On Cycles in Multiset Permutations, Parking Functions, and Related Structures

Calum Buchanan
Department of Mathematics, University of Denver, Denver, CO 80208, USA

Fabian Burghart
Institute of Discrete Mathematics, TU Graz, Steyrergasse 30, 8010 Graz, Austria

Stephan Wagner
Institute of Discrete Mathematics, TU Graz, Steyrergasse 30, 8010 Graz, Austria
Department of Mathematics, Uppsala University, Box 480, 751 06 Uppsala, Sweden

Mei Yin
Department of Mathematics, University of Denver, Denver, CO 80208, USA

Abstract

In this paper we study cycles in multiset permutations and parking functions. As combinatorial objects, multiset permutations are essential building blocks for mappings and permutations, while parking functions lie between mappings and permutations. We take both algebraic and analytic views in our investigation and present exact as well as asymptotic results. We point to a surprising correspondence between two statistics on multiset permutations, terminal closers and cyclic points, shedding light on the combinatorial structure.

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1 Introduction

Let $[n]$ denote the set of positive integers $\{1, \ldots, n\}$. In this paper, we study several related combinatorial objects:

- Let S_n be the set of all *permutations* of $[n]$, so that $|S_n| = n!$.
- A *mapping* of length n is any function f from $[n]$ to $[n]$. Let $\mathcal{F}_n$ denote the set of all mappings on $[n]$, so that $|\mathcal{F}_n| = n^n$.
- *Parking functions* and *prime parking functions* will be defined later in Sections 4 and 5. As we will see, they lie between mappings and permutations.
- Let M be an n-element multiset of integers taken from $[n]$. A *multiset permutation* of M is a sequence $(w_1, \ldots, w_n)$ such that the multiplicity of each element j in M is $|\{i : w_i = j\}|$. Let S_M denote the set of all permutations of M. For example, $(2, 1, 4, 2) \in S_{\{1,2,2,4\}}$.

A common feature of all these sets of combinatorial objects is *invariance under permutation*: if one element belongs to a set, then so does any of its permutations. Multiset permutations can be seen as building blocks of the others in that each of them is a disjoint union of S_M's.

The study of cycles in random mappings and permutations has a rich history. We refer the reader to Arratia, Barbour, and Tavaré [1] and Flajolet and Odlyzko [6] for a comprehensive treatise. Multiset permutations and parking functions are less studied, and thus will be the central focus of our paper. Note that while our investigation into parking functions follows the classical interpretation of digraphs as done for mappings and permutations as in [18] and [22], there is an alternate interpretation of parking function digraphs, investigated in [12] and [14], by considering the vertices as the parking spaces and the directed edges as the one-way streets.

We will take both algebraic and analytic views in our investigation. Sections 2, 3, 4, and 5 respectively present exact results on multiset permutations, mappings and permutations, parking functions, and prime parking functions. Section 6 on the other hand is dedicated to asymptotic analysis, demonstrating the asymptotic *equivalence of ensembles* between parking functions and mappings concerning the number of cyclic points in their associated connected digraphs.

We point to a surprising correspondence between *terminal closers* and *cyclic points* in multiset permutations in Section 2.2, which in turn leads to the correspondence between terminal closers and cyclic points in mappings, permutations, and parking functions. Since parking functions describe a dynamic parking process, the equality in distribution between terminal closers and cyclic points is even more intriguing than in the context of mappings and permutations. Having k terminal closers in a parking function ensures that the first k cars are lucky (a car is *lucky* if it parks at its preferred spot) and that the $(k+1)$st car, having the same preference as one of the first k cars, is unlucky and causes a collision. Thus the expected number of lucky cars in a parking function is lower bounded by the expected number of cyclic points, and moreover, our asympotic results on cyclic points show that the number of lucky cars until the first unlucky car is on average of the order of magnitude $\sqrt{n}$.

2 Multiset permutations

In this section, we introduce relevant terminology and background. Since multiset permutations are essential building blocks for mappings, permutations, and parking functions, results presented in this section will serve as a basis for our derivations throughout the paper.

2.1 Cyclic points

Given a multiset permutation $w = (w_1, \ldots, w_n) \in S_M$, let G_w denote the *digraph* on the vertex set $[n]$ with edges $i \to w_i$. It is well known that the digraph G_w associated to any w can be obtained from a disjoint union of directed cycles on some subset C of $[n]$ (the set of *cyclic points*) by selecting some vertices in C and rooting to them some trees whose edges are all directed towards the root. For $k \geq 1$, a directed k-cycle in G_w is a *cycle of length k in w*, a subset $\{j_1, \ldots, j_k\}$ of $[n]$ such that $w_{j_i} = j_{i+1}$ for each $i \in [k-1]$ and $w_{j_k} = j_1$.

We use the notation $\lambda \vdash m$ to indicate that λ is a partition of m. In other words, if $\lambda = (1^{k_1(\lambda)}, 2^{k_2(\lambda)}, \ldots)$, where $k_i(\lambda)$ denotes the number of i's in λ, then $\sum_i ik_i(\lambda) = m$. Given a partition λ, let $a(M, \lambda)$ be the number of $w \in S_M$ such that G_w has cycle type $\lambda = (1^{k_1(\lambda)}, 2^{k_2(\lambda)}, \ldots)$, i.e., G_w has $k_i(\lambda)$ cycles of length i. Write $a(M, i)$ for $a(M, (i))$, where $\lambda = (i)$ corresponds to $w \in S_M$ where G_w is connected and has a cycle of length i. For G_w with cycle type λ, form a monomial

$$u_w = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n},$$

where p_j is a power sum symmetric function $p_j = \sum_{i\geq 1} x_i^j$. See Figure 1 for the digraph representation of a multiset permutation $w = (6, 1, 2, 4, 1, 9, 1, 6, 8, 4, 2, 10) \in S_{\{1,1,1,2,2,4,4,6,6,8,9,10\}}$ with $u_w = p_1p_3$. Define the augmented cycle index for multiset permutations $Z_{S_M} = \sum_{w\in S_M} u_w$.

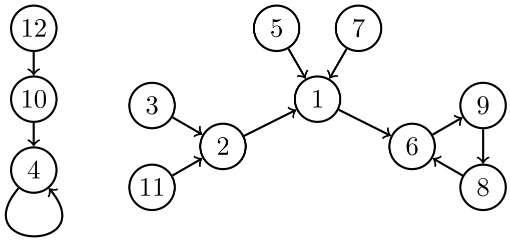

Figure 1 The digraph of the multiset permutation $w = (6, 1, 2, 4, 1, 9, 1, 6, 8, 4, 2, 10)$ in $S_{\{1,1,1,2,2,4,4,6,6,8,9,10\}}$ which consists of one 1-cycle (fixed point) (4), one 3-cycle (6, 9, 8), and tree branches hanging from the cycles.

▶ **Lemma 1.** *For $\lambda \vdash i$, we have*

$$a(M, \lambda) = ia(M, i)/z_\lambda,$$

where $z_\lambda = \prod_{j=1}^{i} j^{k_j} k_j!$ is as in Macdonald's book [15, page 24].

Proof. The digraph G_w corresponding to $a(M, i)$ comprises rooted trees arranged in a length-i cycle. We keep the hanging tree branches intact with their incident vertices on the cycle and decompose the length-i cycle into smaller cycles.

To do this, we write the length-i cycle in one-line notation. There are i possible starting points, which introduces an additional scalar factor i. Then we insert pairs of parentheses from left to right into this one-line notation according to λ, making sure that the lengths of the cycles are non-decreasing. This procedure generates the same digraph z_λ times and we compensate for the overcounting by dividing by z_λ.

We note that our construction is an adaption from counting permutations of cycle type λ in S_i, except that in our case the numerator becomes $ia(M, i)$ instead of $i!$. ◀

▶ **Proposition 2.** *The augmented cycle index for multiset permutations satisfies*

$$Z_{S_M} = \sum_{w\in S_M} u_w = \sum_{i=1}^{n} ia(M, i)h_i,$$

where $h_i = \sum_{1\leq j_1\leq j_2\leq\cdots\leq j_i} x_{j_1}x_{j_2}\cdots x_{j_i}$ is a complete symmetric function (the sum of all monomials of degree i).

Proof. By Corollary 7.7.6 in Stanley's book [23],

$$\sum_{i=1}^{n} ia(M, i)h_i = \sum_{i=1}^{n} ia(M, i)\left(\sum_{\lambda\vdash i} z_\lambda^{-1} p_\lambda\right),$$

where z_λ is as defined earlier and $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$ is a power sum symmetric function for $\lambda = (\lambda_1, \lambda_2, \dots)$. The conclusion then follows from Lemma 1. ◀

▶ **Example 3.** Take $M = \{1, 1, 2\}$ so that

$$S_M = \{(1, 1, 2), (1, 2, 1), (2, 1, 1)\}.$$

We compute that

$$Z_{S_M} = p_1 + p_1^2 + p_2 = h_1 + 2h_2.$$

Take $M = \{1, 1, 2, 2\}$ so that

$$S_M = \{(1, 1, 2, 2), (1, 2, 1, 2), (1, 2, 2, 1), (2, 1, 1, 2), (2, 1, 2, 1), (2, 2, 1, 1)\}.$$

We compute that

$$Z_{S_M} = 2p_1 + 2p_1^2 + 2p_2 = 2h_1 + 4h_2.$$

▶ **Corollary 4.**

$$\sum_{i=1}^{n} ia(M, i) = |S_M|.$$

Proof. We take $x_1 = 1$ and $x_i = 0$ for all $i \geq 2$, which yields that $p_i = h_i = 1$ for all $i \geq 1$ and $u_w = 1$ for all $w \in S_M$. By Proposition 2, the claimed identity is then immediate. ◀

2.2 Terminal closers

Consider a partition P of $[n]$ into parts $P_1, P_2, \ldots, P_n$, some of which may be empty. In the FindStat database [3, St001050], the *closers* of P are defined to be those elements of $[n]$ which are maximal in their parts. A closer i is *terminal* if every $j \in \{i + 1, \ldots, n\}$ is also a closer. Terminal closers can also be interpreted in the context of multiset permutations. Let M be the n-element multiset over $[n]$ consisting of $|P_j|$ copies of j for each j in $[n]$, and define the multiset permutation w of M by

$$w_i = j \quad \text{if and only if} \quad i \in P_j. \tag{1}$$

Notice that the elements $i, \ldots, n$ are terminal closers of P if and only if $w_i \neq w_{i+1} \neq \cdots \neq w_n$. Further, $\{i, \ldots, n\}$ is the set of *all* terminal closers of P if and only if the number $i - 1$ is not maximal in its part, which equates to having $w_{i-1} \in \{w_i, \ldots, w_n\}$.

For example, if $n = 6$, $P_1 = \{2, 4\}$, $P_2 = \{1, 3, 6\}$, $P_3 = \{5\}$, and $P_4 = P_5 = P_6 = \varnothing$, then the closers are 4, 5, and 6, and all are terminal. Under the above correspondence $(P_1, \ldots, P_6) \mapsto (2, 1, 2, 1, 3, 2)$, and the maximal end segment of distinct values is $(\ldots, 1, 3, 2)$. These are the labels of the parts P_j containing terminal closers of the partition. On the other hand, if we move the element 5 from P_3 into P_2, then we obtain the partition $(\{2, 4\}, \{1, 3, 5, 6\}, \varnothing, \varnothing, \varnothing, \varnothing)$. The closers here are 4 and 6, but only 6 is a terminal closer. Indeed, in the corresponding multiset permutation $(2, 1, 2, 1, 2, 2)$, the maximal end segment of distinct values is $(\ldots, 2)$.

It would thus be natural to define the terminal closers of an arbitrary permutation w of an arbitrary n-element multiset M over $[n]$ to be the elements of the maximal end segment of distinct terms in w. On the other hand, we could have instead defined the correspondence (1) between multiset permutations and n-part partitions of $[n]$ by $w_{n+1-i} = j$ if and only if $i \in P_j$, in which case the natural analogue of the terminal closers of P would be the elements of the maximal *initial* segment of distinct terms in w. We choose the latter for the purposes of this paper, defining the set of *terminal closers of* w to be $\{w_1, \ldots, w_k\}$ where $w_1 \neq w_2 \neq \cdots \neq w_k$ but $w_{k+1} = w_i$ for some $i \in [k]$. (By default, w has n terminal closers if $w_1, \ldots, w_n$ are all distinct.)

We proceed to describe a surprising correspondence between cyclic points and terminal closers for multiset permutations. Notice that, if a multiset permutation $w \in S_M$ has exactly k cyclic points with $k < n$, then G_w is not simply a union of cycles. In particular, at least one cyclic point has in-degree at least 2 in G_w. Using this observation and the commonplace tool of Prüfer code [19], we shall associate to w a multiset permutation whose terminal closers are the cyclic points of w. We note that Prüfer code and digraphs associated to multiset permutations (and mappings) have been used numerous times in the literature to provide combinatorial bijections, for instance, by Labelle and Gessel (see, e.g., [8, 9]) in proving various generalizations of the Lagrange inversion formula.

▶ **Theorem 5.** *For any multiset M, the permutations of M with a given set S of cyclic points are in bijection with the permutations of M whose set of terminal closers is S.*

Proof. Let M be an n-element multiset of positive integers taken from $[n]$, and let w be an element of S_M, $w = (w_1, \ldots, w_n)$. It is not hard to see that w has at least one cycle, for the digraph G_w associated to w has n edges and n vertices. Let $\{j_1, \ldots, j_k\}$ be the set of cyclic points of G_w, where $j_i < j_{i+1}$ for each $i \in [k-1]$. We construct an element w' of S_M whose terminal closers are $j_1, \ldots, j_k$ as follows.

1. For each $i \in [k]$, define $w'_i = w_{j_i}$.
2. If w is not a permutation, then $k < n$, and G_w has a leaf. We define $w'_n, w'_{n-1}, \ldots, w'_{k+1}$ recursively. At step i, for $i \in \{0, 1, \ldots, n-k-1\}$, let ℓ_i be the leaf of G_w with the largest label, delete ℓ_i from $G_w - \{\ell_0, \ldots, \ell_{i-1}\}$, and define $w'_{n-i} = w_{\ell_i}$.

Note that each one of $j_1, \ldots, j_k$ is a terminal closer of w'. If $k < n$, then since $G_w - \{\ell_0, \ldots, \ell_{n-k-2}\}$ consists only of cycles and a single leaf ℓ_{n-k-1} adjacent to a cyclic point, we have $w_{\ell_{n-k-1}} = w'_{k+1} \in \{j_1, \ldots, j_k\}$. Thus, $\{j_1, \ldots, j_k\}$ is precisely the set of terminal closers of w'.

The inspiration for item 2 above comes from the Prüfer code [19]. That this construction is a bijection naturally follows. ◀

▶ **Example 6.** For the multiset permutation $w = (6, 1, 2, 4, 1, 9, 1, 6, 8, 4, 2, 10)$ with cyclic points 4, 6, 8, and 9, whose digraph G_w is depicted in Figure 1, the permutation w' having terminal closers $4, 6, 8, 9$ obtained from the bijection in Theorem 5 is $(4, 9, 6, 8, 6, 1, 2, 1, 1, 4, 2, 10)$. The digraph $G_{w'}$ is depicted in Figure 2.

▶ **Corollary 7.** *Let M be a multiset. Let $a(M, k)$ denote the number of permutations w of the multiset M for which G_w is connected and has a cycle of length k. Let $t(M, k)$ denote the number of permutations of M with exactly k terminal closers. Then $a(M, k) = t(M, k)/k$.*

Proof. The claim readily follows once we note that exactly $(k-1)!$ of the $k!$ permutations of S consist of a single cycle, as opposed to a disjoint union of at least two cycles. ◀

Let M be a multiset, which is denoted by $\left(m_1^{n_1}, \ldots, m_{j_M}^{n_{j_M}}\right)$, where j_M is the number of distinct parts of M, $1 \leq m_1 < \cdots < m_{j_M} \leq n$ and $\sum_i n_i = n$. Corollary 7 provides an alternative (and faster) way to calculate $a(M, k)$, as only the values of the initial $k+1$ terms in $w \in S_M$ matter. We present two special cases as examples.

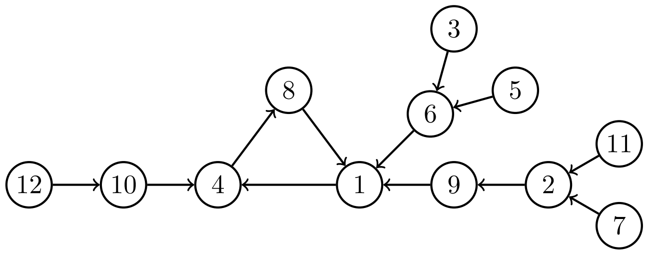

Figure 2 The digraph of the multiset permutation $w' = (4,9,6,8,6,1,2,1,1,4,2,10)$ in $S_{\{1,1,1,2,2,4,4,6,6,8,9,10\}}$ obtained from the multiset permutation $w \in S_{\{1,1,1,2,2,4,4,6,6,8,9,10\}}$ (whose digraph is depicted in Figure 1) via the bijection in Theorem 5. The set $\{4,6,8,9\}$ of terminal closers of w' is the set of cyclic points of w.

▶ **Example 8** ($a(M,1) = t(M,1)$)**.** When $k = 1$, for a permutation $w = (w_1, \ldots, w_n)$ of M to have a single terminal closer, we must have some $n_s \geq 2$ so that $w_1 = w_2$. Following w_1 and w_2, we can have any of the $(n-2)!/(\prod_i n_i'!)$ permutations of the remaining elements of M, where $n_i' = n_i$ for $i \neq s$ and $n_s' = n_s - 2$. Summing over all s such that $n_s \geq 2$, we obtain $t(M,1)$. For instance, when $M = \{1,1,1,1,2,2,2,2,3,3\} = (1^4, 2^4, 3^2)$, we obtain

$$t(M,1) = \frac{8!}{2! \cdot 4! \cdot 2!} + \frac{8!}{4! \cdot 2! \cdot 2!} + \frac{8!}{4! \cdot 4!} = 910.$$

▶ **Example 9** ($a(M, j_M)$)**.** If every m_i in M appears as a terminal closer, then we have $j_M!$ ways to permute the terminal closers and can choose any permutation of $\left(m_1^{n_1-1}, \ldots, m_{j_M}^{n_{j_M}-1}\right)$ to follow. Thus,

$$t(M, j_M) = j_M! \frac{(n - j_M)!}{\prod_i (n_i - 1)!}, \qquad \text{and} \qquad a(M, j_M) = \frac{(n-j_M)!(j_M - 1)!}{\prod_i (n_i - 1)!}.$$

3 Mappings and permutations

Any set of permutation invariant mappings is a union of certain S_M's, where each M is an n-element multiset of positive integers taken from $[n]$. In particular, $\mathcal{F}_n$ is the union of S_M for all n-element multisets M of $[n]$. Permutations are trivially permutation invariant and $S_n = S_M$ when the n-element multiset M exactly equals $[n]$. Note that $S_n \subseteq \mathcal{F}_n$. Given a mapping $f = (f_1, \ldots, f_n) \in \mathcal{F}_n$, let G_f denote the digraph on the vertex set $[n]$ with edges $i \to f_i$.

Mappings and permutations have been well studied. We present some standard results in this section.

▶ **Theorem 10** (see [10, Section 3.3.13])**.** *The number of mappings f of length n where G_f has k_i cycles of length i for every $i \in [n]$ is given by*

$$\frac{1}{\prod_{i=1}^{n} i^{k_i} k_i!} \binom{n}{k} k! \cdot k n^{n-k-1},$$

where $k = \sum_{i=1}^{n} k_i i$ is the number of cyclic points with $1 \leq k \leq n$.

▶ **Theorem 11.** *Take $1 \leq k \leq n$. The number of mappings f of length n for which G_f is connected and has a cycle of length k is given by*

$$A(n,k) = k! \binom{n}{k} n^{n-k-1}.$$

Proof. Theorem 11 is a special case of Theorem 10 where we take $(k_1, \ldots, k_n)$ to be the elementary vector e_k that is 1 in the kth position and 0 elsewhere. ◀

▶ Remark 12. Let $C(n) = \sum_{k=1}^{n} A(n, k)$. Then $C(n)$ counts the number of mappings f of length n for which G_f is connected. The first few terms of $\{C(n)\}_{n\geq 1}$ are given by

$$1, 3, 17, 142, 1569, 21576, 355081, 6805296,$$

which is [17, OEIS A001865].

Following the notation in Theorem 11, $A(n, i)$ is the number of mappings f of length n such that G_f is connected and has a cycle of length i. For G_f with cycle type $\lambda = (1^{k_1(\lambda)}, 2^{k_2(\lambda)}, \ldots)$, form a monomial

$$u_f = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n},$$

where p_j is a power sum symmetric function. Define the augmented cycle index for mappings

$$Z_n = \sum_{f \in \mathcal{F}_n} u_f,$$

where the sum is over all mappings of length n, so Z_n is an inhomogeneous symmetric function and $A(n, i)$ is the coefficient of p_i in Z_n.

▶ **Proposition 13.** *The augmented cycle index for mappings satisfies*

$$Z_n = \sum_{f \in \mathcal{F}_n} u_f = \sum_{i=1}^{n} iA(n, i)h_i,$$

where h_i is a complete symmetric function (the sum of all monomials of degree i).

Proof. Note that mappings are permutation invariant and thus $\mathcal{F}_n$ is a union of certain S_M's. The result then follows readily from Proposition 2. ◀

▶ **Example 14.** Take $n = 3$. We compute that

$$Z_3 = 9p_1 + 6p_2 + 2p_3 + 6p_1^2 + 3p_1p_2 + p_1^3 = 9h_1 + 12h_2 + 6h_3.$$

▶ **Corollary 15.**

$$\sum_{i=1}^{n} iA(n, i) = n^n.$$

Proof. We take $x_1 = 1$ and $x_i = 0$ for all $i \geq 2$, which yields that $p_i = h_i = 1$ for all $i \geq 1$ and $u_f = 1$ for all $f \in \mathcal{F}_n$. By Proposition 13, the claimed identity is then immediate. ◀

Since permutations are mappings with exactly n cyclic points, the following results may be seen as special cases of Theorems 10 and 11 and Proposition 13.

▶ **Theorem 16.** *The number of permutations σ of length n where G_σ has k_i cycles of length i for every $i \in [n]$ is given by*

$$\frac{1}{\prod_{i=1}^{n} i^{k_i} k_i!} n!\,.$$

▶ **Theorem 17.** *The number of permutations σ of length n for which G_σ is connected (and thus has a cycle of length n) is given by*

$$A(n,n) = (n-1)!\,.$$

▶ **Proposition 18.** *The augmented cycle index for permutations satisfies*

$$Z_n = \sum_{\sigma \in S_n} u_\sigma = n! h_n,$$

where h_n is a complete symmetric function (the sum of all monomials of degree n).

▶ **Example 19.** Take $n = 3$. We compute that

$$Z_3 = 2p_3 + 3p_1p_2 + p_1^3 = 6h_3.$$

4 Parking functions

Lemma 1 and Proposition 2 provide the core argument for the augmented cycle index decomposition for any union of multiset permutations; mappings and permutations only serve as special cases.

In this section, we study a combinatorial object that lies between mappings and permutations. This combinatorial object is commonly referred to as a parking function and was introduced by Konheim and Weiss [13] in their study of the hash storage structure.

In the classical parking function scenario, we have n parking spaces on a one-way street, labelled $1, 2, \dots, n$ in consecutive order as we drive down the street. There are n cars $C_1, \dots, C_n$. Each car C_i has a preferred space $1 \le \pi_i \le n$. The cars drive down the street one at a time in order $C_1, \dots, C_n$. The car C_i drives immediately to space π_i and then parks in the first available space. Thus if π_i is empty, then C_i parks there; otherwise C_i next goes to space $\pi_i + 1$, $\pi_i + 2$, etc., until it finds an available space to park in (if no such space exists, then C_i leaves the street unparked). If all cars are able to park, then the sequence $\pi = (\pi_1, \dots, \pi_n)$ is called a *parking function* of length n.

Write PF_n for the set of parking functions of length n, so that $|\mathrm{PF}_n| = (n+1)^{n-1}$. An elegant, unpublished proof of this result using a circular symmetry argument was given by Pollak and is recounted in [7] and [21]. Observe that every permutation is a parking function and every parking function is a mapping, so that $S_n \subseteq \mathrm{PF}_n \subseteq \mathcal{F}_n$.

▶ **Theorem 20.** *The number of parking functions π of length n where G_π has k_i cycles of length i for every $i \in [n]$ is given by*

$$\begin{cases} \frac{1}{\prod_{i=1}^n i^{k_i} k_i!} n! & \text{if } k = n, \\ \frac{1}{\prod_{i=1}^n i^{k_i} k_i!} \binom{n+1}{k} k! \cdot k(n+1)^{n-k-2} & \text{if } 1 \le k < n, \end{cases}$$

where $k = \sum_{i=1}^n k_i i$ is the number of cyclic points.

▶ Remark 21. We note that a parking function π of length n has n cyclic points if and only if the parking preferences constitute a permutation of $1, \dots, n$. Thus in this special case, our formula coincides with the number of permutations in S_n of cycle type $(1^{k_1}, 2^{k_2}, \dots)$.

Given $\pi = (\pi_1, \dots, \pi_n) \in \mathrm{PF}_n$, let G_π denote the digraph on the vertex set $[n]$ with edges $i \to \pi_i$. Let $a(n, \lambda)$ be the number of $\pi \in \mathrm{PF}_n$ such that G_π has cycle type $\lambda = (1^{k_1(\lambda)}, 2^{k_2(\lambda)}, \dots)$, i.e., G_π has $k_i(\lambda)$ cycles of length i. Write $a(n, i)$ for $a(n, (i))$.

▶ **Lemma 22.** *For $\lambda \vdash i$, we have*

$$a(n,\lambda) = ia(n,i)/z_\lambda,$$

where $z_\lambda = \prod_{j=1}^{i} j^{k_j} k_j!$ is as defined earlier.

Proof. Note that parking functions are permutation invariant and thus PF_n is a union of certain S_M's. The result then follows readily from Lemma 1. ◀

Proof of Theorem 20. Let M be a multiset for which $S_M \subseteq \mathrm{PF}_n$. By Corollary 7, we have $t(M,k) = ka(M,k)$. Thus, the number $t(n,k)$ of parking functions in PF_n with exactly k terminal closers is $ka(n,k)$. Lemma 22 then implies that, for any partition $\lambda = (1^{k_1}, 2^{k_2}, \ldots)$ of k, we have

$$t(n,k) = z_\lambda a(n,\lambda),$$

where $z_\lambda = \prod_{i=1}^{k} j^{k_i} k_i!$ is as defined earlier.

The case $k = n$ is precisely Theorem 17, so we assume that $k < n$. Using a variation of Pollak's circular symmetry argument [7, 21], we are able to count $t(n,k)$ relatively easily. Consider n cars $C_1, \ldots, C_n$ parking in spaces numbered $1, 2, \ldots, n+1$ on a circular one-way street. Suppose that cars $C_1, \ldots, C_k$ have distinct preferences, and that car C_{k+1} prefers one of these spots. Cars $C_{k+2}, \ldots, C_n$ may have any preferences in $[n+1]$. Of course, all cars will be able to park, and $C_1, \ldots, C_k$ will park in their desired spots. Of the $n+1$ circular rotations of the labels of the parking spots, exactly one corresponds to a parking function π: the one which leaves spot $n+1$ open. If $C_1, \ldots, C_k$ prefer spots $\pi_1, \ldots, \pi_k$ in π, then these are the terminal closers of π, and any set of k terminal closers can be obtained in this manner. Thus,

$$t(n,k) = \frac{k!\binom{n+1}{k} k(n+1)^{n+1-k-2}}{n+1} = k!\binom{n+1}{k} k(n+1)^{n-k-2}.$$

The statement follows. ◀

▶ **Theorem 23.** *The number of parking functions π of length n for which G_π is connected and has a cycle of length k is given by*

$$a(n,k) = \begin{cases} (n-1)! & \text{if } k = n, \\ k!\binom{n+1}{k}(n+1)^{n-k-2} & \text{if } 1 \le k < n. \end{cases}$$

▶ Remark 24. We note that the digraph G_π associated to a parking function π of length n consists of a single cycle of length n if and only if the parking preferences constitute a cycle of $1, \ldots, n$. Thus in this special case, our formula coincides with the number of permutations in S_n which form an n-cycle.

▶ Remark 25. Let $c(n) = \sum_{k=1}^{n} a(n,k)$. Then $c(n)$ counts the number of parking functions π of length n for which G_π is connected. The first few terms of $\{c(n)\}_{n \ge 1}$ are given by

$$1, 2, 9, 63, 600, 7249, 106344, 1837953,$$

which is not in the OEIS [17].

Proof of Theorem 23. Theorem 23 is a special case of Theorem 20 where we take $(k_1, \ldots, k_n)$ to be the elementary vector e_k that is 1 in the kth position and 0 elsewhere. ◀

For G_π with cycle type $\lambda = (1^{k_1(\lambda)}, 2^{k_2(\lambda)}, \dots)$, form a monomial

$$u_\pi = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n},$$

where p_j is a power sum symmetric function. Define the augmented cycle index for parking functions

$$Z_n = \sum_{\pi \in \mathrm{PF}_n} u_\pi,$$

where the sum is over all parking functions of length n, so Z_n is an inhomogeneous symmetric function and $a(n,i)$ is the coefficient of p_i in Z_n.

▶ **Proposition 26.** *The augmented cycle index for parking functions satisfies*

$$Z_n = \sum_{\pi \in \mathrm{PF}_n} u_\pi = \sum_{i=1}^{n} i a(n,i) h_i,$$

where h_i is a complete symmetric function (the sum of all monomials of degree i).

Proof. Note that parking functions are permutation invariant and thus PF_n is a union of certain S_M's. The result then follows readily from Proposition 2. ◀

▶ **Example 27.** Take $n = 3$. We compute that

$$Z_3 = 4p_1 + 3p_2 + 2p_3 + 3p_1^2 + 3p_2 p_1 + p_1^3 = 4h_1 + 6h_2 + 6h_3.$$

▶ **Corollary 28.**

$$\sum_{i=1}^{n} i a(n,i) = (n+1)^{n-1}.$$

Proof. We take $x_1 = 1$ and $x_i = 0$ for all $i \geq 2$, which yields that $p_i = h_i = 1$ for all $i \geq 1$ and $u_\pi = 1$ for all $\pi \in \mathrm{PF}_n$. By Proposition 26, the claimed identity is then immediate. ◀

5 Prime parking functions

In this section, we focus on a subclass of parking functions called *prime parking functions*. A parking function $\pi = (\pi_1, \dots, \pi_n)$ is a prime parking function if for all $1 \leq j \leq n-1$, at least $j+1$ cars have a parking preference in the first j parking spots. Equivalently, π is a prime parking function if it remains a parking function even after removing a coordinate that equals 1. Let PPF_n be the set of prime parking functions of length n. Kalikow (see [23, pages 141–142]) modified Pollak's circular symmetry argument to show that $|\,\mathrm{PPF}_n\,| = (n-1)^{n-1}$, so that $|\,\mathrm{PPF}_n\,| = |\mathcal{F}_{n-1}|$. Observe that PPF_n and S_n are disjoint subsets of PF_n, that is, $\mathrm{PPF}_n \subseteq \mathrm{PF}_n$ and $S_n \subseteq \mathrm{PF}_n$, but $\mathrm{PPF}_n \cap S_n = \emptyset$.

▶ **Theorem 29.** *The number of prime parking functions π of length n where G_π has k_i cycles of length i for every $i \in [n-1]$ is given by*

$$\frac{1}{\prod_{i=1}^{n-1} i^{k_i} k_i!} \binom{n-1}{k} k! k (n-1)^{n-k-2},$$

where $k = \sum_{i=1}^{n-1} k_i i$ is the number of cyclic points with $1 \leq k \leq n-1$.

▶ Remark 30. We note that a prime parking function π of length n cannot have n cyclic points, since by definition π must have at least two coordinates equalling 1 and no coordinate equalling n.

▶ Remark 31. We note that the number of mappings f of length $n-1$ where G_f has k_i cycles of length i for every $i \in [n-1]$ coincides with the number of prime parking functions π of length n where G_π has k_i cycles of length i for every $i \in [n-1]$.

Recall that G_π denotes the digraph on the vertex set $[n]$ with edges $i \to \pi_i$. Let $a'(n,\lambda)$ be the number of $\pi \in \mathrm{PPF}_n$ such that G_π has cycle type $\lambda = (1^{k_1(\lambda)}, 2^{k_2(\lambda)}, \dots)$, i.e., G_π has $k_i(\lambda)$ cycles of length i. Write $a'(n,i)$ for $a'(n,(i))$.

▶ **Lemma 32.** *For $\lambda \vdash i$, we have*

$$a'(n,\lambda) = ia'(n,i)/z_\lambda,$$

where $z_\lambda = \prod_{j=1}^{i} j^{k_j} k_j!$ is as defined earlier.

Proof. Note that prime parking functions are permutation invariant and thus PPF_n is a union of certain S_M's. The result then follows readily from Lemma 1. ◀

Proof of Theorem 29. The proof proceeds similarly as the proof of Theorem 20 for parking functions, except that our circular symmetry argument is now applied to a circle with $n-1$ spots instead of $n+1$ spots as in the classical case. ◀

▶ **Theorem 33.** *Take $1 \le k \le n-1$. The number of prime parking functions π of length n for which G_π is connected and has a cycle of length k is given by*

$$a'(n,k) = k!\binom{n-1}{k}(n-1)^{n-k-2}.$$

▶ Remark 34. Let $c'(n) = \sum_{k=1}^{n-1} a'(n,k)$. Then $c'(n)$ counts the number of prime parking functions π of length n for which G_π is connected. The first few terms of $\{c'(n)\}_{n\ge1}$ are given by

$$0, 1, 3, 17, 142, 1569, 21576, 355081,$$

which by shifting one term is [17, OEIS A001865]. We note that the number of mappings f of length $n-1$ for which G_f is connected coincides with the number of prime parking functions π of length n for which G_π is connected.

Proof of Theorem 33. Theorem 33 is a special case of Theorem 29 where we take $(k_1,\dots,k_n)$ to be the elementary vector e_k that is 1 in the kth position and 0 elsewhere. ◀

For G_π with cycle type $\lambda = (1^{k_1(\lambda)}, 2^{k_2(\lambda)}, \dots)$, form a monomial

$$u_\pi = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n},$$

where p_j is a power sum symmetric function. Define the augmented cycle index for prime parking functions

$$Z_n = \sum_{\pi \in \mathrm{PPF}_n} u_\pi,$$

where the sum is over all prime parking functions of length n, so Z_n is an inhomogeneous symmetric function and $a(n,i)$ is the coefficient of p_i in Z_n.

▶ **Proposition 35.** *The augmented cycle index for prime parking functions satisfies*

$$Z_n = \sum_{\pi \in \mathrm{PPF}_n} u_\pi = \sum_{i=1}^{n} i a'(n,i) h_i,$$

where h_i is a complete symmetric function (the sum of all monomials of degree i).

Proof. Note that parking functions are permutation invariant and thus PPF_n is a union of certain S_M's. The result then follows readily from Proposition 2. ◀

▶ **Example 36.** Take $n = 3$. We compute that

$$Z_3 = 2p_1 + p_2 + p_1^2 = 2h_1 + 2h_2.$$

▶ **Corollary 37.**

$$\sum_{i=1}^{n} i a'(n,i) = (n-1)^{n-1}.$$

Proof. We take $x_1 = 1$ and $x_i = 0$ for all $i \geq 2$, which yields that $p_i = h_i = 1$ for all $i \geq 1$ and $u_\pi = 1$ for all $\pi \in \mathrm{PPF}_n$. By Proposition 35, the claimed identity is then immediate. ◀

6 Asymptotic analysis

For convenience, we say that a multiset permutation (mapping, parking function, prime parking function, etc.) is *indecomposable (connected)* if its digraph is connected, i.e., the digraph possesses exactly one component. Recall that $C(n) = \sum_{k=1}^{n} A(n,k)$ counts the total number of connected mappings (cf. Remark 12). Katz [11] established that the probability that a random mapping is indecomposable is asymptotically equal to $\sqrt{\pi/(2n)}$, that is, $C(n)/n^n \sim \sqrt{\pi/(2n)}$. Rényi [20] further showed that the number of cyclic points (equivalently, the length of the unique cycle), scaled by $\sqrt{n}$, of an indecomposable random mapping converges in distribution to a folded normal distribution $|N(0,1)|$. See also related discussions in the paper of Mutafchiev and Finch [16].

A unifying theme in the probabilistic study of parking functions is the notion of the *equivalence of ensembles*. Diaconis and Hicks [2] stated that for certain statistics, it is natural to expect the distribution of statistics in the *micro-canonical ensemble* PF_n, to be close to the distribution of statistics in the *canonical ensemble* $\mathcal{F}_n$. Indeed, they showed that the equivalence of ensembles holds for statistics such as the number of repeats, lucky cars, and descents. However, they also showed that it fails for some statistics such as the distribution of the value of the first coordinate.

In this section, we will demonstrate the asymptotic equivalence of ensembles between parking functions and mappings concerning the number of cyclic points in their associated connected digraphs, in the sense that both the probability of a random parking function being connected is asymptotically equal to $\sqrt{\pi/(2n)}$ as in [11], and that the number of cyclic points in a random connected parking function, scaled by $n^{-1/2}$, converges in distribution to $|N(0,1)|$ as in [20]. The same also holds true for connected prime parking functions. Finally, we will extend these techniques to connected regular multiset permutations.

We note that this asymptotic equivalence of ensembles between parking functions and mappings concerning cyclic points does not come as a surprise. Indeed, parking functions and mappings behave similarly not just asymptotically but also combinatorially. Since parking functions and mappings are both unions of multiset permutations, a direct consequence of

Theorem 5 is that the number of parking functions/mappings of length n with k cyclic points respectively equals the number of parking functions/mappings of length n with k terminal closers. The number of parking functions of length n with k terminal closers was computed in the proof of Theorem 20 using a variation of Pollak's circular symmetry argument and was shown to be $k!\binom{n+1}{k}k(n+1)^{n-k-2}$. Following the definition for terminal closers, the number of mappings of length n with k terminal closers may be easily calculated as $k!\binom{n}{k}kn^{n-k-1}$ (see also Theorem 10, where we cited a related result from standard literature). We see that the two numbers and the way they are obtained display striking similarity.

We remark that the natural analogous question for the number of cyclic points in a uniformly random parking function has been answered in [18], where the authors showed that this number is asymptotically Rayleigh-distributed after rescaling by $n^{-1/2}$.

Throughout this section, we use $\mathsf{Poi}(\lambda)$ to denote a Poisson-distributed random variable with parameter $\lambda > 0$, and $\mathsf{NB}(\ell, p)$ to denote a negative binomial random variable, i.e. the random variable counting the number of failures in a sequence of i.i.d. Bernoulli-trials with success probability p before seeing the ℓ-th success. We note here the very well-known central limit theorems

$$\frac{\mathsf{Poi}(n) - n}{\sqrt{n}} \xrightarrow{d} N(0,1) \qquad \text{and} \qquad \frac{\mathsf{NB}(\ell,p) - \ell(1-p)/p}{\sqrt{\ell(1-p)/p^2}} \xrightarrow{d} N(0,1)$$

for $n \to \infty$ and $\ell \to \infty$ with $p \in (0,1)$ constant, respectively. Both of these statements can easily be obtained after noting that $\mathsf{Poi}(n)$ is the sum of n i.i.d. $\mathsf{Poi}(1)$ random variables, and $\mathsf{NB}(\ell, p)$ is the sum of ℓ i.i.d. geometric random variables with parameter p.

6.1 Connected parking functions

Recall that $c(n) = \sum_{k=1}^{n} a(n,k)$ counts the total number of connected parking functions (cf. Remark 25).

▶ **Lemma 38.** *We have $c(n) \sim \sqrt{\pi/2} \cdot en^{n-3/2}$, so that $c(n)/(n+1)^{n-1} \sim \sqrt{\pi/(2n)}$.*

Proof. We first consider $S_n = \sum_{k=1}^{n-1} a(n,k)$. By rewriting the above expression for $a(n,k)$, we obtain

$$\begin{aligned} S_n &= \frac{(n+1)!}{(n+1)^3} \sum_{k=1}^{n-1} \frac{(n+1)^{n+1-k}}{(n+1-k)!} = \frac{(n+1)!}{(n+1)^3} \sum_{j=2}^{n} \frac{(n+1)^j}{j!} \\ &= \frac{(n+1)!}{(n+1)^3} e^{n+1} \mathbb{P}\big(2 \le \mathsf{Poi}(n+1) \le n\big). \end{aligned} \tag{2}$$

By the central limit theorem, the probability converges to $1/2$; combining this with Stirling's formula for $(n+1)!$ yields

$$S_n \sim \frac{\sqrt{2\pi}}{2}(n+1)^{n+3/2-3},$$

which simplifies to the desired expression. It remains to show that $a(n,n) = o(S_n)$. For this, note that

$$\frac{a(n,n)}{S_n} \sim \frac{\sqrt{2\pi}(n-1)^{n-1/2}e^{-(n-1)}}{\sqrt{\pi/2}(n+1)^{n-3/2}} = \frac{2(n-1)}{e^{n-1}}\left(1 - \frac{2}{n+1}\right)^{n-3/2} \sim \frac{2(n-1)}{e^{n+1}} \to 0,$$

therefore $c(n) \sim S_n$, as required. ◀

▶ Remark 39. The sum in (2) is well-studied: the Ramanujan Q-function is defined by $Q(n) = \frac{n!}{n^n}\sum_{k=0}^{n-1}\frac{n^k}{k!}$, which can also be written as $\frac{n!e^n}{n^n}\mathbb{P}\big(\mathsf{Poi}(n) \leq n-1\big)$. We have $S_n = (n+1)^{n-2}Q(n+1) - \frac{(n+2)!}{(n+1)^3}$, and it is known that

$$Q(n) = \sqrt{\frac{\pi n}{2}} - \frac{1}{3} + \frac{1}{12}\sqrt{\frac{\pi}{2n}} - \frac{4}{135n} + O(n^{-3/2}).$$

The Ramanujan Q-function is a special case of the more general Q_r-function, where

$$Q_r(m,n) = \binom{r}{0} + \binom{r+1}{1}\frac{n}{m} + \binom{r+2}{2}\frac{n(n-1)}{m^2} + \cdots$$

is a generalized hypergeometric function of the second kind, and $Q(n) = Q_0(n, n-1)$. See [5] and [4].

Building upon Lemma 38, we can now determine the limiting distribution of L_n, the number of cyclic points in a uniform random connected parking function.

▶ **Theorem 40.** *We have* $n^{-1/2}L_n \xrightarrow{d} |N(0,1)|$ *as* $n \to \infty$.

Proof. Fix $x > 0$ and set $R = R_n = \lfloor x\sqrt{n}\rfloor$. Consider n large enough so that $R < n$. Similarly to (2), we compute

$$\begin{aligned}\mathbb{P}(L_n \leq R) &= \frac{1}{c(n)}\sum_{k=1}^{R} a(n,k) = \frac{1}{c(n)}\frac{(n+1)!}{(n+1)^3}e^{n+1}\mathbb{P}\big(n+1-R \leq \mathsf{Poi}(n+1) \leq n\big)\\ &\sim 2\int_{-\frac{R}{\sqrt{n+1}}}^{-\frac{1}{\sqrt{n+1}}} \frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,\mathrm{d}y \sim \int_{-x}^{x}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,\mathrm{d}y, \qquad (3)\end{aligned}$$

where we again relied on the central limit theorem, using $c(n) \sim \frac{(n+1)!}{(n+1)^3}e^{n+1}/2$ as a consequence of (2). Noting that the left-hand side of (3) equals $\mathbb{P}(n^{-1/2}L_n \leq x)$ while the right-hand side is the cumulative distribution function of $|N(0,1)|$ concludes the proof. ◀

Recall that $c'(n) = \sum_{k=1}^{n-1} a'(n,k)$ counts the total number of connected prime parking functions (cf. Remark 34). Let L_n' be the number of cyclic points in a uniform random connected prime parking function.

▶ **Proposition 41.** *We have* $c'(n) \sim \sqrt{\pi/2}e^{-1}n^{n-3/2}$ *(so that* $c'(n)/(n-1)^{n-1} \sim \sqrt{\pi/(2n)}$*) and* $n^{-1/2}L_n' \xrightarrow{d} |N(0,1)|$.

Proof. The proof works analogously to those of Lemma 38 and Theorem 40. Indeed, imitating the approach of (2), together with the central limit theorem and a Stirling approximation yields

$$\begin{aligned}c'(n) &= (n-2)!\sum_{k=1}^{n-1}\frac{(n-1)^{n-k-1}}{(n-k-1)!} = (n-2)!e^{n-1}\mathbb{P}(0 \leq \mathsf{Poi}(n-1) \leq n-2)\\ &\sim \frac{1}{2}\sqrt{2\pi}(n-1)^{n-1+1/2-1},\end{aligned}$$

which simplifies to the desired asymptotics for $c'(n)$. For the second part, we again fix $x > 0$, set $R = \lfloor x\sqrt{n}\rfloor$ and compute, as before,

$$\begin{aligned}\mathbb{P}(L_n' \leq R) &= \frac{1}{c'(n)}\sum_{k=1}^{R} a'(n,k) = \frac{(n-2)!}{c'(n)}e^{n-1}\mathbb{P}(n-1-R \leq \mathsf{Poi}(n-1) \leq n-2)\\ &\sim 2\int_{-\frac{R}{\sqrt{n-1}}}^{-\frac{1}{\sqrt{n-1}}}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,\mathrm{d}y \sim \int_{-x}^{x}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}\,\mathrm{d}y.\end{aligned}$$ ◀

6.2 Connected regular multiset permutations

Fix $r \geq 2$ and consider permutations of the r-*regular multiset* $M = (1^r, 2^r, \dots, j^r)$. Imitating the calculations in Example 9, we find that

$$t(M,k) = \binom{j}{k} k! \cdot k \cdot \frac{(jr-k-1)!}{r!^{j-k}(r-1)!^{k-1}(r-2)!},$$

which reduces to the special case in Example 9 when $j = k$. Indeed, we first choose k distinct elements from $[j]$ to be our terminal closers. They can appear in any order among the first k elements of the permutation. The $(k+1)$-th element needs to repeat one of these k values. The remaining $jr - k - 1$ positions are filled with the remaining elements from the multiset. By Corollary 7, we thus have

$$a(M,k) = \binom{j}{k} k! \frac{(jr-k-1)!}{r!^{j-k}(r-1)!^{k-1}(r-2)!} = \frac{r-1}{r!^j}\binom{j}{k} k! r^k (jr-k-1)!.$$

Let $c_M(j) := \sum_{k=1}^{j} a(M,k)$ be the total number of connected multiset permutations.

▶ **Theorem 42.** *For $r \geq 2$ fixed and M as above, we have*

$$c_M(j) \sim j^{1/2} \frac{(jr-1)!}{r!^j} \sqrt{\frac{\pi r(r-1)}{2}} \tag{4}$$

for $j \to \infty$. Moreover, let L_j denote the number of cyclic points in a uniform random connected multiset permutation of M. Then

$$j^{-1/2} L_j \xrightarrow{d} \left| N\left(0, \frac{r}{r-1}\right) \right|.$$

Proof. We argue as in Section 6.1, just with a negative binomial distribution instead of Poisson. We have

$$\begin{aligned}
c_M(j) &= \sum_{k=1}^{j} a(M,k) = \sum_{k=1}^{j} \frac{r-1}{r!^j}\binom{j}{k} k! r^k (jr-k-1)! \\
&= \frac{r-1}{r!^j} j! \sum_{k=1}^{j} \frac{r^k (jr-k-1)!}{(j-k)!} = \frac{r-1}{r!^j} j! \sum_{h=0}^{j-1} \frac{r^{j-h}(jr-j+h-1)!}{h!} \\
&= \frac{(r-1) r^j j! (jr-j-1)!}{r!^j} \sum_{h=0}^{j-1} \binom{jr-j+h-1}{h} r^{-h} \\
&= \frac{r^{jr} j! (jr-j-1)!}{(r-1)^{jr-j-1} r!^j} \sum_{h=0}^{j-1} \binom{jr-j+h-1}{h} (1-r^{-1})^{jr-j} r^{-h} \\
&= \frac{r^{jr} j! (jr-j-1)!}{(r-1)^{jr-j-1} r!^j} \mathbb{P}\big(\mathsf{NB}(jr-j, 1-r^{-1}) \leq j-1\big).
\end{aligned}$$

The probability $\mathbb{P}(\cdot)$ tends to $1/2$ as in the previous instances. Applying Stirling's formula, one gets

$$\begin{aligned}
\frac{j!(jr-j-1)!}{(jr-1)!} &\sim \sqrt{\frac{2\pi j(jr-j-1)}{jr-1}} \cdot \frac{j^{j+jr-j-1}}{j^{jr-1}} \frac{\left(r-1-\frac{1}{j}\right)^{jr-j-1}}{\left(r-\frac{1}{j}\right)^{jr-1}} \\
&\sim j^{1/2} \sqrt{\frac{2\pi(r-1)}{r}} \frac{(r-1)^{jr-j-1} e^{-1}}{r^{jr-1} e^{-1}}.
\end{aligned}$$

Plugging this into the full expression for $c_M(j)$ leads to some more cancellations and (4).

For the second part, we again fix $x > 0$, set $R = \lfloor x\sqrt{j} \rfloor$ and compute

$$\begin{aligned}\mathbb{P}(L_j \leq R) = \frac{1}{c_M(j)} \sum_{k=1}^{R} a(M,k) &\sim 2\mathbb{P}\big(j - R \leq \mathsf{NB}(jr - j, 1 - r^{-1}) \leq j - 1\big),\\ &\sim 2\int_{-\frac{R}{\sqrt{j\frac{r}{r-1}}}}^{-\frac{1}{\sqrt{j\frac{r}{r-1}}}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,\mathrm{d}y \sim \int_{-x}^{x} \frac{1}{\sqrt{2\pi\frac{r}{r-1}}} e^{-\frac{y^2}{2\frac{r}{r-1}}}\,\mathrm{d}y,\end{aligned}$$

which is the cumulative distribution function of $\left|N\left(0, \frac{r}{r-1}\right)\right|$. ◀

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