Enumeration of Bipartite Acyclic Digraphs

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Abstract

We consider the asymptotic enumeration of labelled acyclic digraphs (DAGs) with the additional restriction of being bipartite. The analysis leads us to a meromorphic generating function in two variables for the number of bicoloured labelled DAGs whose analysis falls within the scope of analytic combinatorics in several variables. This allows us to obtain asymptotic formulas for the total number of labelled bipartite DAGs with a given number of vertices as well as for the number of such DAGs with a given bipartition (i.e., with prescribed sizes of the two partite sets).

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1 Introduction

The enumeration of acyclic digraphs or DAGs (directed acyclic graphs) is a classical combinatorial problem whose history goes back more than 50 years, to the work of Liskovec [7, 8], Robinson [13, 14], and Stanley [16]. It involves the somewhat unusual notion of a *special generating function* (as opposed to an ordinary or exponential generating function): if we let a_n be the number of DAGs on n labelled vertices, then this special generating function is

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n! 2^{\binom{n}{2}}}, \tag{1}$$

which satisfies

$$A(x) = \frac{1}{\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n! 2^{\binom{n}{2}}}}. \tag{2}$$

The denominator of this function is easily seen to represent an entire function of x, and it follows from general results due to Schur and Pólya [15] that all its zeros are in fact positive real numbers. The smallest of these zeros is $z_0 \approx 1.48808$, the second-smallest is $z_1 \approx 4.88114$. Thus a straightforward application of singularity analysis in the meromorphic setting yields the asymptotic formula

$$a_n \sim A \cdot z_0^{-n} \cdot n! \cdot 2^{\binom{n}{2}} \text{ as } n \to \infty \tag{3}$$

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for a constant $A \approx 1.74106$. This can also be extended to a bivariate generating function involving the number of edges as a second parameter, which allowed Bender, Richmond, Robinson and Wormald [1, 2] to analyse the number of DAGs with a given number of vertices and edges.

Various other statistics of DAGs have been studied in the literature as well, such as the number of sources and sinks [5] and the height [9] of a random DAG. We also mention that different types of DAGs with additional restrictions have been enumerated, such as *essential* [17] and *extensional* [18] acyclic digraphs (with asymptotic behaviour very similar to (3)). The asymptotic enumeration of DAGs resulting from a compression algorithm applied to binary trees was considered in [4], and ordered DAGs in [12].

In this paper, we study a natural graph-theoretical restriction by considering acyclic digraphs that are *bipartite*. As a first step towards the enumeration of bipartite DAGs, one considers bicoloured DAGs, for which a special type of generating function similar to (1) is available. As we will show in Section 3, if $g_{p,q}$ is the number of labelled bicoloured acyclic digraphs with p vertices in one colour and q in the other such that there is no edge between vertices of the same colour, the generating function

$$G(x,y) = \sum_{p,q \geq 0} g_{p,q} \frac{x^p y^q}{(p+q)! 2^{pq}} \tag{4}$$

can be written as

$$G(x,y) = \frac{1}{F(x,y)},$$

where

$$F(x,y) = \sum_{s,t \geq 0} \frac{(-1)^{s+t} x^s y^t}{s! t! 2^{st}}. \tag{5}$$

This represents an entire function in the two variables x and y, in a similar way as the denominator in (2) is an entire function. This means that we can analyse the generating function $G(x,y)$ using the powerful framework of analytic combinatorics in several variables [10, 11].

While analytic combinatorics in several variables is now a well-established field and there are many known examples where results in this area are successfully applied to the asymptotic analysis of rational generating functions, examples of non-rational meromorphic generating functions and their analysis are somewhat scarce in the literature. Thus we hope that this paper makes an interesting contribution to the theory of multivariate analytic combinatorics as well. In particular, some of our techniques to verify the technical conditions that are needed to apply general asymptotic formulas might be of independent interest.

2 Statement of results

Let us now state the main results of this paper concerning the enumeration of bicoloured and bipartite DAGs. We start with the number $g_{p,q}$ of labelled bicoloured DAGs that was already mentioned in the introduction and whose (special) generating function is given by (4).

▶ **Theorem 1.** *The number $g_{p,q}$ of labelled bicoloured DAGs with p vertices in one colour and q vertices in the other is asymptotically given by*

$$g_{p,q} \sim c q^{-1/2} \rho_x^{-p} \rho_y^{-q} (p+q)!\, 2^{pq} \tag{6}$$

as $p, q \to \infty$ and $\frac{p}{q}$ remains constant, where c, ρ_x, ρ_y are positive numbers that depend only on the ratio $\frac{p}{q}$ and that satisfy

(a) *if* $\frac{p}{q}$ *is chosen from any fixed compact interval,* c, ρ_x, ρ_y *are uniformly bounded away from both* 0 *and* ∞ *and the convergence in* (6) *is uniform;*
(b) $c, \rho_x, \rho_y \colon \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous, and* ρ_x, ρ_y *are bijective, with* ρ_x *being strictly increasing and* ρ_y *being strictly decreasing;*
(c) $\rho_x\left(\frac{p}{q}\right) = \rho_y\left(\frac{q}{p}\right)$*;*
(d) $\rho := \rho_x(1) = \rho_y(1) \approx 1.15196$ *is the smallest positive zero of the function* $F(x, x)$*, with* F *as in* (5)*, and* $c(1) \approx 2.12560$*;*
(e) *for every* $r \in \mathbb{R}^+$*, there exists* $\alpha = \alpha(r) > 0$ *such that*

$$\rho_x(r+h) = \big(1 + \alpha h + O(h^2)\big)\rho_x(r) \qquad \text{and} \qquad \rho_y(r+h) = \big(1 - \alpha r h + O(h^2)\big)\rho_y(r)$$

as $h \to 0$*.*

It turns out that most bipartite DAGs are connected. For connected bipartite DAGs, the bipartition is unique up to switching the two partite sets. We thus obtain the following result.

▶ **Theorem 2.** *The number of labelled bipartite DAGs that have a bipartition whose partite sets contain* p *respectively* q *vertices is asymptotically given by*

$$b_{p,q} \sim \begin{cases} \frac{1}{2} c q^{-1/2} \rho_x^{-p} \rho_y^{-q} (p+q)!\, 2^{pq} & \text{for } p = q, \\ c q^{-1/2} \rho_x^{-p} \rho_y^{-q} (p+q)!\, 2^{pq} & \text{otherwise} \end{cases}$$

as $p, q \to \infty$ *and* $\frac{p}{q}$ *remains constant, where* c, ρ_x, ρ_y *are as in Theorem 1.*

This can be exploited to obtain an asymptotic formula for the total number of labelled bipartite DAGs with a given number of vertices.

▶ **Theorem 3.** *The number* b_n *of bipartite DAGs with* n *labelled vertices is asymptotically given by*

$$b_n \sim \gamma_n n^{-1/2} \rho^{-n} \cdot n! \cdot 2^{\frac{n^2}{4}}$$

as $n \to \infty$*, where* $\rho \approx 1.15196$ *and* γ_n *only depends on the parity of* n*, with*

$$\gamma_n \approx \begin{cases} 3.19984 & \text{if } n \text{ is even}, \\ 3.19983 & \text{if } n \text{ is odd}. \end{cases}$$

3 The bivariate generating function

We start by deriving the generating function. Let black and white be the two colours; x marks the black vertices, and y marks the white ones. Let $\mathcal{G}_{p,q}$ be the class of vertex-labelled, bicoloured DAGs with p black vertices and q white vertices, and let $g_{p,q} = |\mathcal{G}_{p,q}|$. Suppose $\mathcal{G}_{0,0}$ contains only the null graph with no vertex. The first few terms of $g_{p,q}$ are as follows:

$p \backslash q$	0	1	2	3	4
0	1	1	1	1	1
1	1	6	27	108	405
2	1	27	474	6810	86865
3	1	108	6810	327660	13326915
4	1	405	86865	13326915	1651166650

Inspired by (1), we define a special generating function and its reciprocal

$$\frac{1}{F(x,y)} := G(x,y) := \sum_{p,q\geq 0} g_{p,q} \frac{x^p y^q}{(p+q)!2^{pq}}.$$

Consider removing some vertices without out-edges (i.e., sinks), say s black ones and t white ones, from a DAG in $\mathcal{G}_{p,q}$ when $p+q>0$. There are $2^{s(q-t)+(p-s)t}$ possible configurations of the in-edges of these vertices. By the inclusion-exclusion principle, we thus obtain the recurrence relation

$$g_{p,q} = \sum_{\substack{0\leq s\leq p \\ 0\leq t\leq q \\ s+t\geq 1}} (-1)^{s+t-1} \binom{p+q}{s+t} \binom{s+t}{s} 2^{s(q-t)+(p-s)t} g_{p-s,q-t},$$

for $p \geq 0, q \geq 0$ and $p+q>0$. That is,

$$\begin{aligned} 1_{p=q=0} &= \sum_{\substack{0\leq s\leq p \\ 0\leq t\leq q}} (-1)^{s+t} \binom{p+q}{s+t} \binom{s+t}{s} 2^{s(q-t)+(p-s)t} g_{p-s,q-t} \frac{1}{2^{pq}} \\ &= (p+q)! \sum_{\substack{0\leq s\leq p \\ 0\leq t\leq q}} \left(\frac{(-1)^{s+t}}{s!t!2^{st}} \right) \cdot \left(g_{p-s,q-t} \frac{1}{((p-s)+(q-t))!2^{(p-s)(q-t)}} \right). \end{aligned}$$

This formula shows exactly the coefficient of the product of two reciprocal formal power series. Hence the reciprocal of $G(x,y)$ is

$$F(x,y) = \sum_{s,t\geq 0} \frac{(-1)^{s+t}}{s!t!2^{st}} x^s y^t.$$

4 Analysis of the generating function

Our goal is to apply the following general result to our function $G(x,y) = 1/F(x,y)$:

▶ **Theorem 4** (Theorem 3.1 of [10]). *Let $A(x,y) = \sum_{s,t\geq 0} a_{s,t} x^s y^t$ be a meromorphic function that is not singular at the origin, and let B, C be entire functions such that $A = B/C$. Define*

$$Q(x,y) := -x^2 (C_x)^2 y C_y - x C_x y^2 (C_y)^2 - x^2 y^2 \big((C_x)^2 C_{yy} + (C_y)^2 C_{xx} - 2 C_x C_y C_{xy} \big).$$

Let U be a compact subset of $\mathbb{C}^2$ such that for each point $(x_0,y_0) \in U$, the following holds:

- *(x_0,y_0) is a pole of A of order 1, i.e. $C(x_0,y_0)=0$, but both $B(x_0,y_0)$ and the gradient $\nabla C(x_0,y_0)$ are nonzero.*
- *A has no singularity $(x_1,y_1) \neq (x_0,y_0)$ with both $|x_1| \leq |x_0|$ and $|y_1| \leq |y_0|$.*
- *$Q(x_0,y_0) \neq 0$.*

Then uniformly over $(x_0,y_0) \in U$, the coefficients $a_{s,t}$ of A for which

$$(s,t) = -\lambda \cdot (x_0 C_x(x_0,y_0), y_0 C_y(x_0,y_0))$$

satisfy the asymptotic formula

$$a_{s,t} \sim \frac{B(x_0,y_0)}{\sqrt{2\pi}} x_0^{-s} y_0^{-t} \frac{1}{\sqrt{\lambda Q(x_0,y_0)}}$$

as $\lambda \to \infty$.

We shall apply this result for $A = G$, $B = 1$, and $C = F$. In the following, we determine the set of zeros of $F(x, y)$ and prove the properties of those zeros listed in Theorem 4.

We begin by observing that the partial derivatives of F at a given point (x, y) can in fact be computed by evaluating F at a different point:

$$\begin{aligned} F_x(x,y) &= \sum_{s\geq 1,t\geq 0} \frac{(-1)^{s+t}}{(s-1)!t!2^{st}} x^{s-1}y^t = -\sum_{s,t\geq 0} \frac{(-1)^{s+t}}{s!t!2^{st}} x^s \left(\frac{y}{2}\right)^t = -F\left(x, \frac{y}{2}\right), \\ F_y(x,y) &= -F\left(\frac{x}{2}, y\right), \end{aligned} \tag{7}$$

where the argument for the second identity is analogous to the first identity.

4.1 The zeros of the denominator

In this subsection, we shall determine the curve of "minimal" singularities of G, within which the compact set U in Theorem 4 will be chosen, and derive its most important properties. Being a generating function of combinatorial objects, $G(x, y) = \frac{1}{F(x,y)}$ has only nonnegative coefficients. By Pringsheim's Theorem, at the radius of convergence, which might be ∞, there must be a singularity. For $x \in \mathbb{R}^+$, let us define $y_0(x) := \inf\{y \in \mathbb{R}^+ \mid F(x, y) = 0\}$.

▶ **Lemma 5.** *For every $x \in \mathbb{R}^+$, $y_0(x)$ defined above is a finite value. Furthermore,*

(a) *$y_0(x)$ is symmetric: If $a = y_0(b)$, then $b = y_0(a)$;*

(b) *$y_0(x) \leq \frac{\rho^2}{x}$, where $\rho \approx 1.15196$ is the smallest positive zero of $F(x, x)$;*

(c) *$y_0(x)$ is strictly decreasing;*

(d) *For sufficiently small x, we have $-2\log x < y_0(x) < -2\log x + \sqrt{x} + \frac{x}{2}$, thus in particular $y_0(x) = -2\log x + O(\sqrt{x})$.*

Proof. Any singularity of G is a zero of F, as F is entire. We rearrange the terms of $F(x, x)$ by summing over $n = s + t$:

$$\begin{aligned} F(x,x) &= \sum_{s,t\geq 0} \frac{(-1)^{s+t}}{s!t!2^{st}} x^{s+t} = \sum_{n\geq 0}\sum_{s=0}^{n} \frac{(-x)^n}{s!(n-s)!2^{s(n-s)}} \\ &\leq \sum_{n=0}^{N}\sum_{s=0}^{n} \frac{(-x)^n}{s!(n-s)!2^{s(n-s)}} + \left|\sum_{n>N}\sum_{s=0}^{n} \frac{(-x)^n}{s!(n-s)!2^{s(n-s)}}\right|. \end{aligned}$$

Since

$$\sum_{s=0}^{n} \frac{1}{s!(n-s)!2^{s(n-s)}} \leq \frac{2}{n!} + \sum_{s=1}^{n-1} \frac{1}{s!(n-s)!2^{n-1}} = \frac{2}{n!} + \frac{2^n-2}{n!2^{n-1}} \leq \frac{4}{n!},$$

we have, for $x \in \mathbb{R}^+$,

$$F(x,x) \leq \sum_{n=0}^{N}\sum_{s=0}^{n} \frac{(-x)^n}{s!(n-s)!2^{s(n-s)}} + \sum_{n>N} \frac{4}{n!} x^n.$$

For $N = 6$, this implies that $F(1.2, 1.2) \leq -0.00681 + 0.00334 < 0$. By the intermediate value theorem, the function $F(x, x)$ has at least one zero in the interval $[0, 1.2]$.

The derivative of $F(x, x)$ can also be estimated similarly:

$$\begin{aligned} \frac{d}{dx}F(x,x) &\leq \sum_{n=1}^{5}\sum_{s=0}^{n} \frac{n(-1)^n x^{n-1}}{s!(n-s)!2^{s(n-s)}} + \sum_{n=6}^{\infty} \frac{4nx^{n-1}}{n!} \\ &= -2 + 3x - \frac{7x^2}{4} + \frac{9x^3}{16} - \frac{47x^4}{384} + \sum_{n=5}^{\infty} \frac{4x^n}{n!}. \end{aligned}$$

The maximum of the fourth degree polynomial $-2+3x-\frac{7x^2}{4}+\frac{9x^3}{16}-\frac{47x^4}{384}$ on $[0,1.2]$ is attained at $x=1.2$ and is equal to -0.20180. On the other hand, $\sum_{n=5}^{\infty}\frac{4x^n}{n!}$ is bounded by $\sum_{n=5}^{\infty}\frac{4\cdot(1.2)^n}{n!}\approx 0.10287$. It follows that $\frac{d}{dx}F(x,x)<0$ for $x\le 1.2$, which means that there is *exactly one* zero ρ of $F(x,x)$ with $0<x<1.2$.

One can determine the value of this zero quite precisely. We continue to use the estimate

$$\left|F(x,x)-\sum_{n=0}^{N}\sum_{s=0}^{n}\frac{(-x)^n}{s!(n-s)!2^{s(n-s)}}\right|\le\sum_{n>N}\frac{4}{n!}x^n.$$

For $0\le x\le 1.2$ and $N=20$, the sum on the right side is bounded above by $4\cdot 10^{-18}$. Thus by establishing that

$$\sum_{n=0}^{N}\sum_{s=0}^{n}\frac{(-x_1)^n}{s!(n-s)!2^{s(n-s)}}\approx 1.02366\cdot 10^{-13}$$

for $x_1=1.151961260017$ and

$$\sum_{n=0}^{N}\sum_{s=0}^{n}\frac{(-x_2)^n}{s!(n-s)!2^{s(n-s)}}\approx -8.32808\cdot 10^{-14}$$

for $x_2=1.151961260018$, we find that the zero lies in the interval $[x_1,x_2]$, so that the decimal expansion must be

$$\rho=1.151961260017\ldots. \tag{8}$$

For any point (x,y) for which $G(x,y)$ converges (absolutely), we have

$$G(x,y)=\frac{1}{2}\sum_{s,t}g_{s,t}\frac{x^sy^t+y^sx^t}{(s+t)!2^{st}}\ge\frac{1}{2}\sum_{s,t}g_{s,t}\frac{2\sqrt{x^{s+t}y^{s+t}}}{(s+t)!2^{st}}=G(\sqrt{xy},\sqrt{xy}).$$

Thus $G(x,y)$ diverges on $xy=\rho^2$. This result also follows directly from the logarithmic convexity of the domain of convergence of complex power series (see for example [6, Chapter II, Proposition 7.4]). This in particular proves that the radius of convergence of G is finite, and since G can only become singular at zeros of F, it shows that $y_0(x)$ exists for every $x>0$ and that $y_0(x)\le\rho^2/x$. The symmetry of y_0 follows directly from the symmetry of $F(x,y)$.

Item (c) follows by implicit differentiation:

$$y_0'(x)=\frac{-F_x(x,y_0(x))}{F_y(x,y_0(x))}=\frac{-F(x,\frac{y_0(x)}{2})}{F(\frac{x}{2},y_0(x))}<0.$$

It remains to prove (d). To that end, we use the representation

$$\begin{aligned}F(x,y)&=\sum_{s,t\ge 0}\frac{(-1)^{s+t}x^sy^t}{s!t!2^{st}}=\sum_{s\ge 0}\frac{(-1)^sx^s}{s!}e^{-y/2^s}\\&=e^{-y}-xe^{-y/2}+\sum_{k\ge 1}\left(\frac{x^{2k}}{(2k)!}e^{-y/2^{2k}}-\frac{x^{2k+1}}{(2k+1)!}e^{-y/2^{2k+1}}\right).\end{aligned}$$

We claim that $F(x,y)>0$ for $0\le x\le 1$ and $0\le y\le -2\log x$. To prove this, let $a\in[0,2]$, possibly depending on x. Then

$$F(x,-a\log x)=x^a-x^{1+a/2}+\sum_{k\ge 1}\phi_k\quad\text{with }\phi_k=\frac{x^{2k+a/2^{2k}}}{(2k)!}-\frac{x^{2k+1+a/2^{2k+1}}}{(2k+1)!}.$$

If $0 \le x \le 1$, then $x^a - x^{1+a/2} \ge 0$ and $\phi_k > 0$ for each $k \ge 1$, and thus $F(x,y) > 0$ for $0 \le y \le -2\log x$. On the other hand,

$$F(x,y) = e^{-y} - xe^{-y/2} + \frac{x^2}{2}e^{-y/4} - \frac{x^3}{6}e^{-y/8} + \sum_{s\ge 4} \frac{(-1)^s x^s}{s!} e^{-y/2^s}$$

and thus by applying Taylor expansion to the exponential terms,

$$\begin{aligned} F(x, -2\log x + \sqrt{x} + x/2) &= x^2 e^{-\sqrt{x}-x/2} - x^2 e^{-\sqrt{x}/2 - x/4} + \frac{x^{5/2}}{2} e^{-\sqrt{x}/4 - x/8} \\ &\quad - \frac{x^{13/4}}{6} e^{-\sqrt{x}/8 - x/16} + O(x^4) \\ &= -\frac{x^{13/4}}{6} + O(x^{7/2}), \end{aligned}$$

which is negative for sufficiently small x. Therefore, $F(x,y)$ is positive for $0 \le x \le 1$ and $0 \le y \le -2\log x$, and it is negative for sufficiently small x and $y = -2\log x + \sqrt{x} + \frac{x}{2}$, so (d) follows by the intermediate value theorem. ◀

Our goal is to apply Theorem 4 to the curve $y_0(x)$; we thus need to know how the expressions xF_x, yF_y behave when (x,y) lies on this curve. In particular, we will be interested in the case $x \to 0$. The partial derivative with respect to x is

$$F_x(x,y) = \sum_{s\ge 1} \frac{(-1)^s x^{s-1}}{(s-1)!} e^{-y/2^s}.$$

The sum over $s \ge 2$ is $O(x^{3/2})$ as $x \to 0$ by Lemma 5(d), and the term corresponding to $s = 1$ dominates. Thus we have

$$F_x(x, y_0(x)) = -e^{-y_0(x)/2} + O(x^{3/2}) = -x + O(x^{3/2}).$$

Similarly,

$$F_y(x,y) = \sum_{s\ge 0} \frac{(-1)^{s+1} x^s}{s! 2^s} e^{-y/2^s}$$

is dominated by the first two terms, while the sum over $s \ge 2$ is $O(x^{5/2})$. We have

$$F_y(x, y_0(x)) = -e^{-y_0(x)} + \frac{x}{2} e^{-y_0(x)/2} + O(x^{5/2}) = -\frac{x^2}{2} + O(x^{5/2}).$$

This implies the following lemma:

▶ **Lemma 6.** *As $x \to 0$, we have*

$$\frac{xF_x(x, y_0(x))}{y_0(x) F_y(x, y_0(x))} \sim \frac{-x^2}{-x^2 y_0(x)/2} = \frac{2}{y_0(x)} \sim -\frac{1}{\log x}.$$

In particular, the quotient tends to 0. Analogously, it tends to ∞ as $x \to \infty$.

Consequently, by the intermediate value theorem, the quotient admits all positive real values.

▶ **Lemma 7.** *For every $r > 0$, there exists $x_0 > 0$ such that*

$$\frac{x_0 F_x(x_0, y_0(x_0))}{y_0(x_0) F_y(x_0, y_0(x_0))} = r.$$

4.2 Ruling out other complex zeros

In this subsection, we rule out the existence of "bad" complex zeros of the function $F(x,y)$, i.e., complex zeros that have the same modulus as the dominant real zeros of the form $(x, y_0(x))$. This turns out to be quite challenging in spite of the explicit nature of the function $F(x,y)$. The main idea is to express the generating function $G(x,y)$ as a quotient where the denominator is of the form $1 - H(x,y)$, and where H only has nonnegative coefficients. The triangle inequality then achieves the desired property, since zeros of $F(x,y)$ automatically become zeros of $1 - H(x,y)$.

Our approach is based on an infinite system of functional equations, akin to what was used by McKay in his study of the height of DAGs [9], and uses ideas from a paper of Drmota, Gittenberger and Morgenbesser [3]. Since [3] only treats the univariate case (albeit with additional auxiliary variables), we present the argument in detail.

Let us first define some more auxiliary quantities and generating functions. We let $g_{p,q}^{(i,j)}$ be the number of labelled bicoloured DAGs with the additional requirement that

- there are p vertices of the first colour and q of the second, and
- there are i sources (vertices of indegree 0) of the first colour and j of the second.

Observe that such an acyclic digraph can be obtained in the following way:

- Choose which of the $p+q$ labels are assigned to the sources of the first colour, and which to the sources of the second colour. The number of possibilities is $\binom{p+q}{i,j,p+q-i-j} = \binom{p+q}{i+j}\binom{i+j}{i}$.
- Choose a labelled bicoloured DAG with $p-i$ vertices of the first colour and $q-j$ of the second. There are $g_{p-i,q-j}^{(k,\ell)}$ such DAGs with k sources of the first colour and ℓ of the second.
- Now connect the $i+j$ new sources to the other vertices. All edges have to be directed away from the sources, and there cannot be any edges between two sources by definition. Moreover, all edges go between vertices of opposite colours. Thus we have $2^{i(q-j)+j(p-i)}$ possibilities.
- We have to rule out those combinations where some of the $k+\ell$ "old" sources remain sources in the new DAG; for each of them, there needs to be at least one edge to it from one of the new sources. Thus we multiply by a factor $(1-2^{-i})^{\ell}(1-2^{-j})^{k}$ (interpreting 0^0 as 1).

Putting everything together and summing over all possible values of k and ℓ, we obtain

$$g_{p,q}^{(i,j)} = \sum_{k,\ell\geq 0} \binom{p+q}{i+j}\binom{i+j}{i} 2^{i(q-j)+j(p-i)}(1-2^{-i})^{\ell}(1-2^{-j})^{k} g_{p-i,q-j}^{(k,\ell)} \tag{9}$$

if $(i,j) \neq (0,0)$. Additionally, we have

$$g_{p,q}^{(0,0)} = \begin{cases} 1 & p=q=0, \\ 0 & \text{otherwise.} \end{cases}$$

Now let

$$G^{(i,j)}(x,y) = \sum_{p,q\geq 0} g_{p,q}^{(i,j)} \frac{x^p y^q}{(p+q)!2^{pq}}$$

be the special generating function associated with $g_{p,q}^{(i,j)}$. The recursion (9) translates to

$$G^{(i,j)}(x,y) = \frac{x^i y^j}{i!j!2^{ij}} \sum_{k,\ell\geq 0} (1-2^{-i})^{\ell}(1-2^{-j})^{k} G^{(k,\ell)}(x,y) \tag{10}$$

if $(i,j) \neq (0,0)$, and $G^{(0,0)}(x,y) = 1$. We can consider this as an infinite linear system of equations indexed by pairs (i,j) of nonnegative integers. Let $\mathbf{G}$ be the infinite-dimensional vector of generating functions $G^{(i,j)}(x,y)$, and let $\mathbf{M}$ be the infinite-dimensional matrix whose entries are given by $\frac{x^i y^j}{i!j!2^{ij}}(1-2^{-i})^{\ell}(1-2^{-j})^k$ in the row indexed by $(i,j) \neq (0,0)$ and column indexed by (k,ℓ), and by 0 everywhere in row $(0,0)$. Then we can write the system as

$$\mathbf{G} = \mathbf{M} \cdot \mathbf{G} + \mathbf{e}, \tag{11}$$

with $\mathbf{e}$ being the unit vector associated with row $(0,0)$ (i.e., the entry in the row indexed $(0,0)$ is 1, all others are 0).

In the following, we will assume that $|x|, |y| \leq K$, where K is fixed, but arbitrary. Now for every pair (x,y), the matrix $\mathbf{M}$ can be regarded as an operator from ℓ^1 to itself. Indeed, note that the entry in row (i,j) of $\mathbf{M} \cdot \mathbf{w}$ is bounded in absolute value by

$$\frac{|x|^i |y|^j}{i!j!2^{ij}} \|\mathbf{w}\|_1, \tag{12}$$

so that

$$\|\mathbf{M} \cdot \mathbf{w}\|_1 \leq \sum_{i,j} \frac{|x|^i |y|^j}{i!j!2^{ij}} \cdot \|\mathbf{w}\|_1 \leq \sum_{i,j} \frac{K^{i+j}}{i!j!2^{ij}} \cdot \|\mathbf{w}\|_1,$$

which is finite since the sum converges for all K. It also follows from this estimate that the operator represented by $\mathbf{M}$ is compact (since we have a uniform bound for the tails of the sum).

We assume now that x, y are nonnegative real numbers such that (x,y) is inside the region where the power series that defines $G(x,y)$ converges. In other words, it lies in the set

$$U = \{(x,y) \in \mathbb{R}^2 \,:\, 0 \leq x, y \leq K, y < y_0(x)\}.$$

Now trivially, $G^{(i,j)}(x,y) \leq G(x,y)$, and by definition, $G(x,y) = \sum_{i,j \geq 0} G^{(i,j)}(x,y) = \|\mathbf{G}\|_1$. In view of (10), we have

$$G^{(i,j)}(x,y) \leq \frac{x^i y^j}{i!j!2^{ij}} G(x,y), \tag{13}$$

cf. (12). Again, we use the fact that

$$\sum_{i,j \geq 0} \frac{x^i y^j}{i!j!2^{ij}} \leq \sum_{i,j \geq 0} \frac{K^{i+j}}{i!j!2^{ij}} < \infty.$$

This implies that there exists a finite set $\mathcal{I} \subseteq \{(i,j) : i,j \geq 0\}$ such that

$$\sum_{\substack{i,j \geq 0 \\ (i,j) \notin \mathcal{I}}} \frac{x^i y^j}{i!j!2^{ij}} \leq \sum_{\substack{i,j \geq 0 \\ (i,j) \notin \mathcal{I}}} \frac{K^{i+j}}{i!j!2^{ij}} < 1.$$

Combined with (13), this means that

$$\sum_{(i,j) \in \mathcal{I}} \frac{G^{(i,j)}(x,y)}{G(x,y)}$$

has to be bounded below by a positive constant on U. Let $m = \max_{(i,j)\in\mathcal{I}} ij$ and note that

$$G^{(1,1)}(x,y) = \frac{xy}{2} \sum_{k,\ell\geq 0} \frac{1}{2^{k\ell}} G^{(k,\ell)}(x,y) \geq \frac{xy}{2^{1+m}} \sum_{(i,j)\in\mathcal{I}} G^{(i,j)}(x,y),$$

so that $\frac{G^{(1,1)}(x,y)}{xyG(x,y)}$ is already bounded below by a positive constant on the set U. Let $\mathbf{V}$ be the vector $\mathbf{G}$ without the component corresponding to $G^{(1,1)}(x,y)$. If we restrict the system (11) to $\mathbf{V}$, we obtain

$$\mathbf{V} = \overline{\mathbf{M}}\mathbf{V} + \mathbf{a}G^{(1,1)}(x,y) + \overline{\mathbf{e}},$$

where $\overline{\mathbf{M}}$ and $\overline{\mathbf{e}}$ are suitably restricted versions of $\mathbf{M}$ and $\mathbf{e}$, respectively, and $\mathbf{a}$ is the column of $\mathbf{M}$ corresponding to $G^{(1,1)}(x,y)$ with the entry for row $(1,1)$ removed. Solving for $\mathbf{V}$ yields

$$\mathbf{V} = (\mathbf{I} - \overline{\mathbf{M}})^{-1}(\mathbf{a}G^{(1,1)}(x,y) + \overline{\mathbf{e}}). \tag{14}$$

Here, the inverse $(\mathbf{I} - \overline{\mathbf{M}})^{-1}$ can be defined a priori with entries in the ring of formal power series by

$$(\mathbf{I} - \overline{\mathbf{M}})^{-1} = \mathbf{I} + \overline{\mathbf{M}} + \overline{\mathbf{M}}^2 + \cdots . \tag{15}$$

Now we consider the equation for $G^{(1,1)}(x,y)$, which takes the form

$$G^{(1,1)}(x,y) = \frac{xy}{8} G^{(1,1)}(x,y) + \mathbf{b}^T\mathbf{V}$$

for a vector $\mathbf{b}^T$ (the row of $\mathbf{M}$ that corresponds to $G^{(1,1)}(x,y)$, again with one entry removed). We can plug in (14) to get

$$G^{(1,1)}(x,y) = \frac{xy}{8} G^{(1,1)}(x,y) + \mathbf{b}^T(\mathbf{I} - \overline{\mathbf{M}})^{-1}(\mathbf{a}G^{(1,1)}(x,y) + \overline{\mathbf{e}}),$$

thus

$$G^{(1,1)}(x,y) = \Big(1 - \frac{xy}{8} - \mathbf{b}^T(\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{a}\Big)^{-1} \cdot \mathbf{b}^T(\mathbf{I} - \overline{\mathbf{M}})^{-1}\overline{\mathbf{e}}. \tag{16}$$

A priori, all of these expressions are at least well-defined within the ring of formal power series, and all infinite-dimensional matrices and vectors in the expression have only nonnegative entries.

Now it becomes relevant that $\frac{G^{(1,1)}(x,y)}{xyG(x,y)}$ is bounded below on U. It implies that the infinite series in (15) must actually remain bounded and thus converge, as one would otherwise get a contradiction with (14). In fact, since the quotient is uniformly bounded by a positive constant on all of U, this extends to the closure $\overline{U}$, thus including the boundary where $y = y_0(x)$. So we find that the spectral radius of the compact operator $\overline{\mathbf{M}}$ needs to be less than 1 on $\overline{U}$. This means that the expression

$$1 - \frac{xy}{8} - \mathbf{b}^T(\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{a}$$

in (16) actually represents a function that is analytic in a domain that contains $\overline{U}$ (in fact, as shown by Drmota, Gittenberger and Morgenbesser [3], this follows from general considerations based purely on the compactness of $\mathbf{M}$). We can write this as

$$1 - H(x,y),$$

where $H(x,y) = \frac{xy}{8} + \mathbf{b}^T(\mathbf{I} - \overline{\mathbf{M}})^{-1}\mathbf{a}$ has only nonnegative coefficients. So $G^{(1,1)}(x,y)$ has a representation of the form $\frac{N(x,y)}{1-H(x,y)}$, where both $H(x,y)$ and $N(x,y)$ have only nonnegative coefficients. In view of (14), this is also the case for all other $G^{(i,j)}(x,y)$ and thus for $G(x,y)$.

Note that zeros of $F(x,y)$ are necessarily also zeros of $1 - H(x,y)$. We observe that $\mathbf{b}^T\mathbf{a}$, and thus also $H(x,y)$, contains terms of the form $x^{1+i}y^{1+j}$ with strictly positive coefficients for all $i,j \geq 1$, $(i,j) \neq (1,1)$. So equality in the triangle inequality $|H(x,y)| \leq H(|x|,|y|)$ can only hold when x and y are positive and real. This proves the following key lemma.

▶ **Lemma 8.** *For every $x_0 \in \mathbb{R}^+$, the point $(x_0, y_0(x_0))$ is the only zero of F in the set $\{(x,y) \in \mathbb{C} : |x| = |x_0|$ and $|y| = |y_0(x_0)|\}$.*

4.3 Non-vanishing of Q

In this subsection, we verify that the auxiliary function Q in Theorem 4 (with $C = F$) is never zero on the curve $y = y_0(x)$. This will be based on the following general lemma whose proof is given in the appendix.

▶ **Lemma 9.** *Suppose the denominator C in Theorem 4 is of the form $C(x,y) = 1 - H(x,y)$, where $H(x,y) = \sum_{m,n\geq 0} h_{m,n}x^m y^n$ has only nonnegative coefficients $h_{m,n}$ and $h_{0,0} = 0$. Then $Q(x,y) > 0$ for all $x, y > 0$ unless $H(x,y)$ can be represented in the form $H(x,y) = K(x^r y^s)$ for a univariate power series K.*

Now, the main lemma of this subsection follows readily from the considerations that also gave us Lemma 8.

▶ **Lemma 10.** *For all positive real x, we have*

$$Q(x, y_0(x)) > 0.$$

Proof. By the proof of Lemma 8, we have an alternative representation of our generating function $G(x,y)$ that has the form $\frac{N(x,y)}{1-H(x,y)}$, where both $H(x,y)$ and $N(x,y)$ have only nonnegative coefficients. Moreover, it was already verified in the proof of Lemma 8 that $H(x,y)$ is not of the form $K(x^r y^s)$. Thus Lemma 9 readily proves the statement. ◀

5 Bicoloured acyclic digraphs

We now have all ingredients to apply Theorem 4 to $G(x,y)$ so as to determine the asymptotic behaviour of $g_{p,q}$.

Proof of Theorem 1. Let U be any compact subset of the point set $\{(x, y_0(x)) \mid x \in \mathbb{R}^+\}$ of the curve of zeros of $F(x,y)$. The conditions of Theorem 4 are now satisfied with $A = G$, $B = 1$, and $C = F$:

- The numerator of $G = 1/F$ is constant, while the denominator is an entire function with $F(0,0) = 1$. Thus, G is meromorphic and not singular at the origin.
- We chose U as a compact set of zeros of F. The constant function $B(x,y) = 1$ is nonzero on U, and so is the gradient of F, since

$$\nabla F(x_0, y_0) \overset{(7)}{=} \begin{pmatrix} -F\left(x_0, \frac{y_0}{2}\right) \\ -F\left(\frac{x_0}{2}, y_0\right) \end{pmatrix},$$

  and the coordinates are nonzero by the minimality and monotonicity of the function $y_0(x)$, see Lemma 5.

- By Pringsheim's Theorem, if there were a point $(x_0, y_0) \in U$ and another zero $(x_1, y_1) \neq (x_0, y_0)$ of F that satisfies both $|x_1| \leq |x_0|$ and $|y_1| \leq |y_0|$, then either all such zeros would need to satisfy both conditions with equality, or there would be a real-valued such zero. By Lemma 8, the former do not exist, while the latter is ruled out by the minimality and monotonicity of $y_0(x)$ as in Lemma 5.
- Finally, Q is nonzero on U by Lemma 10.

Thus, Theorem 4 yields the asymptotic expression

$$\frac{g_{p,q}}{(p+q)!2^{pq}} \sim \frac{1}{\sqrt{2\pi}} x_0^{-p} y_0^{-q} \frac{1}{\sqrt{\lambda Q(x_0, y_0)}} \tag{17}$$

for every $(x_0, y_0) \in U$ and $(p, q) = -\lambda(x_0 F_x(x_0, y_0), y_0 F_y(x_0, y_0))$.

Now let $r = \frac{p}{q}$ be fixed. By Lemma 7, there exists a value $x_0 > 0$ such that

$$\frac{x_0 F_x(x_0, y_0(x_0))}{y_0(x_0) F_y(x_0, y_0(x_0))} = r$$

and thus (17) holds for this value x_0 and $y_0 = y_0(x_0)$. Moreover, x_0 is unique, since $g_{p,q}$ cannot satisfy (17) for two different values of (x_0, y_0). Thus, (17) implies (6) with

$$c = \sqrt{\frac{-y_0 F_y(x_0, y_0)}{2\pi Q(x_0, y_0)}}, \qquad \rho_x = x_0, \qquad \text{and} \qquad \rho_y = y_0(x_0).$$

Since F is an entire function, c, ρ_x, ρ_y are all continuous functions in r. Together with the uniform convergence in Theorem 4, this implies statement (a) of Theorem 1, while (b) follows from Lemma 6. The symmetry of the curve $y_0(x)$ (Lemma 5) and the value of the smallest positive zero of $F(x, x)$ stated in (8) prove (c) and the value ρ in (d). Using $x_0 = y_0 = \rho$ in the formula for c above yields the value $c(1)$.

It remains to prove (e). To that end, write $\rho_x(r + h) = \hat{\rho}_x + \xi$ and $\rho_y(r + h) = \hat{\rho}_y + \eta$, where $\hat{\rho}_x := \rho_x(r)$ and $\hat{\rho}_y := \rho_y(r)$. Since ρ_x, ρ_y are continuous, ξ, η both tend to zero. The Taylor expansion of $F(\hat{\rho}_x + \xi, \hat{\rho}_y + \eta)$ around the origin is

$$F_x(\hat{\rho}_x, \hat{\rho}_y)\xi + F_y(\hat{\rho}_x, \hat{\rho}_y)\eta + O(\xi^2 + \eta^2).$$

Since this is zero for $\eta = y_0(\hat{\rho}_x + \xi) - \hat{\rho}_y$, we deduce that $\eta = -\frac{F_x(\hat{\rho}_x, \hat{\rho}_y)}{F_y(\hat{\rho}_x, \hat{\rho}_y)}\xi + O(\xi^2)$ and hence

$$r + h = \frac{(\hat{\rho}_x + \xi) F_x \left(\hat{\rho}_x + \xi, \hat{\rho}_y - \frac{F_x(\hat{\rho}_x, \hat{\rho}_y)}{F_y(\hat{\rho}_x, \hat{\rho}_y)}\xi + O(\xi^2)\right)}{\left(\hat{\rho}_y - \frac{F_x(\hat{\rho}_x, \hat{\rho}_y)}{F_y(\hat{\rho}_x, \hat{\rho}_y)}\xi + O(\xi^2)\right) F_y \left(\hat{\rho}_x + \xi, \hat{\rho}_y - \frac{F_x(\hat{\rho}_x, \hat{\rho}_y)}{F_y(\hat{\rho}_x, \hat{\rho}_y)}\xi + O(\xi^2)\right)}.$$

The fraction of the first factors in the numerator and denominator, respectively, can be expanded as

$$\frac{\hat{\rho}_x}{\hat{\rho}_y} + \frac{\hat{\rho}_x F_x(\hat{\rho}_x, \hat{\rho}_y) + \hat{\rho}_x F_y(\hat{\rho}_x, \hat{\rho}_y)}{\hat{\rho}_y^2 F_y(\hat{\rho}_x, \hat{\rho}_y)}\xi + O(\xi^2).$$

Using the Taylor expansions of F_x, F_y around the point $(\hat{\rho}_x, \hat{\rho}_y)$, the fraction of the remaining parts can be rewritten as

$$\left.\frac{F_x}{F_y}\right|_{(\hat{\rho}_x, \hat{\rho}_y)} + \left.\frac{F_x^2 F_{yy} + F_y F_{xx} - 2F_x F_y F_{xy}}{F_y^3}\right|_{(\hat{\rho}_x, \hat{\rho}_y)} \xi + O(\xi^2).$$

Taking the product of these two expressions, we find that

$$r+h=\frac{\hat\rho_x F_x(\hat\rho_x,\hat\rho_y)}{\hat\rho_y F_y(\hat\rho_x,\hat\rho_y)}-\frac{Q(\hat\rho_x,\hat\rho_y)}{\hat\rho_x\hat\rho_y^3 F_y(\hat\rho_x,\hat\rho_y)}\xi+O(\xi^2)=r-\frac{Q(\hat\rho_x,\hat\rho_y)}{\hat\rho_x\hat\rho_y^3 F_y(\hat\rho_x,\hat\rho_y)}\xi+O(\xi^2).$$

Vice versa, this means that

$$\xi=-\frac{\hat\rho_x\hat\rho_y^3 F_y(\hat\rho_x,\hat\rho_y)}{Q(\hat\rho_x,\hat\rho_y)}h+O(h^2)$$

and thus

$$\begin{aligned}
\rho_x(r+h)&=\left(1-\frac{\hat\rho_y^3 F_y(\hat\rho_x,\hat\rho_y)}{Q(\hat\rho_x,\hat\rho_y)}h+O(h^2)\right)\hat\rho_x,\\
\rho_y(r+h)&=\left(1+\frac{F_x(\hat\rho_x,\hat\rho_y)}{F_y(\hat\rho_x,\hat\rho_y)}\cdot\frac{\hat\rho_x\hat\rho_y^2 F_y(\hat\rho_x,\hat\rho_y)}{Q(\hat\rho_x,\hat\rho_y)}h+O(h^2)\right)\hat\rho_y\\
&=\left(1+\frac{\hat\rho_y^3 F_y(\hat\rho_x,\hat\rho_y)}{Q(\hat\rho_x,\hat\rho_y)}rh+O(h^2)\right)\hat\rho_y.
\end{aligned}$$

Now (e) follows with

$$\alpha:=-\frac{\hat\rho_y^3 F_y(\hat\rho_x,\hat\rho_y)}{Q(\hat\rho_x,\hat\rho_y)},$$

which is positive, since $\rho_y>0$, $F_y(\hat\rho_x,\hat\rho_y)=-F(\hat\rho_x/2,\hat\rho_y)<0$ by (7), and $Q(\hat\rho_x,\hat\rho_y)>0$ by Lemma 10. ◀

6 From bicoloured to bipartite acyclic digraphs

Let us first observe that almost all bicoloured DAGs are connected. More precisely, we have the following lemma, whose somewhat technical proof is given in the appendix.

▶ **Lemma 11.** *For every constant $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that for all $p,q\ge N_\varepsilon$ with $\varepsilon\le\frac{p}{q}\le\frac{1}{\varepsilon}$, at least $(1-\varepsilon)g_{p,q}$ DAGs in $\mathcal{G}_{p,q}$ are connected.*

Lemma 11 now enables us to prove Theorem 2.

Proof of Theorem 2. Lemma 11 implies that if $\frac{p}{q}$ is bounded away from both 0 and ∞, then almost all DAGs in $\mathcal{G}_{p,q}$ are connected as $p,q\to\infty$. Now Theorem 2 follows from the trivial fact that every connected bipartite DAG has precisely two 2-colourings with specified number of vertices of each colour if its partite sets have the same size, and a unique 2-colouring otherwise. ◀

Proof of Theorem 3. Given a bipartite DAG with n vertices, we denote the sizes of its partite sets by p,q, where $p\ge q$. Assume first that n is even and let $k:=\lfloor n^{2/3}\rfloor$ and $l:=\lceil\frac{n}{3}\rceil$. Then

$$b_n=\sum_{d=0}^{k}b_{\frac n2+d,\frac n2-d}+\sum_{d=k+1}^{l}b_{\frac n2+d,\frac n2-d}+\sum_{d=l+1}^{\frac n2}b_{\frac n2+d,\frac n2-d}.$$

In the range $1 \le d \le k$, Theorem 1(e) and Theorem 2 yield for $h := \frac{\frac{n}{2}+d}{\frac{n}{2}-d} - 1 = O(\frac{d}{n})$

$$\begin{aligned}
b_{\frac{n}{2}+d,\frac{n}{2}-d} &\sim \sqrt{2}cn^{-1/2}\left(1+\alpha h + O(h^2)\right)^{-\frac{n}{2}-d}\left(1-\alpha h+O(h^2)\right)^{-\frac{n}{2}+d}\rho^{-n}n!2^{\frac{n^2}{4}-d^2}\\
&= \sqrt{2}cn^{-1/2}\rho^{-n}n!2^{\frac{n^2}{4}-d^2}\cdot\exp\left(-2\alpha dh + O(h^2n)\right)\\
&= \sqrt{2}cn^{-1/2}\rho^{-n}n!2^{\frac{n^2}{4}-d^2}\cdot\exp\left(O(n^{-1}d^2)\right)\\
&\sim 2^{1-d^2}\exp\left(O(n^{-1}d^2)\right) b_{\frac{n}{2},\frac{n}{2}},
\end{aligned}$$

where the convergence is uniform throughout the range. Thus

$$\sum_{d=0}^{k} b_{\frac{n}{2}+d,\frac{n}{2}-d} \sim \left(1+2\sum_{d=1}^{k}\left(2+O\left(\frac{1}{n}\right)\right)^{-d^2}\right) b_{\frac{n}{2},\frac{n}{2}} \sim \left(1+2\sum_{d=1}^{\infty}2^{-d^2}\right) b_{\frac{n}{2},\frac{n}{2}}.$$

In the range $k+1 \le d \le l$, the fraction $\frac{\frac{n}{2}+d}{\frac{n}{2}-d}$ is bounded and thus ρ_x, ρ_y are bounded by Theorem 1. Let $\tilde{\rho}_x, \tilde{\rho}_y$ be their smallest values. Then Theorem 2 implies

$$b_{\frac{n}{2}+d,\frac{n}{2}-d} \sim \frac{1}{2}c\left(\frac{n}{2}-d\right)^{-1/2}\tilde{\rho}_x^{-\frac{n}{2}-d}\tilde{\rho}_y^{-\frac{n}{2}+d}n!2^{\frac{n^2}{4}-d^2} = \Theta(1)\rho^n\tilde{\rho}_x^{-\frac{n}{2}-d}\tilde{\rho}_y^{-\frac{n}{2}+d}2^{-d^2}\, b_{\frac{n}{2},\frac{n}{2}},$$

which is (more than) exponentially smaller than $b_{\frac{n}{2},\frac{n}{2}}$, since $d^2 = \Omega(n^{4/3})$ and all terms apart from 2^{-d^2} are at most exponential. Thus,

$$\sum_{d=k+1}^{l} b_{\frac{n}{2}+d,\frac{n}{2}-d} = o(b_{\frac{n}{2},\frac{n}{2}}).$$

Finally, for $l+1 \le d$, we use the trivial observation that every pair of a black vertex v and a white vertex w can either have no edge between them, an edge from v to w, or an edge from w to v. This gives us the general upper bound

$$b_{s,t} \le g_{s,t} \le (s+t)!3^{st}, \tag{18}$$

and $b_{\frac{n}{2}+d,\frac{n}{2}-d} \le n!3^{(\frac{n}{2}+d)(\frac{n}{2}-d)} \le n!3^{\frac{5n^2}{36}}$ now yields

$$\sum_{d=l+1}^{\frac{n}{2}} b_{\frac{n}{2}+d,\frac{n}{2}-d} \le \frac{n}{6}\cdot n!3^{\frac{5n^2}{36}} = o(b_{\frac{n}{2},\frac{n}{2}}),$$

since $3^{\frac{5}{36}} < 2^{\frac{1}{4}}$. Thus

$$b_n \sim \left(1+2\sum_{d=1}^{\infty}2^{-d^2}\right) b_{\frac{n}{2},\frac{n}{2}} \sim \gamma_e n^{-1/2}\rho^{-n}\cdot n!\cdot 2^{\frac{n^2}{4}}$$

with $\gamma_e = \frac{c(1)}{\sqrt{2}}\left(1+2\sum_{d=1}^{\infty}2^{-d^2}\right) \approx 3.19984$ and $\rho \approx 1.15196$.

For n odd, we apply analogous estimates to see that

$$b_{\frac{n+1}{2}+d,\frac{n-1}{2}-d} \sim \exp(-d-d^2+O(d^2/n))b_{\frac{n+1}{2},\frac{n-1}{2}}$$

for $0 \le d \le k$. Here we need to use $(\frac{n+1}{2}+d)(\frac{n-1}{2}-d) = \frac{n^2-1}{4} - d - d^2$, which yields the different error term. In total, we have

$$b_n \sim \left(\sum_{d=0}^{\infty}2^{-d-d^2}\right) b_{\frac{n+1}{2},\frac{n-1}{2}} \sim \gamma_o n^{-1/2}\rho^{-n}\cdot n!\cdot 2^{\frac{n^2}{4}}$$

with $\gamma_o \approx 3.19983$. ◀

7 Outlook

This work can be extended in several directions: a natural generalisation would be to consider k-partite and k-coloured DAGs, which leads to generating functions in k variables for which a similar asymptotic study should be feasible.

Second, it would be interesting to say more about the typical shape of bipartite DAGs, and to determine limiting distributions for different statistics associated with bipartite DAGs (for example the number of sources or sinks, which appeared as auxiliary quantities in our proofs).

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A Further proof details

This appendix contains the proofs of two technical lemmas that do not fit the main body of the paper.

Proof of Lemma 9. Note that

$$\begin{aligned}
xC_x(x,y) &= -\sum_{m,n\geq 0} m h_{m,n} x^m y^n,\\
yC_y(x,y) &= -\sum_{m,n\geq 0} n h_{m,n} x^m y^n,\\
x^2C_{xx}(x,y) + xC_x(x,y) &= -\sum_{m,n\geq 0} m^2 h_{m,n} x^m y^n,\\
xyC_{xy}(x,y) &= -\sum_{m,n\geq 0} mn h_{m,n} x^m y^n,\\
y^2C_{yy}(x,y) + yC_y(x,y) &= -\sum_{m,n\geq 0} n^2 h_{m,n} x^m y^n.
\end{aligned}$$

Plugging all these expressions into the formula for $Q(x,y)$, we find that

$$\begin{aligned}
Q(x,y) = &\Big(\sum_{m,n\geq 0} m h_{m,n} x^m y^n\Big)^2 \Big(\sum_{m,n\geq 0} n^2 h_{m,n} x^m y^n\Big)\\
&+ \Big(\sum_{m,n\geq 0} m^2 h_{m,n} x^m y^n\Big)\Big(\sum_{m,n\geq 0} n h_{m,n} x^m y^n\Big)^2\\
&- 2\Big(\sum_{m,n\geq 0} m h_{m,n} x^m y^n\Big)\Big(\sum_{m,n\geq 0} n h_{m,n} x^m y^n\Big)\Big(\sum_{m,n\geq 0} mn h_{m,n} x^m y^n\Big).
\end{aligned}$$

Now observe that by the inequality between the arithmetic and geometric mean, we have

$$\begin{aligned}
&\Big(\sum_{m,n\geq 0} m h_{m,n} x^m y^n\Big)^2 \Big(\sum_{m,n\geq 0} n^2 h_{m,n} x^m y^n\Big)\\
&\quad + \Big(\sum_{m,n\geq 0} m^2 h_{m,n} x^m y^n\Big)\Big(\sum_{m,n\geq 0} n h_{m,n} x^m y^n\Big)^2\\
&\geq 2\Big(\sum_{m,n\geq 0} m h_{m,n} x^m y^n\Big)\Big(\sum_{m,n\geq 0} n h_{m,n} x^m y^n\Big)\\
&\quad \cdot \Big(\sum_{m,n\geq 0} m^2 h_{m,n} x^m y^n\Big)^{1/2}\Big(\sum_{m,n\geq 0} n^2 h_{m,n} x^m y^n\Big)^{1/2}.
\end{aligned}$$

Moreover, the Cauchy–Schwarz inequality yields

$$\Big(\sum_{m,n\geq 0} m^2 h_{m,n} x^m y^n\Big)^{1/2}\Big(\sum_{m,n\geq 0} n^2 h_{m,n} x^m y^n\Big)^{1/2} \geq \Big(\sum_{m,n\geq 0} mn h_{m,n} x^m y^n\Big).$$

Combined, these inequalities show that $Q(x,y) \geq 0$, and equality holds if and only if $\frac{n}{m}$ has the same value (say, $\frac{s}{r}$ in lowest terms) for all m,n for which the coefficient $h_{m,n}$ is not zero. This however means that $x^m y^n = (x^r y^s)^{m/r}$, so the power series H can in fact be expressed as $K(x^r y^s)$. ◀

Proof of Lemma 11. Let $\varepsilon > 0$ be given and let $p,q \geq N_\varepsilon$ be such that $r := \frac{p}{q}$ satisfies $\varepsilon \leq r \leq \frac{1}{\varepsilon}$. Write $m := \min\{p,q\}$. By Theorem 1,

$$g_{p,q} = \hat{c} q^{-1/2} \hat{\rho}_x^{-p} \hat{\rho}_y^{-q} (p+q)!\, 2^{pq}$$

for $\hat{\rho}_x := \rho_x(r)$, $\hat{\rho}_y := \rho_y(r)$ and some constant $\hat{c}$.

Write $d_{p,q}$ for the number of disconnected DAGs in $\mathcal{G}_{p,q}$. If $G \in \mathcal{G}_{p,q}$ is disconnected, then it can be written as the union of two nonempty bicoloured DAGs – one in $\mathcal{G}_{s,t}$ for $s \leq \frac{p}{2}$ and $t \leq q$, and the other in $\mathcal{G}_{p-s,q-t}$. Let I be the set of all permissible pairs (s,t) of indices. Then

$$d_{p,q} \leq \sum_{(s,t)\in I} \binom{p+q}{s+t} g_{s,t} g_{p-s,q-t}. \tag{19}$$

We call s *small* if it is smaller than $\frac{p}{10}$, and *large* otherwise. Analogously, we call t and $q-t$ small or large by comparing them with $\frac{q}{10}$. The value $p-s$ is at least $\frac{p}{2}$ and thus always considered large. Finally, we call s or t *tiny* if it is smaller than $p^{1/3}$ or $q^{1/3}$, respectively.

If at least one of s,t is small, we use the trivial bound (18) again to get

$$g_{s,t} \leq (s+t)! 3^{st} \leq (s+t)! \left(3^{\frac{1}{10}}\right)^{pq}.$$

The same upper bound also holds for $g_{p-s,q-t}$ if $q-t$ is small. If both indices of $g_{s,t}$ or $g_{p-s,q-t}$ are large, the ratio of its indices lies between $\frac{r}{10} \geq \frac{\varepsilon}{10}$ and $10r \leq \frac{10}{\varepsilon}$. This is a compact range, on which Theorem 1 yields a uniform upper bound

$$g_{s,t} \leq \tilde{c} \tilde{\rho}_x^{-s} \tilde{\rho}_y^{-t} (s+t)!\, 2^{st} \quad \text{respectively} \quad g_{p-s,q-t} \leq \tilde{c} \tilde{\rho}_x^{s-p} \tilde{\rho}_y^{t-q} (p+q-s-t)!\, 2^{(p-s)(q-t)},$$

where $\tilde{c}, \tilde{\rho}_x, \tilde{\rho}_y$ are positive constants independent of the choice of p,q,s,t.

If both (s,t) and $(p-s,q-t)$ contain a small index, we have

$$\binom{p+q}{s+t} g_{s,t} g_{p-s,q-t} \leq (p+q)! \left(3^{\frac{1}{5}}\right)^{pq} \leq \frac{\varepsilon}{pq} g_{p,q}$$

if m is large enough. If all four indices are large,

$$\binom{p+q}{s+t} g_{s,t} g_{p-s,q-t} \leq \tilde{c}^2 \tilde{\rho}_x^{-p} \tilde{\rho}_y^{-q} (p+q)! 2^{pq - s(q-t) - t(p-s)} \leq \frac{\tilde{c}^2 q^{1/2} \hat{\rho}_x^p \hat{\rho}_y^q}{\hat{c} \tilde{\rho}_x^p \tilde{\rho}_y^q \left(2^{\frac{9}{50}}\right)^{pq}} g_{p,q},$$

since $s(q-t) + t(p-s)$ is minimised at $s = \frac{p}{10}, t = \frac{q}{10}$ with value $\frac{9}{50}pq$. Recall that $r \in [\varepsilon, \frac{1}{\varepsilon}]$, so that $\hat{\rho}_x, \hat{\rho}_y$ are bounded, and the largest order term in the fraction is $\left(2^{\frac{9}{50}}\right)^{pq}$. Thus the fraction becomes smaller than $\frac{\varepsilon}{pq}$ if m is large enough.

Now assume that (at least) one of s,t is small, but $p-s, q-t$ are both large. Then

$$\binom{p+q}{s+t} g_{s,t} g_{p-s,q-t} \leq \tilde{c} \tilde{\rho}_x^{s-p} \tilde{\rho}_y^{t-q} (p+q)! 3^{st} 2^{(p-s)(q-t)} = \frac{\tilde{c} q^{1/2} \hat{\rho}_x^p \hat{\rho}_y^q 6^{st}}{\hat{c} \tilde{\rho}_x^{p-s} \tilde{\rho}_y^{q-t} 2^{sq+tp}} g_{p,q}.$$

If at least one of s,t is not tiny, 2^{sq+tp} is the largest order term in the fraction, again implying that the fraction is smaller than $\frac{\varepsilon}{pq}$ if m is large enough.

If both s, t are tiny, we need a stronger bound for $g_{p-s,q-t}$. Observe that $\frac{p-s}{q-t}$ lies between $r(1-m^{-2/3})$ and $r(1+2m^{-2/3})$ if m is large enough. We can thus apply Theorem 1(b) and (e) with a function $h = O(m^{-2/3})$ to find $\alpha > 0$ such that

$$\begin{aligned}
g_{p-s,q-t} &\leq \hat{c}\rho_x^{s-p}\rho_y^{q-t}(p+q-s-t)!\,2^{(p-s)(q-t)}, \quad \text{where} \\
\rho_x^{s-p}\rho_y^{q-t} &= \left((1+\alpha h+O(h^2))\hat{\rho}_x\right)^{s-p}\left((1-\alpha rh+O(h^2))\hat{\rho}_y\right)^{t-q} \\
&= \exp\Big((s-p)(\alpha h+O(h^2))+(t-q)(-\alpha rh+O(h^2))\Big)\hat{\rho}_x^{s-p}\hat{\rho}_y^{t-q} \\
&= \exp\left(O\left(\frac{\log^2(p+q)}{p+q}\right)\right)\hat{\rho}_x^{s-p}\hat{\rho}_y^{t-q},
\end{aligned}$$

because $p - qr = 0$. Thus, for m large enough, there is a constant $c' > 0$ such that

$$g_{p-s,q-t} \leq c'\,\hat{\rho}_x^{s-p}\hat{\rho}_y^{t-q}(p+q-s-t)!\,2^{(p-s)(q-t)}$$

and hence

$$\binom{p+q}{s+t}g_{s,t}g_{p-s,q-t} \leq c'\,\hat{\rho}_x^{s-p}\hat{\rho}_y^{t-q}(p+q)!3^{st}2^{(p-s)(q-t)} = \frac{c' q^{1/2}\hat{\rho}_x^{s}\hat{\rho}_y^{t}6^{st}}{\hat{c}2^{sq+tp}}g_{p,q}.$$

The largest order term in this fraction is 2^{sq+tp}, and thus the fraction becomes smaller than $\frac{\varepsilon}{pq}$ if m is large enough.

Finally, the case that s, t are both large and $q - t$ is small is analogous to the previous case, with the only difference being that $p - s$ cannot be tiny.

We have thus shown that if m is large enough, then all summands in (19) are smaller than $\frac{\varepsilon}{pq}g_{p,q}$, which implies that $d_{p,q} < \varepsilon g_{p,q}$, as desired. ◀