding K_5 as a topological minor.*

Note that K_5 is optimal here since K_4-topological-minor-free graphs are precisely those of treewidth ≤ 2, see Theorem 4.19. Moreover, it can also be easily verified that $\mathcal{D}_3^*$ has bounded ∞-admissibility (see [26, 66]) and bounded local treewidth (see [28, 31]), which implies that Conjecture 1.1 also cannot be generalised to those sparsity notions, see Figure 1.

The rest of this section is dedicated to the proof of Lemma 6.1. Essentially, the proof is based on two ingredients: First, we use properties of the homomorphism distinguishing closure established in [64] to reduce Lemma 6.1 to the statement that, if $F \in \mathcal{F}$, then any graph F' obtained from F by contracting edges is in the homomorphism distinguishing closure $\text{cl}(\mathcal{F})$ of $\mathcal{F}$. Second, we employ the so-called *contractors* of Lovász and Szegedy [43] to simulate homomorphism counts from F' by homomorphism counts from graphs obtained from F by replacing the to-be-contracted edges by suitable series-parallel gadgets. When attaching these gadgets, we rely on the assumption $\mathcal{F}$ is closed under 2-sums with K_3.

In order to state these arguments formally, we recall the framework of bilabelled graphs and homomorphism matrices from [46, 35]. We follow the notational conventions of [63, Section 3.2].

A *bilabelled graph* is a tuple $\boldsymbol{F} = (F, u, v)$ where F is a graph and $u, v \in V(F)$. Given a graph G, the *homomorphism matrix* of $\boldsymbol{F}$ with respect to G is $\boldsymbol{F}_G \in \mathbb{N}^{V(G)\times V(G)}$ where $\boldsymbol{F}_G(x, y)$ for $x, y \in V(G)$ is the number of homomorphisms $h \colon F \to G$ such that $h(u) = x$ and $h(v) = y$. For example, the identity matrix is $\boldsymbol{I}_G$ where $\boldsymbol{I} = ((\{u\}, \emptyset), u, u)$ and the adjacency matrix is $\boldsymbol{A}_G$ where $\boldsymbol{A} = ((\{u, v\}, \{uv\}), u, v)$. The all-ones matrix is $\boldsymbol{J}_G$ where $\boldsymbol{J} = ((\{u, v\}, \emptyset), u, v)$, see Figure 4.

Bilabelled graphs admit a wealth of combinatorial operations, which correspond to certain algebraic operations on their homomorphism matrices, see [63, Section 3.2]:

Unlabelling a bilabelled graph $\boldsymbol{F} = (F, u, v)$ corresponds to the *sum-of-entries* of the homomorphism matrix $\boldsymbol{F}_G$ in the sense that $\text{soe}(\boldsymbol{F}_G) := \sum_{x,y\in V(G)} \boldsymbol{F}_G(x, y) = \hom(F, G)$. For this reason, we write $\text{soe}(\boldsymbol{F}) := F$ for the underlying unlabelled graph of $\boldsymbol{F}$.

For two bilabelled graphs $\boldsymbol{F} = (F, u, v)$ and $\boldsymbol{F'} = (F', u', v')$, the *parallel composition* of $\boldsymbol{F}$ and $\boldsymbol{F'}$ is the bilabelled graph obtained from taking the disjoint of F and F', identifying u, u' and v, v', and placing the labels on the two resulting vertices. We write $\boldsymbol{F} \odot \boldsymbol{F'}$ to denote the parallel composition of $\boldsymbol{F}$ and $\boldsymbol{F'}$. This corresponds to taking *entrywise product* of the homomorphism matrices, i.e., $(\boldsymbol{F} \odot \boldsymbol{F'})_G(x, y) = \boldsymbol{F}_G(x, y)\boldsymbol{F'}_G(x, y) =: (\boldsymbol{F}_G \odot \boldsymbol{F'}_G)(x, y)$ for every graph G and every $x, y \in V(G)$.

The *series composition* of $\boldsymbol{F}$ and $\boldsymbol{F'}$ is the bilabelled graph $\boldsymbol{K} = (K, u, v')$ where K is obtained from taking the disjoint of F and F', and identifying the vertices v, u'. We write $\boldsymbol{F} \cdot \boldsymbol{F}'$ to denote the series composition of $\boldsymbol{F}$ and $\boldsymbol{F'}$. Series composition corresponds to taking the *matrix product* of the homomorphism matrices, i.e., $(\boldsymbol{F} \cdot \boldsymbol{F'})_G(x, y) = \sum_{z \in V(G)} \boldsymbol{F}_G(x, z)\boldsymbol{F'}_G(z, y) =: (\boldsymbol{F}_G \cdot \boldsymbol{F'}_G)(x, y)$ for every graph G and every $x, y \in V(G)$.

A bilabelled graph $\boldsymbol{S} = (S, u, v)$, where $u \neq v$, is *series-parallel* if it can be constructed by a sequence of parallel and series compositions from the bilabelled graph $\boldsymbol{A}$. We write $\mathcal{S}$ for the class of bilabelled series-parallel graphs. It is well-known that, if $\boldsymbol{S} = (S, u, v)$ is series-parallel, then there is a tree decomposition (T, β) of S of width ≤ 2 such that $u, v \in \beta(t)$ for some node $t \in V(T)$. In particular, we obtain the following:

▶ **Observation 6.3.** *Let $\mathcal{F}$ be a graph class that is closed under taking subgraphs and 2-sums with K_3. Let $F \in \mathcal{F}$ and $uv \in E(F)$, and set $\boldsymbol{F} = (F, u, v)$. For every $\boldsymbol{S} \in \mathcal{S}$, it holds that* $\operatorname{soe}(\boldsymbol{F} \odot \boldsymbol{S}) \in \mathcal{F}$.

A key tool in the proof of Lemma 6.1 is the notion of a *contractor*. Let G be a graph. A *contractor* [43] for G is a finite linear combination $\sum_{\boldsymbol{F}} \alpha_{\boldsymbol{F}} \boldsymbol{F}$ of bilabelled graphs $\boldsymbol{F} = (F, u, v)$ whose labels do not coincide, i.e. $u \neq v$, such that $\boldsymbol{I}_G = \sum_{\boldsymbol{F}} \alpha_{\boldsymbol{F}} \boldsymbol{F}_G$. Since parallel composition with $\boldsymbol{I}$ amounts to identifying the two labelled vertices, a contractor simulates label identification by homomorphism counts from patterns in which the labels are not contracted. We rely on the following result establishing the existence of series-parallel contractors.

▶ **Lemma 6.4** (Lovász and Szegedy [43, Theorem 1.4], see also [42, Supplement 6.29b])**.** *For all graphs G and H, there exists a finitely-supported coefficient vector $\alpha_{\boldsymbol{S}} \in \mathbb{Q}$ such that*

$$\boldsymbol{I}_G = \sum_{\boldsymbol{S} \in \mathcal{S}} \alpha_{\boldsymbol{S}} \boldsymbol{S}_G \quad \text{and} \quad \boldsymbol{I}_H = \sum_{\boldsymbol{S} \in \mathcal{S}} \alpha_{\boldsymbol{S}} \boldsymbol{S}_H.$$

Equipped with Lemma 6.4, we now conduct the proof of Lemma 6.1.

Proof of Lemma 6.1. We start by showing the following claim.

▷ Claim 6.5. Let $\mathcal{F}'$ be a graph class that is closed under taking subgraphs and 2-sums with K_3. Let $F \in \mathcal{F}'$, and let F' be the graph obtained from F by contracting an edge $uv \in E(F)$. Then $F' \in \operatorname{cl}(\mathcal{F}')$.

Proof. Contrapositively, let G and H be two graphs such that $\hom(F', G) \neq \hom(F', H)$. Let $\alpha_{\boldsymbol{S}} \in \mathbb{Q}$ for $\boldsymbol{S} \in \mathcal{S}$ denote the finitely-supported coefficient vector from Lemma 6.4. Write F^- for the graph obtained from F by deleting the edge uv and let $\boldsymbol{F}^- := (F^-, u, v)$. The parallel composition $\boldsymbol{F}^- \odot \boldsymbol{I}$ of $\boldsymbol{F}^-$ with $\boldsymbol{I}$ amounts to identifying the two vertices u, v, i.e., contracting the edge uv in F. Hence, $\operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{I}) \cong F'$. By Lemma 6.4,

$$\hom(F', G) = \operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{I})_G = \sum \alpha_{\boldsymbol{S}} \operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{S})_G$$

and analogously

$$\hom(F', H) = \operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{I})_H = \sum \alpha_{\boldsymbol{S}} \operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{S})_H.$$

Since $\hom(F', G) \neq \hom(F', H)$, there is some $\boldsymbol{S} \in \mathcal{S}$ such that $\operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{S})_G \neq \operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{S})_H$. Note that $\operatorname{soe}(\boldsymbol{F}^- \odot \boldsymbol{S}) \in \mathcal{F}$ by Observation 6.3. Hence, $G \not\equiv_{\mathcal{F}} H$. We conclude that $F' \in \operatorname{cl}(\mathcal{F}')$. ◁

For $\ell \geq 0$, let $\mathcal{F}_\ell$ denote the class of graphs obtained from $\mathcal{F}$ by performing at most ℓ edge contractions. We show by induction that $\mathcal{F}_\ell \subseteq \mathrm{cl}(\mathcal{F})$ for all $\ell \geq 0$. The base case $\ell = 0$ is trivial since $\mathcal{F}_0 = \mathcal{F}$.

So suppose $\ell \geq 0$ and let $F' \in \mathcal{F}_{\ell+1}$. By definition, there is some $F \in \mathcal{F}_\ell$ such that F' is the graph obtained from F by contracting an edge $uv \in E(F)$. Note that $\mathcal{F}_\ell$ is closed under taking subgraphs and 2-sums with K_3. Hence, by Claim 6.5, we conclude that $F' \in \mathrm{cl}(\mathcal{F}_\ell)$. It follows that

$$\mathcal{F}_{\ell+1} \subseteq \mathrm{cl}(\mathcal{F}_\ell) \subseteq \mathrm{cl}(\mathrm{cl}(\mathcal{F})) = \mathrm{cl}(\mathcal{F}),$$

where the second inclusion holds by the induction hypothesis and Equation (2), and the final equality holds by Equation (1).

Now, let $\mathcal{F}^*$ be the minor closure of $\mathcal{F}$, i.e., $\mathcal{F}^*$ consists of the graphs F^* that are minors of some graph $F \in \mathcal{F}$. Since, for every graph $F^* \in \mathcal{F}^*$, there is some $\ell \geq 0$ such that $F^* \in \mathcal{F}_\ell$, we conclude that $\mathcal{F}^* \subseteq \mathrm{cl}(\mathcal{F})$. In particular, by Equations (1) and (2), $\mathrm{cl}(\mathcal{F}) \subseteq \mathrm{cl}(\mathcal{F}^*) \subseteq \mathrm{cl}(\mathrm{cl}(\mathcal{F})) = \mathrm{cl}(\mathcal{F})$. Hence, $\mathrm{cl}(\mathcal{F}) = \mathrm{cl}(\mathcal{F}^*)$. Finally, by [64, Theorem 8], the homomorphism distinguishing closure of a minor-closed graph class is itself minor-closed. ◀

7 Conclusion

We proved a series of results on the distinguishing power of homomorphism indistinguishability relations over sparse graph classes. Most notably, we proved Conjecture 1.1 for all vortex-free graph classes. In particular, homomorphism indistinguishability over graphs of bounded Euler genus is not the same as isomorphism. As a negative result, we show that Conjecture 1.1 cannot be siginificantly generalised beyond graph classes with excluded minor. Finally, we showed that several graph classes are homomorphism distinguishing closed and separated homomorphism indistinguishability relations defined via the genus or the vortex-free Hadwiger number.

Nevertheless, several questions remain open. Most notably, Conjecture 1.1 remains unresolved for $k \geq 5$. With Theorem 1.2 in mind, it remains to understand the role of vortices. Is homomorphism indistinguishability over planar graphs with one vortex the same as isomorphism? Towards this end, it may be helpful to obtain a purely graph-theoretic proof of Lemma 2.7a, for which the only known proof crucially relies on the characterisation via quantum isomorphisms [46, 8].

Beyond this, it is interesting to investigate homomorphism indistinguishability over classes of dense graphs. To highlight only one possible question, are there non-isomorphic graphs that are homomorphism indistinguishable over all graphs of cliquewidth $\leq k$?

Finally, our results separating homomorphism indistinguishability relations $\equiv_{\mathcal{F}}$ from isomorphism pave the way for studying the computational complexity of determining whether $G \equiv_{\mathcal{F}} H$ for input graphs G, H, and fixed $\mathcal{F}$. This problem has been conjectured to be undecidable for every proper minor-closed graph class $\mathcal{F}$ of unbounded treewidth [62, Conjecture 25]. However, up to this point, this is only known for the class of planar graphs [67, 8, 46].

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