Lee–Yang Phenomena in Edge-Coloured Graph Counting

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Abstract

We describe the accumulation of complex zeros of a univariate polynomial arising from the enumeration of edge-coloured graphs along certain limit curves. The polynomial is a variant of an edge-chromatic polynomial, which specialises to the partition function of the ferromagnetic Ising model on a random regular graph. We call this accumulation behaviour a Lee–Yang phenomenon in analogy with the Lee–Yang theorem from statistical physics. The limiting loci are semialgebraic and arise from anti-Stokes curves of an exponential integral.

2012 ACM Subject Classification Mathematics of computing → Graph enumeration; Mathematics of computing → Graph coloring; Mathematics of computing → Generating functions

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1 Introduction

In this work, which is an extended abstract of the paper [25] by the same author, we study the univariate polynomial

$$A_n^V(\lambda) := \sum_{G \in \mathcal{G}_{-n}} \frac{1}{|\operatorname{Aut}(G)|} \prod_{v \in V_G} \Lambda_{\deg(v)}(\lambda) \in \mathbb{C}[\lambda]. \tag{1}$$

Here, $\mathcal{G}_{-n}$ denotes the set of isomorphism classes of edge-coloured graphs G with Euler characteristic $\chi(G) = |V_G| - |E_G| = -n$, and $\Lambda_{(w_1,\ldots,w_d)}(\lambda)$ are vertex markings which associate to each vertex that is incident to w_i half-edges of the i^{th} colour a univariate polynomial in λ (where d is the number of edge colours). We assume that for $\boldsymbol{w} \in \mathbb{Z}_{\geq 0}^d$, only finitely many $\Lambda_{\boldsymbol{w}}$ are nonzero, and we conveniently store them in the polynomial

$$V(\boldsymbol{x}, \lambda) = \sum_{\boldsymbol{w} \in \mathbb{Z}_{\geq 0}^d,\, |\boldsymbol{w}| \geq 1} \Lambda_{\boldsymbol{w}}(\lambda) \frac{\boldsymbol{x}^{\boldsymbol{w}}}{\boldsymbol{w}!} \in \mathbb{C}[\lambda][x_1, \ldots, x_d].$$

Throughout this work, we consider the case where V is homogeneous of degree k in $\boldsymbol{x}$. This restricts the sum in (1) to a sum over certain k-regular graphs.

The polynomial A_n^V can be regarded as a variant of an edge-chromatic polynomial. Moreover, for a certain choice of V, A_n^V is an average partition function of the ferromagnetic Ising model on a regular graph. This statement is made precise in Section 5.

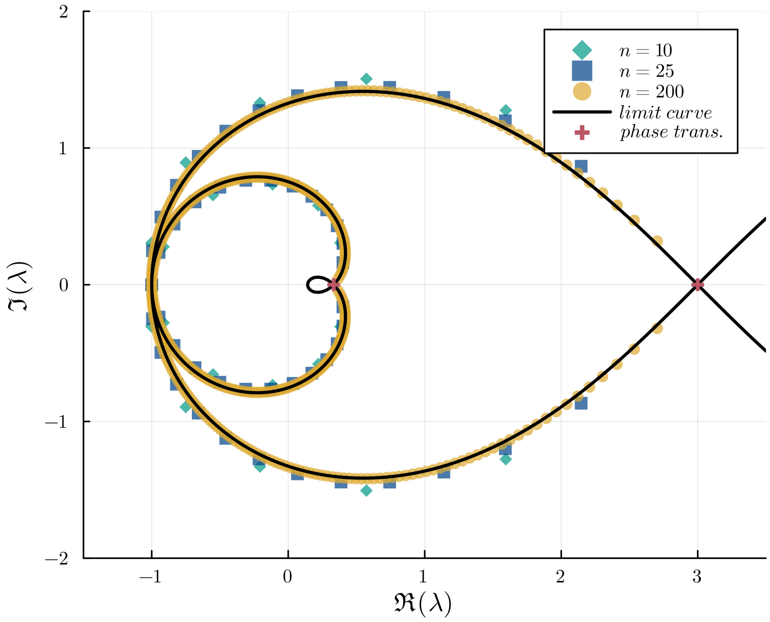


Figure 1 The roots of $A_n^V(\lambda)$ in the complex λ-plane, for $V(x_1, x_2, \lambda) = \frac{x_1^4}{4!} + \lambda \frac{x_1^2 x_2^2}{2! \cdot 2!} + \lambda^2 \frac{x_2^4}{4!}$.

Our main focus is on the zeros of $A_n^V(\lambda)$ for complex λ. As n increases, one observes a rather flummoxing accumulation of the zeros along certain limit curves, as in Figure 1. The red crosses denote phase transitions (points of non-analyticity) of the asymptotics of $A_n^V(\lambda)$ for $n \to \infty$. The goal of this work is the description of the limit curves. Below we state an informal version of our main result, Theorem 14.

▶ **Main Theorem** (informal). *Under mild non-degeneracy assumptions on V, the zeros of A_n^V accumulate along parts of anti-Stokes curves as $n \to \infty$, except for possibly finitely many isolated points. Those limit curves are given as follows. Let $\mathbb{S} = \{\boldsymbol{x} \in \mathbb{C}^d : x_1^2 + \cdots + x_d^2 = 1\}$ and let $\mathrm{crit}_{\mathbb{S}}(V)$ be the set of critical points σ of V restricted to $\mathbb{S}$ such that $V(\sigma) \neq 0$. Then, for $\sigma, \sigma' \in \mathrm{crit}_{\mathbb{S}}(V)$, an anti-Stokes curve is given by the condition*

$$\left\{\lambda \in \mathbb{C} : \Re \log V(\boldsymbol{x}, \lambda)\big|_{\boldsymbol{x}=\sigma} = \Re \log V(\boldsymbol{x}, \lambda)\big|_{\boldsymbol{x}=\sigma'}\right\}.$$

1.1 Context

The accumulation of zeros described above is reminiscent of the celebrated Lee–Yang theorem from statistical physics [17]. It states that the zeros of the ferromagnetic Ising model partition function on any finite graph all lie on the imaginary axis when the partition function is viewed as a function of a complex magnetic field parameter, see Theorem 19 below. As the system size increases, the zeros accumulate on the imaginary axis. The importance of this theorem stems from the general philosophy that partition function zeros reveal the locations of phase transitions [4], i.e. sudden qualitative changes in a physical system, such as the loss of magnetisation above the Curie temperature. Such phase transitions are locations where the partition function becomes non-analytic in the thermodynamic limit where the system size tends to infinity. The insight of Lee and Yang was that phase transitions are located where the complex zeros of the partition function pinch the real axis, and can be approximated by computing the zeros for finite size systems.

Since the 1950s, this theory has been generalised in many directions, for example by considering zeros in different physical parameters, such as the complex temperature leading to Fisher zeros [11], or by extending the class of lattice models [19]. A very general and rigorous treatment of partition function zeros is given in [4].

From a mathematical point of view, one may ask more generally: given a family of polynomials $(f_n)_n$, for instance arising from an enumerative problem, what can be said about the limiting distribution of their zeros as $n \to \infty$? Questions of this kind have also attracted considerable interest from combinatorialists. The zeros of chromatic polynomials and their accumulation were studied in [23], where Sokal constructs a countable family of graphs whose chromatic roots are dense in the complex plane outside a small disc. Moreover, the study of zeros is closely related to the theory of stable polynomials. In fact, Borcea and Brändén use stable polynomials to give an elegant proof of the Lee–Yang theorem [5].

In our framework, the polynomial A_n^V is not attached to a single fixed graph or lattice, but sums over all k-regular graphs with prescribed vertex incidence patterns. Therefore, A_n^V may be regarded as a random graph partition function. The study of such random graph partition functions is also prominent in physics, most notably Kazakov's exact solution of the Ising model on random planar graphs [16]. Depending on the choice of V, our parameter λ can, for example, play the role of a magnetic field parameter, leading to Lee–Yang zeros, or of a temperature-like parameter, leading to Fisher zeros. Our framework thus offers a unified setting for two classical types of partition function zeros.

From the combinatorial perspective, one expects phase transitions at parameter values where the typical large-scale structure of the counted objects changes, together with their asymptotic enumeration. For edge-coloured graphs, such an analysis is carried out in [7].

The methods developed here, combining graph enumeration with exponential integral representations and Picard–Lefschetz theory, also suggest several directions for future work. One natural next step is to extend the analysis from the univariate to a multiparameter setting. This would allow, for example, the study of the full two-parameter family of Ising zeros in (h, J)-space, and hence a more global picture of the phase diagram of the Ising model on a random regular graph. The Picard–Lefschetz-theoretic part of our argument extends naturally to this setting. The main obstacle is a generalisation of Theorem 17 to a higher-dimensional setting, which we leave for future work. It is furthermore expected that the density of zeros and the angle at which they approach a real accumulation point are related to critical exponents of the corresponding phase transition [13]. A finer asymptotic analysis in this direction would be very interesting. More broadly, we expect that the methods used here also apply to other combinatorially defined polynomial sequences whenever an exponential integral representation is available.

1.2 Structure of the argument

The key ingredient for proving the main theorem is an exponential integral representation. In fact, we can rewrite the polynomial $A_n^V(\lambda)$ as the integral

$$A_n^V(\lambda) = \frac{2^{nM+(d-2)/2}\,\Gamma\big(nM + \frac{d}{2}\big)}{(2\pi)^{d/2}(nK)!} \int_{S^{d-1}} V(\boldsymbol{x}, \lambda)^{nK}\, \omega_{S^{d-1}}, \tag{2}$$

where nK and nM are integers depending on the regularity k of the graphs (Proposition 6). For real parameters λ, the limit $\lim_{n\to\infty} A_n^V(\lambda)$ can simply be taken by performing a Laplace expansion around the maxima of $|V(\boldsymbol{x}, \lambda)|$ on S^{d-1}. However, for complex λ, an analytic continuation of (2) by means of Picard–Lefschetz theory becomes necessary in the large-n limit. Such an analytic continuation does not exist globally, but on certain domains $D \subset \mathbb{C}$. On each such domain, the integral from (2) can be written as

$$\sum_{\sigma\in\mathrm{crit}_{\mathbb{S}}(V)} c_\sigma(\lambda)\int_{\Gamma_\sigma} V(\boldsymbol{x},\lambda)^{nK}\,\omega_{S^{d-1}}, \tag{3}$$

where Γ_σ are steepest descent contours associated to each critical point, also known as Lefschetz thimbles, and $c_\sigma(\lambda)$ are complex coefficients. Here, one passes to the complex compactification $\mathbb{S}$ of the real sphere S^{d-1}. Each integral in (3) admits a stationary phase approximation that asymptotically reduces to an evaluation at the critical point in the $n\to\infty$ limit. The coefficients $c_\sigma(\lambda)$ encode topological information and are only analytic within each domain D but can jump if one leaves D. This is referred to as the Stokes' phenomenon, see e.g. [22]. Each domain D is a connected component of the complement of the arrangement of Stokes curves of the integral in (2).

To derive the decomposition (3), one needs to show that the Lefschetz thimbles form a basis of the appropriate homology group. Here, a technical difficulty arises because for many V of interest (in particular, the ones related to the Ising model considered in Section 5), several critical points yield the same evaluation of V, thus violating a commonly found assumption in Picard–Lefschetz theory, see e.g. [8, Hypothesis H4]. To remedy this problem we use a construction by Matsubara [20] to find a basis of Lefschetz thimbles and derive the expression (3). This is done in Section 3, culminating in Theorem 9. The stationary phase formula then gives the asymptotic expression (Corollary 10)

$$\int_{S^{d-1}} V(\boldsymbol{x},\lambda)^n\,\omega_{S^{d-1}} \sim \sum_{\sigma\in\mathrm{crit}_{\mathbb{S}}(V)} \tilde{c}_\sigma(\lambda)e^{n\log V(\boldsymbol{x},\lambda)|_{\boldsymbol{x}=\sigma}}(1+o(n^{-1}))\quad(n\to\infty). \tag{4}$$

A sum decomposition such as (4) is the starting point for the works [4, 23]. In the physics context, the quantities $\log V(\boldsymbol{x},\lambda)\big|_{\boldsymbol{x}=\sigma}$ can be thought of as metastable free energies. Zeros then accumulate along regions of multiphase coexistence, meaning that several free energies have the same magnitude. Indeed, we can apply a generalisation of the Beraha–Kahane–Weiss theorem due to Sokal [23, Theorem 1.5] to conclude that the zeros of $A_n^V(\lambda)$ accumulate along (parts of) the curves defined by

$$\Re\log V(\boldsymbol{x},\lambda)\big|_{\boldsymbol{x}=\sigma} = \Re\log V(\boldsymbol{x},\lambda)\big|_{\boldsymbol{x}=\sigma'}\quad(\Leftrightarrow|V(\sigma,\lambda)|=|V(\sigma',\lambda)|)\,, \tag{5}$$

where σ and σ' are critical points of V on $\mathbb{S}$. Here, we need to slightly modify Sokal's result to accommodate for the fact that analyticity of the coefficients $\tilde{c}_\sigma(\lambda)$ in (4) can only be guaranteed in a one-sided neighbourhood of the accumulation points (Section 4).

The curve defined in (5) is known as an anti-Stokes curve of the integral (2), i.e. a curve where the contribution from different thimble integrals from (3) exchange dominance. As mentioned before, the zeros only accumulate along parts of the curves defined by (5). This comes from the fact that (3) is not globally valid. In fact, not necessarily all critical points $\sigma\in\mathrm{crit}_{\mathbb{S}}(V)$ contribute to (3) for a given λ. The loci where the c_σ (and $\tilde{c}_\sigma$) can potentially become zero are given by the Stokes curves defined by

$$\Im V(\boldsymbol{x},\lambda)\big|_{\boldsymbol{x}=\sigma} = \arg V(\sigma,\lambda) \overset{!}{=} \arg V(\sigma',\lambda) = \Im V(\boldsymbol{x},\lambda)\big|_{\boldsymbol{x}=\sigma'}. \tag{6}$$

The accumulation of zeros is only along the intersection of the curves defined by (5) with those regions in the complement of the curves defined by (6) where both thimbles that exchange dominance on the anti-Stokes curve also contribute to the integral (3). This is illustrated in Figure 2 for the example from Figure 1. The green and purple curves are the anti-Stokes curves, the grey curves are the Stokes curves. The accumulation as in Figure 1 only happens along the intersection of the anti-Stokes curves with the non-shaded regions.

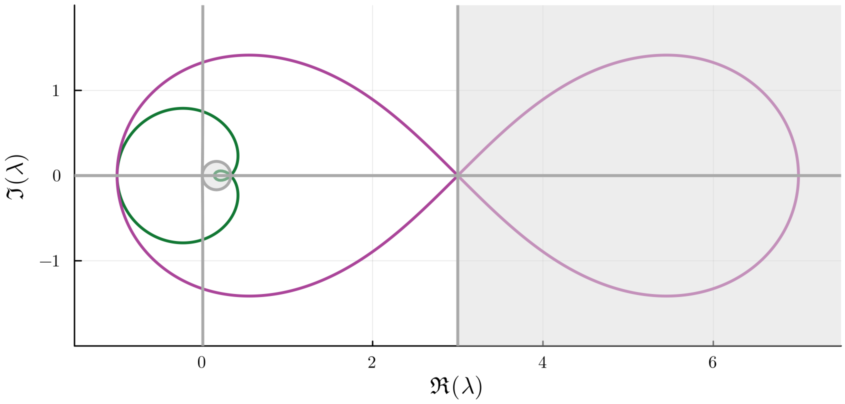


Figure 2 Anti-Stokes curves (green and purple) and Stokes curves (grey) for the example depicted in Figure 1. The roots accumulate along the intersection of the anti-Stokes curves with certain regions in the complement of the Stokes curves (non-shaded regions).

The idea of a relation between Lee–Yang (or Fisher) zeros and anti-Stokes curves has appeared before in different physics contexts: [15] observes such a relation numerically for a partition function in quantum field theory, and [13, 9, 14] use the idea to study zeros of Ising model partition functions. The paper [9] should be highlighted here as it studies partition function zeros for the Ising model on random 3- and 4-regular graphs, thus being closely related to our discussion in Section 5.

Outline of the document

In Section 2 we explain the framework of half-edge labelled coloured graphs, how the polynomial A_n^V naturally arises in the enumeration of such graphs (Corollary 2.4), and we show the representation of A_n^V as an exponential integral (Proposition 6). Section 3 is devoted to the Lefschetz thimble construction. The important results here are the existence of a basis of Lefschetz thimbles (Theorem 9) and the resulting asymptotic expression (Corollary 10). Our main result (Theorem 14) is stated precisely in Section 4. In the final Section 5 we explain how our framework can be seen as a unified approach to Lee–Yang and Fisher zeros of the Ising model on a random regular graph.

2 Edge-coloured graphs

In this section we explain the construction of edge-coloured graphs via half-edge labels. This also appears in [7], where we use it to obtain asymptotic numbers of proper edge-colourings. In the case of bicoloured graphs, the construction has already been described in [6]. For us, a graph G is a finite, one-dimensional CW complex, i.e. we allow for self-loops and multiple edges. Additionally, to each edge of the graph we assign one of d colours. To encode G using only discrete data we use half-edge labelled graphs. This leads to the following definition.

▶ **Definition 1.** *Let $H_1, \dots, H_d$ be disjoint sets of half-edge labels, one for each colour. An $[H_1, \dots, H_d]$-half-edge labelled coloured graph Γ is a tuple $\Gamma = (V, E_{H_1}, \dots, E_{H_d})$ where*

1. *V, the set of vertices, is a set partition of $H_1 \sqcup \dots \sqcup H_d$;*
2. *for all $i = 1, \dots, d$, E_{H_i} is a set partition of H_i into blocks of size two.*

Figure 3 An edge-bicoloured graph G with two connected components.

An edge-coloured graph G is identified with an isomorphism class $[\Gamma]$ of a half-edge labelled coloured representative Γ. Here, an *isomorphism* j between $\Gamma = (V, E_{H_1}, \dots, E_{H_d})$ and $\Gamma' = (V', E'_{H'_1}, \dots, E'_{H'_d})$ is a tuple $j = (j_1, \dots, j_d)$ of bijections $j_i \colon H_i \to H'_i$ such that the canonically induced maps $(j_i)_*$ on the partitions satisfy $(j_i)_*(E_{H_i}) = E'_{H'_i}$ for all $i = 1, \dots, d$, and $(j_1 \sqcup \dots \sqcup j_d)_*(V) = V'$. An *automorphism* of Γ is an isomorphism from Γ to itself. By $|\operatorname{Aut}(G)|$ we denote the cardinality of the automorphism group of Γ, for some representative $G = [\Gamma]$. The set of all isomorphism classes of half-edge labelled coloured graphs is denoted by $\mathcal{G}$. From now on, we might drop the adjective "edge-coloured".

The set of vertices of G is denoted by V_G, while the set of edges is E_G. Below there is an example of how to encode a graph via a half-edge labelled graph.

▶ **Example 2.** Let $H_1 = \{1, 2, \dots, 6\}$ and $H_2 = \{a, b\}$. The partitions

$$\begin{aligned} V &= \{\{1,2,3,4\}, \{5,6,a,b\}\}\,, \\ E_{H_1} &= \{\{1,2\}, \{3,4\}, \{5,6\}\}\,, \\ E_{H_2} &= \{\{a,b\}\}\,, \end{aligned}$$

form an $[H_1, H_2]$-labelled graph representing the graph G depicted in Figure 3. Its automorphism group is isomorphic to $(\mathfrak{S}_2 \times \mathfrak{S}_2 \rtimes \mathfrak{S}_2) \times \mathfrak{S}_2 \times \mathfrak{S}_2$.

A graph G is equipped with a *degree function* $\deg \colon V_G \to \mathbb{Z}^d_{\geq 0}$, where $\deg(v) = (w_1, \dots, w_d)$ if the vertex v is incident to $w_i = |v \cap H_i|$ half-edges of colour i, for $i = 1, \dots, d$.

In the following we make use of some multi-index notation. Bold symbols denote d-tuples, as in $\boldsymbol{w} = (w_1, \dots, w_d)$. The multi-index factorial is $\boldsymbol{w}! = w_1! \dots w_d!$, while $\boldsymbol{x}^{\boldsymbol{w}}$ is shorthand notation for $x_1^{w_1} \dots x_d^{w_d}$. Moreover, we use $|\boldsymbol{w}| = w_1 + \dots + w_d$.

▶ **Proposition 3.** *The generating function for graphs with marked vertex degrees is*

$$\sum_{G \in \mathcal{G}} \frac{\eta^{|E_G|}}{|\operatorname{Aut}(G)|} \prod_{v \in V_G} \Lambda_{\deg(v)} = \sum_{\boldsymbol{s} \geq 0} \eta^{|\boldsymbol{s}|} (2s_1 - 1)!! \cdots (2s_d - 1)!! \cdot [\boldsymbol{x}^{2\boldsymbol{s}}] \exp\left(\sum_{\substack{\boldsymbol{w} \in \mathbb{Z}^d_{\geq 0} \\ |\boldsymbol{w}| \geq 1}} \Lambda_{\boldsymbol{w}} \frac{\boldsymbol{x}^{\boldsymbol{w}}}{\boldsymbol{w}!} \right). \tag{7}$$

Here, $[\boldsymbol{x}^{2\boldsymbol{s}}]$ denotes the coefficient extraction operator, and (7) is an expression in the formal power series ring $\mathbb{Q}[\Lambda_{\boldsymbol{w}} : \boldsymbol{w} \in \mathbb{Z}^d_{\geq 0}][[\eta]]$ with coefficients in the formal variables $\Lambda_{\boldsymbol{w}}$.

A proof of the above statement is given in [25]. It is convenient to shift the generating function such that the index of summation is the *Euler characteristic* $\chi(G)$ of G. Recall that $\chi(G) = |V_G| - |E_G|$.

▶ **Corollary 4.** *Let $\mathcal{G}_{-n}$ be the set of isomorphism classes of half-edge labelled graphs with Euler characteristic $-n$. Then the following identity holds true.*

$$\sum_{\substack{n \geq 0 \\ G \in \mathcal{G}_{-n}}} \frac{z^{\chi(G)}}{|\operatorname{Aut}(G)|} \prod_{v \in V_G} \Lambda_{\deg(v)} = \sum_{\boldsymbol{s} \geq 0} z^{|\boldsymbol{s}|} (2s_1 - 1)!! \cdots (2s_d - 1)!! [\boldsymbol{x}^{2\boldsymbol{s}}] \exp\left(\sum_{\substack{\boldsymbol{w} \in \mathbb{Z}^d_{\geq 0} \\ |\boldsymbol{w}| \geq 1}} z \Lambda_{\boldsymbol{w}} \frac{\boldsymbol{x}^{\boldsymbol{w}}}{\boldsymbol{w}!} \right) \tag{8}$$

The expression (8) is an equality in the formal power series ring $\mathcal{R}[[z]]$ over the polynomial ring $\mathcal{R} = \mathbb{Q}[\Lambda_{\boldsymbol{w}}]$ in infinitely many variables. We will be interested in the case where all the variables $\Lambda_{\boldsymbol{w}}$ are specified to depend on only a *single* parameter λ, and only finitely many are nonzero. In other words, we specialise all $\Lambda_{\boldsymbol{w}}$ in such a way so that

$$V(\boldsymbol{x}, \lambda) = \sum_{\boldsymbol{w} \in \mathbb{Z}^d_{\geq 0},\, |\boldsymbol{w}| \geq 1} \Lambda_{\boldsymbol{w}}(\lambda) \frac{\boldsymbol{x}^{\boldsymbol{w}}}{\boldsymbol{w}!}$$

is a polynomial in $\mathbb{Q}[\lambda][\boldsymbol{x}]$. Then the left-hand side of (8) becomes a generating function for a specific set of graphs with vertex degree markings and allowed vertex-incidences determined by the coefficients and nonzero terms of $V(\boldsymbol{x}, \lambda)$. The coefficient of z^{-n} in (8),

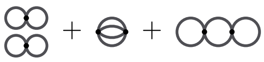
$$A_n^V(\lambda) := \sum_{G \in \mathcal{G}_{-n}} \frac{1}{|\operatorname{Aut}(G)|} \prod_{v \in V_G} \Lambda_{\deg(v)}(\lambda),$$

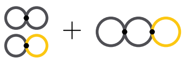
is now a univariate polynomial in λ.

▶ **Example 5.** Consider the case of two colours, $d = 2$, and fix the polynomial

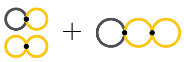
$$V(x_1, x_2, \lambda) = \frac{x_1^4}{4!} + \lambda \frac{x_1^2 x_2^2}{2! \cdot 2!} + \lambda^2 \frac{x_2^4}{4!}. \tag{9}$$

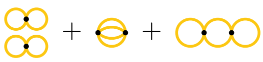
This restricts the generating function (8) to 4-regular graphs where each vertex is incident to four half-edges which are all the same colour, or to two half-edges of the first and two half-edges of the second colour. The parameter λ counts the number of edges of the second colour. For example, the Euler characteristic $-n = -2$ contribution to (8) is given by

 $= \frac{1}{128} + \frac{1}{48} + \frac{1}{16} = \frac{35}{384}$,

 $= \lambda \left(\frac{1}{32} + \frac{1}{8}\right) = \frac{5}{32}\lambda$,

 $= \lambda^2 \left(\frac{1}{64} + \frac{1}{32} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}\right) = \frac{19}{64}\lambda^2$,

 $= \lambda^3 \left(\frac{1}{32} + \frac{1}{8}\right) = \frac{5}{32}\lambda^3$,

 $= \lambda^4 \left(\frac{1}{128} + \frac{1}{48} + \frac{1}{16}\right) = \frac{35}{384}\lambda^4$,

so we obtain $A_2^V(\lambda) = \frac{35}{384}\lambda^4 + \frac{5}{32}\lambda^3 + \frac{19}{64}\lambda^2 + \frac{5}{32}\lambda + \frac{35}{384}$.

From now on, we restrict to the case where V is *homogeneous* of degree k in the $\boldsymbol{x}$ variables. Consequently, we only enumerate k-regular graphs. This assumption lets us rewrite A_n^V in terms of an exponential integral, which will be crucial in the forthcoming section. Note that if a k-regular graph G has Euler characteristic $-n$, then, since $k|V_G| = 2|E_G|$ and $|V_G| - |E_G| = -n$, G is required to have $\frac{2n}{k-2}$ many vertices and $\frac{nk}{k-2}$ many edges.

Using Corollary 4, the homogeneity of V and some standard Gaussian integral identities, we arrive at the following representation of $A_n^V(\lambda)$ as an exponential integral. A detailed proof is provided in the full article [25].

▶ **Proposition 6.** *Let $V \in \mathbb{Q}[\lambda][\boldsymbol{x}]$ be homogeneous of degree k in $\boldsymbol{x}$, and let us denote $M = \frac{k}{k-2}$ and $K = \frac{2}{k-2}$. Then, for any integer $n \geq 0$ such that $nK, nM \in \mathbb{Z}$, we have*

$$A_n^V(\lambda) = \frac{2^{nM+(d-2)/2}\,\Gamma\!\left(nM + \frac{d}{2}\right)}{(2\pi)^{d/2}(nK)!} \int_{S^{d-1}} V(\boldsymbol{x}, \lambda)^{nK}\, \omega_{S^{d-1}}, \tag{10}$$

where $\omega_{S^{d-1}}$ denotes the standard volume form of the $(d-1)$-dimensional sphere S^{d-1}.

In the large-n asymptotics, a saddle point approximation of (10) can be used to derive an asymptotic expression for A_n^V in terms of maxima of $|V(\boldsymbol{x},\lambda)|$ on S^{d-1}, as long as λ is *real*. A detailed analysis of this is carried out in [7]. However, since we are interested also in complex values of λ, we need to analytically continue the $n \to \infty$ limit of (10). This is the content of the next section, using Picard–Lefschetz theory.

3 Lefschetz thimble decomposition

Proposition 6 tells us to study the analytic continuation to $\lambda \in \mathbb{C}$ of the integral

$$I_n^V(\lambda) := \int_{S^{d-1}} V(\boldsymbol{x},\lambda)^n \omega_{S^{d-1}} \tag{11}$$

in the limit $n \to \infty$. To this end, we decompose $I_n^V(\lambda)$ in terms of integrals over *Lefschetz thimbles*. These are integration contours associated to critical points of V along which the real part of $\log V$ decreases; this makes them amenable to a stationary phase approximation. In the large-n limit, the thimble integrals reduce to critical point evaluations. An illustration of how the contour S^{d-1} is decomposed into Lefschetz thimbles can be found in Figure 4.

Such an approach is prominent both in mathematics and theoretical physics, see [8, 26]. To make it precise we need to show that the Lefschetz thimbles define a basis of the appropriate homology group. Here, a technical subtlety arises: one commonly assumes the critical values $\Re \log V(\sigma_j)$ to be *distinct* for different critical points σ_j, see e.g. [8, Hypothesis H4] or [1, Assumption 8]. However, for the applications we have in mind, this assumption is violated, e.g. for V as constructed in Example 5.

We modify the construction of the Lefschetz thimbles in a two-step process which has been laid out in [20, §4]: First, we define thimbles Γ_σ as (un)stable manifolds of a Morse–Smale vector field that is a small perturbation of the gradient vector field used in the standard Lefschetz thimble construction. This ensures transverse intersections of the thimbles. Contours constructed in this way are elements in a *locally finite* homology group.

In a second step, we need to *regularise* these thimbles to obtain honest (twisted) homology classes $\widetilde{\Gamma}_\sigma$. This involves a real oriented blowup construction. For the case of complements of hyperplane arrangements, this process is also described in [1, §3.2.3–3.2.5]. Under a genericity assumption on V expressing the number of critical points as a signed Euler characteristic, the $\widetilde{\Gamma}_\sigma$ form a basis of the twisted homology group. Since the integral (11) can be regarded as a period pairing between this twisted homology group and an algebraic de Rham cohomology group, this then proves the Lefschetz thimble decomposition of the integral (11), which is the content of Theorem 9. By applying a stationary phase approximation, we can then get an asymptotic expression for $I_n^V(\lambda)$ that is valid for almost all $\lambda \in \mathbb{C}$, see Corollary 10.

Let us introduce some notation, mostly following [20]. Let

$$\mathbb{S} := \{\boldsymbol{x} \in \mathbb{C}^d \,:\, x_1^2 + x_2^2 + \cdots + x_d^2 = 1\}$$

be the *complexified sphere* of complex dimension $d-1$. For a fixed λ, let $U_\lambda := \mathbb{S} \setminus \mathcal{V}(V(\boldsymbol{x},\lambda))$ be the locus where V is non-vanishing, equipped with a complete Riemannian metric g. We can view $D_\lambda := \mathcal{V}(V(\boldsymbol{x},\lambda))$ as a divisor on $\mathbb{S}$. Moreover, we denote $F_\lambda(\boldsymbol{x}) := \log V(\boldsymbol{x},\lambda)$. In the following we collect the necessary non-degeneracy assumptions. Crucially, these only fail on an at most zero-dimensional discriminant locus Σ, see Remark 11.

(A1) The critical locus $\mathrm{crit}(F_\lambda) := \{\boldsymbol{x} \in U_\lambda \,:\, \mathrm{d}_{\boldsymbol{x}} F_\lambda(\boldsymbol{x}) = 0\}$ is a zero-dimensional variety. This is true as long as the factorisation of $V(\boldsymbol{x},\lambda)$ is squarefree.

(A2) The divisor D_λ should be simple normal crossing. In particular, this is the case if D_λ is smooth, i.e. $\mathrm{d}_{\boldsymbol{x}} V(\boldsymbol{x},\lambda) \neq 0$ for all $\boldsymbol{x} \in D_\lambda$.

(A3) For any $\sigma \in \operatorname{crit}(F_\lambda)$, F_λ is non-degenerate at σ, i.e. $\det \operatorname{Hess}_{F_\lambda}(\sigma) \neq 0$.

(A4) The number of critical points is $|\operatorname{crit}(F_\lambda)| = (-1)^{d-1}\chi(U_\lambda)$, where χ is the topological Euler characteristic. We denote this number by $\chi^* := |\operatorname{crit}(F_\lambda)|$.

Note that $\operatorname{crit}(F_\lambda)$ can easily be computed by first determining the critical points of V on $\mathbb{S}$ via Lagrange multipliers and then intersecting those with U_λ.

▶ Remark 7. The Euler characteristic of $\mathbb{S}$ is $1 + (-1)^{d-1}$. Indeed, let $f = x_1^2 + \cdots + x_d^2$; then the affine Milnor fibration $\mathbb{C}^d \setminus \{f = 0\} \xrightarrow{f} \mathbb{C}^*$ has fibre $\{f = 1\}$ which is homotopy equivalent to a bouquet of μ (complex) spheres of dimension $d - 1$. Here, μ is the Milnor number of the singularity of f at the origin, which is an A_1-singularity, hence $\mu = 1$. This implies $\chi(\mathbb{S}) = 1 + (-1)^{d-1}$. By the excision property of the Euler characteristic,

$$(-1)^{d-1}\chi(U_\lambda) = (-1)^d\,(\chi(D_\lambda) - 1) + 1. \tag{12}$$

▶ **Example 8.** Consider $V(x_1, x_2, \lambda) = \frac{x_1^4}{4!} + \lambda\frac{x_1^2 x_2^2}{2!\cdot 2!} + \lambda^2\frac{x_2^4}{4!}$; generically, this polynomial has eight roots on $\mathbb{S} = \{x_1^2 + x_2^2 = 1\}$ which has Euler characteristic zero. By (12), one expects $|\operatorname{crit}(F_\lambda)| = 8$. This is indeed the case for generic λ. For $\lambda \in \{1/3, 3\}$, there are only six critical points. If $\lambda = 3 \pm \sqrt{8}$, V has only four roots on $\mathbb{S}$, as well as four critical points.

Let $\nabla_\pm := \mathrm{d} \pm \tau\,\mathrm{dlog}\,V\wedge$ be a *twisted differential* on $\mathbb{S}$ with logarithmic poles along D_λ. The $(d-1)$-form $\omega_{S^{d-1}}$ from (11) is an element of $H^0(U_\lambda, \Omega^{d-1}_{U_\lambda})$. We can also view it as a representative of a cohomology class in an algebraic de Rham cohomology group $H^{d-1}_\pm(\tau) := \mathbb{H}^{d-1}(U_\lambda; (\Omega^\bullet_{U_\lambda}, \nabla_\pm))$. Let $\mathcal{L}^\pm(\tau)$ be the sheaf on U_λ of flat sections of the connection $\nabla_\mp$. Then there is the *period pairing*, which is the perfect pairing

$$\begin{aligned} \langle \bullet, \bullet \rangle_{\text{per}} \;:\; H_{d-1}(U_\lambda; \mathcal{L}^\pm(\tau)) \otimes_{\mathbb{C}} H^{d-1}_\pm(\tau) &\to \mathbb{C} \\ [\widetilde{\Gamma}^\pm] \otimes [\omega_\pm] &\mapsto \int_{\widetilde{\Gamma}^\pm} V(\boldsymbol{x}, \lambda)^{\pm\tau}\omega_\pm. \end{aligned} \tag{13}$$

Here, $H_{d-1}(U_\lambda; \mathcal{L}^\pm(\tau))$ is a *twisted homology group*, taking values in the local system $\mathcal{L}^\pm(\tau)$. The idea is now to construct a basis of $H_{d-1}(U_\lambda; \mathcal{L}^\pm(\tau))$ such that the integral (11) decomposes as a sum of integrals over the basis contours. Under assumptions (A1)–(A4) and for generic $\tau > 0$, this basis is given by regularised Lefschetz thimbles $\{\widetilde{\Gamma}^\pm_\sigma(\tau)\}$, one for each critical point $\sigma \in \operatorname{crit}(F_\lambda)$ (and hence of V). The construction of such regularised thimbles is taken from [20, §4], and we explain it in detail in [25]. As a result, we obtain the following statement.

▶ **Theorem 9.** *For a fixed $\lambda \in \mathbb{C}$ so that assumptions (A1)–(A4) are valid, and for generic $\tau > 0$, the regularised Lefschetz thimbles $\{\widetilde{\Gamma}^\pm_\sigma(\tau)\}_{\sigma \in \operatorname{crit}(F_\lambda)}$ form a basis of the twisted homology $H_{d-1}(U_\lambda; \mathcal{L}^\pm(\tau))$. Via the period pairing* (13)*, the integral $I^V_\tau(\lambda)$ decomposes as*

$$I^V_\tau(\lambda) = \sum_{\sigma \in \operatorname{crit}(F_\lambda)} c_\sigma(\lambda) \int_{\widetilde{\Gamma}^+_\sigma(\tau)} V(\boldsymbol{x}, \lambda)^\tau \omega_{S^{d-1}}, \tag{14}$$

for some coefficients $c_\sigma(\lambda) \in \mathbb{C}$.

The benefit of having this Lefschetz thimble decomposition is that each integral in (14) is amenable to a stationary phase approximation in the large-n limit. This is the content of the next statement, a proof of which can be found in the full article [25].

▶ **Corollary 10.** *Let us fix $\lambda \in \mathbb{C}$ so that assumptions (A1)–(A4) are valid. Then, in the large-n limit, the integral $I_n^V(\lambda)$ is asymptotically given by*

$$I_n^V(\lambda) \sim \left(-\frac{2\pi}{n}\right)^{(d-1)/2} \sum_{\sigma \in \mathrm{crit}(F_\lambda)} \frac{c_\sigma(\lambda)\omega_{S^{d-1}}(\sigma\partial_{\boldsymbol{x}})}{\sqrt{\det \mathrm{Hess}_{F_\lambda}(\sigma)}} e^{nF_\lambda(\sigma)}(1 + o(n^{-1})), \tag{15}$$

where $\sigma\partial_{\boldsymbol{x}}$ denotes the constant vector field on $\mathbb{S}$ with value σ.

We conclude this section with some remarks and an example.

▶ Remark 11. If $V(\boldsymbol{x}, \lambda) \in \mathbb{C}[\boldsymbol{x}, \lambda]$ is irreducible, then assumptions (A1)–(A4) are violated on an at most zero-dimensional variety in $\mathbb{C}_\lambda$. Let us denote this finite *discriminant* set of points by Σ. In case of (A1), the claim follows from Sard's theorem. The divisor D_λ is smooth away from a discriminant hypersurface in the one-dimensional parameter space $\mathbb{C}_\lambda$. Similarly, the vanishing Hessian is a hypersurface in $\mathbb{C}_\lambda$, giving a finite set of exceptional points. The locus where (A4) fails is a version of an *Euler discriminant*, see e.g. [24].

▶ Remark 12. By construction of $\widetilde{\Gamma}_\sigma^+$, each of the integrals appearing in (14) is an analytic function in λ on $\mathbb{C} \setminus \Sigma$, as well as the function $I_n^V(\lambda)$. That is, however, not true for the coefficients $c_\sigma(\lambda)$, which is an instance of *Stokes' phenomenon.* These coefficients can even jump discontinuously along so-called *Stokes curves* of the form

$$S_{\sigma,\sigma'} := \{\lambda \in \mathbb{C} \,:\, \Im F_\lambda(\sigma) = \Im F_\lambda(\sigma') \mod 2\pi\},$$

for $\sigma, \sigma' \in \mathrm{crit}(F_\lambda)$ such that $S_{\sigma,\sigma'}$ is one-dimensional if interpreted as a real algebraic variety in $\mathbb{R}^2 \simeq \mathbb{C}$. If λ lies on such a Stokes curve, the choice of basis of Lefschetz thimbles is not canonical any more. As λ crosses a Stokes line, the basis of Lefschetz thimbles transforms via a unipotent matrix, see e.g. [22]. Note, however, that the $c_\sigma(\lambda)$ possess well-defined one-sided limits to the Stokes curves.

For $\sigma, \sigma' \in \mathrm{crit}(F_\lambda)$ such that $\dim_{\mathbb{R}}(S_{\sigma,\sigma'}) = 1$, we denote by

$$\mathcal{S}^c_{\sigma,\sigma'} := \pi_0\left(\mathbb{R}^2 \setminus S_{\sigma,\sigma'}\right)$$

the connected components in the complement of the Stokes curve. In particular, we have that $c_\sigma(\lambda) \neq 0$ on a union of regions in $\mathcal{S}^c_{\sigma,\sigma'}$, except for possibly isolated points.

▶ **Example 13.** We illustrate the decompostion of the integration contour S^{d-1} into Lefschetz thimbles Γ_σ^+ for $V(x_1, x_2, \lambda) = \frac{x_1^4}{4!} + \lambda\frac{x_1^2 x_2^2}{2!\cdot 2!} + \lambda^2\frac{x_2^4}{4!}$ in Figure 4. The space $\mathbb{S}$ is depicted under the projection $\pi : \mathbb{S} \subset \mathbb{C}^2 \to \mathbb{R}^3, (x_1, x_2) \mapsto (\Re x_1, \Re x_2, \Im x_1 + 2\Im x_2)$ as the yellow surface. It contains the real circle $S^1 \subset \mathbb{S}$, shown in red. For $\lambda = 2$, the eight critical points of V on $\mathbb{S}$,

$$\sigma_{1,2} = (\pm 1, 0) \quad \sigma_{3,4} = (0, \pm 1) \quad \sigma_{5,\dots,8} = \left(\pm\sqrt{\tfrac{2}{7}}, \pm\sqrt{\tfrac{5}{7}}\right),$$

are shown as the light ($\sigma_{1,\dots,4}$) and dark grey ($\sigma_{5,\dots,8}$) dots; they all lie on the real circle. Since $\lambda = 2$ lies on a Stokes curve, we draw the thimbles and the gradient vector field $\mathrm{d}_{\boldsymbol{x}}\Re F_\lambda$ for a slightly perturbed $\lambda = 2 + \mathrm{i}\epsilon$ (grey arrows). The orange dots are the roots of V, i.e. the points where $\Re F_\lambda \to -\infty$. The picture shows both the thimbles Γ_σ^+, and their duals, Γ_σ^-. The contour Γ_σ^+ is a union of two trajectories, both leaving the critical point σ and flowing in the direction of the gradient field towards a root of V. The dual Γ_σ^- flows towards the critical point σ. Thimbles $\Gamma_{\sigma_i}^\pm$ are drawn in blue if $i \in \{1, \dots, 4\}$, and in green if $i \in \{5, \dots, 8\}$. The original contour S^1 can be decomposed as the sum $\pm\Gamma_{\sigma_1}^+ \pm \cdots \pm \Gamma_{\sigma_8}^+$. The coefficient of $\Gamma_{\sigma_i}^+$ in this sum is determined by the intersection number $\langle S^1, \Gamma_{\sigma_i}^-\rangle$; here, they are all ± 1.

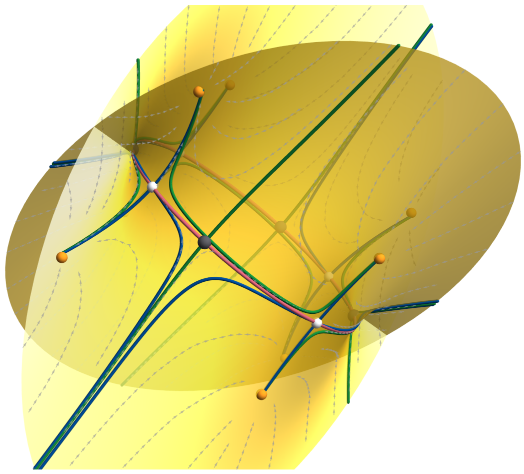


Figure 4 A real projection of the space $\mathbb{S}$ for $d = 2$ (yellow surface) containing the real circle (red curve), the critical locus $\mathrm{crit}(F_\lambda)$ for $F_\lambda = \log(V)$ and $\lambda = 2 + \mathrm{i}\epsilon$ as described in Example 13 (light and dark grey dots), the roots of V on $\mathbb{S}$ (orange dots), the gradient vector field $\mathrm{d}_{\boldsymbol{x}}\Re F_\lambda$ (grey arrows), and the Lefschetz thimbles $\Gamma_\sigma^\pm$ for all $\sigma \in \mathrm{crit}(F_\lambda)$ (blue and green curves).

4 Accumulation of zeros

In this section we state and prove the main theorem of this work. The key idea is to combine Corollary 10 with a result by Sokal on accumulation of zeros which is in turn a generalisation of the Beraha–Kahane–Weiss theorem [3].

Let us first set up some notation following [23]. Let $D \subset \mathbb{C}$ be a domain and $(f_n)_n$ a sequence of function $f_n : D \to \mathbb{C}$. We define the following limit sets of zeros.

$$\liminf \mathcal{V}(f_n) := \left\{ \begin{array}{c} z \in D \,:\, \text{every neighbourhood } U \ni z \text{ has a nonempty} \\ \text{intersection with all but finitely many } \mathcal{V}(f_n) \end{array} \right\}$$

$$\limsup \mathcal{V}(f_n) := \left\{ \begin{array}{c} z \in D \,:\, \text{every neighbourhood } U \ni z \text{ has a nonempty} \\ \text{intersection with infinitely many } \mathcal{V}(f_n) \end{array} \right\}$$

If f_n is of the form $f_n(z) = \sum_{k=1}^m \alpha_k(z)\beta_k(z)^n$, we call an index k *dominant at* z if $|\beta_k(z)| \geq |\beta_l(z)|\ \forall 1 \leq l \leq m$, and write $D_k := \{z \in D : k \text{ is dominant at } z\}$. We call the following assumption the *no-degenerate-dominance condition:*

(B1) There do not exist indices $k \neq k'$ such that $\beta_k \equiv \zeta\beta_{k'}$ for some $\zeta \in \mathbb{C} \setminus \{1\}$ with $|\zeta| = 1$ and such that D_k has nonempty interior.

We are now in the position to formulate our main theorem precisely.

▶ **Theorem 14** (Main result)**.** *Let $V \in \mathbb{C}[\lambda][\boldsymbol{x}]$ be a homogeneous polynomial satisfying assumptions (A1)–(A4) for generic $\lambda \in \mathbb{C}$ (i.e. $\lambda \in \mathbb{C} \setminus \Sigma$ with finite Σ). Moreover, assume that (B1) is satisfied for the functions $V(\sigma)$, $\sigma \in \mathrm{crit}(F_\lambda)$. Then, for $n \to \infty$, the zeros of $A_n^V(\lambda)$ accumulate along segments of anti-Stokes curves. More precisely,*

$$\liminf \mathcal{V}(A_n^V) = \limsup \mathcal{V}(A_n^V) = \mathcal{P} \cup \bigcup_{\substack{\sigma \neq \sigma' \in \mathrm{crit}(F_\lambda) \\ D \in \mathcal{S}^c_{\sigma,\sigma'}}} \{\lambda \in \mathbb{C} : \Re F_\lambda(\sigma) = \Re F_\lambda(\sigma')\} \cap \overline{D}, \quad (16)$$

where $\mathcal{P}$ is a (possibly empty) set of finitely many isolated points and the union ranges over some (not necessarily all) pairs of critical points $\sigma, \sigma' \in \mathrm{crit}(F_\lambda)$ such that the anti-Stokes curve $\{\lambda \in \mathbb{C} : \Re F_\lambda(\sigma) = \Re F_\lambda(\sigma')\}$ is one-dimensional as a real algebraic variety.

▶ Remark 15. Note that the critical points of $F_\lambda = \log V$ agree with the critical points of V restricted to $\mathbb{S}$. Moreover, we have the identities $\Re F_\lambda = \log |V(\lambda)|$ and $\Im F_\lambda = \arg V(\lambda) = \operatorname{atan2}(\Im V(\lambda), \Re V(\lambda))$ (the 2-argument arctangent function), from which one sees that the (anti-)Stokes curves are algebraic curves in $\mathbb{R}^2_{ab}$ where $\lambda = a+ib$. Therefore, the accumulation set of zeros in (16) is semialgebraic in the two real variables a and b.

Before we prove Theorem 14, we illustrate the result with an example.

▶ **Example 16.** Consider $V(x_1,x_2,\lambda) = \frac{x_1^4}{4!} + \lambda \frac{x_1^2 x_2^2}{2!\cdot 2!} + \lambda^2 \frac{x_2^4}{4!}$ as in Example 5. Using Lagrange multipliers the critical points of V restricted to $\mathbb{S}$ can be computed as

$$\sigma_{1,2} = (\pm 1, 0) \quad \sigma_{3,4} = (0,\pm 1) \quad \sigma_{5,\dots,8} = \left(\pm\sqrt{\frac{\lambda^2 - 3\lambda}{\lambda^2 - 6\lambda + 1}}, \pm\sqrt{\frac{-3\lambda+1}{\lambda^2-6\lambda+1}} \right).$$

For generic λ, these are eight distinct critical points, which agrees with $|\chi(U_\lambda)|$, see Example 8. If $\lambda = 3 \pm \sqrt{8}$, $\sigma_{5,\dots,8}$ move off to infinity. However, in this case V has only four roots on $\mathbb{S}$, so assumption (A4) is still satisfied. If $\lambda \in \{1/3, 3\}$, σ_5 and σ_5, respectively σ_7 and σ_8, collide and (A4) is violated. These two points lie in the discriminantal locus $\{1/3, 3\} \subseteq \Sigma$, see Remark 11. The four critical points $\sigma_{1,\dots,4}$ are non-degenerate provided $\lambda \neq 0$. Moreover, $\sigma_{5,\dots,8}$ are non-degenerate as long as $\lambda \notin \{0,1\}$. No critical point lies on the divisor D_λ except if $\lambda = 0$. Therefore, we have $\Sigma = \{0, 1/3, 1, 3\}$ as the finite discriminant locus where assumptions (A1)–(A4) are violated.

There are three distinct anti-Stokes curves:

$$C_1 = \{\Re F_\lambda(\sigma_1) = \Re F_\lambda(\sigma_5)\},\ C_2 = \{\Re F_\lambda(\sigma_3) = \Re F_\lambda(\sigma_5)\},\ C_3 = \{\Re F_\lambda(\sigma_1) = \Re F_\lambda(\sigma_3)\}.$$

The curve C_2 is the purple curve depicted in Figure 2. It is a classical algebraic curve known as the *Bernoulli lemniscate*, following the equation (with $a = \Re\lambda$, $b = \Im\lambda$)

$$\ell(a,b) := ((a-3)^2 + b^2)^2 - 4^2((a-3)^2 - b^2) = 0.$$

The curve C_1 is the green curve from Figure 2. On first sight it looks like a Pascal limaçon (such a limaçon has been found in [9, §6] as the accumulation locus for the zeros of a Potts model on a random graph, and in [2] as the limit curve for the zeros of a generating function of adsorbing Dyck paths). However, that is not quite true for our curve C_1. Instead, it obeys the slightly modified equation $\ell(a,b) - 64((a^2+b^2)^2 - 1) = 0$.

The third anti-Stokes curve C_3 is a unit circle. It does not appear in the accumulation locus (16), see also Figure 1. Finally, the Stokes curve arrangement needs to be determined.

The locus $\{\Im F_\lambda(\sigma_1) = \Im F_\lambda(\sigma_5)\}$ can be identified as the union of the a-axis together with the ellipse defined by $3(a^2+b^2) - a = 0$ which becomes apparent after rewriting as

$$\arctan\left(\frac{2b(a - 3a^2 - 3b^2)}{a^2(1 + (a-6)a) + (2(a-3)a - 1)b^2 + b^4} \right) = 0.$$

Similarly, one identifies the Stokes curve $S_{\sigma_3,\sigma_5} = \{\Im F_\lambda(\sigma_3) = \Im F_\lambda(\sigma_5)\}$ as the union of the three lines $\{a = 0\}$, $\{a = 3\}$, $\{b = 0\}$. Together, the Stokes arrangement is shown as the grey lines in Figure 2. The accumulation of zeros only takes place in the region

$$\overline{D} = \left\{(a,b) \in \mathbb{R}^2 \,:\, a \leq 3 \text{ and } 3(a^2+b^2) - a \geq 0\right\}.$$

In the complement of this region (grey shaded area in Figure 2), the corresponding Lefschetz thimbles do not contribute to the asymptotics (15). Note that the two points $\lambda = 1/3$ and $\lambda = 3$ lie in the (Euclidean) closure of $(C_1 \cap \overline{D}) \setminus \Sigma$, respectively $(C_2 \cap \overline{D}) \setminus \Sigma$, and hence belong to the accumulation locus since this is closed. Our theory does not allow making statements regarding the two isolated points $\lambda \in \{0,1\}$, a subset of which forms $\mathcal{P}$.

In order to prove Theorem 14, we use Sokal's result [23, Theorem 1.5] below.

▶ **Theorem 17** (Generalised Beraha–Kahane–Weiss). *Let D be a domain in $\mathbb{C}$, and let $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$ with $m \geq 2$ be* analytic *functions on D, none of which is identically zero. For each integer $n \geq 0$, let*

$$f_n(z) = \sum_{k=1}^{m} \alpha_k(z)\beta_k(z)^n.$$

Let us assume that (B1) holds. Then $\liminf \mathcal{V}(f_n) = \limsup \mathcal{V}(f_n)$, and a point z lies in this set if and only if either

1. *there is a unique dominant index k at z and $\alpha_k(z) = 0$; or*
2. *there are two or more dominant indices at z.*

Note that the locus where *1.* holds consists of isolated points, whereas *2.* leads to curves in $\mathbb{R}^2 \simeq \mathbb{C}$. Heuristically, it is clear that zeros can only occur along the loci described in *1.* and *2.* since the dominant contributions to f_n need to cancel out in order for a zero to occur. The intuition behind the converse is more intricate: it essentially boils down to a winding number argument of the function f_n/β_j for a loop around z.

A small subtlety arises when one tries to apply Theorem 17 to the expression (15): the coefficients $c_\sigma(\lambda)$ are potentially non-analytic at Stokes curves, see Remark 12. This is the main point that needs to be addressed in the proof below.

Proof of Theorem 14. From Proposition 6 we see that in the $n \to \infty$ limit, all zeros of $A_n^V(\lambda)$ come from zeros of the integral $I_n^V(\lambda) = \int_{S^{d-1}} V(\boldsymbol{x}, \lambda)^n \, \omega_{S^{d-1}}$, since the coefficient in front of the integral does not depend on λ. By Corollary 10, in the large-n limit and under assumptions (A1)–(A4), this integral asymptotically becomes

$$I_n^V(\lambda) \sim \sum_{\sigma \in \mathrm{crit}(F_\lambda)} \tilde{c}_\sigma(\lambda) e^{nF_\lambda(\sigma)} =: \tilde{I}_n^V(\lambda), \tag{17}$$

where we absorbed all coefficients in $\tilde{c}_\sigma(\lambda)$. By assumption and the definition of Σ, this expression is valid for $\lambda \in \mathbb{C} \setminus \Sigma$. The points in Σ potentially lie in $\mathcal{P}$; however, since Σ only consists of finitely many isolated points, we may now assume $\lambda \in \mathbb{C} \setminus \Sigma$. The coefficient c_σ is not identically zero on certain regions $D \in \mathcal{S}^c_{\sigma,\sigma'}$, see Remark 12. If there are several critical points σ, σ' with the same evaluations $F_\lambda(\sigma) = F_\lambda(\sigma')$, we collect the coefficients into one $\alpha_k(\lambda),\ k = 1, \ldots, m$. The result now follows from Theorem 17 except for the potential non-analyticity of α_k along Stokes curves.

The only thing left to show is that if $\lambda_0 \in S_{\sigma,\sigma'} \setminus \bigcup\{\text{anti-Stokes curves}\}$, then $\lambda_0 \notin \limsup \mathcal{V}(A_n^V)$ or λ_0 is an isolated accumulation point (i.e. $\lambda \in \mathcal{P}$). Since λ_0 is not on any anti-Stokes curve, w.l.o.g. let σ be the unique dominant critical point at λ_0. Let $\epsilon > 0$ be small such that $\overline{B}_\epsilon := \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq \epsilon\}$ is contained in a single component of the anti-Stokes curve arrangement complement and does not intersect any other Stokes curve. There are several cases to consider. If c_σ and $c_{\sigma'}$ are both identically zero on $\overline{B}_\epsilon$, we can disregard σ and σ' from (17) and may assume that all the remaining coefficients are analytic on $\overline{B}_\epsilon$, so the claim follows readily from Theorem 17. If λ_0 is an isolated zero of both c_σ and $c_{\sigma'}$, then λ_0 lies potentially in $\mathcal{P}$ and we disregard this case.

Finally, assume that c_σ is nonzero in the intersection of $\overline{B}_\epsilon$ with a *one-sided* neighbourhood of λ_0, i.e. there exists $D \in \mathcal{S}^c_{\sigma,\sigma'}$ such that c_σ is nonzero on $\overline{B}_\epsilon \cap \overline{D}$. Note that even though c_σ might not be analytic at λ_0 it has well-defined one-sided limits. We now adapt Sokal's argument in his proof of Theorem 17. There exist constants $\delta > 0$ and $M < \infty$ such that for all $\lambda \in \overline{B}_\epsilon \cap \overline{D}$ we have $|\tilde{c}_\sigma(\lambda)| \geq \delta$, $|\tilde{c}_{\sigma'}(\lambda)/\tilde{c}_\sigma(\lambda)| \leq M$ for all $\sigma' \neq \sigma$, $|e^{F_\lambda(\sigma)}| \geq \delta$ and $|e^{F_\lambda(\sigma')}/e^{F_\lambda(\sigma)}| \leq 1 - \delta$ for all $\sigma' \neq \sigma$. Then we have

$$|\tilde{I}_n^V(\lambda)| \geq \delta^{n+1} \left(1 - (m-1)M(1-\delta)^n\right)$$

for all $\lambda \in \overline{B}_\epsilon \cap \overline{D}$. Therefore, $\tilde{I}_n^V(\lambda)$ is non-vanishing on $\overline{B}_\epsilon \cap \overline{D}$ for sufficiently large n and hence so is $I_n^V(\lambda)$. Thus, $\lambda \notin \limsup \mathcal{V}(A_n^V)$, which concludes the proof. ◀

We conclude this section with a remark regarding the practicality of Theorem 14.

▶ Remark 18. Even though Theorem 14 is not explicit about which anti-Stokes curves and what parts of them appear as the limit objects, in practice one can decide this computationally via at least two different approaches.

The first is to determine the Stokes structure in detail, i.e. determining which thimbles actually contribute to the sum (17). The integral I_n^V is a holonomic function. Knowing the Lefschetz thimble decomposition at a single point (for example from the asymptotics for real λ [7]), one can then numerically (with high precision) make an analytic continuation to a complex λ using, for example, the software by Mezzarobba [21].

The second approach is to explicitly compute the zeros of A_n^V for some large n and observe which anti-Stokes curves are in proximity to the observed zeros. If one makes a careful analysis of the constants appearing in the proof of Theorem 14 and in Corollary 10 and one chooses n sufficiently large, this heuristic can be turned into a rigorous proof.

5 Connections to the Ising model

In this last section we establish connections to the Ising model and show how the edge-coloured graph counting framework unifies the theories of Lee–Yang and of Fisher zeros. The main focus of this section is on the polynomial

$$W(x_1, x_2, \mathsf{h}, \mathsf{J}) := \sum_{i=0}^{k} \mathsf{J}^i \mathsf{h}^{i \bmod 2} \frac{x_1^i x_2^{k-i}}{i!(k-i)!}. \tag{18}$$

This polynomial specialises to two important instances:

$$V_{\text{Ising}}^{(1)}(x_1, x_2, \lambda) := W(x_1, x_2, \mathsf{h}, \mathsf{J})\big|_{\mathsf{J}=\sqrt{\lambda}, \mathsf{h}=0}\ , \qquad V_{\text{Ising}}^{(2)}(x_1, x_2, \lambda) := W(x_1, x_2, \mathsf{h}, \mathsf{J})\big|_{\mathsf{J}=1, \mathsf{h}=\lambda}.$$

For $k = 4$, $V_{\text{Ising}}^{(1)}(x_1, x_2, \lambda)$ is the polynomial from Example 5. We argue that for V_{Ising}, the polynomial $A_n^{V_{\text{Ising}}}(\lambda)$ can be interpreted as an *average Ising model partition function* on a k-regular graph. The parameter J is a temperature-like parameter, while h corresponds to an external magnetic field parameter. The main theorem applied to $V_{\text{Ising}}^{(2)}$ recovers a version of the original Lee–Yang theorem which we recall below. The main result applied to $V_{\text{Ising}}^{(1)}$ can be interpreted as a result on the accumulation of Fisher zeros of a random regular graph. We explain these connections in the following.

Let G be a (monocoloured) graph with vertices $V_G = \{1, \ldots, |V_G|\}$. We denote an edge between vertices i and j by a multiset $\{i, j\} \in E_G$. The Ising model [18] on G models the properties of a magnet depending on temperature, local interactions and an external magnetic field. To each vertex i one associates a magnetic spin $\sigma_i \in \{\pm 1\}$. The Hamiltonian H of the model accounts for nearest neighbour interactions, as well as external magnetic field parameters at each vertex,

$$H(\sigma) = -\sum_{\{i,j\} \in E_G} J_{ij} \sigma_i \sigma_j - \sum_{i \in V_G} h_i \sigma_i.$$

Here, $J_{ij} \geq 0$ are ferromagnetic coupling parameters and h_i are fugacities determining the external magnetic fields. Then the canonical *partition function* of the Ising model is

$$Z_G(h_1, \ldots, h_{|V_G|}, \beta; J_{ij}) = \sum_{\sigma \in \{\pm 1\}^{|V_G|}} e^{-\beta H(\sigma)},$$

where $\beta = (k_B T)^{-1}$ is the inverse thermodynamic temperature (k_B is the Boltzmann constant). Before we draw the connection to $A_n^{V_{\text{Ising}}}$, let us recall the classic Lee–Yang theorem [17]. We state it in the form as presented in [5], setting $\beta = 1$.

▶ **Theorem 19** (Lee–Yang). *Let $J_{ij} \geq 0$. The partition function $Z_G(h_1, \ldots, h_{|V_G|}, 1; J_{ij})$ does not vanish whenever $\Re(h_i) > 0$ for all $i = 1, \ldots, |V_G|$. In particular, all zeros of the univariate function $Z_G(h, \ldots, h, 1; J_{ij})$ lie on the imaginary axis.*

We now make the common simplification to assume all fugacities h_i to be equal to h. A convenient way to deal with the fugacity is to introduce *Griffiths' ghost vertex* $\mathfrak{g} \notin V_G$ with a fixed spin $\sigma_{\mathfrak{g}} = +1$ and to define the interaction $J_{i\mathfrak{g}} := h$ [12]. Then the original Ising model can be interpreted as an Ising model without external magnetic field on an augmented graph G' with vertices $V_{G'} = V_G \cup \{\mathfrak{g}\}$ and edges $E_{G'} = E_G \cup \{\{i, \mathfrak{g}\} : i \in V_G\}$.

The connection to edge-coloured graph enumeration becomes apparent in the van der Waerden *high energy expansion*, see e.g. [10, §2.2.1]. It expresses $Z_G(\beta, J)$ as

$$Z_G(h, \beta; J_{ij}) = 2^{|V_G|} \left(\prod_{\{i,j\} \in E'} \cosh(\beta J_{ij}) \right) \sum_{\substack{\gamma' \subseteq G' \\ \gamma' \setminus \mathfrak{g} \text{ Eulerian}}} \prod_{\{i,j\} \in E_{\gamma'}} \tanh(\beta J_{ij}). \tag{19}$$

The sum on the right-hand side of (19) ranges over all "almost-Eulerian" subgraphs of G', by which we mean graphs all of whose vertices except $\mathfrak{g}$ have even degree. Let $\gamma' \subseteq G'$ be such a subgraph and let $v \in V_{\gamma'} \setminus \{\mathfrak{g}\}$ be a vertex. Then either v is incident in γ' to an even number of half-edges that belong to the original graph G, or v is incident in γ' to an odd number of half-edges of G and is connected with $\mathfrak{g}$. Therefore, the sum in (19) can be taken over all subgraphs $\gamma \subseteq G$, where a vertex comes with an additional contribution of $\tanh(\beta h)$ if it is incident to an odd number of half-edges in γ.

Given a k-regular graph G and a subgraph γ of G, we can define an edge-colouring on G with two colours, depending on whether an edge belongs to γ. Moreover, the contributions $\tanh(\beta J_{ij})$ can be interpreted as vertex degree markings. Let us assume all interaction parameters J_{ij} to be equal to a single parameter J for all $i, j \in V_G$. Setting $\mathsf{J} = \sqrt{\tanh(\beta J)}$ and $\mathsf{h} = \tanh(\beta h)$, a vertex that is incident to i half-edges of the colour indicating γ contributes $\mathsf{J}^i \mathsf{h}^{i \bmod 2}$ to the sum in (19). Therefore, Z_G simplifies to

$$Z_G(h, \beta; J) = (2h)^{|V_G|} \cosh(\beta J)^{|E_G|} \sum_{\gamma \subseteq G} \prod_{v \in V_G} \mathsf{J}^{\deg_\gamma(v)} \mathsf{h}^{\deg_\gamma(v) \bmod 2},$$

where $\deg_\gamma(v)$ denotes the number of half-edges in γ incident to v. Note that the prefactors $(2h)^{|V_G|} \cosh(\beta J)^{|E_G|}$ could also be absorbed into vertex degree markings, but we ignore this since it does not change the zeros of A_n^W. With this we obtain the following interpretation of A_n^W as an average Ising model partition function.

▶ **Theorem 20.** *Let $W = W(x_1, x_2, \mathsf{J}, \mathsf{h})$ be as in (18). Then the polynomial A_n^W satisfies*

$$A_n^W(\mathsf{h}, \mathsf{J}) = \sum_{\substack{G \text{ } k\text{-regular} \\ \chi(G) = -n}} \frac{1}{|\operatorname{Aut}(G)|} \sum_{\gamma \subseteq G} \prod_{v \in V_G} \mathsf{J}^{\deg_\gamma(v)} \mathsf{h}^{\deg_\gamma(v) \bmod 2}.$$

For the purpose of the automorphism group, G is considered monocoloured.

The proof of this statement is completed in the full article [25].

To apply our main result, we need to specialise the two-parameter polynomial $A_n^W(\mathsf{h},\mathsf{J})$ to a single parameter. There are two natural ways to do that. The first is as follows: we set the fugacity $\mathsf{h}=0$ to zero and consider zeros in the parameter J. For convenience, we also make the substitution $\mathsf{J}^2=\lambda$ to get the polynomial

$$V^{(1)}_{\text{Ising}}(x_1,x_2,\lambda)=W(x_1,x_2,\mathsf{h},\mathsf{J})\big|_{\mathsf{J}=\sqrt{\lambda},\mathsf{h}=0}.$$

With this we obtain the following expression

$$A_n^{V^{(1)}_{\text{Ising}}}(\lambda)=\sum_{\substack{G\ k\text{-regular}\\ \chi(G)=-n}}\frac{1}{|\operatorname{Aut}(G)|}\sum_{\substack{\gamma\subseteq G\\ \gamma\text{ Eulerian}}}\lambda^{|E_\gamma|}.$$

The zeros of $A_n^{V^{(1)}_{\text{Ising}}}(\lambda)$ can be interpreted as *Fisher zeros* [11] of the Ising model on a *random* k-regular graph. Here, by random we mean a graph drawn according to a probability distribution proportional to $|\operatorname{Aut}(G)|^{-1}$. For $k=4$, this case has served as the running example in this work. Another natural way to specialise W is to set $\mathsf{J}=1$ and to consider zeros in the fugacity parameter $\lambda=\mathsf{h}$, so we set

$$V^{(2)}_{\text{Ising}}(x_1,x_2,\lambda)=W(x_1,x_2,\mathsf{h},\mathsf{J})\big|_{\mathsf{J}=1,\mathsf{h}=\lambda}.$$

This brings us to the set-up of the classical Lee–Yang theorem, except that we are now dealing with the Ising model partition function of a random regular graph in the sense above. Indeed, among the critical points of $V^{(2)}_{\text{Ising}}$ on $\mathbb{S}$ there are, for any $k\geq 3$, the two points $\sigma_1=(\sqrt{2}/2,\sqrt{2}/2)$, $\sigma_2=(\sqrt{2}/2,-\sqrt{2}/2)$. After a straightforward computation, the corresponding anti-Stokes curve is seen to be the imaginary axis,

$$\left\{\lambda\in\mathbb{C}\,:\,|V^{(2)}_{\text{Ising}}(\sigma_1,\lambda)|=|V^{(2)}_{\text{Ising}}(\sigma_2,\lambda)|\right\}=\{\lambda\in\mathbb{C}\,:\,\Re\lambda=0\}\,,$$

in accordance with the classical Lee–Yang theorem.

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